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CHARACTERIZATION OF THE PROJECTIONS OF GEODESICS OF THE SASAKIAN METRIC OF TCP^n AND T_1CP^n

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Curves that are projections of geodesics of the Sasakian metric of the tangent and tangent sphere bundles of a complex projective space are considered. The main result is: THEOREM. If Γ is a geodesic of TCP^n (T_1CP^n) then $\pi_0\Gamma$ is a curve in CP^n for which curvatures $k_1, \ldots, k_5$ are constant and $k_6 = \ldots = k_{2n} = 0$.

1. Formulation of Results. Sasaki [1] has noted that the projection of a geodesic of the tangent or tangent sphere bundle of a space of constant curvature to a base is characterized by the fact that its curvatures k_1 and k_2 are constant and $k_3 = \ldots k_n = 0$. Moreover, subsequently it became possible to give a complete description of the geodesics of the tangent and tangent sphere bundles over space forms, that is, over the sphere S^n, plane E^n, and Lobachevski plane L^n [2], [3].

Nagy [4] has shown that for the tangent and tangent sphere bundles of a symmetric space the geodesics are projected onto curves all of whose curvatures $k_1, \ldots, k_n$ are constant.

The goal of this paper consists in proving the following assertion.

THEOREM. If Γ is a geodesic of TCP^n or T_1CP^n, then $\pi_0\Gamma$ is a curve in CP^n for which curvatures $k_1, \ldots, k_5$ are constant and $k_6 = \ldots k_{2n} = 0$.

Since Azo [5] has proved that the geodesics of TM^n and T_1M^n are projected onto the same curves, we consider below geodesics in T_1CP^n.

2. A Remark on Geodesics of Symmetric Spaces. Suppose that $\Gamma(\sigma) = \{x(\sigma), y(\sigma)\}$ is a curve in T_1M^n, σ is its natural parameter, and $\gamma(\sigma) = \pi_0\Gamma(\sigma)$ is the projection of $\Gamma(\sigma)$ to the base. The parameter σ is not natural for the curve γ.

However, if S is the natural parameter of γ, then $dS^2 = (1 - c^2)d\sigma^2$, where c is a constant [2]. Let us denote by "$'$" the covariant derivative with respect to σ. Then the equation of the geodesics in T_1M^n can be written as [1]

$$x'' = R(y, y')x', \quad y'' = -c^2 y, \tag{1}$$

where $c^2 = \|y'\|^2$, $R(y, y')$ is the curvature operator of M^n.

It is easy to verify that c^2 is constant along γ.

Suppose that M^n is a symmetric space. This means that $R'(y, y')x' \equiv 0$.

LEMMA 1. If $\gamma(\sigma) = \pi_0\Gamma$ is the projection of a geodesic of T_1M^n of symmetric space M^n and $\{x(\sigma)\}$ is its parametric equation, then the $p + 1$ derivative of $\gamma(\sigma)$ satisfies

$$\text{a) } x^{(p+1)} = R^p(y, y')x',$$
$$\text{b) } x^{(p+1)} = R(y, y')x^{(p)}.$$

Proof. Let us covariantly differentiate the first equation of the system. We obtain $x''' = R'(y, y')x' + R(y', y')x' + R(y, y'')x' + R(y, y')x''$.

Taking into account the symmetry of M^n, the second equation of (1), and the oblique symmetry of the curvature tensor, we find that

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$$x''' = R(y, y')x''. \tag{2}$$

Repeating this operation $p - 2$ times, we obtain Assertion b) of the lemma.

On the other hand, substituting into (2) the expression for x'' from the first equation of (1), we obtain $x''' = R^2(y, y')x'$, where $R^2(y, y')$ is the square of the curvature operator of M^n. Repeating this operation $p - 2$ times we prove Assertion a) of the lemma.

COROLLARY 1. If $\Gamma(\sigma)$ is a geodesic of T_1M^n of symmetric space M^n, then all the curvatures of $\gamma(\sigma) = \pi_0\Gamma(\sigma)$ are constant.

Proof (see also [4]). Indeed, Lemma 1 (b) implies that $\langle x^{(p+1)}, x^{(p)}\rangle = 0$. This means that $\|x^{(p)}\| = \text{const}$.

Suppose that $\xi_1(s), \ldots, \xi_n(s)$ is the Frenet frame along $x(\sigma)$. Then, taking into account the affine connection of s and σ, we find from Frenet's formulas:

$$x' = (1 - c^2)^{1/2}\xi_1,$$
$$x'' = (1 - c^2)k_1\xi_2.$$

However, $\|x''\| = \text{const}$. Consequently, $k_1 = \text{const}$.

Therefore, $x''' = (1 - c^2)^{3/2}k_1(-k_1\xi_1 + k_2\xi_3)$.

However, $\|x'''\| = \text{const}$. Therefore, $k_1^2 + k_2^2 = \text{const}$. Consequently, $k_2 = \text{const}$. The proof is concluded by continuing this process.

COROLLARY 2. If $\Gamma(\sigma) = \{x(\sigma), y(\sigma)\}$ is a geodesic of T_1M^n and M^n is a symmetric space, then $R(y, y')$ acts as differentiation on the vectors of the Frenet frame of $\gamma(\sigma) = \pi_0\Gamma(\sigma)$: $R(y, y')\xi_p = (1 - c^2)^{1/2}\{-k_{p-1}\xi_{p-1} + k_p\xi_{p+1}\}$.

The proof is conducted by induction on p.

Proposition. If $\Gamma(\sigma) - \{x(\sigma), y(\sigma)\}$ is a geodesic of TM^n and M^n is a symmetric space, then the sectional curvature of M^n along $\pi_0\Gamma$ in the direction of the elementary area element of vectors (y, y') is constant.

Proof. By definition

$$k_{yy'} = \frac{\langle R(y, y')y', y\rangle}{\|y\|^2\|y'\|^2 - \langle y, y'\rangle^2}.$$

However, y is a unit vector field; therefore, $\langle y, y'\rangle = 0$, $\|y'\|^2 = c^2$, $\|y\|^2 = 1$. Consequently, $k_{yy'} = \frac{1}{c^2}\langle R(y, y')y', y\rangle$.

From this we immediately conclude that $k_{yy'}' = 0$.

Remark. This assertion is also true for geodesics in TM.

3. Characterization of the Projections of Geodesics of TCP^n (T_1CP^n). Since CP^n is a symmetric space, all the assertions of Section 2 and, in particular, Lemma 1 are true for the geodesics of TCP^n (T_1CP^n). Let us show that all curvatures, starting with k_6, are zero for the projections of geodesics of TCP^n (T_1CP^n).

The curvature oprator of CP^n has the form $R(x, y)z = \frac{k}{4}(\langle y, z\rangle x - \langle x, z\rangle y + \langle Iy, z\rangle Ix - \langle Ix, z\rangle Iy + 2\langle x, Iy\rangle Iz$, where I is the operator of complex structure for which $I^2 = -E$, $\langle x, Iy\rangle = -\langle Ix, y\rangle$.

Let us denote by $S(x,y)$ an operator of the type of curvature operator of a sphere, that is, $S(x, y)z = \langle y, z\rangle x - \langle x, z\rangle y$.

Then the curvature operator of CP^n can be written as

$$R(x, y)z = \frac{k}{4}\{S(x, y) + S(Ix, Iy) + 2\langle x, Iy\rangle I\}z$$

or

$$R(x, y) = \frac{k}{4}\{S(x, y) + S(Ix, Iy) + 2mI\},$$

where $m = \langle x, Iy\rangle$.

LEMMA 2. Let $S(x,y)$ be the curvature operator of the unit sphere and $p^2 = \|x \wedge y\|^2$ the norm of bivector $x \wedge y$. Then $S^3 + p^2S \equiv 0$.

Proof. Let us introduce the notation $\langle y, z\rangle = a$, $\langle x, z\rangle = b$. Then $S(x, y)z = ax - by$;

$$S^2(x, y) z = S(x, y) S(x, y) z = S(x, y)(ax - by) = \\ = aS(x, y) x - bS(x, y) y;$$

$$S^3(x, y) z = S(x, y) S^2(x, y) z = S(x, y)(aS(x, y) x - \\ - bS(x, y) y) = \langle y, (aS(x, y) x - bS(x, y) y)\rangle x - \langle x, (aS(x, y) x - \\ - bS(x, y) y)\rangle y = a\langle S(x, y) x, y\rangle x + b\langle S(x, y) y, x\rangle y = \\ = -\langle S(x, y) y, x\rangle (ax - by).$$

But $\langle S(x, y) y, x\rangle = \|x \wedge y\|^2$. Consequenly, $S^3 + p^2 S \equiv 0$. The lemma is proved.

Remark. For a space of constant curvature k operator S' satisfies the identity $S^3 + k^2p^2S = 0$.

The proof is analogous to the previous one.

COROLLARY (see also [2], [3]). If $\Gamma(\sigma)$ is a geodesic in the tangent bundle of a space of constant curvature, then the curvatures of $\gamma(\sigma) = \pi_0(\Gamma(\sigma))$, starting with k_3, are zero.

Proof. According to Lemma 1 (a), $x^{(IV)} = S^3(y, y')x'$. Taking into account the fact that the curvature is constant, we find from Frenet's formulas

$$x^{(IV)} = (1 - c^2)^2 \{k_1 k_2 k_3 \xi_4 - k_1 (k_1^2 + k_2^2) \xi_2\}. \tag{3}$$

On the other hand,

$$S^3(y, y') = -p^2 S(x, y); \text{ therefore, } x^{(IV)} = -p^2 S(x, y) x' = -p^2 x'' \tag{4}$$

From Frenet's formulas $x'' = k_1(1 - c^2)\xi_2$. Comparing (3) and (4), we obtain

$$(1 - c^2)^2 k_1 k_2 k_3 \xi_1 + (p^2 k_1 (1 - c^2) - (1 - c^2) k_1 (k_1^2 + k_2^2)) \xi_2 = 0.$$

Since ξ_1 and ξ_2 are linearly independent, we conclude from this that $k_3 = 0$.

LEMMA 3. The curvature operator of CP^n satisfies the identities

$$R^{2s} = a_s R^2 + b_s IR + c_s E, \\ R^{2s+1} = \alpha_s IR^2 + \beta_s R + \gamma_s I,$$

where a_s, b_s, c_s, α_s, β_s, γ_s are coefficients.

To prove the last assertion we need a "table of multiplication" of operators $S(x, y)$, $S(Ix, Iy)$, I. To abbreviate the notation let us denote $S(x, y)$ and $S(Ix, Iy)$ by A and B, respectively.

LEMMA 4. Operators A ($=S(x, y)$), B ($=S(Ix, Iy)$), I are multiplied according to the following table:

	A	B	I
A	A^2	mIB	IB
B	mIA	B^2	IA
I	IA	IB	$-E$

Proof. Let us consider $AB = S(x, y)S(Ix, Iy)$;

$$S(x, y) S(Ix, Iy) z = S(x, y)(\langle Iy, z\rangle Ix - \langle Ix, z\rangle Iy) = \\ = \langle y, \langle Iy, z\rangle Ix - \langle Ix, z\rangle Iy\rangle x - \langle x, \langle Iy, z\rangle Ix, -\langle Ix, z\rangle Iy\rangle y = \\ = \langle y, Ix\rangle \langle Iy, z\rangle x + \langle Ix, z\rangle \langle x, Iy\rangle y = \\ = mI(\langle Iy, z\rangle Ix - \langle Ix, z\rangle Iy) = mIS(Ix, Iy).$$

Thus, $AB = mIB$.

The remaining equalities are proved analogously.

Proof of Lemma 3. Note that the multiplication table implies that $(A + B)I = I(A + B)$, that is, $A + B$ and I commute. Therefore, numerical formulas for raising to a power are true for operator degree $R = \frac{k}{4}\{(A + B) + 2mI\}$, that is, $R^3(x, y) = \left(\frac{k}{4}\right)^3 \{(A + B)^3 - 8m^3 I + 6m(A + B) I (A + B + 2mI)\}$.

Note that $A+B+2mI=\frac{4}{k}R,\ A+B=\frac{4}{k}R-2mI.$

Therefore, $R^3=\left(\frac{k}{4}\right)^3\left\{(A+B)^3+\frac{24}{k}IR^2+12m^2R-8m^3I\right\}$

Moreover, $(A+B)^3=A^3+B^3+AB^2+BA^2+ABA+BAB+A^2B+B^2A.$

Using the table of multiplication, we find that $AB^2=ABB=mIB^2,\ BA^2=mIA^2,\ ABA=-m^2A,\ BAB=-m^2B,\ A^2B=AAB=mAIB=mIB^2,\ B^2A=mIA^2.$

In addition, according to Lemma 2, $A^3=-p^2A,\ B^3=-p^2B.$

Thus,

$$(A+B)^3=-p^2(A+B)+mI\,(B^2+A^2)-m^2(A+B)+ +mI\,(A^2+B^2)=(p^2+m^2)(A+B)+mI(A^2+B^2). \tag{6}$$

Note that $R^2=\left(\frac{k}{4}\right)^2\left\{(A+B)^2-4m^2E+2mI\,(A+B)\right\}=\left(\frac{k}{4}\right)^2\{A^2+B^2+3mI\,(A+B)-4m^2E\}$. The last inequality implies that

$$A^2+B^2=\left(\frac{4}{k}\right)^2R^2-3mI\left(\frac{4}{k}R-2mI\right)-4m^2E=\left(\frac{4}{k}\right)^2R^2-\frac{12m}{k}IR-10m^2E.$$

Substituting this expression into (6), we find that $(A+B)^3$ is a linear combination of operators IR^2, R, and I.

Substituting this linear combination into (5), we obtain that

$$R^3=\alpha_1IR^2+\beta_1R+\gamma_1I, \tag{7}$$

where α_1, β_1, γ_1 are, generally speaking, nonzero coefficients.

Subsequent R are calculated using (7):

$$\begin{aligned}R^4=R^3R=\alpha_1IR^3+\beta_1R^2+\gamma_1IR=\\=\alpha_1I(\alpha_1IR^2+\beta_1R+\gamma_1I)+\beta_1R^2+\gamma_1IR=\\=(-\alpha_1^2+\beta_1)R^2+(\alpha_1\beta_1+\gamma_1)IR-\alpha_1\gamma_1E=\\=a_2R^2+b_2IR+c_2E.\end{aligned}$$

The proof of Lemma 3 is concluded by continuing this process.

Proof of the Main Theorem. According to Lemmas 1 and 3,

$$\begin{aligned}x^{(2s)}&=a_sx'''+b_sIx''+c_sx',\\x^{(2s+1)}&=\alpha_sIx'''+\beta_sx''+c_sIx'.\end{aligned} \tag{8}$$

Let us now consider the first equation of system (8) for $s=2, 3$. According to Frenet's formulas, $x^{(4)}=(1-c^2)^2k_1k_2k_3k_4\xi_5$ + l.c. (ξ_1, ξ_3), where l.c. denotes a linear combination with constant coefficients of vectors ξ_1, ξ_3 of the Frenet frame while

$$x'''=\text{l.c. }(\xi_1),\ x'=\text{l.c. }(\xi_1),\ x''=\text{l.c. }(\xi_2). \tag{9}$$

Consequently, for $s=2$ we obtain from the first equation of (8)

$$\text{l.c. }(\xi_1,\xi_3)+\text{l.c. }(I\xi_2)+k_1\ldots k_4\xi_5=0. \tag{10}$$

If the coefficient of $I\xi_2$ is equal to zero, then $k_4=0$ and the theorem is proved. Suppose that the coefficient of $I\xi_2$ is nonzero.

For $s = 3$, according to Frenet's formulas

$$x^{(6)} = (1-c^2)^3 k_1 \ldots k_6 \xi_7 + \text{ l.c. } (\xi_1, \xi_3, \xi_5).$$

Keeping (9) in mind, we find that

$$\text{l.c. } (\xi_1, \xi_3, \xi_5) + \text{ l.c. } I\xi_2 + k_1 \ldots k_6 \xi_7 = 0. \tag{11}$$

Expressing vector $I\xi_2$ from (10) and substituting it into (11), we obtain l.c. $(\xi_1, \xi_3, \xi_5) + k_1 \ldots k_6 \xi_7 = 0$.

Since the vectors of the Frenet frame are linearly independent, we can conclude from the last equality that in the general case k_6=0 and, consequently, all the remaining curvatures are zero. The theorem is proved.

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