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ON THE SASAKI METRIC OF THE TANGENT AND NORMAL BUNDLE

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The metric on the tangent bundle of a Riemannian manifold, induced by the parallel translation of tangent vectors, was constructed by Sasaki [1] in 1958. Its properties were investigated by Kowalski [2], Yano and Okubo [3], Klingenberg and Sasaki [4], and others. The Sasaki metric was used in the study of geodesic flows on Riemannian manifolds [5] and in proving a theorem on the volume of manifolds with all geodesics closed [6]. The Sasaki metric on the normal bundle of a submanifold in a Riemannian space was considered by Reckziegel [7] and used to study the geometry of immersed manifolds. However, no systematic investigation of its properties has been carried out. In this note we study: a) the tangent and normal bundles whose Sasaki metric has constant null index (§3); and b) the sufficient conditions for the nonnegativity of the sectional curvature of the tangent and normal sphere bundles with Sasaki metric (§4).

1. Let M^n be a Riemannian manifold with metric g, and let (x^i) be local coordinates on it. At each point $Q \in M^n$, the vectors $(\partial/\partial x^i)$ form the natural basis of the tangent space T_QM^n. Denote by v^i the coordinates of a tangent vector $v \in T_QM^n$ in that basis. The system of functions (x^i, v^i) determines the natural induced local coordinates on TM^n. The line element $d\sigma$ of the Sasaki metric on TM^n is defined in these coordinates by the formula [1] $d\sigma^2 = g_{ij}dx^i dx^j + g_{ij}Dv^i Dv^j$, where the $Dv^i = dv^i + \Gamma^i_{jk}v^j dx^k$ are the covariant differentials of v^i in the Riemannian connection on M^n.

2. Let F^l be a submanifold of a Riemannian manifold M^{l+p}. Denote by $\overline{g}$ the Riemannian metric of M^{l+p}. Then $\overline{g}$ induces: a) a Riemannian metric g on F^l, and b) a fiber metric $g^\perp$ on each normal space N_QF^l. Without loss of generality we assume $g^\perp$ to be Euclidean, since F^l has a global orthonormal basis of normal vector fields. Denote by (x^i) the induced local coordinates on F^l, and by (ξ^α) the coordinates of an arbitrary normal vector ξ in the orthonormal basis $\{n_{\alpha|}\}$. The system of functions (x^i, ξ^α) determines the natural local coordinates on NF^l. We define the line element $d\sigma$ of the Sasaki metric on NF^l in these coordinates by

$$d\sigma^2 = g_{ij}dx^i dx^j + \sum_{\alpha=1}^{p}(D^\perp \xi^\alpha),$$

where $D^\perp\xi^\alpha = d\xi^\alpha + \mu_{\alpha\beta|i}\xi^\beta\, dx^i$ is the covariant differential of ξ^α in the normal connection on the submanifold.

3. The *intrinsic null index* $\nu(Q)$ of the point Q of a Riemannian manifold M^n is defined to be the dimension of the maximal linear subspace $L_Q \subset T_QM^n$ such that for any $Y \in L_Q$ and any $X, Z \in T_QM^n$ the curvature tensor of the metric on M^n satisfies $R(X,Y)Z = 0$. The distribution L is called the null distribution on M^n. If the dimension of L is constant, then we call it the intrinsic null index of the metric of M^n.

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THEOREM 1. *If the intrinsic null index $\tilde{\nu}$ of the tangent bundle TM^n with Sasaki metric is equal to k, then k is even and M^n is the metric product of a Riemannian manifold $M^{n-k/2}$ and the Euclidean space $E^{k/2}$, and TM^n is the metric product of $TM^{n-k/2}$ and E^k.*

Kowalski [2] proved that if the Sasaki metric on TM^n is flat, then M^n is also flat. Theorem 1 generalizes this result significantly. One of the main points of the proof is that the requirement on the intrinsic null index of TM^n implies the existence of a $k/2$-dimensional regular distribution L on M^n such that for every $Y \in L$ and any vector fields X, Z, U tangent to M^n the curvature tensor R of M^n satisfies the conditions $R(X,Y)Z = 0$ and $(\nabla_U R)(X,Y)Z = 0$. As shown by Shirokov [8] (see also [9], §28), these conditions guarantee the existence of $k/2$ linearly independent parallel vector fields on M^n, and M^n is the metric product of a Riemannian manifold $M^{n-k/2}$ and the Euclidean space $E^{k/2}$.

The converse theorem is not true, i.e. the strong parabolicity of the metric on M^n does not in general imply the strong parabolicity of the Sasaki metric on TM^n.

We say that a distribution $\tilde{L}$ on NF^l is *vertical* (*horizonal*) if, at each point $\tilde{Q}$, the subspace $\tilde{L}_{\tilde{Q}}$ is tangent (orthogonal) to the fiber. If $\tilde{L}$ is the null distribution on NF^l, then we will call its dimension the *vertical* (*horizontal*) *null index.*

THEOREM 2. a) *If the vertical intrinsic null index of the Sasaki metric on NF^l is equal to ν, then on F^l there exist ν normal vector fields which are parallel in the normal connection.*

b) *Suppose F^l is a surface in a Euclidean space. If the horizontal intrinsic null index of NF^l is equal to k, then F^l is fibered into k-dimensional intrinsically flat totally geodesic submanifolds with flat normal connection in the ambient space.*

REMARK. In the case of a distribution in general position both possibilities are realized, depending on the dimension of the projections of the null distribution on the vertical and horizonal subspaces.

4. If, in each fiber of TM^n, we consider only the vectors of a fixed length ρ, we obtain a subbundle of TM^n called the *tangent sphere bundle*: $T_\rho M^n$. If M^n is compact, then $T_\rho M^n$ is a compact hypersurface in TM^n. On $T_\rho M^n$ we consider the metric induced by the Sasaki metric on TM^n.

Klingenberg and Sasaki [4] showed that the Sasaki metric on $T_1 S^2$, where S^2 is the standard 2-dimensional sphere, has constant sectional curvature equal to 1/4. For $n \geq 3$ it was shown in [10] that the Sasaki metric of $T_\rho S^n$ has nonnegative sectional curvature for $0 < \rho^2 \leq 4/3$. If M^2 is a 2-dimensional Riemannian manifold with Gaussian curvature K, then the sectional curvature of the Sasaki metric on $T_\rho M^2$ is nonnegative if and only if $\Delta_1 K \leq K^3(1 - 3\rho^2 K/4)$, where Δ_1 is the first Beltrami differential parameter [11]. To formulate the result for $n \geq 3$, we need the following notation: g is the Riemannian metric on M^n; $\langle\cdot,\cdot\rangle$ and $||\cdot||$ are the inner product and the norm of tangent vectors in the metric g; $R(X,Y)Z = R^i_{jkm} Z^j X^k Y^m \partial/\partial x^i$ is the curvature tensor of g; and

$$(\nabla_U R)(X,Y)Z = \nabla_s R^i_{jkm} Z^j X^k Y^m U^s \frac{\partial}{\partial x^i}.$$

THEOREM 3. *Suppose X, Y, U, W, and ξ are unit vectors tangent to M^n at an arbitrary point Q, and $\langle X,Y\rangle = \langle U,W\rangle = 0$ and $\langle U,\xi\rangle = \langle W,\xi\rangle = 0$. Let K_{XY} be the sectional curvature of M^n in the direction of the surface element spanned by the vectors*

X *and* Y. *If, at* $Q \in M^n$, *for any fixed* $\xi \in T_Q M^n$ *and all* $X, Y, U, W \in T_Q M^n$,

$$(1)\frac{\langle(\nabla_X R)(\xi,W)X,Y\rangle^2}{||R(\xi,W)X||^2}+\frac{\langle(\nabla_Y R)(\xi,U)Y,X\rangle^2}{||R(\xi,U)Y||^2}$$
$$+\frac{\rho^2}{4}\left[3\langle R(X,Y)W,U\rangle-\rho^2\langle R(\xi,U)X,R(\xi,W)Y\rangle+\frac{\rho^2}{2}\langle R(\xi,U)Y,R(\xi,W)X\rangle\right]^2$$
$$\le K_{XY}-\frac{3\rho^2}{4}||R(X,Y)\xi||^2,$$

then the sectional curvature of the Sasaki metric on $T_\rho M^n$ *is nonnegative.*

REMARK. a) The assumptions of Theorem 3 are satisfied by compact rank 1 symmetric spaces. For these, $K_{XY} > 0$ and inequality (1) is certainly valid for $\rho = 0$. Consequently, it is also true for $\rho > 0$ sufficiently small.

b) For $n = 2$, Theorem 3 yields the aforementioned necessary and sufficient condition.

Let us introduce the notation

$$\mu=\inf_{||X\wedge Y||=1}K_{XY},\qquad M=\sup_{||\xi||=1,||X\wedge Y||=1}||R(X,Y)\xi||,$$
$$M_\nabla=\sup_{\substack{||X||=1,||\xi\wedge W||=1,\\ ||X\wedge Y||=1}}\frac{|\langle(\nabla_X R)(\xi,W)X,Y\rangle|}{||R(\xi,W)X||}.$$

Weakening inequality (1), we obtain the following assertion.

THEOREM 4. a) *If*

$$\rho^2 M\le\frac{4}{3}\left[\sqrt[3]{1+\frac{3}{4}\;\frac{\mu-2M_\nabla^2}{M}}-1\right],$$

then $T_\rho M^n$ *has nonnegative sectional curvature.*

b) *If* $0 \le \mu \le 1/6$, $M^2 \le \mu/6$, *and* $M_\nabla^2 \le \mu/6$, *then* $T_1 M^n$ *has nonnegative sectional curvature.*

The proof of Theorem 3 is based on the analysis of the formula for the sectional curvature of the Sasaki metric of $T_\rho M^n$. It is known that at each point $\overline{Q} = (Q, \rho\xi) \in T_\rho M^n$ the tangent space $T_{\overline{Q}}(T_\rho M^n)$ decomposes into the direct sum of two subspaces $V_{\overline{Q}}(T_\rho M^n)$ and $H_{\overline{Q}}(T_\rho M^n)$ which are orthogonal in the Sasaki metric. The subspace V is tangent to the fiber and is called vertical, and the subspace H is called horizontal.

On $T_{\overline{Q}}(T_\rho M^n)$ there are defined two maps [12]: π_* (the differential of the projection $\pi: T_\rho M^n \to M^n$) and K (the connection map). π_* maps the H-subspace onto $T_Q M^n$, and K maps the V-subspace onto the orthogonal complement of the vector ξ in $T_Q M^n$.

Let $\overline{X} \in T_{\overline{Q}}(T_\rho M^n)$. Put $X_H = \pi_* \overline{X}$ and $X_V = K\overline{X}$.

LEMMA 1. *The sectional curvature* $\overline{K}_{\overline{X}\,\overline{Y}}$ *of the Sasaki metric on* $T_\rho M^n$ *in the direction of the surface element spanned by the orthonormal vectors* $\overline{X}$ *and* $\overline{Y}$ *at the point* $\overline{Q} = (Q, \rho\xi)$ *is*

$$\begin{aligned}\overline{K}_{\overline{X}\,\overline{Y}}=\;&\langle R(X_H,Y_H)Y_H,X_H\rangle-\tfrac{3}{4}\rho^2||R(X_H,Y_H)\xi||^2\\&+3\langle R(X_H,Y_H)Y_V,X_V\rangle-\rho^2\langle R(\xi,X_V)X_H,R(\xi,Y_V)Y_H\rangle\\&+\tfrac{1}{4}\rho^2||R(\xi,Y_V)X_H+R(\xi,X_V)Y_H||^2+\rho\langle(\nabla_{Y_H}R)(X_H,Y_H)\xi,X_V\rangle\\&-\rho\langle(\nabla_{X_H}R)(X_H,Y_H)\xi,Y_V\rangle+\rho^{-2}(||X_V||^2||Y_V||^2-\langle X_V,Y_V\rangle^2).\end{aligned}$$

There are analogs of Theorems 3 and 4 and of Lemma 1 for the Sasaki metric on the normal bundle $N_\rho F^l$ of vectors of fixed length. The following assertion is an analog of the result of Klingenberg and Sasaki [4] for the Sasaki metric on the normal bundle. Let V^2 be the standard Veronese surface in S^4 (in the Euclidean space E^5 its radius vector has the form

$$r = \left(\frac{1}{\sqrt{3}}x_2x_3, \frac{1}{\sqrt{3}}x_1x_3, \frac{1}{\sqrt{3}}x_1x_2, \frac{1}{2\sqrt{3}}(x_1^2 - x_2^2), \frac{1}{6}(x_1^2 + x_2^2 - 2x_3^2)\right),$$
$$x_1^2 + x_2^2 + x_3^2 = 3,$$

where x_1, x_2, x_3 are the Cartesian coordinates in E^3). For $\rho = \sqrt{3}/2$ the sectional curvature of $N_\rho V^2$ is constant and equals $1/12$.

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