THE CURVATURE OF THE SASAKI METRIC OF TANGENT SPHERE BUNDLES

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We study the sectional curvatures K of the Sasaki metric of tangent sphere bundles over spaces of constant curvature $K(T_1(M^n, K))$. We give precise bounds on the variation of the Ricci curvature and a bound on the scalar curvature of $T_1(M^n, K)$ that is uniform on K. In an appendix we calculate and give lower bounds for the lengths of closed geodesics on $T_1 S^n$. Bibliography: 10 titles.

Let M^n be an n-dimensional Riemannian manifold. Under its natural topology the set of all vectors tangent to M^n forms the tangent bundle TM^n with base M^n, fiber E^n, projection π and structure group $GL(n)$.

If we consider tangent vectors only of unit length (resp. of length $\lambda > 0$), we obtain a subbundle T_1M^n (resp. $T_\lambda M^n$) of the tangent bundle TM^n with base M^n, fiber S^{n-1} (resp. S_λ^{n-1}) projection π_1 and structure group $SO(n)$, which is called the *tangent sphere bundle* over M^n. The fiber over $x \in M^n$ is denoted M_x^n.

In 1958 Sasaki [1] constructed a natural Riemannian metric on TM^n and T_1M^n, thereby founding the metric study of tangent sphere bundles of Riemannian manifolds as independent objects. To be specific, if (x^i) are local coordinates in a neighborhood $U \subset M^n$, then the element of length on TM^n is defined by the equality

$$d\sigma^2 = g_{ij}\,dx^i\,dx^j + g_{ij}\,Dv^i\,Dv^j; \quad i,j = 1,\dots,n, \tag{1}$$

where g_{ij} are the components of the metric tensor of M^n and $Dv^i = dv^i + \Gamma^i_{jk}v^j\,dx^k$ are the covariant differentials of the coordinates of the tangent vector in the natural basis $\left(\dfrac{\partial}{\partial x^i}\right)$.

The coordinates (x^i, v^i) are local coordinates in TM^n. As Dombrowski [2] has shown, at each point $Z \in TM^n$ the direct-sum decomposition $TM^n_Z = HTM^n_Z \oplus VTM^n_Z$, where HTM^n_Z and VTM^n_Z are mutually orthogonal subspaces of dimension n, called *horizontal* and *vertical* respectively. The vertical subspace is tangent to the fiber.

To each vector field X on M^n there corresponds a unique pair of vector fields X^h and X^v on TM^n, one of which is horizontal and the other vertical [2]. They are called the *horizontal lift* and the *vertical lift* respectively.

Over T_1M^n there exists an adaptive frame bundle consisting of the frames $\{e_1,\dots,e_{n-1}; e_n; f_1,\dots,f_{n-1}\}$ such that $e_1,\dots,e_n$ are horizontal and $f_1,\dots,f_{n-1}$ are vertical; moreover e_n is the horizontal lift of a given (unit) vector $Z \in M^n_x$, $x = \pi_1(Z)$, and $d\pi_1 e_i = Kf_i$ $(i = 1,\dots,n-1)$, where K is a connection mapping [3, 4]. At the given point $Z \in T_1M^n$ we have the decomposition $T_1M^n_Z = \tilde{H}T_1M^n_Z \oplus L_Z \oplus VT_1M^n_Z$, where $\tilde{H}T_1M^n_Z$ is the horizontal $(n-1)$-subspace with basis $\{e_1,\dots,e_{n-1}\}$; L_Z is the one-dimensional horizontal subspace $\{e_n\}$; and $VT_1M^n_Z$ is the vertical $(n-1)$-dimensional subspace $\{f_1,\dots,f_{n-1}\}$.

If we set $X^i = x^i$ and $X^{n+i} = v^i$, the metric (1) can be represented in the form [1]: $d\sigma^2 = \bar{G}_{IJ}\,dX^I\,dX^J$, $I,J = 1,\dots,2n$, where

$$\begin{gathered}\bar{G}_{ij} = g_{ij} + g_{kl}\Gamma^k_{is}\Gamma^l_{jt}v^s v^t,\\ \bar{G}_{in+j} = \Gamma_{is,j}v^s, \quad \bar{G}_{n+in+j} = g_{ij}, \quad i,j = 1,\dots,n.\end{gathered} \tag{2}$$

T_1M^n is defined as a subbundle of TM^n by the condition that the tangent vector be of unit length:

$$g_{ij}v^i v^j = 1. \tag{3}$$

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The metric induced by the metric (2) and the imbedding (3) is called the *Sasaki metric* of T_1M^n.

We denote the tangent sphere bundle over a manifold M^n of constant curvature K by $T_1(M^n, K)$. The purpose of the present paper is to obtain bounds on the variation of the sectional curvature of the Sasaki metric $T_1(M^n, K)$. We shall prove

THEOREM 1. *The sectional curvature $\tilde{K}$ of the Sasaki metric of the manifold $T_1(M^n, K)$ is nonnegative if and only if $0 \leqslant K \leqslant 4/3$.*

THEOREM 2. *The sectional curvature K of the Sasaki metric of T_1S^n lies within the limits $0 \leqslant \tilde{K} \leqslant 5/4$.*

THEOREM 3. a) *The Ricci curvature $\widetilde{Ric}$ of the Sasaki metric of the manifold $T_1(M^n, K)$ lies within the limits*

$$\begin{aligned} (i)\quad n = 2:\quad & K^2/2 \leqslant \widetilde{Ric} \leqslant K(2-K)/2 \quad \text{for} \quad 0 < K \leqslant 1;\\ & K(2-K) \leqslant \widetilde{Ric} \leqslant K^2/2 \quad \text{for} \quad K \leqslant 0 \quad \text{or} \quad K > 1;\\ (ii)\quad n \geqslant 3:\quad & K(2-K)(n-1)/2 \leqslant \widetilde{Ric} \leqslant K(2(n-1)-K)/2 \quad \text{for} \quad 1 < K \leqslant n-2;\\ & K(2-K)(n-1)/2 \leqslant \widetilde{Ric} \leqslant (K^2 + 2(n-2))/2 \quad \text{for} \quad K \leqslant 1 \quad \text{or} \quad K > n-2. \end{aligned}$$

b) *The scalar curvature $\tilde{R}$ of the Sasaki metric of the manifold $T_1(M^n, K)$ satisfies the inequality $\tilde{R} \leqslant (n-1)(n^2+2n-4)/2$. Equality is attained when $K = n$.*

In the Appendix we give lower bounds for the lengths of closed geodesics of the Sasaki metric of T_1S^n.

1. The Sasaki metric of $T_1(M^n, K)$ and its curvature tensor. Without loss of generality we may assume that $g_{nn} \neq 0$. Then (3) defines a differentiable imbedding $T_1M^n \to TM^n$ given locally by the function

$$v^n = v^n(x^1, \dots, x^n; v^1, \dots, v^{n-1}), \tag{4}$$

and $(x^1, \dots, x^n, v^1, \dots, v^{n-1})$ is a local coordinate system on T_1M^n, which we shall call the *natural* system.

LEMMA 1. *In natural coordinates the metric tensor of T_1M^n has the form:*

$$\begin{aligned} G_{ik} &= g_{ik} + g_{je}\Gamma^j_{is}\Gamma^l_{kt}v^tv^s + \Gamma_{ti,n}v^tA_k + \Gamma_{tk,n}v^tA_i + g_{nn}A_iA_k,\\ G_{in+p} &= \Gamma_{ti,p}v^t + \Gamma_{ti,n}v^tB_p + g_{np}A_i + g_{nn}A_iB_p,\\ G_{n+qn+p} &= g_{qp} + g_{np}B_q + g_{nq}B_p + g_{nn}B_pB_q, \end{aligned} \tag{5}$$

where $B_p = -g_{ip}v^i/g_{in}v^n$ and $A_k = -\dfrac{1}{2g_{in}v^i}\dfrac{\partial g_{st}}{\partial x^k}v^tv^s$ $(i, k, l, s, t = 1, \dots, n; p, q = 1, \dots, n-1)$.

PROOF: Differentiating (3), we obtain $d(g_{ij}v^iv^j) = 0$, i.e.,

$$\left(\frac{\partial g_{ij}}{\partial x^k}v^iv^j + 2g_{in}v^i\frac{\partial v^n}{\partial x^k}\right)dx^k + 2\left(g_{ip}v^i + g_{in}v^i\frac{\partial v^n}{\partial v^p}\right)dv^p = 0.$$

We set

$$A_k = \frac{\partial v^n}{\partial x^k} = -\frac{1}{2g_{in}v^i}\frac{\partial g_{st}}{\partial x^k}v^sv^t, \tag{6}$$

$$B_p = \frac{\partial v^n}{\partial v^p} = -\frac{g_{ip}v^i}{g_{in}v^i}. \tag{7}$$

By definition the metric of T_1M^n is induced by the metric (3). Having in mind (4), (6), and (7), it is easy to verify that

$$\begin{aligned} G_{ik} &= \bar{G}_{ik} + \bar{G}_{2nk}A_i + \bar{G}_{i2n}A_k + \bar{G}_{2n2n}A_iA_k,\\ G_{in+p} &= \bar{G}_{in+p} + \bar{G}_{i2n}B_p + \bar{G}_{2nn+p}A_i + \bar{G}_{2n2n}A_iB_p,\\ G_{n+qn+p} &= \bar{G}_{n+qn+p} + \bar{G}_{2nn+p}B_q + \bar{G}_{n+q2n}B_p + \bar{G}_{2n2n}B_pB_q. \end{aligned}$$

Substituting (2) here, we obtain (5).

The curvature tensor of the metric (2) was computed in [5]. We shall compute the curvature tensor of the metric (5). We agree that $i,j,k,l,\alpha = 1,\dots,n$ and $p,q,r,t = 1,\dots,n-1$. By the symmetry of the fiber the value of the curvature tensor of T_1M^n is independent of the "fiber" coordinates of a point. Therefore without loss of generality we shall calculate the curvature tensor at the point $(x,0) \in T_1M^n$. As local coordinates of the base we choose the Fermi coordinates along a geodesic x^n. Then at the point $x \in x^n$ we shall have

$$\begin{gathered} g_{ij} = \delta_{ij}, \quad \frac{\partial g_{ij}}{\partial x^k} = 0, \quad -\frac{1}{2}\frac{\partial^2 g_{nn}}{\partial x^i \partial x^k} = R_{nink} \\ \Gamma^i_{jk} = \Gamma_{jk,i} = 0, \quad \frac{\partial \Gamma_{ni,n}}{\partial x^k} = -R_{nink}, \end{gathered} \tag{8}$$

where R is the curvature tensor of M^n at the point x and δ_{ij} is the Kronecker symbol. From condition (8) and equality (3) it follows that

$$v^n = 1 \tag{9}$$

at the point $(x,0)$. From (6) and (7), taking account of (8), it is easy to obtain at the point $(x,0)$ the relations

$$\begin{gathered} A_k = 0, \quad B_p = 0, \\ \frac{\partial A_k}{\partial x^j} = R_{nknj}, \quad \frac{\partial B_p}{\partial x^j} = 0, \\ \frac{\partial A_k}{\partial v^q} = 0, \quad \frac{\partial B_p}{\partial v^q} = -\delta_{pq}. \end{gathered} \tag{10}$$

Finally, substituting (8) in (5), we obtain

$$G_{ij} = \delta_{ij}, \quad G_{in+p} = 0, \quad G_{n+qn+p} = \delta_{pq}. \tag{11}$$

In what is to follow, we shall need the first and second partial derivatives of the metric (5) at the point $(x,0)$. Taking account of (8)–(10), it is easy to verify that

$$\begin{gathered} \frac{\partial G_{ik}}{\partial x^j} = 0, \quad \frac{\partial G_{ik}}{\partial v^p} = 0, \\ \frac{\partial G_{in+p}}{\partial x^j} = \frac{\partial \Gamma_{ni,p}}{\partial x^j}, \quad \frac{\partial G_{in+p}}{\partial v^q} = 0, \\ \frac{\partial G_{n+qn+p}}{\partial x^j} = 0, \quad \frac{\partial G_{n+rn+p}}{\partial v^q} = 0, \end{gathered} \tag{12}$$

and

$$\begin{gathered} \frac{\partial^2 G_{ik}}{\partial x^j \partial x^l} = \frac{\partial^2 g_{ik}}{\partial x^j \partial x^l} + \frac{\partial \Gamma^\alpha_{ni}}{\partial x^j}\frac{\partial \Gamma_{nk,\alpha}}{\partial x^l} + \frac{\partial \Gamma^\alpha_{ni}}{\partial x^l}\frac{\partial \Gamma_{nk,\alpha}}{\partial x^j}, \\ \frac{\partial^2 G_{ik}}{\partial x^j \partial v^p} = 0, \quad \frac{\partial^2 G_{ik}}{\partial v^q \partial v^p} = 0, \quad \frac{\partial^2 G_{in+p}}{\partial x^j \partial x^l} = \frac{\partial^2 \Gamma_{ni,p}}{\partial x^j \partial x^l}, \\ \frac{\partial^2 G_{in+p}}{\partial x^i \partial v^q} = \frac{\partial \Gamma_{qi,p}}{\partial x^j}, \quad \frac{\partial^2 G_{in+p}}{\partial v^r \partial v^q} = 0, \\ \frac{\partial^2 G_{n+qn+p}}{\partial x^j \partial x^l} = 0, \quad \frac{\partial^2 G_{n+qn+p}}{\partial x^j \partial v^r} = 0, \\ \frac{\partial^2 G_{n+qn+p}}{\partial v^t \partial v^r} = \delta_{pr}\delta_{qt} + \delta_{pt}\delta_{qr}. \end{gathered} \tag{13}$$

From (12) we find the following expression for the Christoffel symbols $\tilde{\Gamma}$ of the manifold T_1M^n at the point $(x,0)$:

$$\begin{gathered} \tilde{\Gamma}_{ij,k} = 0, \quad \tilde{\Gamma}_{ij,n+p} = \frac{1}{2}\left(\frac{\partial \Gamma_{ni,p}}{\partial x^j} + \frac{\partial \Gamma_{nj,p}}{\partial x^i}\right), \\ \tilde{\Gamma}_{in+p,k} = \frac{1}{2}\left(\frac{\partial \Gamma_{nk,p}}{\partial x^i} - \frac{\partial \Gamma_{ni,p}}{\partial x^k}\right), \\ \tilde{\Gamma}_{n+qn+p,k} = \tilde{\Gamma}_{in+p,n+q} = \tilde{\Gamma}_{n+rn+p,n+q} = 0, \end{gathered} \tag{14}$$

where Γ is the Christoffel symbol of M^n at the point x. From (11), (13), and (14), we obtain by direct computation the following components of the curvature tensor $\tilde{R}$ of the manifold T_1M^n:

$$
\begin{aligned}
\tilde{R}_{ijkl} &= R_{ijkl} + (1/4)(R_{iln\alpha} R^{\alpha}_{nkj} + R_{ikn\alpha} R^{\alpha}_{njl}) + (1/2)R_{ijn\alpha} R^{\alpha}_{nkl}, \\
\tilde{R}_{ijkn+q} &= (1/2)\nabla_k R_{ijnq}, \\
\tilde{R}_{ijn+pn+q} &= R_{ijpq} + (1/4)R_{i\alpha np} R^{\alpha}_{jnq} - (1/4)R_{i\alpha nq} R^{\alpha}_{jnp}, \\
\tilde{R}_{in+pkn+q} &= (1/2)R_{ikpq} - (1/4)R_{i\alpha nq} R^{\alpha}_{knp}, \\
\tilde{R}_{in+pn+rn+q} &= 0, \quad \tilde{R}_{n+tn+pn+rn+q} = \delta_{tr}\delta_{pq} - \delta_{pr}\delta_{tq}.
\end{aligned} \tag{15}
$$

From this we find the nonzero components of the curvature tensor for $T_1(M^n, K)$:

$$
\begin{aligned}
&\tilde{R}_{pqpq} = K, \quad \tilde{R}_{nqnq} = K(1-3K/4), \quad \tilde{R}_{pqn+pn+q} = K(1-K/4), \quad \tilde{R}_{pn+pqn+q} = K/2, \\
&\tilde{R}_{pn+ppn+p} = K^2/4, \quad \tilde{R}_{nn+pnn+p} = K^2/4, \quad \tilde{R}_{qn+pn+qp} = K(1-K/2)/2, \quad \tilde{R}_{n+qn+pn+qn+p} = 1.
\end{aligned} \tag{16}
$$

Indeed, taking account of (8) in this case,

$$R^i_{jkl} = R_{ijkl} = K(g_{ik}g_{jl} - g_{il}g_{jk}) = K(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}).$$

We substitute this value in the first of Eqs. (15):

$$
\begin{aligned}
\tilde{R}_{ijkl} = K(\delta_{ik}\delta_{jl} - \delta_{ij}\delta_{jk}) + (K^2/4)\sum_{\alpha=1}^{n}[(\delta_{in}\delta_{l\alpha} - \delta_{i\alpha}\delta_{ln})(\delta_{\alpha k}\delta_{nj} - \delta_{\alpha j}\delta_{nk}) + \\
(\delta_{in}\delta_{k\alpha} - \delta_{i\alpha}\delta_{kn})(\delta_{\alpha j}\delta_{nl} - \delta_{\alpha l}\delta_{nj}) + 2(\delta_{in}\delta_{j\alpha} - \delta_{i\alpha}\delta_{jn})(\delta_{\alpha k}\delta_{nl} - \delta_{\alpha l}\delta_{nk})] = \\
K(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - (3K^2/4)(\delta_{ik}\delta_{ln}\delta_{jn} + \delta_{jl}\delta_{in}\delta_{kn} - \delta_{il}\delta_{jn}\delta_{kn} - \delta_{jk}\delta_{in}\delta_{ln}).
\end{aligned}
$$

Hence we obtain $\tilde{R}_{pqpq} = K$ and $\tilde{R}_{nqnq} = K(1-3K/4)$. Other combinations of indices give 0. The remaining formulas are obtained similarly.

2. The sectional curvature of $T_1(M^n, K)$. Let X and Y be unit orthogonal vectors tangent to $T_1(M^n, K)$ at the point $(x, 0)$. The sectional curvature $\tilde{K}$ of the manifold $T_1(M^n, K)$ at this point is given by $\tilde{K} = \tilde{R}_{IJKL}X^IY^JX^KY^L$, where $I, J, K, L = 1, \ldots, 2n-1$. Taking account of the symmetry of the curvature tensor, we obtain $\tilde{K} = \tilde{R}_{IJKL}(X^IY^J - X^JY^I)(X^KY^L - X^LY^K)$. We introduce the bivector $S^{IJ} = X^IY^J - X^JY^I$. Then $\tilde{K} = \tilde{R}_{IJKL}S^{IJ}S^{KL} = \tilde{R}_{AB}S^AS^B$, where $A = (I, J)$ and $B = (K, L)$ are collective indices, and

$$\sum_A (S^A)^2 = 1. \tag{17}$$

Thus, using (16) we find

$$
\begin{aligned}
\tilde{K} = \sum_A \tilde{R}_{AA}(S^A)^2 + 2\sum_{A\neq B}\tilde{R}_{AB}S^AS^B = \\
\sum_{p=1}^{n-2}\sum_{q=p+1}^{n-1}[K(S^{pq})^2 + (S^{n+pn+q})^2 + 2K(1-K/4)S^{pq}S^{n+pn+q} + \\
KS^{pn+p}S^{qn+q} - K(1-K/2)S^{pn+q}S^{qn+p}] + \\
\sum_{p=1}^{n-1}[(K^2/2)(S^{nn+p})^2 + (K^2/2)(S^{pn+p})^2 + K(1-3K/4)(S^{pn})^2].
\end{aligned} \tag{18}
$$

If we use one of the conditions for simplicity of the bivector:

$$S^{pq}S^{n+pn+q} - S^{pn+p}S^{qn+q} + S^{pn+q}S^{qn+p} = 0, \tag{18'}$$

one can verify the following fact.

LEMMA 2. *The sectional curvature of the Sasaki metric of the manifold $T_1(M^n, K)$ is calculated from the formula*

$$\tilde{K} = \sum_{p=1}^{n-2} \sum_{q=p+1}^{n-1} [K(S^{pq})^2 + K(3-K)S^{pq}S^{n+pn+q} + (S^{n+pn+q})^2]$$
$$+ (K^2/4)\left(\sum_{p=1}^{n-1} S^{pn+p}\right)^2 + \sum_{p=1}^{n-1}[K(1-3K/4)(S^{pn})2 + (K^2/4)(S^{nn+p})^2], \quad (19)$$

where $S^{IJ} = X^I Y^J - X^J Y^I$ are the components of the simple bivector corresponding to the two-dimensional area of the orthogonal vectors X and Y.

Theorem 1 follows easily from Lemma 2. Indeed, let $0 \leqslant K \leqslant 4/3$. Then $K^2(3-K)^2 < 4K$ and consequently each group of terms in (19) is nonnegative. Conversely, if $K < 0$ or $K > 4/3$, then $\tilde{K} < 0$ on areas for which $S^{pn} = 1$. For example $X = (0, \dots, 0; 1; 0, \dots, 0)$, $Y = (0, \dots, 0, 1, 0, \dots, 0; 0; 0, \dots, 0)$.

REMARK. For $n = 2$ this result was obtained by the author [6].

Consider (19) as a quadratic form on the unit sphere (17). Then the extrema of $\tilde{K}$ coincide with the largest and smallest eigenvalues of the coefficient matrix of (19). As a result one can obtain the following estimates.

The sectional curvature $\tilde{K}$ of the Sasaki manifold $T_1(M^n, K)$ for $n \geqslant 3$ satisfies the following inequalities:

$$\tilde{K} < (K + 1 + \sqrt{(K-1)^2 + K^2(3-K)^2})/2 \quad \text{for} \quad -\infty < K < (11+\sqrt{57})/2;$$
$$\tilde{K} < K^2/2 \quad \text{for} \quad (11+\sqrt{57})/2 \leqslant K < +\infty;$$
$$\tilde{K} \geqslant K(1-3K/4) \quad \text{for} \quad -\infty < K < -2(3+\sqrt{39})/5 \quad \text{and} \quad 4/3 \leqslant K < +\infty;$$
$$\tilde{K} \geqslant 0 \quad \text{for} \quad 0 \leqslant K < 4/3;$$
$$\tilde{K} > (K + 1 - \sqrt{(K-1)^2 + K^2(3-K)^2})/2 \quad \text{for} \quad -2(3+\sqrt{39})/5 \leqslant K < 0.$$

For $n = 2$ there are sharp bounds. To be specific the sectional curvature $\tilde{K}$ of the Sasaki metric of the manifold $T_1(M^2, K)$ lies within the following limits:

$$K(1-3K/4) \leqslant \tilde{K} \leqslant K^2/4 \quad \text{for} \quad K \leqslant 0 \quad \text{or} \quad K \geqslant 1;$$
$$K^2/4 \leqslant \tilde{K} \leqslant K(1-3K/4) \quad \text{for} \quad 0 \leqslant K \leqslant 1.$$

Indeed, for $n = 2$ Eqs. (19) and (17) have the form

$$\tilde{K} = (K^2/4)(S^{23})^2 + (K^2/4)(S^{13})^2 + K(1-3K/4); \quad (S^{12})^2 + (S^{13})^2 + (S^{23})^2 = 1,$$

so that $\tilde{K} = K^2/4 + K(1-K)(S^{12})^2$, from which we obtain the desired bounds immediately.

REMARK. The manifold $T_1(M^2, K)$ is a manifold of constant curvature only for $K = 0$ and $K = 1$. In these cases $\tilde{K} = 0$ and $\tilde{K} = 1/4$ respectively. For $K = 1$ this result was obtained by Sasaki and Klingenberg [7].

Analogous results hold also for $T_\lambda(M^n, K)$. We note only

LEMMA 2′. *The sectional curvature $\tilde{K}$ of the Sasaki metric of the manifold $T_\lambda(M^n, K)$ is calculated from the formula*

$$\tilde{K} = \sum_{p=1}^{n-2} \sum_{q=p+1}^{n-1} [K(S^{pq})^2 + K(3-\lambda^2 K)S^{pq}S^{n+pn+q} + (1/\lambda^2)(S^{n+pn+q})^2] +$$
$$(\lambda^2 K^2/4)\left(\sum_{p=1}^{n-1} S^{pn+p}\right)^2 + \sum_{p=1}^{n-1}[K(1-3\lambda^2 K/4)(S^{pn})^2 + (\lambda^2 K^2/4)(S^{nn+p})^2].$$

THEOREM 1$'$. *The sectional curvature $\tilde{K}$ of the Sasaki metric of the manifold $T_\lambda(M^n, K)$ is nonnegative if and only if $0 \leqslant \lambda^2 K \leqslant 4/3$.*

The proofs are the same as in the case $\lambda = 1$. For $n = 2$ this result was obtained by the author [6]. It is also easy to see that if $n = 2$ and $\lambda^2 K = 1$, then $\tilde{K} = K/4$, as was obtained by Nagy [8].

Let us consider in more detail the case $K = 1$, i.e., when M^n is the unit n-sphere. Then we have

LEMMA 2$''$. *The sectional curvature $\tilde{K}$ of the Sasaki metric of $T_1 S^n$ is*

$$\tilde{K} = \sum_{p=1}^{n-2} \sum_{q=p+1}^{n-1} [S^{pq} + S^{n+pn+q}]^2 + (1/4)\left[\left(\sum_{p=1}^{n-1} S^{pn+p}\right)^2 + \sum_{p=1}^{n-1}(S^{pn})^2 + \sum_{p=1}^{n-1}(S^{nn+p})^2\right]. \tag{20}$$

PROOF OF THEOREM 2: Let X and Y be two orthonormal vectors constituting an elementary tangent area to $T_1 S^n$. As was shown in § 1, it is possible to choose a coordinate system in $T_1 S^n$ so as to be orthogonal at a given point. In addition one can adapt the coordinate system to a given elementary area as follows.

Let $\{e_1, \dots, e_{n-1}; e_n; f_1, \dots, f_{n-1}\}$ be an adapted basis of the fiber $TT_1 S^n$ at the point $(x, 0) \in T_1 S^n$. Then $X = \sum_{p=1}^{n-1} x^p e_p + x^n e_n + \sum_{p=1}^{n-1} V^p f_p = \tilde{X} + x^n e_n + V$. In general $\tilde{X} \neq 0$ and $V \neq 0$, so that one can set $e_1 = \tilde{X}/\|\tilde{X}\|$, $e_1' \in \tilde{H} T_1 S^n$, and $f_1' = (d\pi_1 \cdot e_1')^v$, $f_1' \in V T_1 S^n$.

In the plane of the vectors f_1' and $V = \sum_{p=1}^{n-1} v^p f_p$ we choose a unit vector f_2' orthogonal to f_1'. We set $e_2' = (K f_2')^h$. Then $e_2' \in \tilde{H} T_1 S^n$. In the subspace $\tilde{H} T_1 S^n$ we choose a unit vector e_3' orthogonal to e_2' and e_1'. We set $f_3' = (d\pi_1 e_3')$. Continuing this process, we obtain a basis of $TT_1 S^n$, which we shall call *adapted* to the given elementary area. In this basis

$$\begin{aligned} X &= (x^1, 0, \dots, 0; x^n; v^1, v^2, 0, \dots, 0), \\ Y &= (y^1, y^2, \dots, y^{n-1}; y^n; w^1, w^2, \dots, w^{n-1}). \end{aligned} \tag{21}$$

We shall show that $\tilde{K} \leqslant 5/4$. From condition (17) we find

$$1 = \sum_A (S^A)^2 = \sum_{p=1}^{n-2} \sum_{q=p+1}^{n-1} [(S^{pq})^2 + (S^{n+pn+q})^2 + (S^{pn+q})^2 + (S^{qn+p})^2] + \sum_{p=1}^{n-1}[(S^{pn+p})^2 + (S^{pn})^2 + (S^{nn+p})^2].$$

Therefore

$$\begin{aligned} 5/4 - \tilde{K} = (5/4)\Bigg\{ &\sum_{p=1}^{n-2} \sum_{q=p+1}^{n-1} [(S^{pq})^2 + (S^{n+pn+q})^2 + (S^{pn+q})^2 + (S^{qn+p})^2] + \\ &\sum_{p=1}^{n-1}[(S^{pn+p})^2 + (S^{pn})^2 + (S^{nn+p})^2]\Bigg\} - \sum_{p=1}^{n-2} \sum_{q=p+1}^{n-1} (S^{pq} + S^{n+pn+q})^2 - \\ (1/4)\Bigg[\left(\sum_{p=1}^{n-1} S^{pn+p}\right)^2 &+ \sum_{p=1}^{n-1}(S^{pn})^2 + \sum_{p=1}^{n-1}(S^{nn+p})^2\Bigg] = \sum_{p=1}^{n-2} \sum_{q=p+1}^{n-1} [(1/4)(S^{pq})^2 + (1/4)(S^{n+pn+q})^2 - \end{aligned}$$

$$2S^{pq} S^{n+pn+q} + (5/4)(S^{pn+q})^2 + (5/4)(S^{qn+p})^2 - (1/2) S^{pn+p} S^{qn+q}] + \sum_{p=1}^{n-1}[(S^{pn+p})^2 + (S^{pn})^2 + (S^{nn+p})^2].$$

Using the identity (18$'$), we obtain

$$\begin{aligned} 5/4 - \tilde{K} = \sum_{p=1}^{n-2} \sum_{q=p+1}^{n-1} [(S^{pq} - S^{n+pn+q})^2 + (5/4)(S^{pn+q})^2 + (5/4)(S^{qn+p})^2 + (3/2) S^{pn+q} S^{qn+p}] + \\ \sum_{p=1}^{n-1}(S^{pn+p})^2 - 2\sum_{p=1}^{n-2} \sum_{q=p+1}^{n-1} S^{pn+p} S^{qn+q}. \end{aligned}$$

But it follows from (21) that $S^{pn+p} = 0$ for $p = 3, \ldots, n-1$, whence $\sum_{p=1}^{n-1} (S^{pn+p})^2 - 2\sum_{p=1}^{n-2} \sum_{q=p+1}^{n-1} S^{pn+p} S^{qn+q} = (S^{1n+1})^2 + (S^{2n+2})^2 - 2S^{1n+1} S^{2n+2} = (S^{1n+1} - S^{2n+2})^2$.

It is now obvious that $5/4 - \tilde{K} \geqslant 0$. We remark that $\tilde{K} = 5/4$ in the following circumstances:

$$S^{pq} - S^{n+pp+q} = 0, \quad S^{pn+q} = S^{qn+p} = 0, \quad S^{1n+1} - S^{2n+2} = 0, \quad S^{pn} = S^{nn+p} = 0, \tag{22}$$

which hold, for example, for the vectors

$$X = (1/\sqrt{2}, 0, \ldots, 0; 0; 0, 1/\sqrt{2}, 0, \ldots, 0), \quad \text{and} \quad Y = (0, 1/\sqrt{2}, 0, \ldots, 0; 0; -1/\sqrt{2}, 0, \ldots, 0).$$

The inequality $\tilde{K} \geqslant 0$ is obvious. We note merely that equality is attained when

$$S^{pq} + S^{n+pn+q} = 0, \quad S^{1n+1} + S^{2n+2} = 0, \quad S^{pn} = S^{nn+p} = 0, \tag{23}$$

which hold, for example, for the vectors

$$X = (1/\sqrt{2}, 0, \ldots, 0; 0; 0, 1/\sqrt{2}, 0, \ldots, 0), \quad \text{and} \quad Y = (0, 1/\sqrt{2}, 0, \ldots, 0; 0; 1/\sqrt{2}, 0, \ldots, 0).$$

We shall show that the choice of vectors indicated in the examples is not accidental. One can show, using [9], that $T_1 S^3$ is a totally geodesic submanifold of $T_1 S^n$. Therefore the distribution of the sectional curvature of $T_1 S^n$ coincides with the distribution of the sectional curvature on $T_1 S^3$. It turns out that at each point of $T_1 S^3$ there exists a 4-orthoframe X_1, X_2, X_3, X_4 such that $\tilde{K}(X_1, X_2) = \tilde{K}(X_3, X_4) = 5/4$ and $\tilde{K}(X_1, X_3) = \tilde{K}(X_1, X_4) = \tilde{K}(X_2, X_3) = \tilde{K}(X_2, X_4) = 0$, where $\tilde{K}(X_i, X_j)$ is the sectional curvature of $T_1 S^3$ in the direction of the elementary area of the vectors X_i, X_j $(i, j = 1, \ldots, 4)$.

Indeed, for $T_1 S^3$ Eq. (20) has the form

$$\tilde{K} = (S^{12} + S^{45})^2 + (1/4)[(S^{14} + S^{25})^2 + (S^{13})^2 + (S^{23})^2 + (S^{34})^2 + (S^{35})^2].$$

According to (22) we have $\tilde{K} = 5/4$ if

$$S^{12} - S^{45} = 0, \quad S^{14} - S^{25} = 0, \quad S^{15} = S^{24} = 0, \quad S^{13} = S^{23} = S^{34} = S^{35} = 0. \tag{24}$$

Suppose $X = (x^1, x^2, x^3, v^1, v^2)$ and $Y = (y^1, y^2, y^3, w^1, w^2)$ are the vectors determining the elementary area, i.e.,

$$\begin{aligned} (x^1)^2 + (x^2)^2 + (x^3)^2 + (v^1)^2 + (v^2)^2 &= 1, \\ (y^1)^2 + (y^2)^2 + (y^3)^2 + (w^1)^2 + (w^2)^2 &= 1, \\ x^1 y^1 + x^2 y^2 + x^3 y^3 + v^1 w^1 + v^2 w^2 &= 0. \end{aligned} \tag{25}$$

Then from the fourth of the equalities (24) it is easy to obtain $x^3 = y^3 = 0$. Passing to a basis adapted to the given elementary area $(x^2 = 0)$, we obtain a system of equations composed of the first three of equalities (24) and the third equality of (25):

$$\begin{aligned} x^1 y^2 - v^1 w^2 + v^2 w^1 &= 0, \\ x^1 w^1 - y^1 v^1 + y^2 v^2 &= 0, \\ x^1 y^1 + v^1 w^1 + v^2 w^2 &= 0, \\ x^1 w^2 - y^1 v^2 &= 0, \\ -y^2 v^1 &= 0. \end{aligned} \tag{26}$$

From the last equality of (26) we obtain one of two possibilities:

a) $y^2 = 0$, from which $S^{12} = 0$ and $S^{25} = 0$. The first two of conditions (23) give $S^{45} = S^{14} = 0$, and therefore S^{IJ} is the zero bivector. This is the degenerate case.

b) $v^1 = 0$. In this case we regard the first four equalities of (26) as a homogeneous system of equations with respect to y^1, y^2, w^1, w^2. For this system to have a nontrivial solution it is necessary and sufficient that its determinant Δ vanish. It is easy to verify that $\Delta = (v^2)^4 - (x^1)^4$. From $\Delta = 0$ it follows that $(x^1)^2 - (v^2)^2 = 0$.

1) If $x^1 = v^2$ (both are nonzero, since otherwise we would have $X = 0$), we obtain from the second and third of Eqs. (26) the equations $y^1 = w^2 = 0$. From the first equality of (26) we then find $w^1 = -y^2$. Finally, from the first two equalities of (25) we get $x^1 = v^2 = 1/\sqrt{2}$ and $-y^2 = w^1 = 1/\sqrt{2}$. Thus

$$X_1 = (1/\sqrt{2}, 0; 0; 0, 1/\sqrt{2}) \quad \text{and} \quad X_2 = (0, -1/\sqrt{2}; 0; 1/\sqrt{2}; 0).$$

2) If $x^1 = -v^2$, we proceed as in Case 1), obtaining two other vectors $X_3 = (1/\sqrt{2}, 0; 0; 0, -1/\sqrt{2})$ and $X_4 = (0, 1/\sqrt{2}; 0; 1/\sqrt{2}; 0)$. It is not difficult to verify that the pairs (X_1, X_3), (X_1, X_4), (X_2, X_3), (X_2, X_4) satisfy conditions (23) also and consequently $\tilde{K}(X_1, X_2) = \tilde{K}(X_3, X_4) = 5/4$; $\tilde{K}(X_1, X_3) = \tilde{K}(X_1, X_4) = \tilde{K}(X_2, X_3) = \tilde{K}(X_2, X_4) = 0$. We remark finally that the vectors X_1, X_2, X_3, X_4 form an orthonormal frame at the given point.

3. The Ricci curvature and the scalar curvature of $T_1(M^n, K)$. Using (15) we obtain the following components of the Ricci tensor $\tilde{R}$ of the manifold $T_1 M^n$:

$$\tilde{R}_{ik} = R_{ik} + (3/4)\sum_{l=1}^{n} R_{ilna} R^{\alpha}_{nkl} - (1/4)\sum_{t=1}^{n-1} R_{iant} R^{\alpha}_{knt},$$

$$\tilde{R}_{in+q} = -(1/2)\sum_{l=1}^{n} \nabla_l R_{ilnq},$$

$$\tilde{R}_{n+pn+q} = (n-2)\delta_{pq} - (1/4)\sum_{l=1}^{n} R_{lanq} R^{\alpha}_{lnp}.$$

From this we obtain for $T_1(M^n, K)$:

$$\tilde{R}_{pp} = [2(n-1) - K]K/2, \quad R_{nn} = (2-K)(n-1)K/2, \quad \tilde{R}_{n+pn+p} = [K^2 + 2(n-2)]/2. \tag{27}$$

PROOF OF THEOREM 3: a) For a unit vector $X = (x^1, \ldots, x^{n-1}; x^n; v^1, \ldots, v^{n-1})$, since we have (11) in natural coordinates, we find

$$\widetilde{Ric} = \sum_{p=1}^{n-1} \tilde{R}_{pp}(x^p)^2 + \tilde{R}_{nn}(x^n)^2 + \sum_{p=1}^{n-1} \tilde{R}_{n+pn+p}(v^p)^2.$$

This expression is a quadratic form on the unit sphere. Its extrema are the largest and smallest of its coefficients. It is easy to verify that

$$\tilde{R}_{pp} - \tilde{R}_{nn} = K^2(n-2)/2, \quad \tilde{R}_{pp} - \tilde{R}_{n+pn+p} = -(K-1)[K-(n-2)],$$
$$\tilde{R}_{nn} - \tilde{R}_{n+pn+p} = -(1/2)[nK^2 - 2(n-1)K + 2(n-2)].$$

This implies the following inequalities:

$$\tilde{R}_{pp} \geqslant \tilde{R}_{nn} \quad \text{for} \quad n \geqslant 2;$$
$$\tilde{R}_{pp} \geqslant \tilde{R}_{n+pn+p} \quad \text{for} \quad 0 \leqslant K \leqslant 1, \quad n = 2, \quad \text{and} \quad 1 \leqslant K \leqslant n-2, \quad n \geqslant 3;$$
$$\tilde{R}_{nn} \geqslant \tilde{R}_{n+pn+p} \quad \text{for} \quad 0 \leqslant K \leqslant 1, \quad n = 2;$$
$$\tilde{R}_{pp} < \tilde{R}_{n+pn+p} \quad \text{for} \quad K > 1, \quad K < 0, \quad n = 2, \quad K < 1, \quad K > n-2, \quad n \geqslant 3;$$
$$\tilde{R}_{nn} < \tilde{R}_{n+pn+p} \quad \text{for} \quad n \geqslant 3.$$

Assertion a) follows immediately from this.

b) Using (27) it is easy to see that $\tilde{R} = -(1/2)(n-1)[K^2 + 2Kn - 2(n-2)]$. Hence $\tilde{R}_{max} = (1/2)(n-1)(n^2+2n-4)$ for $K = n$.

Appendix. An estimate of the lengths of the closed geodesics of T_1S^n. A curve Γ in T_1M^n is called *horizontal* (resp. *vertical*) if its tangent vector at each point is horizontal (resp. vertical).

As Sasaki [9] has shown, the geodesics of T_1S^n divide into horizontal, vertical, and umbilic types, which under certain conditions can be closed. Geodesics of umbilic type divide into 3 classes as follows.

Let Γ be a geodesic of T_1S^n, and let $\varkappa_1$ and $\varkappa_2$ be the curvature and torsion of the curve $\gamma = \pi_1\Gamma$; Γ is a geodesic of class:

$$(i),\quad \text{if}\quad \varkappa_k = 0,\quad k = 1,\dots,n;$$
$$(ii),\quad \text{if}\quad \varkappa_1 > 0,\quad \varkappa_k = 0,\quad k = 2,\dots,n;$$
$$(iii),\quad \text{if}\quad \varkappa_1 > 0,\quad \varkappa_2 \neq 0,\quad \varkappa_k = 0,\quad k = 3,\dots,n.$$

Let ξ_1 be a unit vector tangent to γ; let y be a unit vector field along γ defining the curve $\Gamma = (\gamma, y)$ in T_1S^n; let y' be the derivative of y with respect to arc length on Γ. We use the notation $\cos\varphi = \langle \xi_1, y\rangle$; $\cos\psi = (1/c)\langle \xi_1, y'\rangle$, where $c^2 = \langle y', y'\rangle$.

Let (x^1, x^2, x^3, x^4) be cartesian coordinates in E^4. A Clifford torus is by definition a surface $T^2(\alpha)$ satisfying the conditions $(x^1)^2 + (x^2)^2 = \cos^2(\alpha/2)$, $(x^3)^2 + (x^4)^2 = \sin^2(\alpha/2)$, where $0 < \alpha < \pi/2$. If we set $\lambda = \cos(\alpha/2)$ and $\mu = \sin(\alpha/2)$ $(\lambda^2 + \mu^2 = 1)$, then $T^2(\alpha)$ has the parametrization $x^1 = \lambda\cos\theta$, $x^2 = \lambda\sin\theta$, $x^3 = \mu\cos\varphi$, $x^4 = \mu\sin\varphi$. The linear element of $T^2(\alpha)$ is $ds^2 = \lambda^2 d\theta^2 + \mu^2 d\varphi^2$. The curve $C(\alpha, m)$ on $T^2(\alpha)$ defined by the equations

$$x^1 = \lambda\cos t,\quad x^2 = \lambda\sin t,\quad x^3 = \mu\cos(mt),\quad x^4 = \mu\sin(mt),$$

is called a *simple helix* in S^3. A simple helix in S^n is defined to be a simple helix in $S^3 \subset S^n$.

The curvature and torsion of a simple helix are given by

$$\varkappa_1^2 = \lambda^2\mu^2(1-m^2)^2/(\lambda^2 + m^2\mu^2)^2,\quad \varkappa_2^2 = m^2/(\lambda^2 + m^2\mu^2)^2. \tag{28}$$

The condition for a simple helix to be closed is that $m = p/q$ should be a rational number. In this case its length is $2\pi\sqrt{\lambda^2p^2 + \mu^2q^2}$.

THEOREM 4. *Let Γ be a closed geodesic of T_1S^n of length $l(\Gamma)$. Then $l(\Gamma) = 2\pi$ if Γ is of horizontal type, vertical type, or umbilic type of class (ii); $l(\Gamma) = 2\pi(p/q)\sqrt{p^2+q^2} > 2\pi p$ if Γ is of umbilic type and class (i); $l(\Gamma) = 2\pi\sqrt{p^2\lambda^2 + q^2\mu^2}\sqrt{1 + \varkappa_1^2(\cos\varphi + (\varkappa_2/\varkappa_1)^2)^2 + (1 + (\varkappa_2/\varkappa_1)^2)^2\cos^2\psi} > 2\pi\min(p,q)$, where p and q are natural numbers defining the (closed) curve $\gamma = \pi_1\Gamma$; λ and μ are the parameters of the Clifford torus; and $\varkappa_1$ and $\varkappa_2$ are the curvature and torsion of the simple helix $\gamma = \pi_1\Gamma$.*

PROOF: Let σ be arc length of the curve $\Gamma \subset T_1S^n$ and let s be arc length along the curve $\gamma = \pi_1\Gamma$, $\gamma \subset S^n$. If $\gamma = x(s)$, then $\Gamma = (x(s), y(s))$, where $y(s)$ is a unit vector field along $x(s)$. The geometric meaning of the Sasaki metric of T_1S^n [10] is that $d\sigma^2 = ds^2 + d\theta^2$, where θ is the elementary angle of turning of the unit vector field $y(s)$ along the curve $x(s)$.

A geodesic Γ of horizontal type is generated by a parallel vector field along a geodesic of S^n. Then $d\theta = 0$, and $d\sigma^2 = ds^2$, whence $l(\Gamma) = 2\pi$.

A geodesic Γ of vertical type is a great circle of the fiber. Consequently $l(\Gamma) = 2\pi$. A geodesic Γ of umbilic type of class (i) is generated by a vector field $y' = \cos(c\sigma)l_2 + \sin(c\sigma)l_3$ [9] along the geodesic $\gamma \subset S^n$, where l_2 and l_3 are orthonormal parallel vector fields along γ, tangent to S^n and orthogonal to γ; $c = \text{const.}$, $0 < c < 1$.

For the vector field $y(\sigma)$ to be closed it is necessary that $c\sigma = 2\pi p$, where $p = 1, 2, \dots$ In this case the great circle of S^n is traversed possibly q times: $s = 2\pi q$, $q = 1, 2, \dots$ But according to [9] $\dfrac{ds}{d\delta} = \sqrt{1-c^2}$.

Thus Γ is closed if and only if $c/\sqrt{1-c^2} = p/q$. Hence $c = q/\sqrt{p^2+q^2}$ and $\sigma = 2\pi(p/q)\sqrt{p^2+q^2}$. Thus $l(\Gamma) = 2\pi(p/q)\sqrt{p^2+q^2}$, where p and q are mutually prime natural numbers determining a closed geodesic of umbilic type and class (i). It is obvious that $l(\Gamma) > 2\pi p > 2\pi$.

The vector field

$$y(s) = \cos\varphi\xi_1 - (c\cos\psi/\sqrt{1-c^2}\varkappa_1)\xi_2 - (\varkappa_2/\varkappa_1)\xi_3, \tag{29}$$

where (ξ_1, ξ_2, ξ_3) is a Frenet frame of the simple helix, determines a geodesic of T_1S^n of umbilic type and class (iii). Here $\varkappa_1$, $\varkappa_2$, c, $\cos\varphi$, and $\cos\psi$ are constants connected by the relations

$$\varkappa_1^2 + \varkappa_2^2 = c^2/(1-c^2), \quad \cos^2\varphi + \cos^2\psi = (1-c^2)\varkappa_1^2/c^2. \tag{30}$$

It follows from (29) that the geodesic is closed if its simple helix is closed. We set up Fermi coordinates along γ in S^n. Then $g_{ij} = \delta_{ij}$ along γ. Consequently the Sasaki metric is also diagonal along γ, and $l(\Gamma) = \int_0^a \sqrt{1+\dot{y}^2}\,ds$, $a = l(\gamma)$, where $\dot{y}^2 = \varkappa_1^2(\cos\varphi + (\varkappa_2/\varkappa_1)^2)^2 + c^2\cos^2\psi/(1-c^2) + c^2\cos^2\psi(\varkappa_2/\varkappa_1)^2/(1-c^2)$. Taking account of (27) we obtain $\dot{y}^2 = \varkappa_1^2[(\cos\varphi + (\varkappa_2/\varkappa_1)^2)^2 + (1+(\varkappa_2/\varkappa_1)^2)^2\cos^2\psi]$. Taking account of the fact that $\varkappa_1$, $\varkappa_2$, $\cos\varphi$, and $\cos\psi$ are constants, we find

$$l(\Gamma) = 2\pi\sqrt{p^2\lambda^2 + q^2\mu^2}\sqrt{1 + \varkappa_1^2[(\cos\varphi + (\varkappa_2/\varkappa_1)^2)^2 + (1+(\varkappa_2/\varkappa_1)^2)^2\cos^2\psi]}. \tag{31}$$

It is now obvious that

$$l(\Gamma) > 2\pi\sqrt{p^2\lambda^2 + q^2\mu^2} \geqslant 2\pi\min(p,q).$$

If we set $\varkappa_2 = 0$ in (31), we obtain an expression for the length of a closed geodesic of umbilic type and class (ii) [9]. In this case it follows from (28) that $m = p/q = 0$ and $\varkappa_1 = \lambda/\mu$. Hence $p = 0$ and the first closing of a geodesic of class (ii) occurs for $q = 1$. Therefore $l(\Gamma) = 2\pi\mu\sqrt{1+\varkappa_1^2} = 2\pi\sqrt{\lambda^2+\mu^2} = 2\pi$.

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