Journal of Mathematical Sciences, Vol. 72, No. 4, 1994

ON THE STRONGLY SPHERICAL SASAKIAN METRIC OF A SPHERICAL TANGENT BUNDLE

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UDC 514

Let M^n denote an n-dimensional Riemannian manifold. Its metric is called ν-strongly spherical if at every point $Q \in M^n$ there exists a ν-dimensional subspace $\mathcal{L}_Q^\nu \subset T_QM^n$ such that the curvature operator of the metric of M^n satisfies $R(X, Y)Z = k(<Y, Z> X < X, Z > Y)$, where $k = const > 0$, $Y \in \mathcal{L}_Q^\nu$, $X, Z \in T_QM^n$. The number ν is called the index of sphericity and k the exponent of sphericity. The following theorems are proved in the paper. **THEOREM 1.** *Let the Sasakian metric of T_1M^n be ν-strongly spherical with exponent of sphericity k. The following assertions hold: a) $\nu = 1$ if and only if M^2 has constant Gaussian curvature $K \neq 1$ and $k = K^2/4$; b) $\nu = 3$ if and only if M^2 has constant curvature $K = 1$ and $k = 1/4$; c) $\nu = 0$, otherwise.* **THEOREM 2.** *Let the Sasakian metric of T_1M^n ($n \geq M^n$) be ν-strongly spherical with exponent of sphericity k. If $k > 1/3$ and $k \neq 1$, then $\nu = 0$. Let us denote by (M^n, K) a space of constant curvature K.* **THEOREM 3.** *Let the Sasakian metric of $T_1(M^n, K)$ ($n \geq 3$) be ν-strongly spherical with exponent of sphericity k. The following assertions hold: a) $\nu = 1$ if and only if $K = 1/4$; b) $\nu = 0$, otherwise. In dimension $n = 3$ Theorem 2 is true for $k \notin \{1/4, 1\}$.*

Introduction. Let M^n denote an n-dimensional Riemannian manifold. Its metric is called ν-strongly spherical if at every point $Q \in M^n$ there exists a ν-dimensional subspace $\mathcal{L}_Q^\nu \subset T_QM^n$ such that the curvature operator of the metric of M^n satisfies

$$R(X, Y)Z = k(<Y, Z>X - <X, Z>Y), \quad k = const > 0, \tag{1}$$

for every $Y \in \mathcal{L}_Q^\nu$ and any $X, Z \in T_QM^n$. The number ν is called the index of sphericity and k the exponent of sphericity.

For $k = 0$ the metric is called ν-strongly parabolic.

As is known [1], if ν is constant on M^n (and we consider only this case), then subspaces $\mathcal{L}^\nu$ form an integrable distribution on M^n and its integral submanifolds are ν-dimensional, totally geodesic submanifolds in M^n of constant sectional curvature k.

Riemannian manifolds whose tangent bundle with Sasakian metric have a constant index of strong parabolicity were considered in [2]. It was proved that if ν is the index of strong parabolicity of the Sasakian metric of TM^n, then ν is even and metrically $M^n = M_1^{n-\nu/2} \times E^{\nu/2}$ and $TM^n = TM_1^{n-\nu/2} \times E^\nu$.

Borisenko conjectured that an analogous result does not hold for T_1M^n ($n \geq 3$). That is, a strongly spherical distribution on T_1M^n, for $n \geq 3$, is trivial. We prove this conjecture under certain constraints on the exponent of sphericity.

Thus, the goal of this paper in to describe manifolds the Sasakian metric of whose spherical tangent bundle has constant index of sphericity ν.

In dimension $n = 2$ it is known that if M^n is the standard sphere S^2, then T_1S^2 has constant sectional curvature 1/4 [3]. This means that in the sense of definition (1) $\nu = 3$ and $k = 1/4$. In this paper we prove

THEOREM 1. Suppose that the Sasakian metric of T_1M^n is ν-strongly spherical with exponent of sphericity k. The following assertions hold:

Translated from Ukrainskii Geometricheskii Sbornik, No. 35, pp. 150-159, 1992. Original article submitted November 22, 1990.

1072-3374/94/7204-3261$12.50

a) $\nu = 1$ if and only if M^2 has constant Gaussian curvature $K \neq 1$ and $k = K^2/4$;

b) $\nu = 3$ if and only if M^2 has constant curvature $K = 1$ and $k = 1/4$;

c) $\nu = 0$, otherwise.

THEOREM 2. Suppose that the Sasakian metric of T_1M^n ($n \geq 3$) is ν-strongly spherical with exponent of sphericity k. If $k > 1/3$ and $k \neq 1$, then $\nu = 0$.

REMARK. In dimension $n = 3$ Theorem 2 is true assuming that $k \notin \{1/4, 1\}$.

Let us denote by (M^n, K) a space of constant curvature K.

THEOREM 3. Suppose that the Sasakian metric of $T_1(M^n, K)$ ($n \geq 3$) is ν-strongly spherical with exponent of sphericity k. The following assertions hold:

a) $\nu = 1$ if and only if $K = 1$, $k = 1/4$;

b) $\nu = 0$, otherwise.

1. Preliminary Information and Results.

Let (M^n, g) be a Riemannian manifold. The linear element of the Sasakian metric of TM^n has form $d\tau^2 = g_{ik}dq_idq^k + g_{ik}D\xi^i d\xi^k$, where $(q^1, \ldots, q^n)$ are the local coordinates on M^n, ξ^i are the coordinates of the tangent vectors in natural basis $(\partial/\partial q_i)$, and $d\xi^i = d\xi^i + \Gamma_{jk}^i\xi^j dq^k$. Introducing the notation $dq = dq^i\partial/\partial q^i$, $\xi = \xi^i\partial/\partial q^i$, we can rewrite the formula for the linear element of the Sasakian metric as

$$d\tau^2 = \langle dq, dq\rangle + \langle D\xi, D\xi\rangle, \tag{2}$$

where $\langle\cdot, \cdot\rangle$ denotes scalar vector product in metric g.

Let $Q = (q^1, \ldots, q^n)$, $\xi = \{\xi^1, \ldots, \xi^n\}$ be an arbitrary point of TM^n. An arbitrary tangent vector $\tilde{X}$ to TM^n at (Q, ξ) has the form $\tilde{X} = \tilde{X}^i\partial/\partial q^i + \tilde{X}^{n+i}\partial/\partial\xi^i$. Two mappings $\Pi_*: T_{(Q,\xi)}TM^n \to T_QM^n$ and $K: T_{(Q,\xi)}TM^n \to T_QM^c$ acting in local coordinates in the following way are defined [4]: $\Pi_*\tilde{X} = \tilde{X}^i\partial/\partial q^i$, $K\tilde{X} = (\tilde{X}^{n+i} + \Gamma_{jk}^i\tilde{X}^j\xi^k)\partial/\partial q^i$. The decomposition $T_{(Q,\xi)}TM^n = H_{(Q,\xi)}TM^n \oplus V(Q, \xi)TM^n$, where $H_{(Q,\xi)} = \mathrm{Ker}K$, holds with respect to these mappings.

Accordingly, any tangent vector $\tilde{X}$ to TM^n is represented as $\tilde{X} = X^H + U^V$, where $X, U \in T_QM^n$ and $X^H \in H_{(Q,\xi)}TM^n$, $U^V \in V_{(Q,\xi)}TM^n$, moreover, in local coordinates $X^H = X^i\partial/\partial q^i - \Gamma_{jk}^iX^j\xi^k\partial/\partial\xi^i$.

If we denote by $\langle\langle,\rangle\rangle$ the scalar product of vectors tangent to TM^n in Sasakian metric, the we can easily obtain from (2)

$$\langle\langle\tilde{X}, \tilde{Y}\rangle\rangle = \langle\Pi_*\tilde{X}, \Pi_*\tilde{Y}\rangle + \langle K\tilde{X}, K\tilde{Y}\rangle \tag{3}$$

$H_{(Q,\xi)}$ and $V_{(Q,\xi)}$ are called, respectively, a horizontal and a vertical subspace at (Q, ξ) and since $X^H \in H_{(Q,\xi)}$, $U^V = V_{(Q,\xi)}$, X^H is called the horizontal lift of vector X and U^V the vertical lift of vector U at (Q, ξ). Vertical vectors are tangent to the fiber while horizontal vectors are orthogonal to it.

The tangent bundle of the unit vectors of T_1M^n is subbundle TM^n defined by $\langle\xi, \xi\rangle = 1$. The metric on T_1M^n is defined as a metric of a hypersurface in TM^n.

The unit normal to T_1M^n at every $(Q, \xi) \in T_1M^n$ is the vector ξ^V.

Thus, $\tilde{X} \in T_{(Q,\xi)}TM^n$ is tangent to T_1M^n at the same point if and only if $\langle K\tilde{X}, \xi\rangle = 0$. The converse is also true: any vector $\tilde{X}$ of the form $\tilde{X} = X^H + U^V$, where $X \in T_QM^n$ and $u \in L_Q^\perp(\xi)$ is orthogonal to the complement to ξ in T_QM^n, belongs to $T_{(Q,\xi)}T_1M^n$.

In what follows $X, Y, Z\ldots$ denote vectors from T_QM^n while $u, v, x, y\ldots$ denote vectors from $L_Q^\perp(\xi)$.

LEMMA 1 [5]. At every $(Q, \xi) \in T_1M^n$ the curvature tensor $\tilde{R}$ of the Sasakian metric of T_1M^n is determined by

$$\langle\langle\tilde{R}(X^H, Y^H)Z^H, U^H\rangle\rangle = \langle R(X,Y)Z, U\rangle + \frac{1}{4}\langle R(X,U)\xi, R(Z,Y)\xi\rangle +$$
$$+\frac{1}{4}\langle R(X,Z)\xi, R(Y,U)\xi\rangle + \frac{1}{2}\langle R(X,Y)\xi, R(Z,U)\xi\rangle,$$
$$\langle\langle\tilde{R}(X^H, y^H)Z^H, u^V\rangle\rangle = \frac{1}{2}\langle(\nabla_Z R)(X,Y)\xi, u\rangle,$$
$$\langle\langle\tilde{R}(X^H, y^V)Z^H, u^V\rangle\rangle = \frac{1}{2}\langle R(X,Z)y, u\rangle - \frac{1}{4}\langle R(\xi,y)Z, R(\xi,u)X\rangle,$$
$$\langle\langle\tilde{R}(X^H, Y^H)z^V, u^V\rangle\rangle = \langle R(X,Y)z, u\rangle + \frac{1}{4}\langle R(\xi,z)X, R(\xi,u)Y\rangle -$$

$$-\frac{1}{4}\langle R(\xi,z)Y,R(\xi,u)X\rangle,$$
$$\langle \tilde{R}(x^V,y^V)z^V,U^H\rangle=0,$$
$$\langle \tilde{R}(x^V,y^V)z^V,u^V\rangle=\langle y,z\rangle\langle x,u\rangle-\langle x,z\rangle\langle y,u\rangle,$$

where R is the curvature tensor of M^n at Q.

Using the result of Lemma 1, it is easy to obtain the necessary and sufficient conditions for strong sphericity of the Sasakian metric of T_1M^n.

Let $\tilde{\mathfrak{L}}^\nu$ be a strongly spherical distribution on T_1M^n. Then at an arbitrary point (Q, ξ) the vector $\tilde{Y} \in \mathfrak{L}^\nu_{(Q,\xi)}$ can be represented as $\tilde{Y} = h^H + V^V$ assuming that $h \in \Pi_*\mathfrak{L}^\nu_{(Q,\xi)}$, $\nu \in K\mathfrak{L}^\nu_{(Q,\xi)}$, $\langle \nu, \xi \rangle = 0$.

LEMMA 2. For the Sasakian metric of T_1M^n to be ν-strongly spherical it is necessary and sufficient that at every $(Q, \xi) \in T_1M^n$ for every pair of vectors h,ν from $\Pi_*\mathfrak{L}^\nu$ and $K\mathfrak{L}^\nu$, respectively, and any $X, Y, Z \in T_QM^n$; $u, w, x \in L_Q^\perp(\xi)$

$$1)\ \langle R(X,Y)h,Z\rangle-\frac{1}{4}\langle R(Y,h)\xi,R(X,Z)\xi\rangle+\frac{1}{4}\langle R(X,h)\xi,R(Y,Z)\xi\rangle+$$
$$+\frac{1}{2}\langle R(X,Y)\xi,R(h,Z)\xi\rangle-\frac{1}{2}\langle(\nabla_Z R)(X,Y)\xi,\nu\rangle-k(\langle Y,h\rangle\langle X,Z\rangle-\langle X,h\rangle\langle Y,Z\rangle)=0,$$
$$2)\ \frac{1}{2}\langle R(\nu,u)X,Z\rangle-\frac{1}{4}\langle R(\xi,\nu)Z,R(\xi,u)X\rangle+\frac{1}{2}\langle(\nabla_Z R)(\xi,u)X,h\rangle+$$
$$+k\langle u,\nu\rangle\langle X,Z\rangle=0,$$
$$3)\ \frac{1}{2}\langle R(\nu,u)h,Z\rangle-\frac{1}{4}\langle R(\xi,\nu)Z,R(\xi,u)h\rangle+k\langle\nu,u\rangle\langle h,Z\rangle=0,$$
$$4)\ (1-k)(\langle u,\nu\rangle\langle w,x\rangle-\langle u,x\rangle\langle w,\nu\rangle)=0.$$

The proof is obtained by applying the results of Lemma 1 to (1) for various combinations of lifts. Because the proof is simple yet lengthy, we omit its details.

In dimension $n = 2$ the curvature tensor of the Sasakian metric of T_1M^n can be written out especially simply in a special coordinate system on M^n. Let (l_1, l_2) be mutually orthogonal vectors in T_QM^2. We superpose l_2 with ξ and connect to (l_1, l_2) a normal Riemannian coordinate system in a neighborhood of Q on M^2. Let us call such a coordinate system ξ-special.

LEMMA 3 [6]. In a ξ-special coordinate system the curvature tensor of the Sasakian metric of T_1M^2 has the form $\tilde{R}_{1212} = K(1-(3/4)K)$, $\tilde{R}_{1213} = (1/2)\ \partial K/\partial q^1$, $\tilde{R}_{1223} = (1/2)\ \partial K/\partial q^2$, $\tilde{R}_{1313} = K^2/4$, $\tilde{R}_{1323} = 0$, $\tilde{R}_{2323} = K^2/4$ where $K(q^1, q^2)$ is the Gaussian curvature of M^2.

Note that an analgous ξ-special coordinate system can also be chosen on M^n $(n \geq 3)$.

2. Proof of the Main Results.

Proof of Theorem 1. Having chosen on M^2 a ξ-special coordinate system, we can rewrite the conditions of strong sphericity (1) as the following system of equations: $\tilde{R}_{IJLS}\tilde{y}^S = k(\delta_{IL}\delta_{JS} = \delta_{IS}\delta_{JL})\tilde{y}^S$, $I, J, L, S = 1, 2, 3$.

In this system, in view of the special choice of the coordinate system, $\tilde{y}^1 = h^1$, $\tilde{y}^2 = h^2$, $\tilde{y}^3 = \nu^1$ while I, J, L are arbitrary. Using Lemma 3, we write out this system for some combinations of indices (I, J, L):

$$(1,2,1):\ K(1-\frac{3}{4}K)h^2+\frac{1}{2}\frac{\partial K}{\partial q^1}\nu^1=kh^2;$$
$$(1,2,2):\ -K(1-\frac{3}{4}K)h^1+\frac{1}{2}\frac{\partial K}{\partial q^2}\nu^1=-kh^1;$$
$$(1,3,1):\ \frac{1}{2}\frac{\partial K}{\partial q^1}h^1+\frac{K^2}{4}\nu^1=k\nu^2;$$
$$(1,3,3):\ -\frac{K^2}{4}h^1=-kh^1;$$
$$(2,3,3):\ -\frac{K^2}{4}h^2=-kh^2.$$

(1,3,3) and (2,3,3) imply that either i) $K^2/4 = k$ or ii) $h^1 = 0$, $h^2 = 0$. By virtue of the fact that k = const and the point on M^2 was chosen arbitrarily, i) means that $K = 2\sqrt{k}$ = const and, consequently, $\partial K/\partial q^1 = 0$, $\partial K/\partial q^2 = 0$.

Letting $k = K^2/4$ in the system, we reduce it to

$$(1,2,1):\ K(1-K)h^2=0;$$

$$(1,2,2):\ K(1-K)h^1=0.$$

Clearly, for $K = 0$ and $K = 1$ the system is identically satisfied. We discard $K = 0$ since by hypothesis $k > 0$. A so, for $K = 1$, $k = 1/4$ the index of strong sphericity is $\nu = 3$. Assertion b) is proved. If, however, $K \neq 1$, then $\nu =$ Thus, $h^1 = h^2 = 0$ turns out to be one-dimensional, $\nu = 1$, vertical, and the sphericity exponent is $k = K^2/4$. In case (1,2,1) and (1,2,2) imply that if $K = (q^1, q^2) \neq$ const, then $\nu^1 = 0$ and, consequently, $\mathfrak{Z}$ is a zero distribution.

If, however, $K(q^1, q^2) = K =$ const, then (1,3,1) implies $k = K^2/4$. Thus, $\nu = 1$ and $k = K^2/4$. Theorem 1 proved.

Proof of Theorem 2.

Since by hypothesis $k \neq 1$, letting $u = v$, $w = x$ in Equation 4) of Lemma 2 and choosing x orthogonal to $v (n \geq$ we obtain $(1-K)|v|^2|x|^2 = 0$. By virtue of the arbitrariness of x, the latter equality implies $v = 0$.

Thus, if the desired distribution exists, then it is horizontal.

Let $\mathfrak{Z}^\nu$ be the desired horizontal strongly-spherical distribution on T_1M^n.

LEMMA 4. If at every $\tilde{Q} = (Q, \xi)$ subspace $\mathfrak{Z}^\nu$ contains ξ^H, then M^n has constant sectional curvature equal to or and $k = 1/4$.

Proof. Let us consider at $Q \in M^n$ the linear operator $R(\cdot, \xi)\xi$: $L_Q^\perp(\xi) \to L_Q^\perp(\xi)$. It is symmetric, therefore ther exists an orthonormal basis $\{e_1, \ldots, e_{n-1}\}$ at Q consisting of the eigenvectors of this operator. Clearly, its eigenvalues K_α ($= 1, \ldots, n-1$) are the sectional curvatures of M^n in the direction of area elements $(\xi \wedge e_\alpha)$. Thus,

$$R(e_\alpha, \xi)\xi = K_\alpha e_\alpha, \quad \alpha = 1, \ldots, n-1. \tag{4}$$

Let us consider at Q the orthogonal coordinate system $\{e_1, \ldots, e_{n-1}, \xi\}$.

Let h be an arbitrary vector of $\Pi_* \mathfrak{Z}$.

Assume that $z = h$, $w = e_\alpha$, $u = e_\beta$ in Equation 3) of Lemma 2. Then we obtain that $1/4 < R(e_\alpha, \xi)h, R(e_\beta, \xi)h > = k\delta_{\alpha_\beta}$.

Consequently, vectors $f_\alpha = 1/2\sqrt{k}\, R(e_\alpha, \xi)h$ are orthonormal and $<f_\alpha, h> = 0$.

Consider a new orthonormal basis $\{f_1, \ldots, f_{n-1}, h\}$ at Q. Since $\{\{e_\alpha\}, \xi\}$ and $\{\{f_\alpha\}, h\}$ are orthonormal, the transition matrix from one basis to the other is orthogonal. It is easy to see that its elements have the form $a_{\alpha_\beta} = (1/2\sqrt{k}) < R(e_\alpha, \xi)h, e_\beta >$, $a_{\alpha n} = (1/2\sqrt{k}) < R(e_\alpha, \xi)h, \xi >$, $a_{n_\beta} = h^\beta$, $a_{nn} = h^n$.

Because $\begin{bmatrix} a_{\alpha_\beta} & a_{\alpha n} \\ a_{n_\beta} & a_{nn} \end{bmatrix}$ $(\alpha, \beta = 1, \ldots, n-1)$ is orthogonal, we find $\sum_{\alpha=1}^{n-1} a_{\alpha_\beta}{}^2 + a_{\alpha\beta}{}^2 = 1$ or $\sum_{\alpha=1}^{n-1} (1/4k) < R(e_\alpha, \xi)\, h, e_\beta >^2 = 1-(h^\beta)^2$. Taking into account the symmetries of the curvature tensor, we obtain $1/4k \sum_{\alpha=1}^{n-1} < r(e_\beta, h)\xi, e_\alpha >^2 = 1-(h^\beta)^2$. The latter equality, clearly, means that at Q

$$\frac{1}{4k}|R(e_\beta, h)\xi|^2 = 1-(h^\beta)^2, \quad \beta = 1, \ldots, n-1. \tag{5}$$

On the other hand, if we let $z = x = e_\beta$, $y = h$ in Equation 1) of Lemma 2, then we obtain

$$<R(e_\beta, h)h, e_\beta> = -\frac{3}{4}|R(e_\beta, h)\xi|^2 = k(1-(h^\beta)^2). \tag{6}$$

We multiply (5) by $3k$ and add it to (6). We obtain $<R(e_\beta, h)h, e_\beta> = 4k(1-(h^\beta)^2)$.

The last equality means that $K_h \wedge e_\beta = 4k =$ const.

Since h is an arbitrary vector from $\Pi_* \mathfrak{Z}^\nu$ and by hypothesis $\mathfrak{Z}^\nu e \xi^H$, letting $h = \xi$ we find that

$$K_{\xi \wedge e_\beta} = 4k. \tag{7}$$

Suppose now that u is an arbitrary unit vector from $L_Q{}^{\perp}(\xi)$. Then for $K_{\xi} \wedge_u$, taking into account (4) and the orthonormality of $\{e_\alpha\}$, we obtain $K_{\xi} \wedge_u = <R(\xi, u)u, \xi> = <R(\xi, e_\alpha)e_\beta, \xi> u^\alpha u^\beta = \sum_{\alpha=1}^{n-1} K_\alpha(u^\alpha)^2$. Bearing (7) in mind, we obtain $K_\xi \wedge_u = 4k$.

By virtue of the arbitrariness of the choice of ξ, the last inequality means that M^n has constant sectional curvature equal to $4k$.

On the other hand, letting $h = \xi$, $z = \xi$, $u = w = e_\beta$ in Equation 3) of Lemma 2, we obtain $1/4\ |R(\xi, e_\beta)\xi|^2 = k$. Taking the choice of the basis into account, we find that $1/4K_\beta{}^2 = k$. In comparison with (7) it means that $k = 1/4$.

Consequently, the sectional curvature of M^n is equal to one. The lemma is proved.

LEMMA 5. If $k > 1/3$, then $\tilde{\mathfrak{F}}^\nu \ni \xi^H$ and, consequently, $\nu = 0$.

Proof. Let h be an arbitrary vector from $\Pi_* \tilde{\mathfrak{F}}_{\tilde{Q}}{}^\nu$, $\tilde{Q} = (Q, \xi)$.

Let us introduce in a neighborhood of Q the coordinate system described in Lemma 4. Letting $Y = e_\alpha$, $X = Z = \xi$ in Equation 1) of Lemma 2, we obtain $<R(e_\alpha, \xi)\xi, h> -3/4<R(h, \xi)\xi, R(e_\alpha, \xi)\xi> -kh^\alpha = 0$. Taking (4) into account, we arrive at

$$(K_\alpha - \frac{3}{4} K_\alpha^2 - k)h^\alpha = 0 \quad (\alpha = 1, \ldots, n-1). \tag{8}$$

If none of the K_α is a root of

$$t - \frac{3}{4}t^2 - k = 0, \tag{9}$$

then $h^\alpha = 0$, $\alpha = 1, \ldots, n-1$. Since $k > 1/3$, (9) has no solutions. Consequently, $h = \xi$. According to Lemma 4, however, in that case $h = \xi$. We have a contradiction. Hence, $h = 0$ and the distribution is trivial. The Lemma is proved.

The application of Lemmas 4 and 5 proves Theorem 2.

Proof of Theorem 3. Let (M^n, K) $(n \geq 3)$ be a manifold of constant curvature K. Then $R(X, Y)Z = K(<Y, Z>X - <X, Z>Y)$ for any vectors tangent to M^n.

Formulas 4) of Lemma 2 imply that if $k \neq 1$, then $\nu = 0$.

If $k = 1$, then the constancy of the curvature of M^n and, for $X = Z(|Z| = 1)$, $u = v$ Equation 2) of Lemma 2 imply $1/4|R(\xi, v)Z|^2 - |v|^2 = 0$.

Choosing Z to be orthogonal at the same time to both ξ and v, we obtain that $v = 0$.

Thus, $\tilde{\mathfrak{F}}^\nu$ is horizontal. We let $Z = w$ in 3) of Lemma 2 and choose w from the orthogonal complement to the arbitrary vector u. Using the constancy of the curvature, we arrive at $K/2(1-K/2) < u, h > = 0$.

It is can be trivially verified that $K = 0$ implies $k = 0$, which is inadmissible. If $K = 2$, then letting $Z = u$ in 3) of Lemma 2 and assuming w to be orthogonal to u, we obtain $<h, w> = 0$. If $K \neq 0$ and $K \neq 2$, then $<h, u> = 0$ for any u orthogonal to ξ. Consequently, $h = \xi$ and we can apply Lemma 4.

Thus, $\tilde{\mathfrak{F}}^\nu$ is not trivial if $\tilde{\mathfrak{F}}^\nu = \xi^H$, $\nu = 1$, $k = 1/4$, $K = 1$.

It is easy to verify that the converse is also true.

The theorem is proved.

Proof of the Remark. In a neighborhood of Q on M^n we consider the coordinate system described in Lemma 4. Let $K_1 = K_\xi \wedge_{e_1}$, $K_2 = K_\xi \wedge_{e_2}$ be the solutions of (9). Then $K_1 = 2/3(1 + \sqrt{1-3k})$, $K_2 = 2/3\ (1-\sqrt{1-3k})$, where $0 < k \leq 1/3$. Letting $x = z = e_1$, $y = \xi$, and then $x = z = e_2$, $y = \xi$ in 1) of Lemma 2, we obtain $R_{1213}(1-3/4\ K_1)h^2 = 0$, $R_{1223}(1-3/4\ K_2)h^1 = 0$.

The case of $h^1 = h^2 = 0$ is considered in Lemma 4 and leads to $k = 1/4$, which contradicts the hypothesis.

Suppose that $R_{1213} = R_{1223} = 0$. Then letting first $u = w = e_1$, $z = e_2$ and then $u = w = e_2$, $z = e$ in 3) of Lemma 2, we obtain $kh^2 = 0$, $kh^1 = 0$, which brings us back under the conditions of Lemma 4. Finally, if $R_{1213} = 0$ and $R_{1223} \neq 0$, then $h^2 = 0$. Then letting $x = z = e_2$, $y = \xi$ in 1) of Lemma 2, we obtain $R_{1223}(1-3/4\ K_2)h^1 = 0$. Consequently, $h^1 = 0$ and we again find ourselves in a situation where Lemma 4 is applicable. The case of $R_{1223} = 0$, $R_{1213} \neq 0$ is considered analogously.

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