THE SECTIONAL CURVATURE OF THE SASAKI METRIC OF T_1M^n

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We study the tangent bundle of vectors of fixed length on a Riemannian manifold. We give sufficient conditions for the sectional curvature of the Sasaki metric on the tangent bundle of vectors of fixed length to be nonnegative. Bibliography: 6 titles.

Let TM^n denote the tangent bundle of an n-dimensional Riemannian manifold M^n with metric g. There is a natural Riemannian metric (the Sasaki metric) on TM^n, which we denote Tg. If in each fiber we restrict ourselves to vectors of unit length (or length $\lambda > 0$), we obtain a subbundle T_1M^n (resp. $T_\lambda M^n$) called the spherical tangent bundle. The metric G induced on $T_\lambda M^n$ by the metric Tg is called the Sasaki metric of $T_\lambda M^n$. The purpose of the present article is to obtain sufficient conditions for the sectional curvature of the metric G to be nonnegative.

INDEX CONVENTION. All indices denoted by lower-case letters (except p and q) assume values from 1 to n. The indices p and q assume values from 1 to $n-1$, and $A, B, C, D = 1, \dots, 2n$; $I, J = 1, \dots, 2n-1$.

§ 1. Statement of the Results.

THEOREM 1. *Let X, Y, U, W, and ξ be unit vectors tangent to M^n at the point Q; and let $\langle X, Y\rangle = \langle U, W\rangle = 0$, and $\langle U, \xi\rangle = \langle W, \xi\rangle = 0$. If*

$$\frac{\langle(\nabla_X R)(\xi, W)X, Y\rangle^2}{|R(\xi, W)X|^2} + \frac{\langle(\nabla_Y R)(\xi, U)Y, X\rangle^2}{|R(\xi, U)Y|^2} + \frac{\lambda^2}{4}\Big[3\langle R(X,Y)W, U\rangle - \lambda^2\langle R(\xi,U)X, R(\xi,W)Y\rangle + \frac{\lambda^2}{2}\langle R(\xi,U)Y, R(\xi,W)X\rangle\Big]^2 \leqslant K_{XY} - \frac{3\lambda^2}{4}|R(X,Y)\xi|^2,$$

where R is the curvature tensor of M^n and K_{XY} is its sectional curvature, then $T_\lambda M^n$ has nonnegative sectional curvature at the point $\bar{Q} = (Q, \lambda\xi)$.

REMARK. For $n = 2$ this condition is necessary and sufficient.

We introduce the notation

$$M = \sup_{\substack{|X\wedge Y|=1\\ |\xi|=1}} |R(X,Y)\xi|, \quad M_\nabla = \sup_{\substack{|X\wedge Y|=1\\ |U\wedge W|=1\\ |\xi|=1}} \frac{|\langle(\nabla_X R)(\xi, W)X, Y\rangle|}{|R(\xi, W)X|}, \quad \mu = \inf_{|X\wedge Y|=1} K_{XY}.$$

THEOREM 2. a) *If $\lambda^2 M \leqslant \left[\frac{4}{3}\sqrt[3]{1 + \frac{3}{4}\frac{m - 2M_\nabla^2}{M}} - 1\right]$, then $T_\lambda M^n$ has nonnegative sectional curvature.*

b) *If 1) $0 \leqslant \mu \leqslant 1/6$, 2) $M^2 \leqslant \mu/6$, and 3) $M_\nabla^2 \leqslant \mu/6$, then T_1M^n has nonnegative sectional curvature.*

§ 2. Preliminary information.

Let (x^i) be a local coordinate system in a neighborhood of an arbitrary fixed point $Q \in M^n$. Then the vectors $(\partial/\partial x^i)_Q$ constitute the natural basis of the tangent space T_QM^n

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to M^n. If $\xi \in T_Q M^n$, then $\xi = \xi^i(\partial/\partial x^i)_Q$, and the set of quantities (x^i, ξ^i) gives the local coordinates, called the natural induced coordinates, of a point $\tilde{Q} = (Q, \xi) \in TM^n$. Regarding x^i and ξ^i as independent functions, we obtain a local coordinate system in some neighborhood of the point $\tilde{Q}$. The natural basis of $T_{\tilde{Q}}TM^n$ consists of the vectors$(\partial/\partial x^i, \partial/\partial \xi^i)$, and any vector $\tilde{X} \in T_{\tilde{Q}}TM^n$ has the form $\tilde{X} = \tilde{X}^i \dfrac{\partial}{\partial x^i} + \tilde{X}^{n+i} \dfrac{\partial}{\partial \xi^i}$. Sasaki [1] has defined a Riemannian metric on TM^n having the form

$$\begin{cases} Tg_{ij} &= g_{ij} + \Gamma^\alpha_{is}\Gamma_{jt,\alpha}\,\xi^i\xi^s, \\ Tg_{in+j} &= \Gamma_{is,j}\,\xi^s, \\ Tg_{n+i\,n+j} &= g_{ij}, \end{cases}$$

in the local coordinates (x^i, ξ^i), where g_{ij} is the metric tensor of M^n and Γ^α_{is} are the Christoffel symbols of the Riemannian connection of M^n. We shall denote by $\langle\langle\cdot,\cdot\rangle\rangle$ the scalar product of the vectors in the metric Tg. It is not difficult to verify that

$$\langle\langle \tilde{X}, \tilde{Y}\rangle\rangle = g_{ij}\tilde{X}^i\tilde{Y}^j + g_{\alpha\beta}(\tilde{X}^{n+\alpha} + \Gamma^\alpha_{s\nu}\tilde{X}^s\xi^\nu)(\tilde{Y}^{n+\beta} + \Gamma^\beta_{t\mu}\tilde{Y}^t\xi^\mu) = \langle \pi_*\tilde{X}, \pi_*\tilde{Y}\rangle + \langle K\tilde{X}, K\tilde{Y}\rangle,$$

where $\pi_*\tilde{X} = \tilde{X}^i\partial/\partial x^i$ is the differential of the projection $\pi : TM^n \to M^n$ and $K\tilde{X} = (\tilde{X}^{n+i} + \tilde{\Gamma}^i_{s\nu}\tilde{X}^s\xi^\nu)\partial/\partial x^i$ is the connection mapping of [2].

The imbedding of T_1M^n into TM^n is given by the condition that the tangent vector be a unit vector: $g_{ij}\xi^i\xi^j = 1$, making it possible to consider one of the coordinates, for example, ξ^n as a function of the other coordinates $x^1, \ldots, x^n, \xi^1, \ldots, \xi^{n-1}$. Let y^A be local coordinates on TM^n and x^I local coordinates on T_1M^n. Then the imbedding of T_1M^n into TM^n can be written as follows:

$$y^1 = x^1, \ldots, y^n = x^n, y^{n+1} = x^{n+1}\,(= \xi^1), \ldots, y^{2n-1} = x^{2n-1}\,(= \xi^{n-1})$$
$$y^{2n} = y^{2n}(x^1, \ldots, x^n, x^{n+1}, \ldots, x^{2n-1}).$$

It is not difficult to verify that in this case the Sasaki metric of T_1M^n has the form [3]:

$$\begin{cases} G_{ik} = g_{ik} + \Gamma^\alpha_{is}\Gamma_{kt,\alpha}\,\xi^i\xi^s + \Gamma_{\alpha i,n}\,\xi^\alpha A_k + \Gamma_{\alpha k,n}\,\xi^\alpha A_i + g_{nn}A_iA_k, \\ G_{in+p} = \Gamma_{\alpha i,p}\,\xi^\alpha + \Gamma_{\alpha i,n}\,\xi^\alpha B_p + g_{nn}A_iB_p, \\ G_{n+q\,n+p} = g_{pq} + g_{np}B_q + g_{nq}B_p + g_{nn}B_pB_q, \end{cases}$$

where $A_k = \partial y^{2n}/\partial x^k$; $B_p = \partial y^{2n}/\partial x^{n+p}\,(= \partial y^{2n}/\partial\xi^p)$. A_k and B_p can be easily found by taking the covariant derivative of the equality $g_{ij}\xi^i\xi^j = 1$. It then results that $A_k = -\dfrac{1}{\xi_n}\Gamma^i_{jk}\xi^j\xi_i$ and $B_p = -\dfrac{\xi_p}{\xi_n}$, where $\xi_i = g_{is}\xi^s$.

It is known [4] that the unit normal to T_1M^n in M^n at the point (Q, ξ) is the vector $N = \xi^V$—the vertical lift of the vector ξ to the point $\tilde{Q} = (Q, \xi)$. Consequently the vector $\tilde{X} \in T_{\tilde{Q}}TM^n$ is tangent to T_1M^n if and only if $\langle K\tilde{X}, \xi\rangle = 0$, i.e., $K\tilde{X} \in L^\perp_Q(\xi)$. We denote by K_1 the restriction of the connection mapping of K to vectors tangent to T_1M^n. Then $K_1 : T_{\bar{Q}}T_1M^n \to L^\perp_Q(\xi)$. Let $\bar{X}$ and $\bar{Y}$ be elements of $T_{\bar{Q}}TM^n$. If we denote the scalar product of vectors in the metric G by $\langle\langle\cdot,\cdot\rangle\rangle_1$, we obtain

$$\langle\langle \bar{X}, \bar{Y}\rangle\rangle_1 = \langle \pi_*\bar{X}, \pi_*\bar{Y}\rangle + \langle K_1\bar{X}, K_1\bar{Y}\rangle.$$

We define the *horizontal lift* of a vector $X \in T_QM^n$ to the point $\bar{Q} = (Q, \xi) \in T_1M^n$ to be the horizontal lift of the vector X to the point $\bar{Q}$ in the ambient space TM^n. Similarly for a vector $X \in L^\perp_Q(\xi)$ we shall define the *vertical lift* to $T_{\bar{Q}}T_1M^n$ as the vertical lift in TM^n. To simplify the notation we shall distinguish the lifts into TT_1M^n and $\tilde{T}TM^n$ only by context.

§ 3. The second quadratic form of $T_1M^n \subset TM^n$ and the curvature tensor of T_1M^n. To calculate the second quadratic form of the imbedding $T_1M^n \subset TM^n$ we need the components of the Riemannian connection of the metric Tg.

LEMMA 1 [1]. *The Christoffel symbols of the Riemannian connection of the Sasaki metric of TM^n are*

$$\tilde{\Gamma}^a_{n+b\,n+c} = 0, \quad \tilde{\Gamma}^a_{n+b\,c} = -\frac{1}{2} R^a_{cb\lambda}\, \xi^\lambda,$$
$$\tilde{\Gamma}^{n+a}_{n+b\,c} = \Gamma^a_{bc} + \frac{1}{2}\Gamma^a_{\mu h} R^h_{cb\lambda}\, \xi^\lambda \xi^\mu,$$
$$\tilde{\Gamma}^a_{bc} = \Gamma^a_{bc} - \frac{1}{2}(R^a_{bh\mu}\Gamma^h_{\lambda c} + R^a_{ch\mu}\Gamma^h_{\lambda b})\xi^\lambda \xi^\mu$$
$$\tilde{\Gamma}^{n+a}_{bc} = \frac{1}{2}\Big(R^a_{bc\lambda} + R^a_{cb\lambda} + 2\frac{\partial \Gamma^a_{bc}}{\partial x^\lambda}\Big)\xi^\lambda - \frac{1}{2}\Gamma^a_{\nu h}\left(R^h_{b\mu l}\Gamma^l_{\lambda c} + R^h_{c\mu l}\Gamma^l_{\lambda b}\right)\xi^\lambda \xi^\mu \xi^\nu,$$

where $R^a_{bc\lambda}$ is the curvature tensor of the metric g and Γ^a_{bc} are the Christoffel symbols of the Riemannian connection of the metric g.

LEMMA 2. *The matrix Ω of the second quadratic form of the imbedding $T_1M^n \subset TM^n$ has the form*

$$\Omega = \left[\begin{array}{c|c} g_{ij} - G & -G_{i\,n+q} \\ \hline -G_{n+p\,j} & -G_{n+p\,n+q} \end{array}\right].$$

COROLLARY 1. *If $\bar{X}$ and $\bar{Y}$ are tangent vectors to T_1M^n, then $\Omega(\bar{X},\bar{Y}) = -\langle K_1\bar{X}, K_1\bar{Y}\rangle$.*

COROLLARY 1'. *If $\bar{X}$ and $\bar{Y}$ are tangent vectors to $T_\lambda M^n$, then $\Omega_\lambda(\bar{X},\bar{Y}) = -\frac{1}{\lambda}\langle K_1\bar{X}, K_1\bar{Y}\rangle$.*

PROOF: As already noted in § 2, the local coordinates $(x^1,\dots,x^{2n-1})$ of T_1M^n can be chosen so that if $y^1,\dots,y^{2n}$ are local coordinates of TM^n, then the imbedding $T_1M^n \subset TM^n$ assumes the form

$$y^1 = x^1,\dots,y^n = x^n; y^{n+1} = x^{n+1}\,(=\xi^1),\dots,$$
$$y^{2n-1} = x^{2n-1}\,(=\xi^{n-1}), y^{2n} = y^{2n}(x^1,\dots,x^{2n-1}),$$

and $\dfrac{\partial y^{2n}}{\partial x^k} = -\dfrac{1}{\xi_n}\Gamma^i_{jk}\,\xi_i\,\xi^j$ and $\dfrac{\partial y^{2n}}{\partial x^{n+p}} = -\dfrac{\xi_p}{\xi_n}$, where $\xi_i = g_{is}\,\xi^s$.

The following equalities can be verified by direct computation.

$$\frac{\partial^2 y^{2n}}{\partial x^k \partial x^m} = -\frac{1}{\xi_n}\Big\{G_{km} - g_{km} + \frac{\partial \Gamma^i_{jk}}{\partial x^m}\xi_i\xi^j + \Gamma^i_{jk}\Gamma_{im,s}\,\xi^s\xi^j + (\Gamma_{nm,s}A_k + \Gamma_{nk,s}A_m)\xi^s\Big\};$$
$$\frac{\partial^2 y^{2n}}{\partial x^m \partial x^{n+p}} = -\frac{1}{\xi_n}\{G_{m\,n+p} + (\Gamma_{nm,s}B_p + \Gamma_{pm,s})\xi^s\};$$
$$\frac{\partial^2 y^{2n}}{\partial x^{n+p}\partial x^{n+q}} = -\frac{1}{\xi_n}G_{n+p\,n+q}\,.$$

We shall denote covariant differentiation in the Riemannian connection of the metric Tg by $\tilde{\nabla}$. Then the components of the unknown second quadratic form are [5]:

$$\Omega_{IJ} = Tg_{AB}\,\tilde{\nabla}_I\Big(\frac{\partial y^A}{\partial x^J}\Big)N^B = \tilde{\nabla}_I\Big(\frac{\partial y^A}{\partial x^J}\Big)N_A,$$

where N^B are the coordinates of the unit normal to T_1M^n in TM^n and $N_A = Tg_{AB}\,N^B$. Since the unit normal to T_1M^n in TM^n at the point $\bar{Q} = (Q,\xi)$ is the vector $\xi^V = (0,\dots,0;\xi^1,\dots,\xi^n)$, it is easy to verify that $\{N_A\} = \{\Gamma^b_{a\lambda}\,\xi^\lambda\,\xi_b;\,\xi_a\}$. Using Lemma 1, we obtain

$$\Omega_{ij} = \Big[\frac{\partial^2 y^a}{\partial x^i \partial x^j} + \tilde\Gamma^a_{CD}\,\frac{\partial y^C}{\partial x^i}\,\frac{\partial y^D}{\partial x^j}\Big] N_a + \Big[\frac{\partial^2 y^{n+a}}{\partial x^i \partial x^j} + \tilde\Gamma^{n+a}_{CD}\,\frac{\partial y^C}{\partial x^i}\,\frac{\partial y^D}{\partial x^j}\Big] N_{n+a} =$$

$$\Big[\Gamma^a_{ij} - \frac{1}{2}\big(R^a_{ih\mu}\,\Gamma^h_{\lambda j} + R^a_{jh\mu}\,\Gamma^h_{\lambda i}\big)\xi^\lambda\xi^\mu - \frac{1}{2}R^a_{jn\lambda}\,\xi^\lambda\,\frac{\partial y^n}{\partial x^i} - \frac{1}{2}R^a_{in\lambda}\,\xi^\lambda\,\frac{\partial y^{2n}}{\partial x^j}\Big]\Gamma^s_{ta}\,\xi^t\,\xi_s +$$

$$\Big[-\frac{1}{2}\big(R^a_{i\lambda j} + R^a_{j\lambda i}\big)\xi^\lambda + \frac{\partial \Gamma^a_{ij}}{\partial x^\lambda}\xi^\lambda - \frac{1}{2}\Gamma^a_{\nu h}\big(R^h_{i\mu l}\,\Gamma^l_{\lambda j} + R^h_{j\mu l}\,\Gamma^l_{\lambda i}\big)\xi^\lambda\xi^\mu\xi^\nu +$$

$$\Big(\Gamma^a_{nj} + \frac{1}{2}\Gamma^a_{\mu h}\,R^h_{jn\lambda}\,\xi^\lambda\xi^\mu\Big)\frac{\partial y^{2n}}{\partial x^i} + \Big(\Gamma^a_{ni} + \frac{1}{2}\Gamma^a_{\mu h}\,R^h_{in\lambda}\,\xi^\lambda\xi^\mu\Big)\frac{\partial y^{2n}}{\partial x^j}\Big]\xi_a + \frac{\partial^2 y^{2n}}{\partial x^i \partial x^j}\xi_n =$$

$$-\frac{1}{2}\big(R^a_{i\lambda j} + R^a_{j\lambda i}\big)\xi^\lambda\xi_a + \Big[\frac{\partial \Gamma^a_{ij}}{\partial x^\lambda} - \frac{\partial \Gamma^a_{i\lambda}}{\partial x^j} + \Gamma^a_{\lambda s}\Gamma^s_{ij} - \Gamma^a_{js}\Gamma^s_{i\lambda}\Big]\xi^a\xi^\lambda + g_{ij} - G_{ij} =$$

$$\frac{1}{2}\big[R^a_{i\lambda j} - R^a_{j\lambda i}\big]\xi^\lambda\xi_a + g_{ij} - G_{ij} = g_{ij} - G_{ij};$$

$$\tilde\Omega_{in+p} = \Big[\frac{\partial^2 y^a}{\partial x^i \partial x^{n+p}} + \tilde\Gamma^a_{CD}\,\frac{\partial y^C}{\partial x^i}\,\frac{\partial y^D}{\partial x^{n+p}}\Big] N_a + \Big[\frac{\partial^2 y^{n+a}}{\partial x^i \partial x^{n+p}} + \tilde\Gamma^{n+a}_{CD}\,\frac{\partial y^C}{\partial x^i}\,\frac{\partial y^D}{\partial x^{n+p}}\Big] N_{n+a} =$$

$$\Big[-R^a_{ip\lambda}\,\xi^\lambda - \frac{1}{2}R^a_{in\lambda}\,\xi^\lambda\Big]\Gamma^s_{ta}\,\xi^t\,\xi_s + \Big[\Gamma^a_{ip} + \frac{1}{2}\Gamma^a_{h\mu}\,R^h_{ip\lambda}\,\xi^\lambda\xi^\mu + \Gamma^a_{in}\,B_p + \frac{1}{2}\Gamma^a_{h\mu}\,R^h_{in\lambda}\,\xi^\lambda\xi^\mu B_p\Big]\xi_a +$$

$$\frac{\partial^2 y^{2n}}{\partial x^i \partial x^{n+p}}\xi_n = \big(\Gamma_{ip,\lambda} + \Gamma_{in,\lambda}\,B_p\big)\xi^\lambda - \big(G_{i\,n+p} + \Gamma_{ip,\lambda} + \Gamma_{in,\lambda}\,B_p\big)\xi_\lambda\big) = -G_{i\,n+p};$$

$$\tilde\Omega_{n+p\,n+q} = \frac{\partial^2 y^{2n}}{\partial x^{n+p}\,\partial x^{n+q}} = -G_{n+p\,n+q}\,.$$

The lemma is now proved.

We omit the proof of the corollaries in view of their simplicity.

The curvature tensor $\tilde R$ of the Sasaki metric of TM^n is calculated in [6]. Consequently the curvature tensor $\bar R$ of the Sasaki metric of $T_\lambda M^n$ can be calculated from Gauss' formula:

$$\langle\langle \bar R(\bar X,\bar Y)\bar Z,\bar U\rangle\rangle_1 = \langle\langle \tilde R(\bar X,\bar Y)\bar Z,\bar U\rangle\rangle + \Omega(\bar X,\bar U)\Omega(\bar Y,\bar Z) - \Omega(\bar X,\bar Z)\Omega(\bar Y,\bar U).$$

We remark that if one of the vectors is horizontal, then $\Omega(\bar X,\bar Z) = 0$. Indeed if $\bar X = X^H$, then $\Omega(X^H,\bar Z) = -\dfrac{1}{\lambda}\langle K_1 X^H, K_1\bar Z\rangle = 0$, since $K_1 X^H = 0$. Using [6], we can state

LEMMA 3. *The curvature tensor $\bar R$ of the Sasaki metric of $T_\lambda M^n$ at the point $\bar Q = (Q,\lambda\xi)$ is*

$$\langle\langle \bar R(X^H,Y^H)Z^H,U^H\rangle\rangle_1 = \langle\langle \tilde R(X^H,Y^H)Z^H,U^H\rangle\rangle =$$

$$\langle R(X,Y)Z,U\rangle + \frac{\lambda^2}{4}\langle R(Z,Y)\xi, R(X,U)\xi\rangle +$$

$$\frac{\lambda^2}{4}\langle R(X,Z)\xi, R(Y,U)\xi\rangle + \frac{\lambda^2}{2}\langle R(X,Y)\xi, R(Z,U)\xi\rangle;$$

$$\langle\langle \bar R(X^H,Y^H)Z^H,U^V\rangle\rangle_1 = \langle\langle \tilde R(X^H,Y^H)Z^H,U^V\rangle\rangle = \frac{\lambda}{2}\langle(\nabla_Z R)(X,Y)\xi,U\rangle;$$

$$\langle\langle \bar R(X^H,Y^H)Z^V,U^V\rangle\rangle_1 = \langle\langle \tilde R(X^H,Y^H)Z^V,U^V\rangle\rangle = \langle R(X,Y)Z,U\rangle +$$

$$\frac{\lambda^2}{4}\langle R(\xi,Z)X, R(\xi,U)Y\rangle - \frac{\lambda^2}{4}\langle R(\xi,Z)Y, R(\xi,U)X\rangle;$$

$$\langle\langle \bar R(X^H,Y^V)Z^H,U^V\rangle\rangle_1 = \langle\langle \tilde R(X^H,Y^V)Z^H,U^V\rangle\rangle =$$

$$\frac{1}{2}\langle R(X,Z)Y,U\rangle - \frac{\lambda^2}{4}\langle R(\xi,Y)Z, R(\xi,U)X\rangle;$$

$$\langle\langle \bar R(X^V,Y^V)Z^V,U^H\rangle\rangle_1 = \langle\langle \tilde R(X^V,Y^V)Z^V,U^H\rangle\rangle = 0;$$

$$\langle\langle \bar R(X^V,Y^V)Z^V,U^V\rangle\rangle = \frac{1}{\lambda^2}\big(\langle Y,Z\rangle\langle X,U\rangle - \langle X,Z\rangle\langle Y,U\rangle\big),$$

where $\langle R(X,Y)Z,U\rangle$ is the curvature tensor of M^n and

$$\langle(\nabla_Z R)(X,Y)\xi,U\rangle = \nabla_s R_{ijkl} X^k Y^l \xi^j U^i Z^s.$$

§ 4. Proofs of the theorems.

PROOF OF THEOREM 1: Let $\bar{X}$ and $\bar{Y}$ be mutually orthogonal unit vectors tangent to $T_\lambda M^n$ at the point $\bar{Q} = (Q, \lambda\xi)$ (ξ is a unit vector). Then the sectional curvature $K_{\bar{X}\bar{Y}}$ in the direction of the element of area of the vectors $\bar{X}$ and $\bar{Y}$ is

$$\bar{K}_{\bar{X}\bar{Y}} = \langle\langle R(X,Y)Y,X\rangle\rangle_1.$$

We introduce the following notation: $X_H = \pi_* \bar{X}$, $Y_H = \pi_* \bar{Y}$, $X_V = K_1\bar{X}$, $Y_V = K_1\bar{Y}$. It is easy to verify that any vector $\bar{X}$ can be represented in the form $\bar{X} = (X_H)^H + (X_V)^V$. Using this decomposition and the result of Lemma 3, we obtain the following expression for the sectional curvature of $T_1 M^n$:

$$\begin{gathered}\bar{K}_{\bar{X}\bar{Y}} = \langle R(X_H,Y_H)Y_H,X_H\rangle - \frac{3\lambda^2}{4}|R(X_H,Y_H)\xi|^2 + 3\langle R(X_H,Y_H)Y_V,X_V\rangle - \\ \lambda^2\langle R(\xi,X_V)X_H, R(\xi,Y_V)Y_H\rangle + \frac{\lambda^2}{4}|R(\xi,Y_V)Y_H + R(\xi,X_V)Y_H|^2 + \lambda\langle(\nabla_{Y_H}R)(X_H,Y_H)\xi,X_V\rangle - \\ \lambda\langle(\nabla_{X_H}R)(X_H,Y_H)\xi,X_V\rangle + \frac{1}{\lambda^2}(|X_V|^2|Y_V|^2 - \langle X_V,Y_V\rangle^2).\end{gathered} \tag{1}$$

We shall require $\bar{K}_{\bar{X}\bar{Y}} \geqslant 0$ and study the case of an area in general position, i.e., such that $X_H \neq 0$, $Y_H \neq 0$, $X_V \neq 0$, $Y_V \neq 0$. We introduce the following notation: $X = X_H/|X_H|$, $Y = Y_H/|Y_H|$, $U = X_V/|X_V|$, $W = Y_V/|Y_V|$, $\alpha = |X_V|/|X_H|$, $\beta = |Y_V|/|Y_H|$.

We remark that in the plane of the orthonormal vectors $\bar{X}\bar{Y}$ the vectors $\bar{X}$ and $\bar{Y}$ can be mapped by an orthogonal transformation to vectors with orthogonal projections

$$\langle X_H, Y_H\rangle = 0, \quad \langle X_V, Y_V\rangle = 0. \tag{2}$$

For that reason we shall assume that the given vectors $\bar{X}$ and $\bar{Y}$ satisfy (2).

Under these conditions Eq. (1) assumes the form

$$\begin{gathered}\bar{K}_{\bar{X}\bar{Y}} = |X_H|^2|Y_H|^2\{K_{XY} - \frac{3\lambda^2}{4}|R(X,Y)\xi|^2 + \\ [3\langle R(X,Y)W,U\rangle - \lambda^2\langle R(\xi,U)X, R(\xi,W)Y\rangle + \frac{\lambda^2}{2}\langle R(\xi,U)Y, R(\xi,W)X\rangle]\alpha\beta + \\ \frac{\lambda^2}{4}|R(\xi,U)Y|^2\alpha^2 + \frac{\lambda^2}{4}|R(\xi,W)X|^2\beta^2 + \lambda\langle(\nabla_Y R)(X,Y)\xi,U\rangle\alpha - \lambda\langle(\nabla_X R)(X,Y)\xi,W\rangle\beta + \frac{1}{\lambda^2}\alpha^2\beta^2\}.\end{gathered}$$

We regard the expression in braces as a polynomial in α and β. In the corresponding notation it has the form

$$a_0 + a_1\alpha\beta + a_2\alpha^2 + a_3\beta^2 + a_4\alpha + a_5\beta + \frac{1}{\lambda^2}\alpha^2\beta^2 =$$
$$\frac{1}{\lambda^2}\left(\alpha\beta + \frac{\lambda^2 a_1}{2}\right)^2 + a_2\left(\alpha + \frac{a_4}{2a_2}\right)^2 + a_3\left(\beta + \frac{a_5}{2a_3}\right)^2 + a_0 - \frac{\lambda^2 a_1^2}{4} - \frac{a_4^2}{4a_2} - \frac{a_5^2}{4a_3}.$$

Consequently to assure that the sectional curvature is nonnegative in the case of general position it suffices to require that

$$a_0 - \frac{\lambda^2 a_1^2}{4} - \frac{a_4^2}{4a_2} - \frac{a_5^2}{4a_3} \geqslant 0.$$

Recalling the notation we are using, we obtain the assertion of the theorem, since it is easy to ascertain that the more special cases for the position of an elementary area are corollaries of the inequality just obtained.

PROOF OF THEOREM 2: a) We coarsen the inequality of Theorem 1 by taking account of the notation introduced before the statement of Theorem 2. We obtain

$$2M_{\nabla}^2 + \frac{\lambda^2}{4}\left[3M + \frac{3}{2}\lambda^2 M^2\right]^2 + \frac{3\lambda^2}{4}M^2 \leqslant \mu. \tag{3}$$

We set $\lambda^2 M = t$. The result is a chain of inequalities:

$$3t\left(1+\frac{1}{2}t\right)^2 + t \leqslant \frac{4(\mu - 2M_{\nabla}^2)}{3M}; \quad \frac{3}{4}t^3 + 3t^2 + 4t \leqslant \frac{4(\mu - 2M_{\nabla}^2)}{3M};$$

$$\frac{3}{4}\left(\left(t+\frac{4}{3}\right)^3 - \frac{64}{27}\right) \leqslant \frac{4}{3}\frac{(\mu - 2M_{\nabla}^2)}{M}; \quad t \leqslant \frac{4}{3}\left[\sqrt[3]{1 + \frac{3}{4}\frac{\mu - 2M_{\nabla}^2}{M}} - 1\right],$$

which was to be proved.

b) For $\lambda = 1$ the sufficient condition for the sectional curvature of T_1M^n to be nonnegative has the form

$$2M_{\nabla}^2 + \left(\frac{3M^2}{4}\right)\left[1 + \left(1+\frac{M}{2}\right)^2\right] \leqslant \mu. \tag{4}$$

It follows from condition 3) that $2M_{\nabla}^2 \leqslant \mu/3$. It follows from conditions 1) and 2) that $M^2 \leqslant 1/36$. Therefore $(3M^2/4)[1 + (1 + M/2)^2] \leqslant (\mu/8)[1 + (1 + 1/12)^2] = (\mu/8)(313/144) < 2\mu/3$.

Thus condition (4) is a consequence of conditions 1)–3). The theorem is now proved.

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