ON THE SASAKI METRIC OF THE NORMAL BUNDLE OF A SUBMANIFOLD IN A RIEMANNIAN SPACE

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ABSTRACT. On the normal bundle of a submanifold in a Riemannian space a natural Riemannian metric is introduced. The structure of surfaces with strongly parabolic normal bundle metric is determined. It is shown that the Sasaki metric of the normal bundle of vectors of fixed length of a two-dimensional Veronese surface has constant sectional curvature.

Bibliography: 16 titles.

Introduction

A metric on the tangent bundle of a Riemannian manifold induced by parallel translation of tangent vectors was constructed by Sasaki [1]. The aim of this work is to construct and study the properties of a Sasaki metric on the normal bundle of a submanifold in a Riemannian space, using local coordinates suitable for concrete calculations and subsequent applications. In the construction of this metric, parallel transport of normals in the normal connection of the submanifold is used.

§1. Basic definitions and results

We denote by F^l a submanifold in a Riemannian space M^{l+p}. At each point $Q \in F^l$ there is a decomposition of the tangent space $T_Q M^{l+p}$ as a direct sum of two subspaces: $T_Q F^l$ and $N_Q F^l$. $T_Q F^l$ is tangent to F^l and $N_Q F^l$ is normal to F^l in the M^{l+p} metric.

By the normal bundle of the submanifold F^l in the Riemannian space M^{l+p} is meant the union of $N_Q F^l$ for all $Q \in F^l$ with structure group $GL(n, \mathbf{R})$.

Let us agree for the duration of the entire article that indices will assume the following values: $i, j, k, m = 1, \dots, l$; $\alpha, \beta, \tau, \sigma, \lambda, \mu, \nu = 1, \dots, p$; $a, b, c, d = 1, \dots, l+p$.

Let $(y^1, \dots, y^{l+p})$ be local coordinates for M^{l+p} in a neighborhood of a point $Q \in F^l$, while $(x^1, \dots, x^l)$ are the induced local coordinates for F^l in a neighborhood of Q. Then F^l locally is given by a system of equations $y^a = y^a(x^1, \dots, x^l)$.

We denote by $\tilde{g}$ the Riemannian metric of M^{l+p}. Then there are induced: a) a Riemannian metric g on F^l with components

$$g_{ij} = \tilde{g}_{ab} \frac{\partial y^a}{\partial x^i} \frac{\partial y^b}{\partial x^j};$$

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and b) a fiber metric $g^\perp$ on each fiber $N_Q F^l$:

$$g_{\alpha\beta} = g_{ab} n^a_{\alpha|} n^b_{\beta|},$$

where $\{n_{\alpha|}\}$ is some basis for $N_Q F^l$.

We note that in each fiber it is possible to choose the basis $\{n_{\alpha|}\}$ to be orthonormal. Thus without loss of generality we will assume that $g^\perp_{\alpha\beta} = \delta_{\alpha\beta}$.

In the normal bundle a normal connection $\nabla^\perp$ is defined. If X is a vector field tangent to F^l and ξ a vector field normal to F^l, then

$$\nabla^\perp_X \xi = X^i \left(\frac{\partial \xi^\alpha}{\partial x^i} + \mu^\alpha_{\beta|i} \xi^\beta \right) n_{\alpha|},$$

where the ξ^α are the coordinates of the vector field ξ in the orthonormal basis $n_{\alpha|}$ and $\mu^\alpha_{\beta|i}$ $(= \mu_{\alpha\beta|i})$ are the torsion coefficients, which are also called the components of the normal connection. If $\bar\nabla$ is the Riemannian connection of M^{l+p}, then

$$\mu_{\alpha\beta|i} = \langle \bar\nabla_i n_{\beta|}, n_{\alpha|} \rangle_g.$$

The connection $\nabla^\perp$ is metric with respect to the fiber metric $g^\perp$, i.e., $\nabla^\perp_X g^\perp = 0$ for an arbitrary tangent vector field X. This means that

$$\frac{\partial g^\perp_{\alpha\beta}}{\partial x^i} = g^\perp_{\alpha\tau} \mu^\tau_{\beta|i} + g^\perp_{\beta\tau} \mu^\tau_{\beta|i}.$$

Taking into account that $g^\perp_{\alpha\beta} = \delta_{\alpha\beta}$, we observe that $\mu_{\alpha\beta|i} + \mu_{\beta\alpha|i} = 0$ (see [2] or [3], §47).

If $\nabla^\perp_X \xi = 0$ for an arbitrary tangent vector field X, we say that the normal vector field ξ is parallel in the normal connection of F^l.

Let ∇ be the Riemannian connection of F^l. The curvature tensors $R(X, Y)Z$ and $N(X, Y)\xi$ of the connections ∇ and $\nabla^\perp$, where X, Y, Z are tangent vector fields and ξ is a normal vector field on F^l, are defined in the standard way.

If $R(X, Y)Z = 0$, then F^l is called a submanifold with intrinsically flat metric.

If $N(X, Y)\xi = 0$, then F^l is called a submanifold with flat normal connection.

The normal connection of a surface in a space of constant curvature is flat if and only if the matrices of all the quadratic forms are reduced simultaneously at each point to diagonal form [2].

The points of the normal bundle NF^l are pairs (Q, ξ), where Q is a point of F^l and $\xi \in N_Q F^l$. Let (x^i) be local coordinates of the point Q, and (ξ^α) coordinates of the normal ξ in the (orthonormal) basis $\{n_{\alpha|}\}$. The set of functions (x^i, ξ^α) determines naturally induced local coordinates for NF^l.

The line element du^2 of the Sasaki metric of NF^l in the naturally induced coordinates is defined by

$$du^2 = g_{ij}\, dx^i\, dx^j + g^\perp_{\alpha\beta} D^\perp \xi^\alpha D^\perp \xi^\beta = g_{ij}\, dx^i\, dx^j + \sum_{\alpha=1}^{p} (D^\perp \xi^\alpha)^2, \tag{1}$$

where $D^\perp \xi^\alpha = d\xi^\alpha + \mu^\alpha_{\beta|i} \xi^\beta\, dx^i$ are the covariant differentials of the coordinates of the normal ξ in the normal connection.

THEOREM 1. *The Sasaki metric of NF^l is flat if and only if F^l is a manifold with intrinsically flat metric embedded in M^{l+p} with a flat normal connection.*

This theorem is the analogue of a result of Kowalski [7] for the Sasaki metric on the tangent bundle. He proved that the Sasaki metric is flat if and only if the metric of the base space is flat. The Sasaki metric on NF^l was determined in an invariant manner by Reckziegel. Namely, let $\pi\colon NF^l \to F^l$ be the map associating to each normal space $N_Q F^l$ the point Q (the projection of the bundle NF^l). Its differential π_* is a fiber-preserving linear transformation from TNF^l onto TF^l. If $\tilde X = (\tilde X^i, \tilde X^{l+\alpha})$ are local coordinates of a vector tangent to NF^l at the point $\tilde Q = (Q, \xi)$ in the natural basis $(\partial/\partial x^i, \partial/\partial\xi^\alpha)$, then $\pi_*\tilde X = \tilde X_i \partial/\partial x^i$. The connection map K is a fiber-preserving linear transformation from TNF^l onto NF^l, given in local coordinates by

$$K\tilde X = (\tilde X^{l+\alpha} + \mu^\alpha_{\beta|i}\xi^\beta \tilde X^i) n_{\alpha|}.$$

The inner product of vectors $\tilde X$ and $\tilde Y$ tangent to NF^l at the point $\tilde Q = (Q, \xi)$ in the Sasaki metric has the form [4]

$$\langle\langle \tilde X, \tilde Y\rangle\rangle = \langle \pi_*\tilde X, \pi_*\tilde Y\rangle + \langle K\tilde X, K\tilde Y\rangle_\perp,$$

where $\langle\ ,\ \rangle$ and $\langle\ ,\ \rangle_\perp$ denote the inner products of vectors in the metrics g and $g^\perp$ respectively.

The kernels of the mappings π_* and K at each point $\tilde Q$ are linear subspaces of the tangent space to NF^l and are called the vertical subspace VNF^l and the horizontal subspace HNF^l. A decomposition holds [4]:

$$T_{\tilde Q} NF^l = V_{\tilde Q} NF^l \oplus H_{\tilde Q} NF^l.$$

The vertical and horizontal subspaces are orthogonal in the Sasaki metric. The vertical subspace is tangent to the fiber.

Let $X = X^i\partial/\partial x^i$ and $\eta = \eta^\alpha n_{\alpha|}$ be a tangent vector and a normal to F^l at the point Q. The vectors

$$\begin{cases} X^H = X^i \dfrac{\partial}{\partial x^i} - \mu^\alpha_{\beta|i}\xi^\beta X^i \dfrac{\partial}{\partial \xi^\alpha}, \\ \eta^V = \eta^\alpha \dfrac{\partial}{\partial\xi^\alpha} \end{cases} \tag{2}$$

are called horizontal and vertical lifts of the vectors X and η respectively. We have

$$X^H \in H_{(Q,\xi)} NF^l, \qquad \eta^V \in V_{(Q,\xi)} NF^l;$$

moreover,

$$\begin{cases} \pi_* X^H = X, & \pi_*\eta^V = 0, \\ K X^H = 0, & K\eta^V = \eta. \end{cases}$$

The intrinsic null-index $\nu(Q)$ of the point $Q \in F^l$ is defined to be the dimension of the maximal linear subspace $L_Q \subset T_Q F^l$ such that for arbitrary $X, Z \in T_Q F^l$ and arbitrary $Y \in L_Q$ the curvature tensor of F^l satisfies the equality (see [5])

$$R(X, Y)Z = 0. \tag{3}$$

If for every Q the null-index $\nu(Q) \geq k$, then the metric of F^l is called strongly k-parabolic [6].

If $\nu(Q)$ is constant on F^l, then the distribution L is holonomic and the integral submanifolds are totally geodesic subspaces of F^l, locally isometric to Euclidean space E^ν [5].

We shall say that a distribution $\tilde L$ on NF^l is vertical (horizontal) if at each point $\tilde Q \in NF^l$ the subspace $\tilde L_{\tilde Q}$ is vertical (horizontal).

Let a vertical (horizontal) distribution $\tilde L$ satisfy condition (3). The dimension of the linear subspace $\tilde L_{\tilde Q}$ is called the vertical (horizontal) null-index of the point $\tilde Q$.

THEOREM 2. a) *If the vertical intrinsic null-index of NF^l in the Sasaki metric is equal to ν, then on F^l there are ν linearly independent normal vector fields parallel in the normal connection.*

b) *Let F^l be a surface in Euclidean space E^{l+p}. If the horizontal intrinsic null-index of NF^l in the Sasaki metric is equal to k, then F^l can be stratified into k-dimensional intrinsically flat, totally geodesic submanifolds with flat normal connection in the ambient space.*

For the Sasaki metric of the tangent bundle TM^n of a Riemannian manifold M^n the following results hold:

a) The Sasaki metric of TM^n is flat if and only if the metric of M^n is flat [7].

b) If the intrinsic null-index of the Sasaki metric of TM^n is equal to k, then k is even and M^n is the metric product of a Riemannian manifold $M^{n-k/2}$ with Euclidean space $E^{k/2}$, while TM^n is the metric product of $TM^{n-k/2}$ with Euclidean space E^k [8].

The normal bundle with the Sasaki metric is used for the study of extrinsic geometric properties of surfaces in a Riemannian space.

A surface F^l in a Riemannian space M^{l+p} is called k-parabolic if the second quadratic form of the surface after reduction to diagonal form has at least k null coefficients. In other words, the rank of the second quadratic form of the surface

$$r(Q) = \max_{\xi \in N_Q F^l} r(Q, \xi),$$

where $N_Q F^l$ is the normal space at the point Q and $r(Q, \xi)$ is the rank of the second quadratic form of the surface with respect to the normal ξ, satisfies at each point the inequality $r(Q) \leq l - k$ [9].

By $r^*(Q, \xi)$ we denote the maximal rank of the second quadratic form of the surface for points close to Q and normals close to ξ.

Let ξ be a normal at the point Q for which the rank $r(Q, \xi) = r(Q) = l - k$. Then it is constant for normals close to ξ. For the normal ξ we consider the null subspace of its second quadratic form. It satisfies

$$\sum_{\alpha=1}^{p} \xi^\alpha A^\alpha_{ij} X^i = 0, \tag{4}$$

where the A^α_{ij} are the coefficients of the second quadratic form with respect to the orthogonal basis of normals $n_{\alpha|}$. Since the rank of system (4) is constant, the solution space $L^k_{(Q,\xi)}$ depends regularly on the points and on the normals. We effect a horizontal lifting of the k-dimensional planes $L^k_{(Q,\xi)}$ to the points $\tilde{Q} = (Q, \xi)$ of the normal bundle. The horizontal lifting of the planes $L^k_{(Q,\xi)}$ in the neighborhood of $\tilde{Q}$ forms a differentiable distribution $\tilde{L}^k_{\tilde{Q}}$ on the normal bundle.

THEOREM (see [16]). *The differentiable horizontal distribution $\tilde{L}^k$ on the normal bundle of a k-parabolic surface F^l in a Riemannian space M^{l+p} is holonomic if at the points of the surface the curvature operator of M^{l+p} satisfies the condition $R(X, Y)\xi = 0$, where $X, Y \in T_Q F^l$ and ξ is an arbitrary normal to the surface at the point Q.*

The fiber $\tilde{R}^k(\tilde{Q}_0)$ is a totally geodesic submanifold of the normal bundle with Sasaki metric. If $\tilde{Q}$ is a boundary point of the fiber, then $r(Q, \xi) \geq r(Q_0, \xi)$. If the surface is complete and

$$r(Q_0, \xi) = r_0 = \max_{Q \in F^l} r(Q),$$

then $\tilde{R}^k(\tilde{Q}_0)$ is a complete Riemannian manifold. The projection of the fiber $\tilde{R}^k(Q_0)$ on the surface F^l is isometric and the image under the projection is a totally geodesic submanifold of the ambient space M^{l+p}, along which the normal ξ is parallel in the normal connection of the surface.

If in each fiber of NF^l normals are restricted to be of constant length $\rho > 0$, then we obtain the subbundle $N_\rho F^l$, which is a hypersurface in NF^l. The metric of $N_\rho F^l$ induced by the Sasaki metric of NF^l is called the Sasaki metric of $N_\rho F^l$. $N_\rho F^l$ will be called a normal sphere bundle.

By definition (1), $N(X, Y)\xi$ is a normal vector field on F^l. If η is another normal vector field, it is possible to compute the inner product $\langle N(X, Y)\xi, \eta\rangle_\perp$. We define the adjoint $\hat{N}(\xi, \eta)X$ by the equality

$$\langle \hat{N}(\xi, \eta)X, Y\rangle = \langle N(X, Y)\xi, \eta\rangle_\perp. \tag{5}$$

$\hat{N}(\xi, \eta)X$ is a vector field tangent to F^l, and in spaces of constant curvature $\hat{N}(\xi, \eta)X = [A_\xi, A_\eta]X$, where $[A_\xi, A_\eta]$ is the commutator of the matrices of the second quadratic forms with respect to the normals ξ and η. (For more details about $\hat{N}$ and its derivatives, see §2.)

Let X and Y be unit tangent vectors to F^l at the point Q; and let ξ, η, and ζ be unit normals to F^l at the point Q, with $\langle X, Y\rangle = 0$, $\langle \eta, \zeta\rangle_\perp = 0$, and $\langle \eta, \xi\rangle_\perp = \langle \zeta, \xi\rangle_\perp = 0$. Let K_{XY} be the sectional curvature of F^l in the direction of the area element of the vectors (X, Y).

THEOREM 3. *If at each point $Q \in F^l$ for any fixed $\xi \in N_Q F^l$ the following inequality holds:*

$$\frac{\langle (\nabla_X \hat{N})(\xi, \eta)X, Y\rangle^2}{\|\hat{N}(\xi, \eta)X\|^2} + \frac{\langle (\nabla_Y \hat{N})(\xi, \zeta)Y, X\rangle^2}{\|\hat{N}(\xi, \zeta)Y\|^2} + \frac{\rho^2}{4}\left[3\langle \hat{N}(\eta, \zeta)X, Y\rangle - \rho^2\langle \hat{N}(\xi, \zeta)X, \hat{N}(\xi, \eta)Y\rangle + \frac{\rho^2}{2}\langle \hat{N}(\xi, \zeta)Y, \hat{N}(\xi, \eta)X\rangle\right]^2 \le K_{XY} - \frac{3\rho^2}{4}\|N(X, Y)\xi\|_\perp^2,$$

then $N_\rho F^l$ with the Sasaki metric is a manifold with nonnegative sectional curvature.

For the Sasaki metric of the tangent sphere bundle $T_\rho M^n$ an analogous theorem holds [11]. However, here we state only consequences of it. Let $R(X, Y)Z$ be the curvature tensor of M^n, and K_{XY} the sectional curvature of M^n in the two-dimensional direction (X, Y). We introduce some notation:

$$M = \sup_{\substack{\|X\wedge Y\|=1\\ \|Z\|=1}} \|R(X, Y)Z\|, \qquad M_\nabla = \sup_{\substack{\|Y\wedge Z\|=1,\\ \|X\wedge U\|=1,\\ \|X\|=1}} \frac{|\langle (\nabla_X R)(Y, Z)X, U\rangle|}{\|R(Y, Z)X\|},$$

$$\mu = \inf_{\|X\wedge Y\|=1} K_{XY}.$$

If a) $0 \le \mu \le 1/6$, b) $M^2 \le \mu/6$, c) $M_\nabla^2 \le \mu/6$, then $T_1 M^n$ with the Sasaki metric is a manifold with nonnegative sectional curvature.

For a two-dimensional surface F^2 in a four-dimensional Euclidean space E^4 the condition for nonnegative sectional curvature of $N_\rho F^2$ has a simpler form. We denote by κ the Gaussian torsion of the surface F^2:

$$\kappa = \frac{\langle N(X,Y)\xi, \eta\rangle_\perp}{\sqrt{\|X\|^2\|Y\|^2 - \langle X,Y\rangle^2}\sqrt{\|\xi\|_\perp^2\|\eta\|_\perp^2 - \langle \xi,\eta\rangle_\perp^2}} = \frac{N_{12|12}}{\sqrt{g}},$$

where $N_{12|12}$ is the curvature tensor of the normal connection and $g = \det(g_{ij})$. The Gaussian torsion is equal to the product of the lengths of the principal semiaxes of the normal curvature ellipse [12].

Let K be the Gaussian curvature of the surface F^2.

THEOREM 4. *The sectional curvature of the Sasaki metric of $N_\rho F^2$ ($F^2 \subset E^4$) is nonnegative if and only if*

$$\Delta_1 K \le \kappa^2 \left(K - \frac{3\rho^2}{4}\kappa\right),$$

where Δ_1 is the first differential Beltrami parameter on F^2:

$$\Delta_1 \kappa = g^{ij}\frac{\partial\kappa}{\partial x^i}\frac{\partial\kappa}{\partial x^j}.$$

For the Sasaki metric of a tangent sphere bundle $T_\rho M^2$ the corresponding necessary and sufficient condition has the form (see [11])

$$\Delta_1 K \le K^3 \left(1 - \frac{3\rho^2}{4}K\right).$$

With regard to the tangent sphere bundle $T_1 S^n$ of the standard n-dimensional sphere S^n, it is known that the Sasaki metric of $T_1 S^2$ has constant sectional curvature $1/4$ (see [12]), while for $n \ge 3$ the sectional curvature $\tilde{K}$ of the Sasaki metric of $T_1 S^n$ varies within the bounds $0 \le \tilde{K} \le 5/4$ (see [13]).

Let us consider the normal sphere bundle $N_\rho V^2$ of the two-dimensional Veronese surface V^2 in the spherical space $S^4(1/3)$. Let $(u_1, \dots, u_5)$ be Cartesian coordinates in E^5, and (x_1, x_2, x_3) Cartesian coordinates in E^3. The surface whose radius vector has the form

$$u = \left(\frac{1}{\sqrt{3}}x_1x_2, \frac{1}{\sqrt{3}}x_2x_3, \frac{1}{\sqrt{3}}x_1x_2, \frac{1}{2\sqrt{3}}(x_1^2 - x_2^2), \frac{1}{6}(x_1^2 + x_2^2 - 2x_3^2)\right)$$

under the condition that $x_1^2 + x_2^2 + x_3^2 = 1$ lies on the four-dimensional sphere $S^4(1/3)$ is called the Veronese surface.

It has been proved that the Sasaki metric of $N_\rho V^2$ for $\rho = 1/2$ has constant sectional curvature $1/4$.

REMARK. Let F^2 be a compact surface with Gaussian curvature of constant sign in E^4. Then on the surface F^2 there are a point Q and a unit normal $\xi \in N_Q F^2$ such that the sectional curvature of $N_\rho F^2$ at the point $(Q, \rho\xi)$ along some area element is equal to zero. This means that, in contrast to spherical space, in a Euclidean space the curvature of $N_\rho F^2$ cannot be strictly of one sign.

§2. Main lemmas

Let the Γ^i_{jk} be the Christoffel symbols of the metric of F^l, $\mu^\alpha_{\beta|i}$, $(= \mu_{\alpha\beta|i})$ the components of the normal connection (i.e., the torsion coefficients), R^i_{jkm} the components of the curvature tensor of the metric of F^l, and $N^\alpha_{\beta|ij}$ $(= N_{\alpha\beta|ij})$ the components of the curvature tensor of the normal connection.

By the definition (5), $\langle \tilde{N}(\xi,\eta)X, Y\rangle = \langle N(X,Y)\xi, \eta\rangle_\perp$. Since the fiber metric $g^\perp$ is Euclidean, then $\tilde{N}^i_{j|\alpha\beta} = g^{is}N_{\alpha\beta|sj}$. By $Ng_{\alpha\beta}$ we denote the components of the Sasaki metric of NF^l. By definition of the line element (1) we easily get

LEMMA 1. *The covariant components of the Sasaki metric of NF^l are*

$$Ng_{ij} = g_{ij} + \mu^{\alpha}_{\tau|i}\mu_{\alpha\sigma|j}\xi^{\tau}\xi^{\sigma},$$
$$Ng_{il+\beta} = \mu_{\beta\tau|i}\xi^{\tau}, \qquad Ng_{l+\alpha\, l+\beta} = \delta_{\alpha\beta}.$$

LEMMA 2. *The Christoffel symbols of the first kind of the Riemannian connection of the Sasaki metric of NF^l are*

$$\tilde{\Gamma}_{ij,k} = \Gamma_{ij,k} + \frac{1}{2}\left[\frac{\partial}{\partial x^j}(\mu^{\alpha}_{\tau|i}\mu_{\alpha\sigma|k}) + \frac{\partial}{\partial x^i}(\mu^{\alpha}_{\tau|j}\mu_{\alpha\sigma|k}) - \frac{\partial}{\partial x^k}(\mu^{\alpha}_{\tau|i}\mu_{\alpha\sigma|j})\right]\xi^{\tau}\xi^{\sigma},$$

$$\tilde{\Gamma}_{ij,l+\sigma} = \frac{1}{2}\left[\frac{\partial\mu_{\sigma\tau|i}}{\partial x^j} + \frac{\partial\mu_{\sigma\tau|j}}{\partial x^i} + \mu_{\sigma\alpha|i}\mu^{\alpha}_{\tau|j} + \mu_{\sigma\alpha|j}\mu^{\alpha}_{\tau|i}\right]\xi^{\tau},$$

$$\tilde{\Gamma}_{l+\alpha\, i,j} = \frac{1}{2}[N_{\sigma\tau|ij} + 2\mu_{\sigma\tau|i}\mu^{\sigma}_{\tau|j}]\xi^{\tau},$$

$$\tilde{\Gamma}_{i\, l+\tau,l+\sigma} = \mu_{\sigma\tau|i}, \quad \tilde{\Gamma}_{l+\tau\, l+\beta,i} = 0, \quad \tilde{\Gamma}_{l+\tau\, l+\beta,l+\sigma} = 0.$$

LEMMA 3. *The contravariant components of the Sasaki metric of NF^l are*

$$Ng^{ij} = g^{ij}, \qquad Ng^{i\, l+\alpha} = -g^{is}\mu^{\alpha}_{\tau|s}\xi^{\tau},$$
$$Ng^{l+\alpha\, l+\beta} = \delta^{\alpha\beta} + g^{ij}\mu^{\alpha}_{\tau|i}\mu^{\beta}_{\sigma|j}\xi^{\tau}\xi^{\sigma}.$$

LEMMA 4. *The Christoffel symbols of the second kind of the Riemannian connection of the Sasaki metric of NF^l are*

$$\tilde{\Gamma}^k_{ij} = \Gamma^k_{ij} + \frac{1}{2}[\hat{N}^k_{i|\nu\alpha}\mu^{\alpha}_{\lambda|j} + \hat{N}^k_{j|\nu\alpha}\mu^{\alpha}_{\lambda|i}]\xi^{\nu}\xi^{\lambda},$$

$$\begin{aligned}\tilde{\Gamma}^{l+\sigma}_{ij} = {} & \frac{1}{2}\mu^{\sigma}_{\lambda|t}[\hat{N}^t_{i|\beta\nu}\mu^{\beta}_{\tau|j} + \hat{N}^t_{j|\beta\nu}\mu^{\beta}_{\tau|i}]\xi^{\mu}\xi^{\nu}\xi^{\lambda} \\ & + \frac{1}{2}\left[\frac{\partial\mu^{\sigma}_{\lambda|i}}{\partial x^j} + \frac{\partial\mu^{\sigma}_{\lambda|j}}{\partial x^i} + \mu^{\sigma}_{\alpha|i}\mu^{\alpha}_{\lambda|j} + \mu^{\sigma}_{\alpha|j}\mu^{\alpha}_{\lambda|i} - 2\Gamma^t_{ij}\mu^{\sigma}_{\lambda|t}\right]\xi^{\lambda},\end{aligned}$$

$$\tilde{\Gamma}^k_{l+\alpha\, j} = \tfrac{1}{2}\hat{N}^k_{j|\nu\alpha}\xi^{\nu}, \qquad \tilde{\Gamma}^{l+\sigma}_{l+\alpha\, j} = \mu^{\sigma}_{\alpha|j} + \tfrac{1}{2}\mu^{\sigma}_{\lambda|t}\hat{N}^t_{j|\alpha\nu}\xi^{\nu}\xi^{\lambda},$$
$$\tilde{\Gamma}^k_{l+\alpha\, l+\beta} = 0, \qquad \tilde{\Gamma}^{l+\sigma}_{l+\alpha\, l+\beta} = 0.$$

LEMMA 5. *Let X and Y be tangent vector fields and η and ζ normal vector fields on F^l. For the Lie brackets of their horizontal and vertical lifts at the point $\tilde{Q} = (Q, \xi)$ the following equalities hold:*

$$[\eta^V, \zeta^V] = 0; \qquad [X^H, \eta^V] = (\nabla^{\perp}_X\eta)^V;$$
$$\pi_*[H^H, Y^H] = [X, Y]; \qquad K[X^H, Y^H] = -N(X, Y)\xi.$$

LEMMA 6. *If X and Y are tangent vector fields and η and ζ normal vector fields on F^l, then at each point $\tilde{Q} = (Q, \xi)$*

$$\tilde{\nabla}_{\eta^V}\zeta^V = 0, \qquad \tilde{\nabla}_{X^H}\eta^V = (\nabla^{\perp}_X\eta)^V + \tfrac{1}{2}(\hat{N}(\xi, \eta)X)^H,$$

$$\tilde{\nabla}_{\eta^V}Y^H = \tfrac{1}{2}(\hat{N}(\xi, \eta)Y)^H, \qquad \tilde{\nabla}_{X^H}Y^H = (\nabla_X Y)^H - \tfrac{1}{2}(N(X, Y)\xi)^V,$$

where $\tilde{\nabla}$ is covariant differentiation in the Riemannian connection of the Sasaki metric of NF^l.

With the aid of the definitions of the maps π_* and K and also the definitions of the horizontal and vertical lifts (2), it is easy to see that for any vector $\tilde{X}$ tangent to NF^l there is a decomposition

$$\tilde{X} = (\pi_* \tilde{X})^H + (K\tilde{X})^V. \tag{6}$$

Consequently, for computation of the curvature tensor of the Sasaki metric of NF^l, it suffices to calculate it for various combinations of horizontally and vertically lifted vector fields.

We define covariant derivatives of the tensors N and $\hat{N}$ by the equalities

$$(\nabla_Z^{\perp} N)(X, Y)\xi = \left(\frac{\partial N^\alpha_{\beta|ij}}{\partial x^k} + \mu^\alpha_{\tau|k} N^\tau_{\beta|ij} - \mu^\tau_{\beta|k} N^\alpha_{\tau|ij} - \Gamma^s_{ik} N^\alpha_{\beta|sj} - \Gamma^s_{jk} N^\alpha_{\beta|is}\right) \xi^\beta Z^k X^j Y^i n_{\alpha|},$$

$$(\nabla_X \hat{N})(\xi, \eta)Z = \left(\frac{\partial \hat{N}^i_{j|\alpha\beta}}{\partial x^k} + \Gamma^i_{sk} \hat{N}^s_{j|\alpha\beta} - \Gamma^s_{jk} \hat{N}^i_{s|\alpha\beta} - \mu^\tau_{\alpha|k} \hat{N}^i_{j|\tau\beta} - \mu^\tau_{\beta|k} N^i_{j|\alpha\tau}\right) X^k Z^j \xi^\alpha \eta^\beta \frac{\partial}{\partial x^i}$$

(X, Y, and Z are tangent vector fields and ξ and η normal vector fields on F^l).

With the aid of the definition of the curvature tensor and Lemmas 5 and 6, it is simple to demonstrate the following statement.

LEMMA 7. *The curvature tensor of the Sasaki metric of NF^l at the point $\tilde{Q} = (Q, \xi)$ is determined by*

$$\begin{aligned}\tilde{R}(X^H, Y^H)Z^H = {} & [R(X, Y)Z + \tfrac{1}{4}\hat{N}(\xi, N(Z, Y)\xi)X \\ & + \tfrac{1}{4}\hat{N}(\xi, N(X, Z)\xi)Y + \tfrac{1}{2}\hat{N}(\xi, N(X, Y)\xi)Z]^H \\ & + \tfrac{1}{2}[(\nabla_Z^{\perp} N)(X, Y)\xi]^V,\end{aligned}$$

$$\begin{aligned}\tilde{R}(X^H, Y^H)\eta^V = {} & \tfrac{1}{2}[(\nabla_X \hat{N})(\xi, \eta)Y - (\nabla_Y \hat{N})(\xi, \eta)X]^H \\ & + [N(X, Y)\eta + \tfrac{1}{4}N(\hat{N}(\xi, \eta)Y, X)\xi - \tfrac{1}{4}N(\hat{N}(\xi, \eta)X, Y)\xi]^V,\end{aligned}$$

$$\tilde{R}(X^H, \eta^V)Z^H = \tfrac{1}{2}[(\nabla_X \hat{N})(\xi, \eta)Z]^H + \left[\tfrac{1}{2}N(X, Z)\eta + \tfrac{1}{4}N(\hat{N}(\xi, \eta)Z, X)\xi\right]^V,$$

$$\tilde{R}(X^H, \eta^V)\zeta^V = -\left[\tfrac{1}{2}\hat{N}(\eta, \zeta)X + \tfrac{1}{4}\hat{N}(\xi, \eta)\hat{N}(\xi, \zeta)X\right]^H,$$

$$\tilde{R}(\eta^V, \zeta^V)V^H = \left[\hat{N}(\eta, \zeta)Z + \tfrac{1}{4}\hat{N}(\xi, \eta)\hat{N}(\xi, \zeta)Z - \tfrac{1}{4}\hat{N}(\xi, \zeta)\hat{N}(\xi, \eta)Z\right]^H,$$

$$\tilde{R}(\eta^V, \zeta^V)\varphi^V = 0.$$

We consider a normal bundle of vectors of fixed length. A local embedding $N_\rho F^l \subset NF^l$ is given by the condition

$$\sum_{\alpha=1}^{p} (\xi^\alpha)^2 = \rho^2.$$

Let the indices $\varphi, \psi, \theta, \kappa, \tau = 1, \dots, p-1$. From the last equality it follows that

$$\xi^p = \sqrt{\rho^2 - \sum_{\theta=1}^{p-1}(\xi^\theta)^2}. \tag{7}$$

As natural local coordinates for $N_\rho F^l$ we choose the coordinates $(x^1, \dots, x^l, \xi^1, \dots, \xi^{p-1})$, where $(x^1, \dots, x^l)$ are local coordinates on F^l, and $(\xi^1, \dots, \xi^{p-1})$ are the coordinates of a normal of fixed length.

If $y^1, \dots, y^{l+p}$ are natural induced coordinates on NF^l, then the embedding $N_\rho F^l \subset NF^l$ has the form

$$y^i = x^i, \quad y^{l+\varphi} = \xi^\varphi, \quad y^{l+p} = \sqrt{\rho^2 - \sum_{\theta=1}^{p-1}(\xi^\theta)^2}.$$

From (7) it follows that

$$\frac{\partial \xi^p}{\partial x^i} = 0, \qquad \frac{\partial \xi^p}{\partial \xi^\theta} = -\frac{\xi^\theta}{\xi^p}.$$

The derivative $\partial\xi^p/\partial\xi^\theta$ hereafter will be denoted by B_θ.

LEMMA 8. *In natural induced coordinates the components of the Sasaki metric of $N_\rho F^l$ are*

$$N_1 g_{ij} = g_{ij} + \mu^\alpha_{\lambda|i}\mu_{\alpha\nu|j}\xi^\lambda\xi^\nu,$$

$$N_1 g_{i\ l+\theta} = (\mu_{\theta\lambda|i} + \mu_{p\lambda|i}B_\theta)\xi^\lambda, \qquad N_1 g_{l+\varphi\ l+\theta} = \delta_{\varphi\theta} + B_\varphi B_\theta.$$

LEMMA 9. *The contravariant components of the Sasaki metric of $N_\rho F^l$ are*

$$N_1 g^{ij} = g^{ij}, \qquad N_1 g^{i\ l+\theta} = -g^{is}\mu^\theta_{\lambda|s}\xi^\lambda,$$

$$N_1 g^{l+\varphi\ l+\theta} = \delta^{\varphi\theta} + g^{is}\mu^\varphi_{\lambda|i}\mu^\theta_{\nu|s}\xi^\lambda\xi^\nu - \xi^\theta\xi^\varphi.$$

We denote by $\bar\Gamma$ the Christoffel symbols of the Sasaki metric of $N_\rho F^l$. If $\tilde\Gamma$ denotes the Christoffel symbols of the Sasaki metric of NF^l (see Lemmas 2 and 4), and Ω is the matrix of the second quadratic form of the embedding $N_\rho F^l \subset NF^l$ (see Lemma 13), then the following statements are true.

LEMMA 10. *The Christoffel symbols of the first kind of the Sasaki metric of $N_\rho F^l$ are*

$$\bar\Gamma_{ij,k} = \tilde\Gamma_{ij,k}, \qquad \bar\Gamma_{ij,l+\theta} = \tilde\Gamma_{ij,l+\theta} + \tilde\Gamma_{ij,l+p}B_\theta,$$

$$\bar\Gamma_{i\,l+\varphi,k} = \tilde\Gamma_{i\,l+\varphi,k} + \tilde\Gamma_{i\,l+p,k}B_\varphi,$$

$$\bar\Gamma_{i\,l+\varphi,l+\theta} = \tilde\Gamma_{i\,l+\varphi,l+\theta} + \tilde\Gamma_{i\,l+\varphi,l+p}B_\theta + \tilde\Gamma_{i\,l+p,l+\theta}B_\varphi,$$

$$\bar\Gamma_{l+\kappa\ l+\theta,k} = \frac{1}{\xi_p}\mu_{p\lambda|k}(\delta_{\kappa\theta} + B_\kappa B_\theta)\xi^\lambda,$$

$$\bar\Gamma_{l+\kappa\ l+\varphi,l+\theta} = -\frac{1}{\xi_p}(\delta_{\kappa\varphi} + B_\kappa B_\varphi)B_\theta.$$

LEMMA 11. *The Christoffel symbols of the second kind of the Sasaki metric of $N_\rho F^l$ are*

$$\bar\Gamma^k_{ij} = \tilde\Gamma^k_{ij}, \qquad \bar\Gamma^{l+\theta}_{ij} = \tilde\Gamma^{l+\theta}_{ij} - \xi^\theta\Omega_{ij},$$

$$\bar\Gamma^k_{i\ l+\varphi} = \tilde\Gamma^k_{i\ l+\varphi} + \tilde\Gamma^k_{i\ l+p}B_\varphi, \qquad \bar\Gamma^{l+\theta}_{i\ l+\varphi} = \tilde\Gamma^{l+\theta}_{i\ l+\varphi} + \tilde\Gamma^{l+\theta}_{i\ l+p}B_\varphi - \xi^\theta\Omega_{i\ l+\varphi},$$

$$\bar{\Gamma}^{k}_{l+\kappa\ l+\varphi} = 0, \qquad \bar{\Gamma}^{l+\theta}_{l+\kappa\ l+\varphi} = -\xi^{\theta}\Omega_{l+\kappa\ l+\varphi}.$$

For calculation of the curvature tensor of the Sasaki metric of $N_\rho F^l$ Lemmas 12 and 13 are used.

LEMMA 12. *Let $\bar{Q} = (Q, \rho\xi)$ (ξ a unit vector) be an arbitrary point of $N_\rho F^l$. A unit normal to $N_\rho F^l$ in NF^l at the point $\bar{Q}$ is the vertical lift of the unit normal ξ to the point $\bar{Q}$ in the ambient space NF^l.*

Lemma 12 signifies that a vector $\tilde{X}$ tangent to NF^l is tangent to $N_\rho F^l$ if and only if $\langle\langle \tilde{X}, \xi^V\rangle\rangle = 0$. This means that $\langle K\tilde{X}, \xi\rangle_\perp = 0$, where K is the connection map of NF^l. We define a connection map K_1 by restricting K to vectors tangent to $N_\rho F^l$: $K_1 = K|_{TN_\rho F^l}$.

We denote by $N^\perp_Q(\xi)$ the orthogonal complement of the normal ξ in $N_Q F^l$. Then $K_1 \colon T_{\bar{Q}} N_\rho F^l \to N^\perp_Q(\xi)$. If $\langle\langle\ ,\ \rangle\rangle_1$ denotes the inner product of vectors tangent to $N_\rho F^l$ in the Sasaki metric of $N_\rho F^l$, then

$$\langle\langle \bar{X}, \bar{Y}\rangle\rangle_1 = \langle \pi_* \bar{X}, \pi_* \bar{Y}\rangle + \langle K_1\bar{X}, K_1\bar{Y}\rangle_\perp.$$

If $X \in T_Q F^l$, its horizontal lift X^H to the point $\bar{Q} = (Q, \rho\xi)$ is tangent to $N_\rho F^l$ at $\bar{Q}$, and also $\langle\langle X^H, \xi^V\rangle\rangle = 0$. If $\eta \in N^\perp_Q(\xi)$, its vertical lift to the point $\bar{Q} = (Q, \rho\xi)$ is also tangent to $N_\rho F^l$ at the point $\bar{Q}$, and moreover $\langle\langle \eta^V, \xi^V\rangle\rangle = \langle \eta, \xi\rangle_\perp = 0$. Therefore, the horizontal and vertical lifts to $TN_\rho F^l$ and TNF^l will not be distinguished, assuming that in a vertical lift to the point $\bar{Q} = (Q, \rho\xi)$ the vectors are taken from $N^\perp_Q(\xi)$.

Similarly to (6), for any vector $\bar{X}$ tangent to $N_\rho F^l$ at the point $\bar{Q} = (Q, \rho\xi)$ there is a decomposition

$$\bar{X} = (\pi_*\bar{X})^H + (K_1\bar{X})^V. \tag{8}$$

LEMMA 13. a) *The matrix Ω of the second quadratic form of the embedding $N_\rho F^l \subset NF^l$ has the form*

$$\Omega = \frac{1}{\rho}\begin{bmatrix} g_{ij} - N_1 g_{ij} & -N_1 g_{i\ l+\theta} \\ -N_1 g_{l+\varphi\ j} & -N_1 g_{l+\varphi\ l+\theta}\end{bmatrix}.$$

b) *If $\bar{X}$ and $\bar{Y}$ are vectors tangent to $N_\rho F^l$, then*

$$\Omega(\bar{X}, \bar{Y}) = -\frac{1}{\rho}\langle K_1\bar{X}, K_1\bar{Y}\rangle_\perp.$$

Let $\bar{R}$ and $\tilde{R}$ be the curvature tensors of the Sasaki metrics of $N_\rho F^l$ and NF^l respectively. If $\bar{X}$, $\bar{Y}$, $\bar{Z}$, and $\bar{U}$ are tangent to $N_\rho F^l$ at the point $\bar{Q} = (Q, \rho\xi)$, then by Gauss's equation we have

$$\langle\langle \bar{R}(\bar{X}, \bar{Y})\bar{Z}, \bar{U}\rangle\rangle_1 = \langle\langle \tilde{R}(\bar{X}, \bar{Y})\bar{Z}, \bar{U}\rangle\rangle + \Omega(\bar{X}, \bar{U})\Omega(\bar{Y}, \bar{Z}) - \Omega(\bar{X}, \bar{Z})\Omega(\bar{Y}, \bar{U}).$$

Using Lemma 7, we find that the following is true.

LEMMA 14. *Let $X, Y, Z, U \in T_Q F^l$, let ξ be a unit normal vector to F^l at the point Q, and let $\varphi, \psi, \eta, \zeta \in N^\perp_Q(\xi)$.*

At the point $\bar{Q} = (Q, \rho\xi)$ the curvature tensor $\bar{R}$ of the Sasaki metric of $N_\rho F^l$ is determined by the following formulas:

$$\begin{aligned}\langle\langle \bar{R}(X^H, Y^H)Z^H, U^H\rangle\rangle_1 = \langle R(X, Y)Z, U\rangle &+ \frac{\rho^2}{4}\langle N(X, U)\xi, N(Z, Y)\xi\rangle_\perp \\ &+ \frac{\rho^2}{4}\langle N(X, Z)\xi, N(Y, U)\xi\rangle_\perp + \frac{\rho^2}{2}\langle N(X, Y)\xi, N(Z, U)\xi\rangle_\perp,\end{aligned}$$

For the proof it suffices to use the decomposition of $\tilde{X}$ and $\tilde{Y}$ into components, the linearity of the curvature tensor, and Lemma 14.

§3. Proofs of the theorems

PROOF OF THEOREM 1. The sufficiency of the conditions is evident. We shall show the necessity.

Let the Sasaki metric of NF^l be flat. Then by Lemma 7 with $\tilde{Z} = Z^H$, $\tilde{X} = X^H$, $\tilde{Y} = \eta^V$ $(X, Z = T_Q F^l,\ \eta \in N_Q F^l)$ we get

$$\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z} = \left[\tfrac{1}{2}N(X, Z)\eta + \tfrac{1}{4}N(\hat{N}(\xi, \eta)Z, X)\xi\right]^V + \left[\tfrac{1}{2}(\nabla_X \hat{N})(\xi, \eta)Z\right]^H \equiv 0.$$

For $\xi = \eta$ this implies that $N(X, Z)\eta = 0$ for any X, Z, η. Thus the normal connection is flat.

For $\tilde{X} = X^H$, $\tilde{Y} = Y^H$, and $\tilde{Z} = Z^H$ (taking into account that the normal connection is flat), by Lemma 7 we get

$$\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z} = [R(X, Y)Z]^H \equiv 0.$$

Consequently, the metric of F^l is flat.

The theorem is proved.

PROOF OF THEOREM 2. a) Let $\tilde{L}$ be a vertical distribution on NF^l of dimension ν such that for any $\tilde{Y} \in \tilde{L}_{(Q,\xi)}$ and any $\tilde{X}, \tilde{Z} \in T_{(Q,\xi)}NF^l$ we have

$$\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z} = 0. \tag{9}$$

We consider a subdistribution $\tilde{L}_{(Q,\lambda\xi)}$ for fixed ξ and $\lambda \in \mathbf{R}$. Since $\tilde{L}_{(Q,\lambda\xi)}$ is vertical, for any $\tilde{Y} \in \tilde{L}_{(Q,\lambda\xi)}$ we have $\tilde{Y} = (\eta_{\lambda\xi})^V$, where $\eta_{\lambda\xi} = K\tilde{Y}$ and lift is effected to the point $(Q, \lambda\xi)$. Using the decomposition (6) $\tilde{X} = (X_H)^H + (X_V)^V$, $\tilde{Z} = (Z_H)^H + (Z_V)^V$, we write (9) in the form:

$$\begin{aligned}&\left[\frac{\lambda}{2}(\nabla_{X_H}\hat{N})(\xi, \eta_{\lambda\xi})Z_H - \frac{1}{2}\hat{N}(\eta_{\lambda\xi}, Z_V)X_H - \frac{\lambda^2}{4}\hat{N}(\xi, \eta_{\lambda\xi})\hat{N}(\xi, Z_V)X_H\right.\\ &\qquad\left. + \hat{N}(X_V, \eta_{\lambda\xi})Z_H + \frac{\lambda^2}{4}\hat{N}(\xi, X_V)\hat{N}(\xi, \eta_{\lambda\xi})Z_H - \frac{\lambda^2}{4}\hat{N}(\xi, \eta_{\lambda\xi})\hat{N}(\xi, X_V)Z_H\right]^H\\ &\qquad + \left[\frac{1}{2}N(X_H, Z_H)\eta_{\lambda\xi} + \frac{\lambda^2}{4}N(\hat{N}(\xi, \eta_{\lambda\xi})Z_H, X_H)\xi\right]^V,\end{aligned} \tag{10}$$

where lifts are effected to the point $(Q, \lambda\xi)$. For $\lambda = 0$ the distribution $\tilde{L}_{(Q,0)}$ is a distribution on F^l as a submanifold in NF^l given by the null cross-section. The distribution $L = K\tilde{L}_{(Q,0)}$ is a distribution of ν-dimensional normal subspaces on F^l.

By virtue of the regularity of the distribution $\tilde{L}$, the vector $\eta_0 = \lim_{\lambda\to 0}\eta_{\lambda\xi}$ lies in L_Q. From (10) it follows that for any $X, Z \in T_Q F^l$ we have a) $N(X, Z)\eta_0 = 0$, and b) $\lim_{\lambda\to 0} N(X, Z)\eta_{\lambda\xi}/\lambda = 0$.

Since $\langle N(X, Z)\eta, \zeta\rangle_\perp = \langle \hat{N}(\eta, \zeta)X, Z\rangle$ for any $X, Z \in T_Q F^l$ and $\eta, \zeta \in N_Q F^l$, b) implies that $\lim_{\lambda\to 0}\hat{N}(\eta_{\lambda\xi}, \zeta)/\lambda = 0$. Dividing the horizontal projection (10) by λ and passing to the limit as $\lambda \to 0$, we get

$$(\nabla_X \hat{N})(\xi, \eta_0)Z = 0 \tag{11}$$

for any $X, Z \in T_Q F^l$, $\xi \in N_Q F^l$, and $\eta_0 \in L_Q$.

It is not hard to verify that

$$\langle(\nabla_X \hat{N})(\xi, \eta)Z, Y\rangle = \langle(\nabla_X^\perp N)(Z, Y)\xi, \eta\rangle_\perp. \tag{12}$$

From (11) and (12) it follows that

$$(\nabla^{\perp}_{X}N)(Y,Z)\eta_0 = 0$$

for any $X, Y, Z \in T_QF^l$ and $\eta_0 \in L$.

Thus, for the curvature tensor of the normal connection the following two equalities hold:

$$\begin{gathered} N(Z,Y)\eta = 0, \\ (\nabla^{\perp}_{X}N)(Z,Y)\eta = 0 \end{gathered} \tag{13}$$

as soon as $\eta \in L$ and for any $X, Z, Y \in T_QF^l$.

Equalities (13) are similar to the conditions imposed on the curvature tensor of a Riemannian manifold which guarantee existence on the manifold of a field of parallel vectors ([3], §23). Namely, let U be a tangent vector field on the Riemannian manifold M^n, and R^i_{jkm} the curvature tensor of M^n. If the equations

$$\frac{\partial U^i}{\partial x^k} + \Gamma^i_{jk}U^j = 0$$

and the first q ($q \geq 0$) of the systems of equations

$$\begin{gathered} R^i_{jkm}U^j = 0, \\ \nabla_{s_1}R^i_{jkm}U^j = 0, \\ \nabla_{s_2}\nabla_{s_1}R^i_{jkm}U^j = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{gathered}$$

admit a complete system of solutions which also satisfy the $(q+1)$th system of these equations, then there exists a collection of parallel vector fields depending on arbitrary constants.

As a simple analysis shows, with R^i_{jkm} replaced by $N^{\alpha}_{\beta|ij}$, Γ^i_{jk} by $\mu^{\alpha}_{\beta|i}$, and the tangent vector field U by a normal vector field η, the proof of this fact is carried over to normal vector fields.

In our case the dimension of L is equal to ν and each of ν basis vector fields satisfies (13), which (with $q = 0$) guarantees the existence of ν linearly independent normal vector fields parallel in the normal connection.

b) Let F^l be a surface in the Euclidean space E^{l+p}. Let $\tilde{L}$ be the horizontal distribution on NF^l of dimension k indicated in the condition. Since F^l is a submanifold in NF^l, given by the null cross-section ($\xi = 0$), we consider the restriction $\tilde{L}|_{F^l}$. Then $L = \pi_*(\tilde{L}|_{F^l})$ is a distribution on F^l, and each vector $\tilde{Y} \in \tilde{L}|_{F^l}$ is the horizontal lift of a vector $Y = \pi_*\tilde{Y}$ to the point $(Q,0)$. Since for $\tilde{L}|_{F^l}$ equality (9) is valid, then, representing the vectors $\tilde{X}, \tilde{Z} \in T_{\tilde{Q}}NF^l$ and $\tilde{Y} \in \tilde{L}_{\tilde{Q}}$ at the point $\tilde{Q} = (Q,0)$ in the form

$$\tilde{X} = (X_H)^H + (X_V)^V, \quad \tilde{Z} = (Z_H)^H + (Z_V)^V, \quad \tilde{Y} = Y^H,$$

where $X_H, Z_H \in T_QF^l$, $X_V, Z_V \in N_QF^l$, and $Y \in L_Q$, we get

$$\begin{aligned} \tilde{R}(\tilde{X},\tilde{Y})\tilde{Z} &= \tilde{R}((X_H)^H + (X_V)^V, Y^H)((Z_H)^H + (Z_V)^V)|_{\xi=0} \\ &= [R(X_H,Y)Z_H + \tfrac{1}{2}\tilde{N}(X_V,Z_V)y]^H \\ &\quad + [-\tfrac{1}{2}N(Y,Z_H)X_V + N(X_H,Y)Z_V]^V = 0. \end{aligned} \tag{14}$$

Since $\tilde{X}$ is an arbitrary vector, if we set $X_V = 0$ for any $Z, X \in T_QF^l$ in the horizontal projection, we get $R(X,Y)Z = 0$, as soon as $Y \in L$. Consequently, the intrinsic null-index of F^l is equal to k. As Chern and Kuiper showed [5], in this case F^l can be stratified into k-dimensional intrinsically flat submanifolds F^k totally geodesic in F^l.

Moreover, with $X_V = 0$ in the vertical projection of (14) we get $N(X, Y)\zeta = 0$ for $X \in T_Q F^l$, $\zeta \in N_Q F^l$, and $Y \in L_Q$. With the aid of (5), it is easy to see that in this case $\hat{N}(\eta, \zeta)Y = 0$ for any $\eta, \zeta \in N_Q F^l$ and $Y \in L_Q$. But in a space of constant curvature $\hat{N}(\zeta, \eta)Y = [A_\zeta, A_\eta]Y$, where A_ζ and A_η are the matrices of the second quadratic forms of $F^l \subset E^{l+p}$ relative to the normals $\zeta, \eta \in N_Q F^l$. Consequently, $[A_\zeta, A_\eta] = 0$ as soon as $Y \in L$. Since the matrices A_ζ and A_η are symmetric, the last equality signifies that there exists an orthogonal basis in which the restrictions of all second quadratic forms A_ζ on F^k relative to the normals in E^{l+p} are simultaneously reduced to diagonal form. But F^k is totally geodesic in F^l. Consequently, the second quadratic forms of F^k relative to normals in F^l are identically zero. So over F^k the matrices of all second quadratic forms are simultaneously reduced to diagonal form. This means that the normal connection of F^k in the ambient space is flat.

PROOF OF THEOREM 3. Let $\bar{A}$, $\bar{B}$ be a pair of orthonormal vectors tangent to $N_\rho F^l$ at the point $\bar{Q} = (Q, \rho\xi)$.

We decompose $\bar{A}$ and $\bar{B}$ into components (8):

$$\bar{A} = A^H + a^V, \qquad \bar{B} = B^H + b^V,$$

where $A = \pi_* \bar{A}$, $a = K_1 \bar{A}$, $B = \pi_* \bar{B}$, and $b = K_1 \bar{B}$. By assumption $\langle\langle \bar{A}, \bar{B} \rangle\rangle_1 = 1$. This means that

$$\langle A, B \rangle + \langle a, b \rangle_\perp = 0.$$

We show that in the two-dimensional plane of the vectors $\bar{A}$, $\bar{B}$ there is a pair of vectors $\bar{A}_1$, $\bar{B}_1$ for which

$$\langle A_1, B_1 \rangle = \langle a_1, b_1 \rangle_\perp = 0.$$

In fact, in the two-dimensional plane of the vectors $\bar{A}$, $\bar{B}$ we can pass to a new basis

$$\bar{A}_1 = \bar{A}\cos\varphi + \bar{B}\sin\varphi, \qquad \bar{B}_1 = -\bar{A}\sin\varphi + \bar{B}\cos\varphi.$$

Then

$$A_1 = \pi_* \bar{A}_1 = A\cos\varphi + B\sin\varphi, \qquad B_1 = \pi_* \bar{B}_1 = -A\sin\varphi + B\cos\varphi.$$

We require that $\langle A_1, B_1 \rangle = 0$. We have

$$\begin{aligned}\langle A_1, B_1 \rangle &= \langle A, B \rangle \cos^2\varphi + (\|B\|^2 - \|A\|^2)\sin\varphi\cos\varphi - \langle A, B \rangle \sin^2\varphi \\ &= \langle A, B \rangle \cos 2\varphi + \tfrac{1}{2}(\|B\|^2 - \|A\|^2)\sin 2\varphi = 0.\end{aligned}$$

Since we assumed that originally $\langle A, B \rangle \neq 0$, we will thus find the rotation angle:

$$\cot 2\varphi = \frac{\|A\|^2 - \|B\|^2}{2\langle A, B \rangle}.$$

The condition $\langle a_1, b_1 \rangle_\perp = 0$ is fulfilled automatically.

Thus, without loss of generality we assume that $\bar{A}$ and $\bar{B}$ satisfy the conditions

$$\langle A, B \rangle = 0, \qquad \langle a, b \rangle_\perp = 0.$$

We introduce the following notation:

$$X = A/\|A\|, \quad Y = B/\|B\|, \quad \zeta = a/\|a\|_\perp,$$
$$\eta = b/\|b\|_\perp, \quad \alpha = \|a\|_\perp/\|A\|, \quad \beta = \|b\|_\perp/\|B\|.$$

Then X, Y, ζ, and η are unit vectors satisfying

$$\langle X, Y \rangle = 0, \quad \langle \eta, \zeta \rangle_\perp = 0, \quad \langle \eta, \xi \rangle_\perp = \langle \zeta, \xi \rangle_\perp = 0.$$

Using (5) and Lemma 15, we write the sectional curvature of $N_\rho F^l$ in the form

$$\begin{aligned}\tilde{K}_{\tilde{A}\tilde{B}} = \|A\|^2\|B\|^2 \Big\{ & K_{XY} - \frac{3\rho^2}{4}\|N(X,Y)\xi\|_\perp^2 \\ & + \Big[3\langle \hat{N}(\eta,\zeta)X, Y\rangle - \rho^2\langle \hat{N}(\xi,\eta)Y, \hat{N}(\xi,\zeta)X\rangle \\ & \qquad + \frac{\rho^2}{4}\langle \hat{N}(\xi,\eta)X, \hat{N}(\xi,\zeta)Y\rangle\Big]\alpha\beta \\ & + \frac{\rho^2}{4}\|\hat{N}(\xi,\eta)X\|^2\beta^2 + \frac{\rho^2}{4}\|\hat{N}(\xi,\zeta)Y\|^2\alpha^2 \\ & - \rho\langle(\nabla_Y\hat{N})(\xi,\zeta)Y, X\rangle\alpha - \rho\langle(\nabla_X\hat{N})(\xi,\eta)(\xi,\eta)X, Y\rangle\beta + \frac{1}{\rho^2}\alpha^2\beta^2\Big\}.\end{aligned}$$

The expression in curly brackets is to be regarded as a polynomial in α and β. We denote the free term and the coefficients of $\alpha\beta$, α^2, β^2, α, β by m_0, m_1, m_2, m_3, m_4, m_5 respectively. Then

$$\begin{aligned}\tilde{K}_{\tilde{A}\tilde{B}} &= \|A\|^2\|B\|^2\left\{m_0 + m_1\alpha\beta + m_2\alpha^2 + m_3\beta^2 + m_4\alpha + m_5\beta + \frac{1}{\rho^2}\alpha^2\beta^2\right\} \\ &= \|A\|^2\|B\|^2\left\{\frac{1}{\rho^2}\left(\alpha\beta + \frac{\rho^2 m_1}{2}\right)^2 + m_2\left(\alpha + \frac{m_4}{2m_2}\right)^2\right. \\ &\qquad \left. + m_3\left(\beta + \frac{m_5}{2m_3}\right)^2 + m_0 - \frac{\rho^2 m_1^2}{4} - \frac{m_4^2}{4m_2^2} - \frac{m_5^2}{4m_3^2}\right\}.\end{aligned}$$

Consequently, for nonnegativity of $\tilde{K}_{\tilde{A}\tilde{B}}$ it suffices to require that

$$m_0 - \frac{\rho^2}{4}m_1^2 - \frac{m_4^2}{4m_2^2} - \frac{m_5^2}{4m_3^2} \geq 0.$$

Restoring the substance of the notation, we get the required inequality.

The theorem is proved.

PROOF OF THEOREM 4. Let $\tilde{X}$ and $\tilde{Y}$ be mutually orthogonal unit vectors tangent to $N_\rho F^2$ at the point $\tilde{Q} = (Q, \rho\xi)$. Let $X_V = K_1\tilde{X}$ and $Y_V = K_1\tilde{Y}$. Then $X_V, Y_V \in N_Q^\perp(\xi)$. But $N_Q^\perp(\xi)$ is one-dimensional. Consequently X_V and Y_V are collinear. On the other hand,

$$\langle\langle \tilde{X}, \tilde{Y}\rangle\rangle_1 = \langle X_H, Y_H\rangle + \langle X_V, Y_V\rangle_\perp = 0,$$

where one may assume that $\langle X_V, Y_V\rangle_\perp = 0$ and $\langle X_H, Y_H\rangle = 0$. Consequently, one of the vectors (for example, X_V) may be assumed to equal the zero vector. Thus, without loss of generality we get that

$$\tilde{X} = X^H, \qquad \tilde{Y} = (Y_H)^H + (Y_V)^V, \tag{15}$$

where $X, Y_H \in T_Q F^2$ and $Y_V \in N_Q^\perp(\xi)$.

By Lemma 15, using (15), we have

$$\begin{aligned}\tilde{K}_{\tilde{X}\tilde{Y}} = {} & \langle R(X, Y_H)Y_H, X\rangle - \frac{3\rho^2}{4}\|N(X, Y_H)\xi\|_\perp^2 \\ & + \rho\langle(\nabla_{\tilde{\lambda}}^\perp N)(Y_H, X)\xi, Y_V\rangle_\perp + \frac{\rho^2}{4}\|\hat{N}(\xi, Y_V)X\|^2.\end{aligned}$$

Carrying out an argument analogous to that in the proof of Theorem 3, we get a necessary and sufficient condition for nonnegativity of the sectional curvature of $N_\rho F^2$ in the given case

$$\langle(\nabla_X^\perp N)(Y,X)\xi,\eta\rangle_\perp^2 \le \|\hat{N}(\xi,\eta)X\|^2\left(\langle R(X,Y)Y,X\rangle - \frac{3\rho^2}{4}\|N(X,Y)\xi\|_\perp^2\right),$$

where X and Y are mutually orthogonal unit vectors and ξ and η are mutually orthogonal unit normals to F^2.

Since $\hat{N}(\xi,\eta)X$ is orthogonal to X, then

$$\|\hat{N}(\xi,\eta)X\|^2 = \langle\hat{N}(\xi,\eta)X,Y\rangle^2 = \langle N(X,Y)\xi,\eta\rangle_\perp^2.$$

Similarly,

$$\|N(X,Y)\xi\|_\perp^2 = \langle N(X,Y)\xi,\eta\rangle_\perp^2.$$

Therefore the necessary and sufficient condition takes the form

$$\langle(\nabla_X^\perp N)(Y,X)\xi,\eta\rangle_\perp^2 \le \langle N(X,Y)\xi,\eta\rangle_\perp^2\left[\langle R(X,Y)Y,X\rangle - \frac{3\rho^2}{4}\langle N(X,Y)\xi,\eta\rangle_\perp^2\right].$$

Using the fact that the fiber metric is Euclidean and (X,Y) and (ξ,η) constitute unit bivectors, we arrive at the inequality

$$\langle X,\operatorname{grad}\kappa\rangle^2 \le \kappa^2\left[K - \frac{3\rho^2}{4}\kappa^2\right]$$

for any unit X, where

$$\operatorname{grad}\kappa = \left(g^{1i}\frac{\partial\kappa}{\partial x^i}, g^{2i}\frac{\partial\kappa}{\partial x^i}\right).$$

Letting $X = \operatorname{grad}\kappa/\|\operatorname{grad}\kappa\|$, we get the required inequality.

The theorem is proved.

EXAMPLE. We consider the embedding of $E^3(x_1,x_2,x_3)$ into $E^5(u_1,u_2,u_3,u_4,u_5)$ given by the radius vector

$$u = \left(\frac{1}{\sqrt{3}}x_1x_3, \frac{1}{\sqrt{3}}x_2x_3, \frac{2}{\sqrt{3}}x_1x_2, \frac{1}{2\sqrt{3}}(x_1^2-x_2^2), \frac{1}{6}(x_1^2+x_2^2-2x_3^2)\right). \tag{16}$$

We consider in E^3 the sphere $S^2(1)\colon x_1^2+x_2^2+x_3^2=1$. Then (16) gives an isometric immersion $S^2(1)\to S^4(1/3)$ by which the points (x_1,x_2,x_3) and $(-x_1,-x_2,-x_3)$ are transformed into the same point. In other words, (16) gives an isometric embedding of the real projective plane into $S^4(1/3)$, which is called the Veronese surface $V^2\subset S^4(1/3)$ (see [2]).

We shall show that the Sasaki metric of $N_\rho V^2$ for $\rho = 1/2$ has a constant sectional curvature equal to $1/4$.

Let F^2 be a two-dimensional surface in a four-dimensional Riemannian space M^4. Let K be the Gaussian curvature of the F^2-metric. Since the codimension of the embedding $F^2\subset M^4$ equals 2, the curvature tensor of the normal connection has exactly one component: $N_{12|12}$. In a coordinate system satisfying the conditions of Lemma 14′, the curvature tensor of the Sasaki metric takes the form

$$\begin{gathered}\bar{R}_{1212} = R_{1212} - \frac{3\rho^2}{4}(N_{12|12})^2,\quad \bar{R}_{1213} = \frac{\rho}{2}\frac{\partial N_{12|12}}{\partial x_1},\quad \bar{R}_{1223} = \frac{\rho}{2}\frac{\partial N_{12|12}}{\partial x_2},\\ \bar{R}_{1312} = \frac{\rho}{2}\frac{\partial N_{12|12}}{\partial x_1},\quad \bar{R}_{1313} = \frac{\rho^2(N_{12|12})^2}{4},\quad \bar{R}_{1323} = 0,\\ \bar{R}_{2312} = \frac{\rho}{2}\frac{\partial N_{12|12}}{\partial x_2},\quad \bar{R}_{2313} = 0,\quad \bar{R}_{2323} = \frac{\rho^2(N_{12|12})^2}{4},\end{gathered} \tag{17}$$

where (x_1,x_2) are local coordinates on F^2 and $R_{1212} = K$.

Let $\tilde{Q} = (Q, \rho\xi)$ be an arbitrary point of $N_\rho V^2$. Without loss of generality we can assume that ξ coincides with the second basic normal vector and that the origin of the local coordinate system of V^2 is situated at the point Q.

As local coordinates for V^2 we choose the coordinates (x_1, x_2) induced on $S^2(1)$ from E^3 by the embedding $x_1^2 + x_2^2 + x_3^2 = 1$. Then at the origin of coordinates $(x_1 = x_2 = 0)$ the Christoffel symbols of the V^2-metric vanish, and the metric itself will be diagonal.

We show that at this point both the torsion coefficients $\mu_{12|1}$ and $\mu_{12|2}$ vanish.

Normals to V^2 in S^4 are conveniently considered in the ambient space E^5. We find these normals ($n_{1|}$ and $n_{2|}$) from the conditions

$$\langle n_{1|}, u\rangle = 0, \quad \langle n_{2|}, u\rangle = 0, \quad \left\langle n_{1|}, \frac{\partial u}{\partial x_1}\right\rangle = 0,$$
$$\left\langle n_{2|}, \frac{\partial u}{\partial x_1}\right\rangle = 0, \quad \left\langle n_{1|}, \frac{\partial u}{\partial x_2}\right\rangle = 0, \quad \left\langle n_{2|}, \frac{\partial u}{\partial x_2}\right\rangle = 0,$$

where $\langle\ ,\ \rangle$ is the Euclidean inner product, and u is given by (16). Solving these systems, we find that

$$\begin{aligned} n_{1|} &= (x_2(1 + x_3^2 - 2x_1^2), x_1(1 + x_3^2 - 2x_2^2), -x_3(1 + x_3^2), 0, 2\sqrt{3}x_1x_2x_3), \\ n_{2|} &= (2x_1(1 - x_1^2), -2x_2(1 - x_2^2), 0, -x_3(1 + x_3^2), \sqrt{3}(x_1^2 - x_2^2)x_3). \end{aligned} \tag{18}$$

We introduce an orthonormal basis of normals $(\xi_{1|}, \xi_{2|})$, obtained from (18) by a standard orthogonalization. Then at the origin of coordinates $(x_1 = x_2 = 0,\ x_3 = 1)$ the following equalities hold:

$$\begin{aligned} &\xi_{1|} = (0, 0, -1, 0, 0), \quad \xi_{2|} = (0, 0, 0, -1, 0), \quad \frac{\partial \xi_{1|}}{\partial x_2} = (0, 1, 0, 0, 0), \\ &\frac{\partial \xi_{2|}}{\partial x_1} = (1, 0, 0, 0, 0), \quad \frac{\partial \xi_{1|}}{\partial x_2} = (1, 0, 0, 0, 0), \quad \frac{\partial \xi_{2|}}{\partial x_2} = (0, -1, 0, 0, 0). \end{aligned} \tag{19}$$

Hence at the origin of coordinates we have $\mu_{12|1} = \mu_{12|2} = 0$. This means that the conditions of Lemma 14′ are fulfilled and the calculation of the curvature tensor of $N_\rho V^2$ can be carried out by (17). Calculating $N_{12|12}$ at the origin of coordinates, we get

$$N_{12|12} = -2. \tag{20}$$

The derivatives of $N_{12|12}$ at the origin of coordinates vanish, a fact that one may verify by a direct computation, using (18) and (19). Thus at the point Q we get

$$\frac{\partial N_{12|12}}{\partial x_2} = 0, \qquad \frac{\partial N_{12|12}}{\partial x_1} = 0. \tag{21}$$

Substituting (20) and (21) into (17), we find that

$$\tilde{R}_{1\,2\,1\,2} = 1 - 3\rho^2, \quad \tilde{R}_{1\,3\,1\,3} = \rho^2, \quad \tilde{R}_{2\,3\,2\,3} = \rho^2,$$
$$\tilde{R}_{1\,2\,1\,3} = \tilde{R}_{1\,2\,2\,3} = \tilde{R}_{1\,3\,2\,3} = 0.$$

Setting $\rho = 1/2$, we get the required result.

We prove the remark. a) Let the Gaussian curvature of the surface F^2 be strictly positive: $K > 0$. Let ξ be the normal at the point Q for which the Gaussian curvature

$$K(\xi) = \frac{A_{11}A_{22} - A_{12}^2}{g_{11}g_{22} - g_{12}^2}$$

is maximal (the A_{ij} are the coefficients of the second quadratic form of F^2 relative to the normal ξ). Since $K > 0$, $K(\xi) > 0$. Let τ_1 and τ_2 be the principal directions of the second quadratic form, $K_1(\xi)$ and $K_2(\xi)$ the principal curvatures. If for all points we have $K_1(\xi) \neq K_2(\xi)$, then the directions τ_1 and τ_2 make up a regular net, but in this case the Euler characteristic $\chi(F^2) = 0$ (see [14]).

On the other hand, by virtue of the positivity of the Gaussian curvature, $\chi(F^2) \neq 0$. Thus there is a point $Q_0 \in F^2$ for which $K_1(\xi) = K_2(\xi)$. Consequently, the second quadratic form of F^2 relative to the normal ξ is umbilic at the point Q_0. Hence for any normal η orthogonal to ξ the commutator $[A_\xi, A_\eta]$ is 0. By Lemma 15, with $\tilde{X} = X^H$ and $\tilde{Y} = \eta^V$ at the point $\tilde{Q} = (Q_0, \rho\xi)$ the sectional curvature of $N_\rho F^2$ is equal to

$$\tilde{K}_{\tilde{X}\tilde{Y}} = \frac{\rho^2}{4}\|[A_\xi, A_\eta]X\|^2 = 0.$$

b) Let $K \leq 0$. Then in the direction of the horizontal area element of $\tilde{X} = X^H$, $\tilde{Y} = Y^H$ at the point $\tilde{Q} = (Q, \rho\xi)$ we have

$$\tilde{K}_{\tilde{X}\tilde{Y}} = K - \frac{3\rho^2}{4}\|N(X, Y)\xi\|_\perp^2 \leq 0,$$

and in the direction of the horizontal-vertical area element of $\tilde{X} = X^H$, $\tilde{Y} = \eta^V$ $(\eta \perp \xi)$ at the point $(Q, \rho\xi)$ we have

$$\tilde{K}_{\tilde{X}\tilde{Y}} = \frac{\rho^2}{4}\|[A_\xi, A_\eta]X\|^2 \geq 0.$$

Consequently, at $(Q, \rho\xi)$ there is an area element $(\tilde{X}, \tilde{Y})$ in whose direction $\tilde{K}_{\tilde{X}\tilde{Y}} = 0$.

If $K < 0$, a stronger statement is true: the sectional curvature of $N_\rho F^2$ has alternating sign. We demonstrate this. In the direction of a horizontal area, $\tilde{K}_{\tilde{X}\tilde{Y}} < 0$. In the direction of a horizontal-vertical area if only at one point $(Q, \rho\xi)$ the sectional curvature $\tilde{K}_{\tilde{X}\tilde{Y}} > 0$. Let this not be the case. Then $[A_\xi, A_\eta] \equiv 0$ for any $\xi, \eta \in N_Q F^2$ and any $X \in T_Q F^2$ at each point of F^2. This means that the normal connection of F^2 is flat. Then for all normals the principal directions coincide. Since the Gaussian curvature of F^2 is negative, on the surface there are no umbilical points and the principal directions form a regular net. Thus the Euler characteristic $\chi(F^2) = 0$; but on such a surface it is impossible to assign a metric with negative curvature [10]. Compact surfaces of negative Gaussian curvature do exist in E^4 [15].

The remark is proved.

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