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Riemannian geometry of fibre bundles

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CONTENTS

Introduction

This survey contains a summary of results on the Riemannian geometry of fibre bundles.

The paper is divided into seven sections. In the first section we consider the Sasaki metric [93] of the tangent and the normal bundles, and the spherical tangent and normal bundles respectively.

Riemannian submersions are a generalization of fibre bundles. Since Riemannian submersions are nowadays the object of study of many geometers, we deem it appropriate to include in the second section a survey of papers by O'Neill [84] and Gray [57]. Thus we fill a substantial gap in the Russian geometric bibliography on these themes.

In the third section we include a survey of results in which is posed, and in a sense solved, the first and natural problem of the Riemannian geometry of fibre bundles: what is the connection between geometric features of the bundle and analogous features of the fibres and the base?

Let us perform a parallel translation of the tangent vector $(\xi^i + d\xi^i)$ in the sense of Levi–Civitá from $(x^i + dx^i)$ to (x^i) along the natural geodesic joining these two points. Let us denote by $d\theta$ the angle between the result of the parallel transport and the vector (ξ^i). Then we define the square of the differential of the distance $d\sigma^2$ between the points (x^i, ξ^i) and $(x^i + dx^i, \xi^i + d\xi^i)$ by $d\sigma^2 = ds^2 + |\xi|^2 d\theta^2$. In the local coordinates (x^i, ξ^i) we obtain the following expression for the linear element of TM:

$$d\sigma^2 = g_{ik}dx^i dx^k + g_{ik}D\xi^i D\xi^k,$$

where the $D\xi^i = d\xi^i + \Gamma^i_{jk}\xi^j dx^k$ are the covariant differentials of the coordinates of the tangent vector.

Let us denote by Tg_{IJ} the components of the Sasaki metric of TM. Then $d\sigma^2 = Tg_{ij}dx^i dx^j + 2Tg_{i n+j}dx^i d\xi^j + Tg_{n+i n+j}d\xi^i d\xi^j$. From this we find the expression of Tg_{IJ} in the local coordinates (x^i, ξ^i):

$$Tg_{ij} = g_{ij} + g_{\alpha\beta}\Gamma^\alpha_{\mu i}\Gamma^\beta_{\nu j}\xi^\mu\xi^\nu,$$

$$Tg_{i\, n+j} = \Gamma_{i\lambda, j}\xi^\lambda, \quad Tg_{n+i\, n+j} = g_{ij}.$$

The contravariant components of the Sasaki metric have the form [93]

$$Tg^{ij} = g^{ij}, \quad Tg^{i\, n+j} = -\Gamma^j_{\mu s}g^{is}\xi^\mu,$$

$$Tg^{n+i\, n+j} = g^{ij} + g^{\alpha\beta}\Gamma^i_{\mu\alpha}\Gamma^j_{\nu\beta}\xi^\mu\xi^\nu.$$

Another approach to the definition of this metric was proposed by Dombrowski [50]. Namely, if $g_{ij}dx^i dx^j$ is the metric form of M, then the scalar product of the vectors X and Y on M, with coordinates $\{X^1, \ldots, X^n\}$ and $\{Y^1, \ldots, Y^n\}$ respectively in the natural basis $(\partial/\partial x^i)$, is computed by the formula $\langle X, Y\rangle = g_{ij}X^i Y^j$.

Let us consider from this point of view the scalar product $\langle\langle \tilde{X}, \tilde{Y}\rangle\rangle$ of two vectors $\tilde{X}$ and $\tilde{Y}$ tangent to TM and having in the natural basis $(\partial/\partial x^i, \partial/\partial\xi^i)$ the coordinates

$$\{\tilde{X}^1, \ldots, \tilde{X}^n; \tilde{X}^{n+1}, \ldots, \tilde{X}^{2n}\},$$

and

$$\{\tilde{Y}^1, \ldots, \tilde{Y}^n; \tilde{Y}^{n+1}, \ldots, \tilde{Y}^{2n}\}$$

respectively. Then

$$\langle\langle \tilde{X}, \tilde{Y}\rangle\rangle = g_{ij}\tilde{X}^i\tilde{Y}^j + g_{ij}(\tilde{X}^{n+i} + \Gamma^i_{\lambda k}\xi^\lambda\tilde{X}^k)(\tilde{Y}^{n+j} + \Gamma^j_{\mu s}\xi^\mu\tilde{Y}^s).$$

Let us define the maps

$$\begin{cases} K\colon TTM \to TM, \\ \pi_*\colon TTM \to TM \end{cases}$$

by the formulae

$$K\tilde{X} = (\tilde{X}^{n+i} + \Gamma^i_{\lambda k}\xi^\lambda \tilde{X}^k)\,\partial/\partial x^i,$$
$$\pi_*\tilde{X} = \tilde{X}^i\partial/\partial x^i.$$

Dombrowski called K *the connection map*, whereas π_* is the differential of the projection $\pi : TM \to M$. Thus, the formula for the computation of the scalar product of $\tilde{X}$ and $\tilde{Y}$ in the Sasaki metric becomes

$$\langle\langle \tilde{X},\ \tilde{Y}\rangle\rangle = \langle \pi_*\tilde{X},\ \pi_*\tilde{Y}\rangle + \langle K\tilde{X},\ K\tilde{Y}\rangle.$$

The vertical subspace is $\mathscr{V}TM$ = Ker π_*, and the horizontal subspace is $\mathscr{H}TM$ = Ker K, and evidently these subspaces are orthogonal in the Sasaki metric (see [93], [50], and [1*]).

Observe that the formula $\langle\langle \tilde{X}, \tilde{Y}\rangle\rangle = \langle \pi_*\tilde{X}, K\tilde{Y}\rangle + \langle K\tilde{X}, \pi_*\tilde{Y}\rangle$ also defines a (pseudo-) Riemannian metric. This is the so-called Vilms metric [101].

Using the definitions of π_* and K one can introduce the concept of horizontal and vertical lifts to TM of a given vector field Z on M. Namely (see [50]), the vector fields Z^H and Z^V on TM are called respectively the *horizontal* and the *vertical lifts* of the field Z if

$$\pi_* Z^H_{(Q,\xi)} = Z_Q, \quad KZ^H_{(Q,\xi)} = 0,$$
$$\pi_* Z^V_{(Q,\xi)} = 0, \quad KZ^V_{(Q,\xi)} = Z_Q$$

at each point $(Q, \xi) \in TM$.

In local coordinates the definition on the lifts is as follows. If $Z = \{Z^1, ..., Z^n\}$ is a vector in T_QM, then its horizontal lift Z^H and its vertical lift Z^V at (Q, ξ) have coordinates

$$Z^H = \{Z^1,\ \ldots,\ Z^n;\ -\Gamma^1_{jk}Z^j\xi^k,\ \ldots,\ -\Gamma^n_{jk}Z^j\xi^k\},$$
$$Z^V = \{0,\ \ldots,\ 0;\ Z^1,\ \ldots,\ Z^n\}.$$

Incidentally, we observe the following technically important lemma on the brackets of the lifts of vector fields.

***Lemma* 1.1** [50]. *At each point $(Q, \xi) \in TM$ we have*

$$[X^V,\ Y^V] = 0,$$
$$[X^H,\ Y^V] = (\nabla_X Y)^V,$$
$$\pi_*[X^H,\ Y^H] = [X,\ Y],$$
$$K[X^H,\ Y^H] = -R\,(X,\ Y)\,\xi,$$

where ∇ is, generally speaking, an arbitrary affine connection on M and $R(X, Y)\xi$ is its curvature tensor.

An analogue for the frame bundle was proved in [47].

***Lemma* 1.2** [93]. *The Christoffel symbols of the first kind of the Levi–Cività connection of the Sasaki metric have the form*

$$\widetilde{\Gamma}_{n+j\,n+k,H} = 0,\ \ \widetilde{\Gamma}_{n+j\,k,h} = \frac{1}{2}\left(R_{j\lambda kh} + 2g_{\alpha\beta}\Gamma^{\alpha}_{\lambda h}\Gamma^{\beta}_{jk}\right)\xi^{\lambda},\quad \Gamma_{n+j\,k,n+h} = \Gamma_{j,kh},$$

$$\widetilde{\Gamma}_{jk,h} = \Gamma_{jk,h} + \frac{1}{2}\left[\frac{\partial}{\partial x^{k}}\left(g_{\alpha\beta}\Gamma^{\alpha}_{\lambda j}\Gamma^{\beta}_{\mu h}\right) + \frac{\partial}{\partial x^{j}}\left(g_{\alpha\beta}\Gamma^{\alpha}_{\lambda h}\Gamma^{\beta}_{\mu k}\right) - \frac{\partial}{\partial x^{h}}\left(g_{\alpha\beta}\Gamma^{\alpha}_{\lambda j}\Gamma^{\beta}_{\mu k}\right)\right]\xi^{\mu}\xi^{\lambda},$$

$$\widetilde{\Gamma}_{jk,n+h} = \frac{1}{2}\left[\frac{\partial\Gamma_{\lambda j,h}}{\partial x^{k}} + \frac{\partial\Gamma_{\lambda k,h}}{\partial x^{j}} - g_{\alpha\beta}\Gamma^{\alpha}_{hj}\Gamma^{\beta}_{\lambda k} - g_{\alpha\beta}\Gamma^{\alpha}_{\lambda j}\Gamma^{\beta}_{hk}\right]\xi^{\lambda}.$$

***Lemma* 1.3** [96]. *The Christoffel symbols of the second kind of the Levi–Cività connection of the Sasaki metric have the form*

$$\widetilde{\Gamma}^{I}_{n+j\,n+k} = 0,\ \ \widetilde{\Gamma}^{i}_{n+j\,k} = \frac{1}{2}R^{i}_{k\,\lambda j}\xi^{\lambda},$$

$$\widetilde{\Gamma}^{n+i}_{n+j\,k} = \Gamma^{i}_{jk} - \frac{1}{2}\Gamma^{i}_{\mu h}R^{h}_{k\lambda j}\xi^{\mu}\xi^{\lambda},$$

$$\widetilde{\Gamma}^{i}_{jk} = \Gamma^{i}_{jk} + \frac{1}{2}\left[R^{i}_{k\mu h}\Gamma^{h}_{\lambda j} + R^{i}_{j\mu h}\Gamma^{h}_{\lambda k}\right]\xi^{\lambda}\xi^{\mu},$$

$$\widetilde{\Gamma}^{n+i}_{jk} = \frac{1}{2}\left[R^{i}_{jk\lambda} + R^{i}_{kj\lambda} + 2\frac{\partial\Gamma^{i}_{jk}}{\partial x^{\lambda}}\right]\xi^{\lambda} + \frac{1}{2}\Gamma^{i}_{\nu h}\left[R^{h}_{ks\mu}\Gamma^{s}_{\lambda j} + R^{h}_{js\mu}\Gamma^{s}_{\lambda k}\right]\xi^{\lambda}\xi^{\mu}\xi^{\nu}.$$

Using these lemmas one can find the covariant derivatives of compositions of lifts of vector fields.

***Lemma* 1.4** [65].

$$\widetilde{\nabla}_{X^V}Y^V = 0,\quad \widetilde{\nabla}_{X^H}Y^V = (\nabla_X Y)^V + \frac{1}{2}\left[R(\xi, Y)X\right]^H,$$

$$\widetilde{\nabla}_{X^V}Y^H = \frac{1}{2}\left[R(\xi, X)Y\right]^H,\quad \widetilde{\nabla}_{X^H}Y^H = (\nabla_X Y)^H - \frac{1}{2}\left[R(X, Y)\xi\right]^V.$$

An analogue of this lemma for the frame bundle is given in [47].

It is also appropriate to observe here that the construction of the Sasaki metric does not require that the connection be Riemannian, and consequently this construction is applicable to any fibre bundles with connection over a Riemannian manifold.

Let us consider from this point of view the normal bundle of a surface F^l in a Riemannian manifold M^{l+p}. At each point $Q \in F^l$ we have the splitting of the tangent space $T_Q M^{l+p}$ to M^{l+p} into the direct sum of two subspaces $T_Q F^l$ and $N_Q F^l$, of which the first is tangent to F^l, and the second is orthogonal to $T_Q F^l$ in the metric of M^{l+p}.

By a *normal bundle space* we mean the disjoint union of the $N_Q F^l$ over all $Q \in F^l$. If $Q \in F^l$ and ξ is the normal to F^l at Q, then the pair $\widetilde{Q} = (Q, \xi)$ is a point of the normal bundle NF^l. Let us denote by $(x^1, \ldots, x^l)$ the local coordinates of F^l, and by $(\xi^1, \ldots, \xi^p)$ the coordinates of the normal ξ in a basis of normals $n_{1|}, \ldots, n_{p|}$, which in what follows will be assumed to be orthonormal. Then the symbol $(x^1, \ldots, x^l; \xi^1, \ldots, \xi^p)$ defines local coordinates on NF^l, which are called the *natural induced coordinates*, by analogy with the tangent bundle.

Let $\bar{g}$ and $\bar{\nabla}$ be a metric and a covariant derivative respectively on M^{l+p}, and let g be the induced metric on F^l. If X is a vector field tangent to F^l and ξ is a vector field normal to F^l, then the *covariant derivative in the normal connection* $\nabla^{\perp}_X\xi$ of the vector field ξ along the direction of X is defined as the projection of the vector field $\bar{\nabla}_X\xi$ onto the normal subspace to F^l.

In local cordinates we have

$$\nabla^{\perp}_i n_{\alpha|} = \mu^{\tau}_{\alpha|i} n_{\tau|} \quad (i = 1, \ldots, l;\ \alpha = 1, \ldots, p),$$

where $\mu^{\tau}_{\alpha|i}$ ($=\mu_{\tau\alpha|i}$) are the torsion coefficients. Hence, for vector fields we have

$$\nabla^{\perp}_X\xi = X^i\left(\frac{\partial\xi^{\alpha}}{\partial x^i} + \mu^{\alpha}_{\tau|i}\xi^{\tau}\right) n_{\alpha|}.$$

The covariant differential $D^{\perp}$ of the normal vector field ξ is defined as the normal covariant derivative of ξ along the direction of the vector field $dX = \{dx^1, \ldots, dx^l\}$:

$$D^{\perp}\xi = (d\xi^{\alpha} + \mu^{\alpha}_{\tau|i}\xi^{\tau}dx^i)\, n_{\alpha|}.$$

Let us define a linear element $d\sigma$ of the Sasaki metric of NF^l in the natural induced coordinates (x^i, ξ^{α}) by means of the equality [5]

$$d\sigma^2 = g_{ij}dx^i dx^j + \delta_{\alpha\beta}D^{\perp}\xi^{\alpha}D^{\perp}\xi^{\beta},$$

where $\delta_{\alpha\beta}$ is the Kronecker delta (corresponding to the fibre Euclidean metric).

For the components of the Sasaki metric Ng of the normal bundle we obtain the following expression [31]:

$$\begin{aligned} Ng_{ij} &= g_{ij} + \delta_{\alpha\beta}\mu^{\alpha}_{\tau|i}\mu^{\beta}_{\sigma|j}\xi^{\tau}\xi^{\sigma}, \\ Ng_{i\ l+\beta} &= \mu_{\beta\tau|i}\xi^{\tau}, \\ Ng_{l+\alpha\ l+\beta} &= \delta_{\alpha\beta}. \end{aligned}$$

The contravariant components have the form

$$\begin{aligned} &Ng^{ij} = g^{ij}, \quad Ng^{i\ l+\beta} = -\mu^{\beta}_{\tau|s}g^{si}\xi^{\tau}, \\ &Ng^{l+\alpha\ l+\beta} = \delta^{\alpha\beta} + g^{ts}\mu^{\alpha}_{\tau|t}\mu^{\beta}_{\sigma|s}\xi^{\tau}\xi^{\sigma}. \end{aligned}$$

For NF^l we define in a natural way the map π_* and the connection map K (see [89]). If $\tilde{X}$ and $\tilde{Y}$ are tangent to NF^l at $\tilde{Q} = (Q, \xi)$ and in the natural basis $(\partial/\partial x^i; \partial/\partial\xi^{\alpha})$ the vectors $\tilde{X}$ and $\tilde{Y}$ have coordinates $\{\tilde{X}^i; \tilde{X}^{l+\alpha}\}$ and $\{\tilde{Y}^i; \tilde{Y}^{l+\alpha}\}$ respectively, then, denoting by $\langle\langle\ \rangle\rangle$ the scalar products of vectors in the Sasaki metric, and by $\langle\,,\rangle$ and $\langle\,,\rangle_{\perp}$ the scalar products of vectors in the metric g and in the fibre (Euclidean) metric respectively, we find that

$$\begin{aligned} \langle\langle\tilde{X}, \tilde{Y}\rangle\rangle &= g_{ij}\tilde{X}^i\tilde{Y}^j + \delta_{\alpha\beta}(\tilde{X}^{l+\alpha} + \mu^{\alpha}_{\tau|i}\tilde{X}^i\xi^{\tau})(\tilde{Y}^{l+\beta} + \mu^{\beta}_{\sigma|j}\tilde{Y}^j\xi^{\sigma}) = \\ &= \langle\pi_*, \tilde{X}, \pi_*\tilde{Y}\rangle + \langle K\tilde{X}, K\tilde{Y}\rangle_{\perp}, \end{aligned}$$

that is, π_* and K in local coordinates are written as

$$\pi_*\tilde{X} = \tilde{X}^i\partial/\partial x^i, \quad K\tilde{X} = (\tilde{X}^{l+\alpha} + \mu^{\alpha}_{\tau|i}\tilde{X}^i\xi^{\tau})\, n_{\alpha|}.$$

The vertical subspace is $\mathscr{V}_{\tilde{Q}}NF^l = \operatorname{Ker} \pi_*$, and the horizontal subspace is $\mathscr{H}_{\tilde{Q}}NF^l = \operatorname{Ker} K$.

We define in a natural way the horizontal and the vertical lifts of a tangent vector X and a normal vector η in TNF^l at (Q, ξ). Namely [89],

$$X^H = \{X^i;\ -\mu^{\alpha}_{\tau|k}X^k\xi^\tau\}, \quad \eta^V = \{0;\ \eta^\alpha\}.$$

Dombrowski's lemma has the following analogue (see [6] and [31]).

***Lemma* 1.5.** *At each point $\tilde{Q} = (Q, \xi) \in NF^l$ we have*

$$[\zeta^V, \eta^V]_{\tilde{Q}} = 0, \quad [X^H, \eta^V]_{\tilde{Q}} = (\nabla^{\perp}_X \eta)^V_Q,$$

$$\pi_* [X^H, Y^H]_{\tilde{Q}} = [X, Y]_Q, \quad K\, [X^H, Y^H]_{\tilde{Q}} = -(N\,(X, Y)\,\xi)_Q,$$

where $N(X, Y)\xi$ is the curvature vector of the normal connection.

We recall that

$$N_{\alpha\beta|ij} = \frac{\partial \mu_{\alpha\beta|j}}{\partial x^i} - \frac{\partial \mu_{\alpha\beta|i}}{\partial x^j} + \mu_{\alpha\tau|i}\mu^{\tau}_{\beta|j} - \mu_{\alpha\tau|j}\mu^{\tau}_{\beta|i}$$

is the curvature tensor of the normal connection, and $N(X, Y)\xi = N^{\alpha}_{\beta|ij}X^iY^j\xi^\beta n_{\alpha|}$. We have the invariant expression

$$N\,(X, Y)\,\xi = \nabla^{\perp}_X\nabla^{\perp}_Y\xi - \nabla^{\perp}_Y\nabla^{\perp}_X\xi - \nabla^{\perp}_{[X,Y]}\xi.$$

Other technical results to be mentioned are the expression of the Christoffel symbols of the metric NF^l and an analogue of Koval'skii's lemma on covariant derivatives (see [6] and [31]).

***Lemma* 1.6.** *The Christoffel symbols of the first kind of the Riemannian connection of the Sasaki metric NF^l are equal to*

$$\tilde{\Gamma}_{ij,k} = \Gamma_{ij,k} + \frac{1}{2}\left[\frac{\partial}{\partial x^j}(\mu_{\alpha\tau|i}\mu^{\alpha}_{\sigma|k}) + \frac{\partial}{\partial x^i}(\mu_{\alpha\tau|j}\mu^{\alpha}_{\sigma|k}) - \frac{\partial}{\partial x^k}(\mu_{\alpha\tau|i}\mu^{\alpha}_{\sigma|j})\right]\xi^\tau\xi^\sigma,$$

$$\tilde{\Gamma}_{ij,l+\beta} = \frac{1}{2}\left[\frac{\partial \mu_{\beta\tau|i}}{\partial x^j} + \frac{\partial \mu_{\beta\tau|j}}{\partial x^i} + \mu_{\beta\alpha|i}\mu^{\alpha}_{\tau|j} + \mu_{\beta\alpha|j}\mu^{\alpha}_{\tau|i}\right]\xi^\tau,$$

$$\tilde{\Gamma}_{l+\beta\, i,j} = \frac{1}{2}\,[N_{\beta\tau|ij} + 2\mu_{\alpha\beta|i}\mu^{\alpha}_{\tau|j}]\,\xi^\tau,$$

$$\tilde{\Gamma}_{i\, l+\alpha,\, l+\beta} = \mu_{\beta\alpha|i}, \quad \tilde{\Gamma}_{l+\alpha\, l+\beta,\, i} = 0, \quad \tilde{\Gamma}_{l+\alpha\, l+\beta,\, l+\tau} = 0.$$

***Lemma* 1.7.** *The Christoffel symbols of the second kind of the Riemannian connection of the Sasaki metric of F^l have the form*

$$\tilde{\Gamma}^k_{ij} = \Gamma^k_{ij} + \frac{1}{2}\,[\hat{N}^k_{i|\tau\alpha}\mu^{\alpha}_{\sigma|j} + \hat{N}^k_{j|\tau\alpha}\mu^{\alpha}_{\sigma|i}]\,\xi^\tau\xi^\sigma,$$

$$\tilde{\Gamma}^{l+\sigma}_{ij} = \frac{1}{2}\,\mu^{\sigma}_{\tau|t}\,(\hat{N}^t_{i|\alpha\beta}\mu^{\alpha}_{\lambda|j} + \hat{N}^t_{j|\alpha\beta}\mu^{\alpha}_{\lambda|i})\,\xi^\tau\xi^\beta\xi^\lambda +$$

$$+ \frac{1}{2}\left[\frac{\partial \mu^{\sigma}_{\lambda|i}}{\partial x^j} + \frac{\partial \mu^{\sigma}_{\lambda|j}}{\partial x^i} + \mu^{\sigma}_{\alpha|i}\mu^{\alpha}_{\lambda|j} + \mu^{\sigma}_{\alpha|j}\mu^{\alpha}_{\lambda|i} - 2\Gamma^t_{ij}\mu^{\sigma}_{\lambda|t}\right]\xi^\lambda,$$

$$\tilde{\Gamma}^k_{l+\alpha\, j} = \frac{1}{2}\,\hat{N}^k_{j|\lambda\alpha}\xi^\lambda, \quad \tilde{\Gamma}^{l+\sigma}_{l+\alpha\, j} = \mu^{\sigma}_{\alpha|j} + \frac{1}{2}\,\mu^{\sigma}_{\lambda|t}\hat{N}^t_{j|\alpha\nu}\xi^\lambda\xi^\nu,$$

$$\tilde{\Gamma}^k_{l+\alpha\, l+\beta} = 0, \quad \tilde{\Gamma}^{l+\sigma}_{l+\alpha\, l+\beta} = 0.$$

Lemma 1.8. *If X and Y are tangent vector fields on $F^l \subset M^{l+p}$ and η and ζ are normal vector fields on F^l, then the following equalities hold at each point $\tilde{Q} = (Q, \xi)$:*

$$\tilde{\nabla}_{\eta^V}\xi^V = 0, \quad \tilde{\nabla}_{X^H}\eta^V = (\nabla^{\perp}_X\eta)^V + \frac{1}{2}[\hat{N}(\zeta, \eta)X]^H,$$

$$\tilde{\nabla}_{\eta^V}Y^H = \frac{1}{2}[\hat{N}(\xi, \eta)Y]^H, \quad \tilde{\nabla}_{X^H}Y^H = (\nabla_X Y)^H - \frac{1}{2}[N(X, Y)\xi]^V.$$

The tensor $\hat{N}$ appears in the two lemmas. In local coordinates, $\hat{N}^i_{j|\alpha\beta} = g^{is}N_{\alpha\beta|sj}$, and in invariant form $\hat{N}(\xi, \eta)X$ is determined by the equality $\langle \hat{N}(\xi, \eta)X, Y\rangle = \langle N(X, Y)\xi, \eta\rangle_{\perp}$. Thus, if $n(X, Y)$ is an antisymmetric linear transformation of $N_Q F^l$, then $\hat{N}(\xi, \eta)$ is an antisymmetric linear transformation of $T_Q F^l$.

Incidentally, Ricci's equation for immersed manifolds

$$\langle \bar{R}(X, Y)\xi, \eta\rangle_{\bar{g}} = \langle N(X, Y)\xi, \eta\rangle_{\perp} - \langle [A_\xi, A_\eta]X, Y\rangle$$

implies that in spaces of constant curvature (in which case $\langle \bar{R}(X, Y)\xi, \eta\rangle_{\bar{g}}$ is zero) one has $\hat{N}(\xi, \eta)X = [A_\xi, A_\eta]X$, where A_ξ and A_η are the matrices of the second quadratic forms of the surface F^l with respect to the normals ξ and η respectively.

If in each fibre of TM we restrict ourselves to vectors of specified length $\rho > 0$, we get a subbundle $T_\rho M$, which is a hypersurface in TM. If M is compact, then $T_\rho M$ is a compact submanifold in TM. Its embedding is given by the condition $g_{ik}\xi^i\xi^k = \rho^2$. The vector ξ^V is a normal to $T_\rho M$ at $(Q, \xi) \in T_\rho M$, where the lift is regarded in the sense of TM.

$T_\rho M$ is called the *spherical tangent bundle*. For $\rho = 1$ it is also called the *unit vector bundle*. Apart from $T_\rho M$, one can consider other types of surfaces in TM (see §5).

The connection map of $T_\rho M$ is defined as the restriction of the connection map of TM : $K_\rho = K|_{T_\rho M}$. We have $K_\rho : T_{(Q,\xi)}, T_\rho M \to L^{\perp}_Q(\xi)$, where $L^{\perp}_Q(\xi)$ is the orthogonal complement of ξ in $T_Q M$.

The components of the metric tensor $T_\rho M$ may be computed, for instance, as follows.

Let us regard the equality $g_{ik}\xi^i\xi^k = \rho^2$ as an equation with respect to ξ^n. Then $T_\rho M$ is given explicitly, and

$$\frac{\partial \xi^n}{\partial x^k} = -\frac{1}{\xi_n}\Gamma^j_{ik}\xi^i\xi_j \quad (= A_k),$$

$$\frac{\partial \xi^n}{\partial \xi^p} = -\frac{\xi_p}{\xi_n} \quad (= B_p),$$

where $\xi_i = g_{is}\xi^s$, $p = 1, \ldots, n-1$.

After standard computations we find that

$$(T_\rho g)_{ij} = g_{ij} + \Gamma^s_{i\lambda}\Gamma_{j\mu,s}\xi^\lambda\xi^\mu + \Gamma_{\lambda i,n},\ \xi^\lambda A_j + \Gamma_{\lambda j,n}\xi^\lambda A_i + g_{nn}A_iA_j,$$
$$(T_\rho g)_{i\,n+p} = \Gamma_{\lambda i,p},\ \xi^\lambda + \Gamma_{\lambda i,n},\ \xi^\lambda B_p + g_{np}A_i + g_{nn}A_iB_p,$$
$$(T_\rho g)_{n+p\,n+q} = g_{pq} + g_{np}B_q + g_{nq}B_p + g_{nn}B_pB_q,$$
$$(p,\ q = 1, \ldots, n-1)$$

Yamaguchi and Kawabata [107] obtained a somewhat different form for the components of the metric of T_1M (that is, for $\rho = 1$).

Let us define T_1M in TM parametrically:

$$x^i = u^i, \quad \xi^i = \xi^i\,(u^i,\ t^p).$$

Let $\nabla_i\xi$ be the covariant derivative of ξ in the connection of M with respect to u^i, and let $\partial_p\xi$ be the usual derivative of ξ with respect to the parameter t^p. Then the components of the Sasaki metric T_1M become

$$(T_1g)_{ij} = g_{ij} + \langle\nabla_i\xi,\ \nabla_j\xi\rangle,$$
$$(T_1g)_{i\,n+p} = \langle\nabla_i\xi,\ \partial_p\xi\rangle,$$
$$(T_1g)_{n+p\,n+q} = \langle\partial_p\xi,\ \partial_q\xi\rangle.$$

If in each fibre of the normal bundle NF^l we restrict ourselves to vectors of a specified length, we obtain the spherical normal bundle $N_\rho F^l$.

From a geometric point of view it is interesting to study the geometry of the normal bundle of unit vectors, that is, for $\rho = 1$.

Since along F^l one can choose an orthonormal basis of normals, the normalization condition of a normal vector becomes

$$\sum_{\alpha=1}^{p} (\xi^\alpha)^2 = 1,$$

and the natural embedding $N_1F^l \subset NF^l$ is given as follows:

$$y^i = x^i, \quad y^{l+\varphi} = \xi^\varphi,$$
$$y^{l+p} = \xi^p\,(\xi^1, \ldots, \xi^{p-1}) = \sqrt{1 - \sum_{\theta=1}^{p-1} (\xi^\theta)^2}\ ,$$

where (y) are coordinates on NF^l, and $(x^i,\ \xi^\varphi)$ are coordinates on N_1F^l, $i = 1, \ldots, l$, $\varphi = 1, \ldots, p-1$. Let us set

$$B_\theta = \frac{\partial\xi^p}{\partial\xi^\theta} = -\frac{\xi^\theta}{\xi^p}\ .$$

In these coordinates we have the following lemmas.

Lemma 1.9. *The covariant components of the metric tensor of the Sasaki metric of N_1F^l are equal to*

$$(N_1g)_{ij} = g_{ij} + \mu^{\alpha}_{\tau|i}\mu_{\alpha\nu|j}\xi^{\tau}\xi^{\alpha},$$
$$(N_1g)_{i\,l+\varphi} = (\mu_{\varphi\lambda|i} + \mu_{p\lambda|i}B_{\varphi}^{*})\,\xi^{\lambda},$$
$$(N_1g)_{l+\theta\,l+\varphi} = \delta_{\theta\varphi} + B_0B_{\varphi}$$
$$(\alpha, \tau, \lambda = 1, \ldots, p;\ \varphi, \theta = 1, \ldots, p-1;\ i, j = 1, \ldots, l).$$

Lemma 1.10. *The contravariant components of the metric tensor are equal to*

$$(N_1g)^{ij} = g^{ij},\ (N_1g)^{i\,l+\varphi} = -g^{is}\mu^{\varphi}_{\lambda|s}\xi^{\lambda},$$
$$(N_1g)^{l+\theta\,l+\varphi} = \delta^{\theta\varphi} - \xi^{\theta}\xi^{\varphi} + g^{is}\mu^{\theta}_{\tau|i}\mu^{\varphi}_{\lambda|s}\xi^{\tau}\xi^{\lambda}$$
$$(\alpha, \tau, \lambda = 1, \ldots, p;\ \varphi, \theta = 1, \ldots, p-1;\ i, j = 1, \ldots, l).$$

Mok [73] constructed an analogue of the Sasaki metric on the frame bundle FM and the orthonormal frame bundle OM.

The geometric meaning of this metric consists in the following. Let $Q = (x^1, \ldots, x^n)$ be a point on M, and let $F = \{\xi_1, \ldots, \xi_n\}$ be a linear frame in T_QM. Then the pair (Q, F) constitutes a point of the frame bundle FM. Let us consider two near points of this bundle: $(Q+dQ, F+dF)$ and (Q, F). Let us perform a parallel transport, in Levi–Cività's sense, of each vector of the frame $F+dF$ into the point Q, along the geodesic joining Q and $Q+dQ$. Let $d\theta^i$ be the angle between the results of the parallel transport of $\xi_i + d\xi_i$ and the vector ξ_i, and let ds be the length of the geodesic segment $(Q, Q+dQ)$. Then the linear element $d\sigma$ of the Sasaki metric of FM is defined by

$$d\sigma^2 = ds^2 + \sum_{i=1}^{n} |\xi_i|^2 (d\theta^i)^2.$$

The paper also contains direct analogues of Lemmas 1.2 and 1.3 and the computation of the curvature tensor of the Sasaki metric of FM. Since it is not our intention to lay stress on heavy index notations, we refer the reader to the original paper [73], and observe that the formulae obtained therein qualitatively differ little from the corresponding ones for TM.

Going back to the tangent bundle, we shall indicate an expression for other types of metrics on TM connected with the Riemannian (pseudo-Riemannian) metric and the connection of M.

Let us consider the quadratic forms [105]:

I. $g_{ij}dx^idx^j$,

II. $2g_{ij}dx^iD\xi^j$,

III. $g_{ij}D\xi^iD\xi^j$.

Then the metric form of the Sasaki metric can be written as the sum I + III. The form II, regarded as a metric form on TM, determines the so-called *complete lift metric*. The metric I + III, that is, the Sasaki metric, is sometimes called the *diagonal lift metric*. The forms I + II and II + III also determine a (pseudo-) Riemannian metric on TM. Their properties were studied in [105] and [100]. Shirkov proposed a construction of a synthetic metric and a synthetic connection on TM [25].

In general, it is possible to indicate a procedure for lifting from M onto TM any tensor fields, not only metrics. One singles out horizontal, vertical, and complete lifts of tensor fields. However, we shall not consider these questions (see [8], [74], and others), since this could be the theme of a separate survey.

Thus, on the tangent bundle of a Riemannian manifold one can construct in various ways a (pseudo-) Riemannian metric starting from the (pseudo-) Riemannian metric of the given Riemannian manifold. Thus, the projection map $\pi : TM \to M$ becomes a map between two (pseudo-) Riemannian manifolds, and π_* is a surjection and preserves the lengths of horizontal vectors. It is likely that this was the basis for the definition of the concept of a Riemannian submersion, which will be considered next.

§2. Riemannian submersions

2.1. Main equations of a Riemannian submersion.

Let M and B be Riemannian manifolds. A *Riemannian submersion* $\pi : M \to B$ is a map from M onto B satisfying the following axioms A1 and A2.

A1. *π has maximal rank, that is, rank π_* = dim B, and therefore $\pi^{-1}(b)$ for every $b \in B$ is a smooth embedding of a submanifold of M of dimension* dim $M -$ dim B.

The submanifolds $\pi^{-1}(b)$ are called *fibres*. The vector fields on M tangent to the fibres are called *vertical*. The vector fields on M orthogonal to the fibres in the metric of M are called *horizontal*.

A2. *π_* preserves the length of horizontal vectors.* M is called the *space of the submersion* and B the *base of the submersion*. O'Neill [84] found analogues of the Gauss–Codazzi levels of an isometric immersion for the case of Riemannian submersions.

Observe that the space of a Riemannian submersion is a particular case of an almost product manifold, whose geometry has been studied by Gray [57]. His method is analogous to that of O'Neill, both in concept and results. We, however, regard the Riemannian submersion as a more natural object of study, and the subsequent exposition follows O'Neill [84] (see also [2*]).

Let $\mathscr{H}$ and $\mathscr{V}$ denote the projections of the tangent space to M onto the spaces of horizontal and vertical vectors respectively. The letters U, V, W will always denote vertical vector fields, whereas X, Y, Z will denote horizontal ones.

The second quadratic form of the fibres is defined as a tensor field T of type (1, 2) on M determined by arbitrary vector fields E and F on M by the formula (cf. [57])

$$T_E F = \mathscr{H}\nabla_{\mathscr{V}E}(\mathscr{V}F) + \mathscr{V}\nabla_{\mathscr{V}E}(\mathscr{H}F),$$

where ∇ is the covariant derivative on M.

T has the following properties:

1. At each point T_E is an antisymmetric linear operator on the tangent space to M and takes horizontal vectors into vertical ones, and conversely.
2. T is vertical, in the sense that $T_E = T_{\mathscr{V}E}$.
3. T is symmetric on vertical vector fields: $T_V W = T_W V$.

To define another tensor, A, we interchange the projections $\mathscr{H}$ and $\mathscr{V}$ in the definition of T:

$$A_E F = \mathscr{V}\nabla_{\mathscr{H}E}(\mathscr{H}F) + \mathscr{H}\nabla_{\mathscr{H}E}(\mathscr{V}F).$$

A is a (1, 2)-tensor with the following properties:

1′. At each point $A_E F$ is an antisymmetric linear operator on the tangent space to M and carries horizontal vectors into vertical ones, and conversely.

2′. A is horizontal, in the sense that $A_E = A_{\mathscr{H}E}$.

3′. For horizontal vector fields we have $A_X Y = -A_Y X$.

A is the integrability tensor of the horizontal distribution, since

$$A_X Y = \frac{1}{2}\mathscr{V}[X, Y].$$

Let us denote by $\overline{\nabla}$ the connetion induced on fibres, that is,

$$\overline{\nabla}_V W = \mathscr{V}\nabla_V W.$$

The following lemma yields an analogue of the Gauss decomposition for a Riemannian submersion.

***Lemma* 2.1** [84].

1) $\nabla_V W = T_V W + \overline{\nabla}_V W,$

2) $\nabla_V X = \mathscr{H}\nabla_V X + T_V X,$

3) $\nabla_X V = A_X V + \mathscr{V}\nabla_X V,$

4) $\nabla_X Y = \mathscr{H}\nabla_X Y + A_X Y$

(cf. [65] and [57]).

Let us denote by $\langle \overline{R}(V_1, V_2)V_3, V_4\rangle$ the curvature tensor of a fibre, and by $\langle R^*(h_1, h_2)h_3, h_4\rangle$, the tensor defined by

$$\langle R^*(h_1, h_2)h_3, h_4\rangle = \langle R^*(h_1^*, h_2^*)h_3^*, h_4^*\rangle,$$

where R^* is the curvature tensor on B, and $h_i^* = \pi_*(h_i)$.

If A_XY in local coordinates is written as $A^s_{ij}X^iY^j$, then we denote by $(\nabla_Z A)_X Y$ the expression $(\nabla_k A^s_{ij})Z^kX^iY^j$. Then the analogues of the Gauss–Codazzi equations of a Riemannian submersion are as follows (cf. [57]):

$$\langle R(X,Y)Z,H\rangle = \langle R^*(X,Y)Z,H\rangle - 2\langle A_XY, A_ZH\rangle + \langle A_YZ, A_XH\rangle + \langle A_ZX, A_YH\rangle,$$

$$\langle R(X,Y)Z,V\rangle = \langle (\nabla_Z A)_X Y, V\rangle + \langle A_XY, T_VZ\rangle - \langle A_YZ, T_VX\rangle - \langle A_ZX, T_VY\rangle,$$

$$\langle R(X,Y)V,W\rangle = \langle (\nabla_V A)_X Y, W\rangle - \langle (\nabla_W A)_X Y, V\rangle + \langle A_XV, A_YW\rangle - \langle A_XW, A_YV\rangle - \langle T_VX, T_WY\rangle + \langle T_WX, T_VY\rangle,$$

$$\langle R(X,V)Y,W\rangle = \langle (\nabla_X T)_V W, Y\rangle + \langle (\nabla_V A)_X Y, W\rangle - \langle T_VX, T_WY\rangle + \langle A_XV, A_YW\rangle,$$

$$\langle R(U,V)W,X\rangle = \langle (\nabla_V T)_U W, X\rangle - \langle (\nabla_U T)_V W, X\rangle,$$

$$\langle R(U,V)W,F\rangle = \langle \bar R(U,V)W,F\rangle + \langle T_UF, T_VW\rangle - \langle T_UW, T_VF\rangle.$$

Here the covariant derivatives of the fields A and T are described by

$$(\nabla_V A)_W = -A_{T_V W}, \quad (\nabla_X T)_Y = -T_{A_X Y},$$

$$(\nabla_X A)_W = -A_{A_X W}, \quad (\nabla_V T)_Y = -T_{T_V Y}.$$

From the Gauss–Codazzi equations one easily obtains the following result (cf. [57]).

Corollary. *Let $\pi : M \to B$ be a Riemannian submersion, and let K, K_*, and $\bar K$ be the sectional curvatures of M, B, and the fibres respectively.*
Then

$$K(V,W) = \bar K(V,W) - \frac{\langle T_VV, T_WW\rangle - |T_VW|^2}{|V\wedge W|^2},$$

$$K(X,V) = \frac{1}{|X|^2|V|^2}\left(\langle (\nabla_X T)_V V, X\rangle + |A_XV|^2 - |T_VX|^2\right),$$

$$K(X,Y) = K_*(X_*Y_*) - \frac{3|A_XY|^2}{|X\wedge Y|^2}, \quad X_* = \pi_*(X).$$

The last formula implies that along horizontal surface elements the sectional curvature of a submersion space is not greater than the curvature of the base ([57], [12]).

The following examples have been considered: a) the Hopf bundles $\pi : S^{2n+1} \to \mathbb{C}P^n$; b) a Riemannian homogeneous space G/K and $\pi : G \to G/K$; and c) the frame bundle FB and $\pi : FB \to B$.

In addition to these applications, Gray [57] considered the Hopf bundles $S^{4n+3} \to \mathbb{H}P^n$ and obtained an expression for the sectional curvature of the quaternionic projective space.

In these examples a crucial step was the determination of an expression for the tensor A of the Riemannian submersion, so that $T \equiv 0$ (these are Riemannian submersions with totally geodesic fibres).

Let us write down the tensor A for the Hopf bundles. Consider $\pi : S^{2n+1} \xrightarrow{S^1} \mathbb{C}P^n$ [84]. Let N denote the unit exterior normal to the unit sphere $S^{2n+1} \subset \mathbb{R}^{2n+2} \subset \mathbb{C}^{n+1}$. Let J be the natural quasi-complex structure on $\mathbb{C}^{n+1}$. The fibres of the submersion $\pi : S^{2n+1} \to \mathbb{C}P^n$ are one-dimensional and are integral curves of the vector field JN, which, in turn, are great circles of S^{2n+1}.

Thus, the vertical space of the submersion $\pi : S^{2n+1} \to \mathbb{C}P^n$ coincides with JN, and the horizontal space coincides with its orthogonal complement.

If X and Y are horizontal vector fields on S^{2n+1}, then the formulae

$$A_X Y = \langle X, JY\rangle JN, \quad A_X(JN) = JX$$

completely determine the tensor A.

From the Corollary given earlier we obtain a formula for the sectional curvature of $\mathbb{C}P^n$:

$$K_*(X_*, Y_*) = 1 + \frac{3\langle X, JY\rangle^2}{|X \wedge Y|^2}.$$

Let us consider the quaternionic Hopf bundle $\pi : S^{4n+3} \to \mathbb{H}P^n$ [57].

On the tangent bundle to S^{4n+3} we have the operators of the quaternionic-Kähler structure, $I, J, K : I^2 = K^2 = J^2 = -E$, and $IJ = K$. Let N be the unit normal to S^{4n+3} in R^{4n+4}. Then

$$A_X Y = -\langle IX, Y\rangle IN - \langle JX, Y\rangle JN - \langle KX, Y\rangle KN,$$

where X and Y are horizontal vectors of the Hopf bundle (that is, they are horizontal lifts of tangent vectors to $\mathbb{H}P^n$ in the submersion space).

Let us define an operator Q by $QX = XI \wedge XJ \wedge XK$. Although, generally speaking XI, XJ, XK are not basic, QX is, that is, there exists a vector field Q_*X_* on $\mathbb{H}P^n$ such that $Q_*X_* = \pi_*(QX)$, and $QX = (Q_*X_*)^H$.

Then

$$K_*(X_*, Y_*) = 1 + 3\,(\langle X_* \wedge Q_*X_*, Y_* \wedge Q_*Y_*\rangle^{1/2} - \langle X_*, Y_*\rangle^2\,|X_* \wedge \wedge Y_*|^{-2} = 1 + 3\sin(\varphi + \theta)\sin(\varphi - \theta)(\sin\varphi)^{-2},$$

where θ and φ are determined from the conditions

$$\langle X \wedge QX, \; Y \wedge QY\rangle = \cos^4\theta\,|X|^4\,|Y|^4,$$
$$\langle X, Y\rangle = \cos\varphi\,|X|\,|Y|.$$

Thus, the Hopf bundles $S^3 \xrightarrow{S^1} S^2$, $S^{2n+1} \xrightarrow{S^1} \mathbb{C}P^n$, $S^{4n+3} \xrightarrow{S^3} \mathbb{H}P^n$ are Riemannian submersions with totally geodesic fibres. It turns out that the converse theorem also holds; it was proved by Escobales [51].

***Theorem* 2.1.** *Let $\pi : S^m \to B$ be a Riemannian submersion with connected totally geodesic fibres, for which the dimension of the fibre lies between* 1 *and* $m-1$. *Then π is a submersion of one of the following five types*:

a) $\pi\colon\ S^{2n+1} \xrightarrow{S^1} \mathbb{C}P^n\ (n \geqslant 2)$,
b) $\pi\colon\ S^{4n+3} \xrightarrow{S^3} \mathbb{H}P^n\ (n \geqslant 2)$,
c) $\pi\colon\ S^3 \xrightarrow{S^1} S^2\ (1/2)$,
d) $\pi\colon\ S^7 \xrightarrow{S^3} S^4\ (1/2)$,
e) $\pi\colon\ S^{15} \xrightarrow{S^7} S^8\ (1/2)$.

In the cases a) *and* b), *B is isometric to a complex and a quaternionic projective space of sectional curvature* $1 \leqslant K_* \leqslant 4$ *respectively. In the cases* c), d), *and* e), *B is isometric to a sphere with curvature* $K_* = 4$.

We recall that the tangent bundle of a Riemannian manifold with Sasaki metric is also a Riemannian submersion with totally geodesic fibres.

At each point $(Q, \xi) \in TM$ the tensor A is completely determined by the following formulae:

$$A_{X^H}Y^H = -\frac{1}{2}\,[R(X, Y)\,\xi]^H,$$
$$A_{X^H}Y^V = \frac{1}{2}\,[R(\xi, Y)\,X]^H,$$

where R is the curvature tensor of the manifold at $Q \in M$.

This implies, in particular, that the sectional curvatures of TM along horizontal, vertical and mixed surface elements at a point (Q, ξ) are equal to [57]:

a) $K\,(X^V, Y^V) = 0$,
b) $K\,(X^V, Y^H) = \frac{1}{4}\,|\,R\,(\xi, X)\,Y\,|^2/|\,X\,|^2\,|\,Y\,|^2$,
c) $K\,(X^H, Y^H) = K_*\,(X, Y) - \frac{3}{4}\,|\,R\,(X, Y)\,\xi\,|^2/|\,X \wedge Y\,|^2$,

where $X \wedge Y$ is a simple bivector on the vectors $X, Y \in T_QM$.

Bergery and Bourguignon [38] studied the connection of the Laplacians on the space of a Riemannian submersion with the Laplacians of the fibres and the base of the submersion with totally geodesic fibres. More precisely, let $\pi : M \to B$ be a Riemannian submersion with totally geodesic fibres. Let $F_m = \pi^{-1}(\pi(m))$ be the fibre passing through $m \in M$. Let us denote by Δ^M the Laplacian on the space of the Riemannian submersion.

If $f : M \to \mathbb{R}$ is a function of class C^2, then we denote by $f \downarrow F_m$ the restriction of f to the fibre F_m, and by Δ^{F_m} the Laplacian of the fibre regarded in the metric induced from M.

By the *vertical Laplacian* Δ_v we mean a second-order differential operator defined on C^2-functions on M by

$$(\Delta_v f)\ (m) = (\Delta^{F_m}\ (f \downarrow F_m))\ (m).$$

The *horizontal Laplacian* Δ_h is defined as the differential operator

$$\Delta_h = \Delta^M - \Delta_v.$$

We say that the operators A and B *commute* if $AB-BA$ is the zero operator.

Theorem 2.2 [38]. *If the fibres of a Riemannian submersion $\pi : M \to B$ are totally geodesic, then the operators Δ^M, Δ_v, and Δ_h are pairwise commutative.*

Using this fact, Bergery and Bourguignon observe that if M is compact and connected, then the spectrum of Δ_v (like that of Δ^M) is discrete, which is not true for Δ_h. However, if the multiplicity of each eigenvalue of Δ^M is finite in the situation under consideration, then for the eigenvalues of Δ_v this is not true, generally speaking.

In general the spectrum of Δ_v contains the spectrum of Δ^B of the base manifold, but does not coincide with it.

If we denote by $L^2(M)$ the Hilbert space of (real-valued) L^2-functions on M, then it turns out that $L^2(M)$ admits a basis consisting of common eigenfunctions of Δ^M and Δ_v.

The connection between the eigenvalues of Δ^M, Δ_v, and Δ_h is described as follows. Let

$$H(b, \varphi) = \{f \in L^2(M) \mid \Delta_h f = bf,\ \Delta_v f = \varphi f\}.$$

If $f \in H(b, \varphi)$, then $\Delta^M f = (b+\varphi) f$. However, the converse is not true, that is, the eigenvalues of Δ^M are not all possible sums of eigenvalues of Δ_h and Δ_v when M is not a direct product. Among other results in this paper we shall mention connections such as inequalities between the diameters of the submersion space, the fibre, and the base. Let us write diam(F/G) for the diameter of the metric space F/G, where F is the fibre with the metric induced from M, and G is the isometry group of the fibre. Let us denote by diam$_h(M)$ the horizontal diameter of M:

$$\operatorname{diam}_h M = \sup_{p,q\in M} \inf\{\text{length of a horizontal geodesic joining } p \text{ and } q, \text{ if this is possible}\},$$

where the inf is taken along all possible horizontal curves.

Then

$$\operatorname{diam}^2 B + \operatorname{diam}^2 (F/G) \leqslant \operatorname{diam}^2 M,$$

$$\operatorname{diam}^2 M \leqslant \operatorname{diam}^2 B + \operatorname{diam}^2 F,$$

$$\operatorname{diam}^2 M \leqslant \operatorname{diam}_h^2 M + \operatorname{diam}^2 (F/G).$$

It seems interesting to study the connection of the geometry of a submanifold in the space of a Riemannian submersion with the geometries of the fibres and of the base.

The fibres are a natural class of submanifolds in M. These are integral submanifolds of a vertical distribution. O'Neill found a connection between the sectional curvatures of M and F and the second quadratic forms of a fibre in the space of a submersion (see above).

The horizontal distribution of a Riemannian submersion is not integrable, except for the case of a flat base. However, this does not prevent one from studying horizontal submanifolds in M whose dimension is smaller than that of the base. These submanifolds were studied by Reckziegel [90]. Let us make the definitions more precise and give a statement of his result.

Let $\pi : M \to B$ be a pseudo-Riemannian submersion. A map $g : N \to M$ is called a *horizontal isometric immersion* if the tangent space to $g(N) \subset M$ is horizontal at each point.

Let us denote by f the composition of the projection of a submersion and an immersion: $f = \pi \circ g$.

The main result of [90] may be stated as follows.

***Theorem* 2.3.** *If $g : N \to M$ is a horizontal isometric immersion of a pseudo-Riemannian manifold N into the space M of a pseudo-Riemannian submersion, then*

a) *$f = \pi \circ g$ is an isometric immersion $N \to B$;*

b) *the second quadratic form h^g of the immersion g is horizontal and $\pi_* h^g = h^f$, where h^f is the second quadratic form of $f : N \to B$;*

c) *let $\perp(g)$ and $\perp(f)$ be the normal bundles of the immersions g and f respectively, and let $\nabla^\perp$ be the normal connection of $\perp(g)$ and $\perp(f)$. For every normal vector field $\eta \in \perp(g)$, $\pi_*\eta$ is a normal vector field from $\perp f$. If η is horizontal, then*

$$2V\nabla^\perp_X\eta = A\ (g_*X,\ \eta), \quad \pi_*\nabla^\perp_X\eta = \nabla^\perp_X\pi_*\eta$$

for every vector field X on N.

(We recall that A is the integrability tensor of the horizontal distribution.)

Corollary. *The immersion $f : N \to B$ is totally geodesic, totally umbilical, or pseudo-umbilical if and only if $g : N \to M$ has the same property.*

In [69], two-dimensional Chebyshev surfaces are considered in the space of a Riemannian submersion. Two cases are studied of the position of such a surface: a) when the projection onto the base is one-dimensional and the projection onto the fibre is two-dimensional; b) when the projections onto the base and onto the fibre are one-dimensional. One of the results is as follows: if the intersection of a surface with the fibre is a line of its Chebyshev net, then the projection of the net onto the fibre is a Chebyshev net of the fibre.

Ikuta [61] stated several results on the geometry of Riemannian submersions whose fibres are not totally geodesic. Thus, for example, he gave an expression for the curvature tensor of the normal connection of the bundle in terms of the integrability tensor of the horizontal distribution and the second quadratic forms of the fibres. He proved the following result.

Theorem **2.4.** *Let $\pi : M^{n+k}(c) \to B^n(a)$ be a Riemannian submersion of a space of constant curvature c onto a space of constant curvature a. If*
a) $c = a$, *then the horizontal distribution is integrable;*
b) $c \neq a$ *and the normal connection of the fibre is flat, then n is even.*

Incidently, assertion a) is a simple consequence of O'Neill's formulae relating the curvature of the base with that of the submersion space. Moreover, the integrability of a horizontal submersion implies that M is a metric product of the base and the fibre. Taking into account the fact that M has constant curvature, this means that M is flat, that is, that $c = a = 0$.

In [80] an interesting problem is stated regarding Riemannian submersions. Let $\pi : M \to B$ be a Riemannian submersion, and let N be a submanifold in M. Then $\pi(N)$ is a submanifold in B. What is the connection between the properties of $N \subset M$ and those of $\pi(N) \subset B$? In [80] it is proved that if N is a locally symmetric submanifold in M, then, under certain assumptions, $\pi(N)$ will be a locally symmetric submanifold in B. Namely, it is assumed that $f : N \to M$ is an isometric immersion of N into a space M of constant curvature; that $\pi : M \to B$ and $\pi : N \to B' \subset B$ are Riemannian submersions with totally geodesic fibres, and the diagram

$$\begin{array}{ccc} N & \xrightarrow{f} & M \\ \pi\downarrow & & \pi\downarrow \\ B' & \xrightarrow{f'} & B \end{array}$$

is commutative; that f is a diffeomorphism on the fibres; that $A_E F = 0$ for a horizontal F tangent to N and E and orthogonal to N. An example (possibly the only one, given all these assumptions) is the Hopf bundle $\pi : S^{2n+1} \to \mathbb{C}P^n$, where $N = S^{2m+1}$ is an $(m < n)$-submanifold in $M = S^{2n+1}$ and $\pi(N) = \mathbb{C}P^m$ is locally symmetric in $\mathbb{C}P^n$. One can obtain more interesting examples of submanifolds in the submersion space by simplifying the situation and considering the tangent bundle of a Riemannian manifold and a surface in it.

2.2. Geodesics in Riemannian submersions.

The equations of geodesics for the tangent bundle of a Riemannian manifold with the Sasaki metric were obtained in [93] for the particular case of a Riemannian submersion. The case of a general Riemannian submersion was considered in detail by O'Neill [85], whose work will now be reviewed.

The main goal of [85] consists in comparing the geodesics of M and B for a Riemannian submersion $\pi : M \to B$ and finding the connection between conjugacy and the index of geodesics in M and B. In particular, the equations of the geodesics in the submersion space were obtained.

More precisely, let $\mathscr{H}$ and $\mathscr{V}$ be the horizontal and vertical projection operators in the tangent space to M, and let E', E'', ... be the covariant derivatives in the Riemann connection of M of a vector field E tangent to M. Let $E_{\mathscr{H}} = \mathscr{H}E$ and $E_{\mathscr{V}} = \mathscr{V}E$. Then for every vector field E we have $E = E_{\mathscr{H}} + E_{\mathscr{V}}$.

Theorem 2.5. *Let $\pi : M \to B$ be a submersion, and let $E = E_{\mathscr{H}} + E_{\mathscr{V}}$ be a vector field on a curve $\gamma(t) \subset M$. Then*

$$\mathscr{H}(E') = E_*' + A_{E_{\mathscr{H}}}(\mathscr{V}\gamma') + A_{\mathscr{H}\gamma'}(E_{\mathscr{V}}) + T_{\mathscr{V}\gamma'}(E_{\mathscr{V}}),$$
$$\mathscr{V}(E') = A_{\mathscr{H}\gamma'}(E_{\mathscr{H}}) + T_{\mathscr{V}\gamma'}(E_{\mathscr{H}}) + \mathscr{V}(E_{\mathscr{V}}'),$$

where γ' is the tangent vector field to $\gamma(t)$, $E_* = \pi_*(E)$ is the projection of E into the tangent vector field to the base, and E_* is simultaneously regarded as the horizontal lift of the field $\pi_*(E)$.

Theorem 2.6. *Let γ be a curve in M, let $X = \mathscr{H}\gamma'$, and $U = \mathscr{V}\gamma'$. Then*

$$\mathscr{H}\gamma'' = \gamma_*'' + 2A_X U,$$
$$\mathscr{V}(\gamma'') = T_U X + \mathscr{V}(U'),$$

where γ_*'' is the horizontal lift of the vector of the second covariant derivative of the curve $\pi \circ \gamma$ onto the base B.

Setting $\mathscr{H}(\gamma'') = 0$ and $\mathscr{V}(\gamma'') = 0$ we obtain a condition for γ to be a geodesic on M. In particular, if γ is horizontal, that is, $\mathscr{V}\gamma'' = 0$, then $\gamma_* = \pi \circ \gamma$ is a geodesic on B, and conversely, the horizontal lift of a geodesic of B to M is a geodesic on M.

Corollary. *If $\pi : M \to B$ is a Riemannian submersion with totally geodesic fibres, then the equation of a geodesic γ on M has the form*

$$\gamma_*'' = -2A_{\mathscr{H}\gamma'}(\mathscr{V}\gamma'), \qquad \mathscr{V}(\mathscr{V}\gamma')' = 0.$$

To state other results we give several definitions.

Let γ be a geodesic of a Riemannian manifold M joining the points a and b in M. A vector field ξ along γ is called a *Jacobi field* if ξ satisfies the Jacobi equation

$$\xi'' + R(\gamma', \xi)\gamma' = 0.$$

The points a and b are said to be *conjugate along* γ if there is a non-zero Jacobi field ξ along γ such that $\xi(a) = \xi(b) = 0$.

Let a and b be conjugate points on a geodesic. The dimension of the space of solutions of the Jacobi equation is called the *multiplicity* of the conjugate point, or the *order of conjugacy*.

The existence of conjugate points on a geodesic segment indicates that the geodesic is not the unique shortest line joining the two given points.

The main result of [85] is stated as follows.

Theorem 2.7. *Let $\gamma : [a, b] \to M$ be a horizontal geodesic segment on the space of a Riemannian submersion $\pi : M \to B$. If I is an index (Morse) form on M and I_B is an index (Morse) form on B, then*

$$I(E, F) = I_B(E_*, F_*) + \int_a^b \langle DE, DF \rangle \, dt,$$

where $DE = \mathcal{V}(E'_{\mathcal{V}}) - T_{E_{\mathcal{V}}}\gamma' + 2A_{\gamma'}E_{\mathcal{H}}$ is the derivative vector field.(1)

A consequence of this theorem is that conjugate points on γ appear no earlier than those on $\pi \circ \gamma$.

As an application of these results, consider a submersion $\pi : M \to B$, where M is compact and has constant sectional curvature 1.

Let

$$A = \sup_{|X \wedge Y|=1} |A_X Y| \quad (X, Y \in \mathcal{H})$$

be the norm of the integrability tensor of the horizontal distribution. (In the case of the tangent bundle we have $A_X Y = R(X, Y)\xi$, where $R(X, Y)$ is the curvature operator.) Let $\beta(s)$ be a geodesic in B with the natural parametrization. Then:

1) for every integer m the point $\beta(m\pi)$ is conjugate to $\beta(0)$ with order $\dim B - 1$;

2) all the other conjugate points $\beta(t)$ for $\beta(0)$ lie in the intervals $m\pi + d \leqslant t \leqslant (m+1)\pi - d$, where $d = \pi(1 + 3A^2(\nu))^{-1/2}$;

3) the order of conjugacy on each such interval is at least equal to the dimension of the fibre.

A more detailed description of non-horizontal geodesics is given in [79].

§3. Connection between the geometric features of the tangent bundle (the normal bundle) and the base

One of the most important features of the Riemannian metric is its sectional curvature. Hence, we will begin the study of the connections between the geometry of the tangent bundle (the normal bundle) and the base with the curvature of the Sasaki metric. We shall carry out the exposition both in invariant form and in local coordinates.

Let us denote by $R(X, Y)Z$ the field of the curvature tensor of the base, and by $\widetilde{R}(\widetilde{X}, \widetilde{Y})\widetilde{Z}$ that of the curvature tensor of the tangent bundle.

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(1) See the definition of the Morse index form in S. Kobayashi and K. Nomizu, Foundations of differential geometry, Vol. 2, Interscience, New York 1969. Translation: *Osnovy differentsial'noi geometrii*, Mir, Moscow 1990, Vol. 2, p. 81.

***Lemma* 3.1.** *The curvature tensor $\tilde{R}$ of the Sasaki metric of TM is determined at each point $\tilde{Q} = (Q, \xi)$ by the following formulae:*

$$\tilde{R}(X^V, Y^V)Z^V = 0,$$

$$\tilde{R}(X^V, Y^V)Z^H = \Big[R(X,Y)Z + \frac{1}{4}R(\xi, X)R(\xi, Y)Z - \frac{1}{4}R(\xi, Y)R(\xi, X)Z\Big]^H,$$

$$\tilde{R}(X^H, Y^V)Z^H = \Big[\frac{1}{4}R(R(\xi, Y)Z, X)\xi + \frac{1}{2}R(X,Z)Y\Big]^V + \Big[\frac{1}{2}(\nabla_X R)(\xi, Y)Z\Big]^H,$$

$$\tilde{R}(X^H, Y^V)Z^V = -\Big[\frac{1}{2}R(Y,Z)X + \frac{1}{4}R(\xi, Y)R(\xi, Z)X\Big]^H,$$

$$\tilde{R}(X^H, Y^H)Z^V = \Big[R(X,Y)Z + \frac{1}{4}R(R(\xi, Z)Y, X)\xi - \frac{1}{4}R(R(\xi, Z)X, Y)\xi\Big]^V,$$

$$\tilde{R}(X^H, Y^H)Z^H = \Big[R(X,Y)Z + \frac{1}{4}R(\xi, R(Z,Y)\xi)X + \frac{1}{4}R(\xi, R(X,Z)\xi)Y + \frac{1}{2}R(\xi, R(X,Y)\xi)Z\Big]^H + \Big[\frac{1}{2}(\nabla_Z R)(X, Y)\xi\Big]^V,$$

where X, Y, Z are tangent vectors at the point $Q = \pi(\tilde{Q})$.

An analogue of this lemma for the frame bundle was proved in [47].

In induced local coordinates the components of the curvature tensor at (Q, ξ) of the Sasaki metric of TM are expressed as follows:

$$\tilde{R}^i_{jkm} = R^i_{jkm} + \frac{1}{4}R^i_{m\lambda\alpha}R^\alpha_{\mu kj}\xi^\lambda\xi^\mu + \frac{1}{4}R^i_{k\lambda\alpha}R^\alpha_{\mu jm}\xi^\lambda\xi^\mu + \frac{1}{2}R^i_{j\lambda\alpha}R^\alpha_{\mu km}\xi^\lambda\xi^\mu,$$

$$\tilde{R}^i_{j\,k\,n+m} = \frac{1}{2}\nabla_k R^i_{j\lambda m}\xi^\lambda,$$

$$\tilde{R}^i_{j\,n+k\,n+m} = R^i_{jkm} + \frac{1}{4}R^i_{\alpha\lambda k}R^\alpha_{j\mu m}\xi^\lambda\xi^\mu - \frac{1}{4}R^i_{\alpha\lambda m}R^\alpha_{j\mu k}\xi^\lambda\xi^\mu,$$

$$\tilde{R}^i_{n+j\,k\,n+m} = \frac{1}{2}R^i_{kjm} - \frac{1}{4}R^i_{\alpha\lambda m}R^\alpha_{k\mu j}\xi^\lambda\xi^\mu,$$

$$\tilde{R}^i_{n+j\,n+k\,n+m} = 0, \quad \tilde{R}^{n+i}_{n+j\,n+k\,n+m} = 0.$$

An analogue of this assertion for the frame bundle is given in [73].

The following assertions follow easily from this lemma.

***Theorem* 3.2** [65]. *The tangent bundle TM with the Sasaki metric is locally symmetric if and only if M is flat.*

***Theorem* 3.3** [65]. *The tangent bundle TM with the Sasaki metric is flat if and only if M is flat.*

***Theorem* 3.4** [34]. *If the sectional curvature of TM is bounded, then M is flat, and therefore so is TM.*

Word-for-word analogues of these results also hold for the frame bundle (see [73] and [47]).

Theorem 3.2 can be considerably strengthened. To give the corresponding statement, let us introduce the concept of intrinsic nullity index. The *intrinsic nullity index* $\nu(Q)$ of a point $Q \in M$ is defined as the dimension of the maximal linear subspace $L_Q \subset T_Q M$ such that for $Y \in L_Q$ and every $X, Z \in T_Q M$ we have $R(X, Y)Z = 0$ for the curvature tensor of M. If $\nu(Q) \geqslant k$ for every $Q \in M$, then the metric on M is called *strongly k-parabolic*. Clearly, the case $\nu = n$ corresponds to the flat metric of M. The geometric structure of manifolds with constant intrinsic nullity index has been described by Hartman [9*] and Maltz [10*]. If ν is constant, then the distribution L is holonomic and the integral submanifolds are totally geodesic in M and locally isometric to the Euclidean space E^ν.

***Theorem* 3.4′** [4]. *If the intrinsic nullity index $\tilde{\nu}$ of the tangent bundle TM^n with the Sasaki metric is equal to k, then k is even and M^n is the metric (Riemannian) product of a Riemannian manifold $M^{n-k/2}$ and the Euclidean space $E^{k/2}$, and TM^n is the metric (Riemannian) product of $TM^{n-k/2}$ and E^k.*

The proof relies on the construction, based on the assumptions of the theorem, of $k/2$ parallel linearly independent vector fields on M.

A substantially larger number of results on the curvature of the Sasaki metric have been obtained for spherical tangent bundles.

The first publication on this theme was a small paper by Klingenberg and Sasaki [64]. They considered the Sasaki metric on T_1S^2 and proved that its sectional curvature is constant and equal to 1/4. Grimaldi [58] proved that among the two-dimensional manifolds the sphere is the only one for which the unit vector bundle with the Sasaki metric is a symmetric space. Considering $T_\rho(M^n, K)$, where (M^n, K) denotes a manifold of constant curvature K, in the case $n = 2$, Tanno [98] and independently Nagy [75] proved that for $\rho^2 = 1/K$ the Sasaki metric of $T_\rho(M^2, K)$ has constant sectional curvature $K/4$. This result is a particular case of the following theorem.

***Theorem* 3.5** [32]. *The extremal values $\overline{K}_{\max}$ and $\overline{K}_{\min}$ of the sectional curvature of the Sasaki metric of $T_1(M^n, K)$ are as follows:*

a) *when* $n = 2$,

$$\overline{K}_{\max} = \begin{cases} K^2/4, & K \in]-\infty, 0] \cup]1, +\infty[, \\ K(1-3K/4), & K \in]0, 1]; \end{cases}$$

$$\overline{K}_{\min} = \begin{cases} K(1-3K/4), & K \in]-\infty; 0] \cup]1, \infty[, \\ K^2/4, & K \in]0, 1]; \end{cases}$$

b) *when* $n \geqslant 3$,

$$\bar{K}_{\max} = \begin{cases} K + \dfrac{K^2(K-5)^2}{4(K^2-4K-1)}, & K \in]-\infty, (3-\sqrt{17})/2], \\ 1, & K \in](3-\sqrt{17})/2, 2/3], \\ K + \dfrac{K^2}{4(2K-1)}, & K \in]2/3, (5+\sqrt{17})/2], \\ K^2/4, & K \in](5+\sqrt{17}/2, +\infty[, \end{cases}$$

$$\bar{K}_{\min} = \begin{cases} K(1-3K/4), & K \in]-\infty, 0] \cup]4/3, \infty[, \\ 0, & K \in]0, 4/3]. \end{cases}$$

The proof is based on a thorough analysis of the formula for the sectional curvature of the Sasaki metric of $T_\rho M$ given below, in the case of a manifold of constant curvature and for $\rho = 1$.

***Lemma* 3.1′** [32]. *Let* $\bar{X}$ *and* $\bar{Y}$ *be perpendicular unit vectors tangent to* $T_\rho M$ *at the point* $\bar{Q} = (Q, \rho\xi)$ $(|\xi| = 1)$. *The sectional curvature* $\bar{K}(\bar{X}, \bar{Y})$ *of the Sasaki metric of* $T_\rho M$ *in the two-dimensional direction* $(\bar{X}, \bar{Y})$ *is equal to*

$$\begin{aligned} \bar{K}(\bar{X}, \bar{Y}) = {} & \langle R(X_H, Y_H)Y_H, X_H\rangle - (3\rho^2/4)\,|R(X_H, Y_H)\xi|^2 + \\ & + 3\langle R(X_H, Y_H)Y_V, X_V\rangle - \rho^2\langle R(\xi, X_V)X_H, R(\xi, Y_V)Y_H\rangle + \\ & + (\rho^2/4)\,|R(\xi, Y_V)X_H + R(\xi, X_V)Y_H|^2 + \rho\langle(\nabla_{X_H}R)(X_H, Y_H)\xi, X_V\rangle - \\ & - \rho\langle(\nabla_{Y_H}R)(X_H, Y_H)\xi, Y_V\rangle + \frac{1}{\rho^2}(|X_V|^2|Y_V|^2 - \langle X_V, Y_V\rangle^2). \end{aligned}$$

It is not hard to observe that if M has constant curvature, the sectional curvatures of $T_\rho(M, K)$ and $T_1(M, K)$ are connected by the relation $\rho^2\bar{K}(\rho, K) = \bar{K}(1, \rho^2 K)$. Therefore, Theorem 3.5 has a trivial generalization to $T_\rho(M, K)$. Namely, let us denote by $\varkappa$ the curvature of the fibre. Then $\varkappa = 1/\rho^2$ and $\bar{K}(\rho, K) = \varkappa\bar{K}(1, k/\varkappa)$. Therefore

$$\bar{K}_{\min}(\rho, K) = \varkappa\bar{K}_{\min}(1, K/\varkappa),$$
$$\bar{K}_{\max}(\rho, K) = \varkappa\bar{K}_{\max}(1, K/\varkappa).$$

Theorem 3.5 implies also that for $T_1(M, K)$ the sectional curvature of the Sasaki metric cannot be non-positive, whereas it is non-negative for $K \in [0, 4/3]$. The question naturally arises of finding necessary and sufficient conditions for the sectional curvature of T_1M or $T_\rho M$ to be non-negative in the general case. The following result holds.

***Theorem* 3.6** [5]. *Let* X, Y, U, W, ξ *be unit vectors tangent to* M *at the point* Q, *and* $\langle X, Y\rangle = \langle U, W\rangle = 0$, $\langle U, \xi\rangle = \langle W, \xi\rangle = 0$. *Let* $K(X, Y)$ *be the* *sectional curvature of M in the direction of the surface element spanned by the vectors (X, Y). At $\overline{Q} = (Q, \xi)$ the sectional curvature of $T_\rho M$ is non-negative if*

$$\frac{\langle(\nabla_X R)(\xi, W)X, Y\rangle^2}{|R(\xi, W)X|^2} + \frac{\langle(\nabla_Y R)(\xi, U)Y, X\rangle^2}{|R(\xi, U)Y|^2} + \\ + (\rho^2/4)[3\langle R(X, Y)W, U\rangle - \rho^2\langle R(\xi, U)X, R(\xi, W)Y\rangle + \\ + (\rho^2/2)\langle R(\xi, U)Y, R(\xi, W)X\rangle]^2 \leqslant K(X, Y) - \frac{3\rho^2}{4}|R(X, Y)\xi|^2$$

for every X, Y, U, W.

The theorem gives a sufficient condition which is close to necessary, in the sense that for $n = 2$ it becomes necessary. Namely, we have the following result.

***Theorem* 3.7** [29]. *For $T_\rho M^2$ to have a non-negative sectional curvature it is necessary and sufficient that*

$$\Delta_1 K \leqslant K^3(1 - 3(\rho^2/4)K),$$

where K is the Gaussian curvature of M^2 and Δ_1 is the first differential Beltrami parameter.

The assumptions of Theorem 3.6 are also satisfied by compact symmetric spaces of rank 1. Indeed, they satisfy $K(X, Y) > 0$ and for $\rho = 0$ the assumptions of the theorem are fulfilled. Consequently, this condition also holds for certain $\rho > 0$.

In order to elucidate the geometric meaning of Theorem 3.6 we shall introduce the following notation:

$$M = \sup_{\substack{|X\wedge Y|=1 \\ |\xi|=1}} |R(X, Y)\xi|, \quad \mu = \inf_{|X\wedge Y|=1} K(X, Y),$$

$$M_\nabla = \sup_{\substack{|X\wedge Y|=1 \\ |\xi\wedge W|=1}} \frac{|\langle(\nabla_X R)(\xi, W)X, Y\rangle|}{|R(\xi, W)X|}.$$

Then a crude form of the inequalities of the theorem yields the following conclusions.

***Theorem* 3.8** [5]. a) *If*

$$\rho^2 M \leqslant \frac{4}{3}\left[\sqrt[3]{1 + \frac{3}{4}\frac{\mu - 2M_\nabla}{M}} - 1\right],$$

then $T_\rho M$ has non-negative curvature.

b) *If $0 \leqslant \mu \leqslant 1/6$, $M^2 \leqslant \mu/6$, and $M_\nabla^2 \leqslant \mu/6$, then $T_1 M$ has non-negative sectional curvature.*

The problem of the necessary value of ρ for $\mathbb{C}P^n$, $\mathbb{H}P^n$, $\mathbb{C}aP^2$ remains open.

Together with the sectional curvature, one considers in Riemannian geometry the Ricci curvatures and the scalar curvature of the Riemannian metric. The *Ricci tensor* R_{ij} is defined as the contraction of the curvature tensor, that is, $R_{ij} = R^s_{isj}$. The *Ricci curvature* in the direction of a unit tangent vector X is defined as the scalar

$$\operatorname{Ric}(X) = R_{ij}X^iX^j.$$

The *scalar curvature of the metric* at a given point is defined as the contraction of the Ricci tensor: $R = g^{is}R_{is}$.

A Riemannian manifold is called *Einsteinian* if its Ricci curvture depends neither on the point nor on the direction.

***Theorem* 3.9** [104]. *If TM is Einsteinian with the Sasaki metric, then M is flat.*

A word-for-word analogue of this assertion for FM has been proved in [47]. For spherical tangent bundles the situation is more interesting. Thus, in the two-dimensional case Grimaldi [58] proved the equivalence of the following three assertions:

a) $T_\rho M^2$ is locally symmetric;

b) $T_\rho M^2$ is an Einstein space;

c) M^2 is isometric to a Euclidean sphere of radius ρ.

Buzzanca [39] studied the problem of the eigenvectors of the Ricci operator on $T_\rho M^2$. One of the results consists in the fact that if $\operatorname{Ric}(X) = \lambda X$ and X is a horizontal (vertical) vector field such that supp $X = T_\rho M^2$, then the Gaussian curvature of M^2 is constant, and $\lambda = K(2-\rho^2K)/2$ (respectively, $\lambda = \rho^2K^2/2$).

In higher dimensions there are results for spaces of constant curvature.

***Theorem* 3.10** [30]. a) *The Ricci curvature* $\widetilde{\operatorname{Ric}}$ *of the Sasaki metric* $T_1(M^n, K)$ *lies between the following limits:*

(i) $n = 2$:

$$K^2/2 \leqslant \widetilde{\operatorname{Ric}} \leqslant K(2-K)/2 \quad \text{for } 0 < K \leqslant 1,$$
$$K(2-K)/2 \leqslant \widetilde{\operatorname{Ric}} \leqslant K^2/2 \quad \text{for } K \leqslant 0, \ K > 1;$$

(ii) $n \geqslant 3$:

$$(n-1)K(2-K)/2 \leqslant \widetilde{\operatorname{Ric}} \leqslant (2(n-1)-K)K/2, \quad 1 < K \leqslant n-2,$$
$$(n-1)K(2-K)/2 \leqslant \widetilde{\operatorname{Ric}} \leqslant (K^2+2(n-2))/2, \quad K \leqslant 1, \ K > n-2.$$

b) *The scalar curvature* $\widetilde{Sc}$ *of the Sasaki metric of* $T_1(M^n, K)$ *is equal to*

$$\widetilde{Sc} = (n-1)(n^2+2n-4-(K-n)^2)/2$$

and, in particular, $\widetilde{Sc} \leqslant (n-1)(n^2+2n-4)/2$ *for every K.*

For $K = 1$ the scalar curvature and the Ricci curvature have also been computed in [107].

The proof of this theorem relies on the following lemma.

***Lemma* 3.2** [30]. *The non-zero components of the Ricci tensor of $T_1(M^n, K)$ are equal to*

$$\bar{R}_{pp} = K\,(n-1) - K^2/2,$$
$$\bar{R}_{nn} = K\,(n-1) - (n-1)\,K^2/2,$$
$$\bar{R}_{n+p\;n+p} = n - 2 + K^2/2, \quad (p = 1, \ldots, n-1).$$

Remark. The system of coordinates on T_1M was chosen in such a way that at the point $(Q, \xi) \in T_1M$ under consideration the nth coordinate corresponds to the direction of the vector ξ, and at Q we have $g_{ij} = \delta_{ij}$ and $\Gamma^i_{jk} = \Gamma_{jk,i} = 0$. (See [107] for another expression for the Ricci tensor of T_1S^n.)

An (invariant) expression of the Ricci tensor for the frame bundle has been obtained in [47].

It is also interesting that T_1M, as a hypersurface in TM, has constant mean curvature [92]. Namely, the mean curvature vector $\bar{H}$ at each point $(Q, \xi) \in T_1M$ has the form

$$\bar{H} = -\frac{n-1}{2n-1}\,\xi^V,$$

where ξ^V is the vertical lift (in the sense of TM) of ξ, that is, the unit normal to T_1M in TM at the point $(Q, \xi) \in T_1M$.

Let us now consider analogous results for the normal bundle of a surface in a Riemannian space.

Let us denote by $N(X, Y)\xi$ the field of the normal curvature tensor of $F^l \subset M^{l+p}$, and by $\hat{N}(\xi, \eta)X$ the field of the conjugate tensor. (We recall that in a constant curvature space $\hat{N}(\xi, \eta)X = [A_\xi, A_\eta]X$, where X and Y are tangent vector fields, and ξ and η are normal vector fields.) Let $\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z}$ be the field of the curvature tensor of the normal bundle NF^l with the Sasaki metric, and let $R(X, Y)Z$ be the field of the curvature tensor of F^l.

***Lemma* 3.3** [6]. *At each point $\tilde{Q} = (Q, \xi)$ the curvature tensor $\tilde{R}$ of the Sasaki metric of NF^l is determined as follows*:

$$\tilde{R}(X^H, Y^H)Z^H = [R(X,Y)Z + \tfrac{1}{4}\hat{N}(\xi, N(Z,Y)\xi)X + \tfrac{1}{4}\hat{N}(\xi, N(X,Z)\xi)Y + \tfrac{1}{2}\hat{N}(\xi, N(X,Y)\xi)Z]^H + \left[\tfrac{1}{2}(\nabla^{\perp}_Z N)(X,Y)\xi\right]^V,$$

$$\tilde{R}(X^H, Y^H)\zeta^V = \tfrac{1}{2}[(\nabla_X\hat{N})(\xi,\zeta)Y - (\nabla_Y\hat{N})(\xi,\zeta)X]^H + \left[N(X,Y)\zeta + \tfrac{1}{4}N(\hat{N}(\xi,\zeta)Y, X)\xi - \tfrac{1}{4}N(\hat{N}(\xi,\zeta)X, Y)\xi\right]^V,$$

$$\widetilde{R}(X^H, \eta^V) Z^H = \Big[\frac{1}{2}(\nabla_X \widehat{N})(\xi, \eta)\, Z\Big]^H + \\ + \Big[\frac{1}{2} N(X, Z)\,\eta + \frac{1}{4} N(\widehat{N}(\xi, \eta) Z, X)\,\xi\Big]^V,$$

$$\widetilde{R}(X^H, \eta_V)\zeta^V = -\Big[\frac{1}{2}\widehat{N}(\eta, \zeta) X + \frac{1}{4}\widehat{N}(\xi, \eta)\widehat{N}(\xi, \zeta) X\Big]^H,$$

$$\widetilde{R}(\varphi^V, \eta^V) Z^H = \Big[\widehat{N}(\varphi, \eta) Z + \frac{1}{4}\widehat{N}(\xi, \varphi)\widehat{N}(\xi, \eta) Z - \\ - \frac{1}{4}\widehat{N}(\xi, \eta)\widehat{N}(\xi, \varphi) Z\Big]^H, \quad \widetilde{R}(\varphi^V, \eta^V)\zeta^V = 0,$$

where all the lifts are carried out at $\widetilde{Q} = (Q, \xi)$ $(Q \in F^l;\ \xi, \varphi, \eta, \zeta \in N_Q F^l;\ X, Y, Z \in T_Q F^l)$.

In a special coordinate system in a neighbourhood of $Q \in F^l$, namely, one such that $g_{ij} = \delta_{ij}$, $\mu_{\alpha\beta|i} = 0$, and $\Gamma_{ij,k} = \Gamma_{jk,i} = 0$ at the given point, the result of the previous lemma may be written as follows.

Lemma **3.3′** [6].

$$\widetilde{R}_{ijkm} = R_{ijkm} + \sum_{\alpha=1}^{p}\Big(\frac{1}{4} N_{\mu\alpha\,|\,im} N_{\alpha\nu\,|\,kj} + \frac{1}{4} N_{\mu\alpha\,|\,ik} N_{\alpha\nu\,|\,jm} + \\ + \frac{1}{2} N_{\mu\alpha\,|\,ij} N_{\alpha\nu\,|\,km}\Big)\xi^\mu\xi^\nu,$$

$$\widetilde{R}_{ijk\,l+\sigma} = \frac{1}{2}\nabla_k N_{\nu\sigma\,|\,ij}\xi^\nu,$$

$$\widetilde{R}_{ij\,l+\tau\,l+\sigma} = N_{\tau\sigma\,|\,ij} + \frac{1}{4}\sum_{t=1}^{l}(N_{\mu\tau\,|\,it}N_{\nu\sigma\,|\,tj} - N_{\mu\sigma\,|\,it}N_{\nu\tau\,|\,tj})\,\xi^\mu\xi^\nu,$$

$$\widetilde{R}_{i\,l+\beta\,k\,l+\sigma} = \frac{1}{2} N_{\beta\sigma\,|\,ik} - \frac{1}{4}\sum_{t=1}^{l} N_{\mu\sigma\,|\,it}N_{\nu\beta\,|\,tk}\xi^\mu\xi^\nu,$$

$$\widetilde{R}_{i\,l+\beta\,l+\tau\,l+\sigma} = 0, \quad \widetilde{R}_{l+\alpha\,l+\beta\,l+\tau\,l+\sigma} = 0,$$

where $i, j, k, m = 1, \ldots, l;\ \alpha, \beta, \tau, \sigma = 1, \ldots, p;\ p = n - l.$

Let us denote by $\overline{R}$ the curvature tensor of the Sasaki metric of $N_\rho F^l$. If the system of coordinates in a neighbourhood of Q is chosen so that at this point we have $g_{ij} = \delta_{ij}$, $\mu_{\alpha\beta|i} = 0$, $\Gamma_{ij,k} = 0$, and the unit normal ξ is taken as the pth basis vector of $N_Q F^l$, then at the point $\overline{Q} = (Q, \rho\xi)$ the curvature tensor of the Sasaki metric of $N_\rho F^l$ has the following form.

Lemma **3.4** [6].

$$\overline{R}_{ijkm} = R_{ijkm} + \frac{\rho^2}{4}\sum_{\alpha=1}^{p}[N_{p\alpha\,|\,im}N_{\alpha p\,|\,kj} + N_{p\alpha\,|\,ik}N_{\alpha p\,|\,jm}] + \frac{\rho^2}{2}\sum_{\alpha=1}^{p} N_{p\alpha\,|\,ij}N_{\alpha p\,|\,km},$$

$$\overline{R}_{ij\,l+\varkappa} = \frac{\rho}{2}\nabla_k N_{p\varkappa\,|\,ij},$$

$$\overline{R}_{ij\,l+\theta\,l+\varkappa} = N_{\theta\varkappa\,|\,ij} + \frac{\rho^2}{4}\sum_{t=1}^{l}[N_{p\theta\,|\,it}N_{p\varkappa\,|\,tj} - N_{p\varkappa\,|\,it}N_{p\theta\,|\,jt}],$$

$$\bar{R}_{i\,l+\psi\,k\,l+\varkappa} = \frac{1}{2} N_{\psi\varkappa\,|\,ik} - \frac{\rho^2}{4} \sum_{t=1}^{l} N_{p\varkappa\,|\,it} N_{p\psi\,|\,tk},\ \bar{R}_{i\,l+\psi\,l+\theta\,l+\varkappa} = 0,$$

$$\bar{R}'_{l+\varphi\,l+\psi\,l+\theta\,l+\varkappa} = \frac{1}{\rho^2}(\delta_{\varphi\theta}\delta_{\psi\varkappa} - \delta_{\varphi\varkappa}\delta_{\psi\theta}),$$

where R_{ijkm} *is the curvature tensor of* F^l $(i, j, k, m = 1, \ldots, l;$ $\varkappa, \theta, \varphi, \psi = 1, \ldots, p-1)$.

Let us state analogues of Theorems 3.2–3.4 above.

***Theorem* 3.11** [6]. a) *The Sasaki metric of* NF^l *is flat if and only if* F^l *is a submanifold with intrinsic flat metric embedded in* M^{l+p} *with a flat normal connection.*

b) NF^l *is locally symmetric if and only if* F^l *is a symmetric space embedded in* M^{l+p} *with flat normal connection.*

A distribution $\tilde{L}$ on NF^l will be called *horizontal* (*vertical*) if at each point $\tilde{Q} \in NF^l$ the subspace $\tilde{L}_{\tilde{Q}}$ is horizontal (vertical).

***Theorem* 3.12** [6]. a) *If the Sasaki metric of* NF^l *is vertically strongly* ν*-parabolic, then on* F^l *there are* ν *parallel normal vector fields in the normal connection.*

b) *Let* F^l *be a surface in the Euclidean space* E^{l+p}. *If the Sasaki metric of* NF^l *is horizontally strongly k-parabolic, then* F^l *can be stratified into k-dimensional intrinsically flat submanifolds totally geodesic in* F^l *with flat normal connection in the ambient space.*

The sectional curvature of the Sasaki metric of $N_\rho F^l$ is given by the following result.

***Lemma* 3.5** [6]. *Let* $\bar{X} = X^H + \zeta^V$ *and* $\bar{Y} = Y^H + \eta^V$ *be perpendicular unit vectors tangent to* $N_\rho F^l$ *at the point* $\bar{Q} = (Q, \rho\xi)$. *Then*

$$\begin{aligned}\bar{K}(\bar{X}, \bar{Y}) = \langle R(X, Y)Y, X\rangle - (3\rho^2/4)\,|\,N(X, Y)\xi\,|_\perp^2 + \\ + 3\langle N(X, Y)\eta, \zeta\rangle_\perp - \rho^2\langle \hat{N}(\xi, \zeta)X, \hat{N}(\xi, \eta)Y\rangle + \\ + \frac{\rho^2}{4}\,|\hat{N}(\xi, \eta)X + \hat{N}(\xi, \zeta)Y\,|^2 + \rho\langle(\nabla_Y^\perp N)(X, Y)\xi, \zeta\rangle_\perp - \\ - \rho\langle\nabla_X N\rangle(X, Y)\xi, \eta\rangle_\perp + \frac{1}{\rho^2}(|\,\eta\,|_\perp^2\,|\,\zeta\,|_\perp^2 - \langle\eta, \zeta\rangle_\perp^2),\end{aligned}$$

where $\langle\,,\rangle$ *and* $\langle\,,\rangle_\perp$ *are scalar products in the metric of* F^l *and the fibre (Euclidean) metric respectively.*

If the curvature of F^l is positive, then for small ρ the curvature of $N_\rho F^l$ may be positive. An example is the Veronese surface.

Let us consider the embedding $E^3 \to E^5$ whose radius-vector has the form

$$U = \left\{\frac{1}{\sqrt{3}}x_1x_3, \frac{1}{\sqrt{3}}x_2x_3, \frac{1}{\sqrt{3}}x_1x_2, \frac{1}{2\sqrt{3}}(x_1^2 - x_2^2), \frac{1}{6}(x_1^2 + x_2^2 - 2x_3^2)\right\}.$$

If $x_1^2+x_2^2+x_3^2 = 3$, then $u_1^2+u_2^2+u_3^2+u_4^2+u_5^2 = 1$. Thus, we obtain an isometric immersion $S^2(\sqrt{3}) \to S^4(1)$ for which the points (x_1, x_2, x_3) and $(-x_1, -x_2, -x_3)$ are carried into the same point, that is, we have an embedding of $\mathbb{R}P^3$ into $S^4(1)$, which is precisely the *Veronese surface* V^2.

***Theorem* 3.13** [6]. *The sectional curvature of the Sasaki metric of* $N_\rho V^2$ *for* $\rho = \sqrt{3}/2$ *is constant, positive, and equal to* $1/12$.

This example yields an analogue of a result in [64] obtained in the study of T_1S^2.

§4. Geodesic lines in the tangent and normal bundles. Totally geodesic submanifolds

4.1. The Sasaki geodesic metrics of the tangent bundle.

Sasaki [93] obtained the equations of the geodesics of TM. Let (x^i, y^i) be the natural induced coordinates in TM. Then $C(t) = (x^i(t), y^i(t))$ is the equation of a curve in TM. Clearly, a curve in TM can be regarded as a vector field $y(t)$ along the curve $x(t)$ on the base manifold. The curve $C(t)$ is called *horizontal* (*vertical*) if for every value of the parameter t the vector dC/dt is horizontal (vertical).

If $y(t)$ is a tangent vector field of the curve $x(t)$, then the curve $C(t) = (x(t), y(t))$ is said to be the *lift* of $x(t)$. The lift of a curve is always a horizontal curve, and the converse also holds. Thus, a curve $C(t)$ on TM is horizontal if and only if it is the lift of a curve on M.

Let σ be the "arc length" parameter of $C(\sigma)$ on TM. The curve $C(\sigma) = (x(\sigma), y(\sigma))$ is a geodesic on TM if $x(\sigma)$ and $y(\sigma)$ satisfy the following differential equations [93]:

$$\begin{cases} \dfrac{d^2x^i}{d\sigma^2} + \Gamma^i_{jk}\dfrac{dx^j}{d\sigma}\dfrac{dx^k}{d\sigma} = R^i_{j\mu\lambda}\dfrac{dx^j}{d\sigma}y^\lambda\dfrac{Dy^\mu}{d\sigma}, \\ \dfrac{D^2y^i}{d\sigma^2} = 0, \end{cases}$$

where $Dy^i/d\sigma$ is the covariant derivative of the vector field along $x(\sigma)$. Let us denote the covariant derivative along $x(\sigma)$ by a prime ($'$). Then the equation of a geodesic in TM in vector form is written as follows:

$$\begin{cases} x'' = R(y', y)x', \\ y'' = 0. \end{cases}$$

The parametrization of $C(\sigma)$ is natural and hence

$$\left\| \frac{dC(\sigma)}{d\sigma} \right\|^2 = |x'(\sigma)|^2 + |y'(\sigma)|^2 = 1.$$

Let us set $|y'(\sigma)| = c$. Along $x(\sigma)$ we have $(c^2)' = 2\langle y', y''\rangle = 0$, that is, $c = \text{const}$. Therefore, $|x'(\sigma)| = \sqrt{1-c^2}$ is constant along $x(\sigma)$. If s is the "arc length" parameter for $x(\sigma)$, then this equality easily implies that

$$ds/d\sigma = \sqrt{1-c^2}.$$

Therefore, the lift of a geodesic in M is a geodesic in TM [93]. These geodesics are orthogonal to the fibres and are called *horizontal geodesics*. Thus, the horizontal geodesics of TM are generated by parallel vector fields along geodesics of M. An analogous theorem for the frame bundle has been proved in [73].

***Theorem* 4.1** [93]. *Each line on the fibre of the tangent bundle TM is a geodesic in TM. (This means that the fibres of TM are totally geodesic.)*

The proof follows immediately from the equations of the geodesics of TM. Clearly, these lines are vertical and they form the class of *vertical geodesics*. The same is also true for FM [73].

The other geodesics are called *geodesics in general position*.

An important property of the geodesics consists in the fact that if a geodesic is horizontal at a point, then it is everywhere horizontal. In other words, if a geodesic is orthogonal to one fibre, it is orthogonal to all the fibres that it meets. This assertion holds also for Riemannian submersions with totally geodesic fibres [85]. Moreover, in [108] this result is strengthened and generalized. Namely, a *Riemannian manifold with a bundle-like metric* is considered. Typical examples of such metrics are, in particular, the Sasaki metric on the tangent bundle of a Riemannian manifold, the metric of a Riemannian submersion, and the metric of a Riemannian manifold with an isometry group action such that all the orbits have the same dimension. Riemannian manifolds with a bundle-like metric are called *foliated*.

Let $\gamma(s)$ be geodesics on a foliated Riemannian manifold M with the arc-length parametrization. We shall say that $\gamma(s)$ makes a *constant angle with the leaves* if along the geodesic the length of the projection of the vector $\gamma'(s)$ onto the tangent space to the fibre is constant.

***Theorem* 4.2** [108]. *Let M be a foliated manifold with a foliation E of codimension q $(= n-p)$ and Riemannian metric $\langle\cdot,\cdot\rangle$. Assume that the fibres are totally geodesic.*

(i) *If the metric $\langle\cdot,\cdot\rangle$ is a bundle-like metric with respect to E, then any geodesic in M makes a constant angle with the leaves.*

(ii) *If all the geodesics of M make constant angles with the leaves, then $\langle\cdot,\cdot\rangle$ is a bundle-like metric with respect to E.*

For constant curvature manifolds it turns out to be possible to give an exhaustive description of all the geodesics on their tangent manifolds.

Namely, geodesics in general position are divided into three classes: (i) the class of geodesics in general position over geodesics of the base; (ii) the class of geodesics over curves of constant first curvature and zero second curvature; (iii) the class of geodesics over curves of constant positive first curvature, non-zero second curvature, and zero third curvature. In each case, the vector fields determining a geodesic of each type on the sphere S^n, the Euclidean space E^n, and the hyperbolic space H^n have been written down explicitly [96].

This classification relies on a lemma in [93] stating that for a constant curvature manifold the projection of any geodesic of its tangent bundle onto the base is a space curve, that is, it is a curve for which the curvature $k_i = 0$ for $i \geqslant 3$.

We note that when one knows curves and vector fields along them determining geodesics in the tangent bundle of a space form, it is natural to pose the question of totally geodesic submanifolds in this bundle. Up to now, however, there are no results in this direction, apart from several general results asserting that for every Riemannian manifold M the following manifolds are totally geodesic in TM: a) the fibre [93]; b) the base, embedded in TM by means of the zero vector field [67]; c) the image of the base in TM given by a parallel vector field of constant length on M [102]. In the last case the base is necessarily a metric product, at least of the form $M_1^{n-1} \times E^1$.

On the unit tangent bundle T_1M the equations of the geodesics parametrized by arc length have the form [94]

$$\frac{d^2x^i}{d\sigma^2} + \Gamma^i_{jk}\frac{dx^j}{d\sigma}\frac{dx^k}{d\sigma} = R^i_{j\mu\lambda}\frac{dx^j}{d\sigma}y^\lambda\frac{Dy^\mu}{d\sigma},$$
$$\frac{D^2y^i}{d\sigma^2} = -c^2y^i,$$

where $c = |y'|$ is a constant.

In invariant form these equations are written as follows:

$$\begin{cases} x'' = R(y', y)\,x', \\ y'' = -c^2 y, \end{cases}$$

where, as before, $(')$ denotes the covariant derivative of the vector field $y(\sigma)$ along the curve $x(\sigma)$ on the base.

One distinguishes in a natural way three classes of geodesics in T_1M: the vertical, the horizontal, and the general ones. One checks trivially that every horizontal geodesic is given by a parallel vector field along a geodesic of M, that is, the "sets" of horizontal geodesics of TM and of T_1M are in a sense the same. The vertical geodesics are those of the fibre. It is not hard to see that these are great circles of the fibre, and thus one establishes that the fibres of T_1M are totally geodesic.

An exhaustive description of the general geodesics in T_1M was given by Sasaki [95] for constant curvature spaces (space forms of S^n, E^n, and H^n).

Namely, the *general geodesics* can be divided into three classes:

(i) The class of geodesics over geodesics of M. Each geodesic of this class is described by a unit vector field changing in helical form along a geodesic of M. The geodesics of this kind may be closed.

(ii) The class of geodesics over curves of constant curvature k_1 and zero curvature (torsion) k_2. On S^n these are small circles; on H^n they are equidistants, horocycles, and regular circles, depending on whether $k_1 < 1$, $k_1 = 1$, or $k_1 > 1$. For T_1S^2 any geodesic of this kind is given by a unit vector field along a small circle, making a contant angle with it [64]. These geodesics are closed.

(iii) The class of geodesics over curves of constant first curvature k_1 (> 0), constant second curvature k_2 ($\neq 0$), and zero third curvature $k_3 = 0$.

On T_1S^n these geodesics are closed, under certain conditions. On T_1H^n there are no closed geodesics of this kind.

As for the tangent bundle, for T_1M, and even for T_1S^n and T_1H^n, the problem on totally geodesic manifolds remains virtually open.

In a sense, one can also give a geometric description of the geodesics on T_1M in the general case [79].

4.2. The geodesics of the Sasaki metric of the normal bundle.
Let $C(\sigma) = (x(\sigma), y(\sigma))$ be a geodesic of the Sasaki metric of NF^l parametrized by arc length. Here $x(\sigma)$ is a curve on F^l (the projection of $C(\sigma)$ onto the base) and $y(\sigma)$ is a normal vector field on F^l along $x(\sigma)$. Let $N_{\alpha\beta|ij}$ be the curvature tensor of the normal connection of $F^l \subset M^{l+p}$, let $\hat{N}^i_{j|\alpha\beta} = g^{ik}N_{\alpha\beta|kj}$ be the adjoint tensor, and let $D^\perp$ be the covariant differentiation in the normal connection.

The equations of the geodesics of the Sasaki metric of NF^l have the form

$$\begin{cases} \dfrac{d^2x^i}{d\sigma^2} + \Gamma^i_{jk}\dfrac{dx^j}{d\sigma}\dfrac{dx^k}{d\sigma} = \hat{N}^i_{j\,|\,\alpha\beta}\dfrac{dx^j}{d\sigma}\,y^\beta\,\dfrac{D^\perp y^\alpha}{d\sigma}, \\ \dfrac{(D^\perp)^2 y^\alpha}{d\sigma^2} = 0 \end{cases}$$

$$(i, j = 1, \ldots, l;\ \alpha, \beta = 1, \ldots, p).$$

Here we also have three natural types of geodesics:

a) horizontal geodesics—normal vector fieids along the geodesics of the base, parallel in the normal connection;

b) vertical geodesics—straight lines in the fibres, that is, in the subspaces normal to the base;

c) geodesics in general position.

Since the fibres of the normal bundle are Euclidean, clearly the fibres are totally geodesic in NF^l. It is also clear that the base is embedded in NF^l via the zero section as a totally geodesic submanifold. Moreover, it is easy to show that if y is a parallel vector field on F^l in the normal connection, then $y(F^l)$—the image of the base in the normal bundle given by the field y—is a totally geodesic submanifold in NF^l. There are no further results in this direction. We only observe that for the surface F^l in a constant curvature space the equations of the geodesics may be written in a simple form:

$$\begin{cases} x'' = -[A_y, A_{\dot{y}}]\, x', \\ \ddot{y} = 0, \end{cases}$$

where $(')$ denotes the covariant derivative in the connection of F^l, $(\dot{\ })$ is the covariant derivative in the normal connection, and $[A_y, A_{\dot{y}}]$ is the commutator of the matrices of the second quadratic forms with respect to the fields y and $\dot{y}$.

§5. Surfaces in the tangent bundle of a Riemannian manifold

A natural and, as we have seen, well-studied surface in TM is the spherical tangent bundle $T_\rho M$.

Another natural type of surface in TM is the following. Let us consider a surface F^l in a Riemannian space M^{l+p}. Then its tangent bundle TF^l is a surface in the Riemannian space TM^{l+p} endowed with the Sasaki metric. Clearly, a Riemannian metric on TF^l may be defined in two ways. On the one hand, one can construct a Sasaki metric starting from the metric of F^l and, on the other, one can consider the metric induced by the Sasaki metric in TM^{l+p}. Generally speaking, these metrics do not coincide (are not isometric).

As an example, one can consider a cylinder F^2 in E^3. The Sasaki metric on TF^2 is flat, whereas the metric induced from $TE^3 = E^6$ on TF^2 has non-zero curvature. These metrics coincide if and only if F^l is totally geodesic in M^{l+p}.

In what follows we shall consider the metric induced on TF^l by the Sasaki metric on TM^{l+p}. It is easy to establish that if F^l is a cylinder in the Euclidean space E^{l+p} with a k-dimensional ruling, then TF^l is a cylinder in $E^{2(l+p)}$ with a $2k$-dimensional ruling. This assertion can be substantially strengthened.

The exterior nullity index $\mu(Q)$ of a point Q of a surface F^l in the Euclidean space E^{l+p} is defined as the dimension of the maximal linear subspace $L_Q \subset T_Q F^l$ such that for every $Y \in L_Q$, the matrix A_η of the second quadratic form of $F^l \subset E^{l+p}$ with respect to any normal η at Q satisfies

$$A_\eta Y = 0.$$

If $\mu(Q) \geqslant k$ for every $Q \in F^l$, then the surface is called *strongly k-parabolic*.

A cylinder with a k-dimensional ruling is a strongly k-parabolic surface.

***Theorem* 5.1** [4]. *If the exterior nullity index $\tilde{\mu}$ of TF^l is equal to k, then the exterior nullity index μ of the surface F^l satisfies the inequality $\mu \geqslant k/2$.*

Moreover,

a) if $\mu = k/2$, then F^l is a cylinder with a $k/2$-dimensional ruling;

b) if $\mu = s \leqslant k$, then F^l is a cylinder with a $(k-s)$-dimensional ruling and TF^l is a cylinder with a $2(k-s)$-dimensional ruling.

The converse theorem also holds.

***Theorem* 5.2** [4]. *If the exterior nullity index μ of a surface $F^l \subset E^{l+p}$ is equal to $k/2$, then the exterior nullity index $\tilde{\mu}$ of TF^l satisfies the inequalities $k/2 \leqslant \tilde{\mu} \leqslant k$.*

The proof of these theorems relies on the analysis of the second quadratic forms of TF^l in TM^{l+p}. It is not hard to show that if $n_{\beta|}$ is a basis of normals to F^l at Q, then a basis of normals to TF^l in TM^{l+p} at (Q, ξ) is provided by the vectors

$$\{N_{\beta|} = n^H_{\beta|}, N_{l+\beta} = n^V_{\beta|} + (\overline{\nabla}_\xi n_{\beta|})^H\},$$

where $\overline{\nabla}$ is the covariant derivative in M^{l+p}, $\beta = 1, \ldots, p$. Let A^β_{ij} be the components of the second quadratic forms of F^l with respect to the orthonormal basis $n_{\beta|}$ at Q.

***Lemma* 5.1** [4]. *The matrices $\tilde{A}^\beta$ and $\tilde{A}^{l+\beta}$ of the second quadratic forms of $TF^l \subset TM^{l+p}$ with respect to the basis of the normals $\{N_{\beta|}, N_{l+\beta|}\}$ at (Q, ξ) have the form*

$$\tilde{A}^\beta = \left[\begin{array}{c|c} A^\beta_{ij} - \frac{1}{2}(R^{l+\beta}_{il+\alpha s}A^\alpha_{jt} + R^{l+\beta}_{j\,l+\alpha\,s}A^\alpha_{it})\xi^t\xi^s & -\frac{1}{2}R^{l+\beta}_{tjt}\xi^t \\ \hline -\frac{1}{2}R^{l+\beta}_{jit}\xi^t & 0 \end{array}\right],$$

$$\tilde{A}^{l+\beta} = \Lambda^\beta_\alpha \left[\begin{array}{c|c} \begin{array}{l}\nabla_\xi A^\alpha_{ij} + \frac{1}{2}((R^{l+\alpha}_{ijm} + R^{l+\alpha}_{jim} + (R^k_{i\,l+\gamma\,s}A^\gamma_{mj} + \\ \qquad + R^k_{j\,l+\gamma\,s}A^\gamma_{mi})A^\alpha_{kt}\xi^t\xi^s)\xi^m\end{array} & A^\alpha_{ij} + \frac{1}{2}R^k_{ijs}A^\alpha_{kt}\xi^t\xi^s, \\ \hline A^\alpha_{ji} + \frac{1}{2}R^k_{jis}A^\alpha_{kt}\xi^t\xi^s & 0 \end{array}\right],$$

where Λ is the Gram matrix of the subsystem of normals $\{N_{l+\beta|}\}$, $i, j, m, s, t, k = 1, \ldots, l$; $\alpha, \beta, \gamma = 1, \ldots, p$.

This lemma immediately implies that if F^l is totally geodesic in M^{l+p} ($A^\beta_{ij} \equiv 0$), then TF^l is totally geodesic in TM^{l+p}. The converse is also true.

***Theorem* 5.3** [4]. *TF^l is totally geodesic in TM^{l+p} if and only if F^l is totally geodesic in M^{l+p}.*

Actually this theorem, with one additional condition, is true for all Riemannian submersions with totally geodesic fibres [52]. For the tangent bundle this condition holds automatically.

Now let M and N be Riemannian manifolds. The map $f : M \to N$ induces a map $f_* : TM \to TN$. Then the map $F : TM \to TN$ given by $F(x, \xi) = (f(x), f_*(\xi))$ is a natural map between the tangent bundles TM and TN as differentiable manifolds. Naturally, the problem arises of comparing the properties of f and F.

A map $f : M \to N$ is called *geodesic* if each geodesic of M is mapped under f into a geodesic of N.

A map $f : M \to N$ is called *harmonic* if the map f_*, regarded as a 1-form on M, has zero divergence.

If $\dim M < \dim N$, then in the first case $f(M)$ is a totally geodesic submanifold in N, and in the second $f(M)$ is a minimal submanifold in N.

Let us introduce the Sasaki metric on TM and TN.

***Theorem* 5.4** [91]. *For $F : TM \to TN$ to be totally geodesic it is necessary and sufficient that $f : M \to N$ be totally geodesic.*

***Theorem* 5.5** [91]. *If $f : M \to N$ is a harmonic map and N is flat, then $F : TM \to TN$ is harmonic.*

Let D be the covariant differential in the Levi–Cività connection of M, and let R^N be the curvature tensor of N. Let $\{e_i\}$ be an orthonormal basis on M.

***Theorem* 5.6** [91]. *Let $f: M \to N$ be a harmonic map. The map $F : TM \to TN$ of the tangent bundles with Sasaki metrics is harmonic if and only if the following conditions are fulfilled for every point $(Q, \xi) \in TM$:*

$$\begin{cases} R^N (Df_* (\xi, e_i), f_*\xi) f_* e_i = 0, \\ \operatorname{div} (Df_*) = 0. \end{cases}$$

If $\dim M < \dim N$, the second quadratic form of the immersion f is $Df_*(X, Y)$.

Thus, for an immersion $f : M \to N$ we have: $F : TM \to TN$ is harmonic if and only if $f : M \to N$ is totally geodesic. In other words, if $M \subset N$, then TM in TN is minimal if and only if M is totally geodesic in N.

Now let $f : M \to N$ be an isometric immersion. Let us consider $F_1 : T_1M \to T_1N$—the map of the spherical tangent bundles with the Sasaki metric. F_1 is the restriction to T_1M of the map $F : TM \to TN$ defined earlier. The following result applies.

***Theorem* 5.7** [92]. *If $f : M \to N$ is an isometric immersion of M into a space N of constant curvature c, then $F_1 : TM \to TN$ is harmonic if and only if either*

a) *$f(M)$ is an Einstein minimal submanifold in N and $c = 0$; or*

b) *$f(M)$ is a totally umbilical submanifold in N and $c =$* $\dim M$.

In [92] it is proved in passing that T_1M is a hypersurface in TM of constant mean curvature. Namely, at each point $(Q, \xi) \in T_1M$ the vector ξ^V is a unit vector of the normal to T_1M in TM.

Theorem **5.8** [92]. *The mean curvature vector H of the hypersurface T_1M in TM is equal to*

$$H = -\frac{n-1}{2n-1}\xi^V, \quad n = \dim M.$$

Another natural type of surface in TM is the image of the base given by a smooth vector field on M. In local coordinates this embedding is as follows. Let $(y^1, \dots, y^n; \xi^1, \dots, \xi^n)$ be natural induced coordinates of TM. Then $y^i = x^i$, $\xi^i = \xi^i(x^1, \dots, x^n)$ defines a natural embedding of M into the tangent bundle. Let us denote this manifold by $\xi(M)$.

If TM is endowed with the Sasaki metric, then the induced metric $\widetilde{G}$ on $\xi(M)$ has components

$$\widetilde{G}_{ij} = g_{ij} + g_{st}\nabla_i\xi^t\nabla_j\xi^s,$$

where g_{ij} and ∇_i are the metric and the covariant differentiation on M respectively.

If the vector field $\xi(x)$ under consideration is a unit field, then $\xi(M)$ is an n-dimensional submanifold in T_1M. *The volume of the vector field* is defined as the volume of this manifold in the Sasaki metric of T_1M. It may be expressed as [55]

$$\operatorname{Vol}(\xi(M)) = \int_M \sqrt{\det(E + (\nabla\xi)^t(\nabla\xi))}\, d\operatorname{Vol}_M,$$

where the covariant derivative $\nabla\xi$ is interpreted as the matrix of a linear transformation of the tangent space onto itself, and ${}^t(\nabla\xi)$ is its transpose.

Gluck and Ziller [55] proved that the unit vector field of minimal volume on S^3 is precisely the unit vector field on S^3 tangent to the fibres of the Hopf bundle $S^3 \xrightarrow{S^1} S^2$.

Submanifolds of the form $\xi(M)$ are totally geodesic in TM if ξ is parallel on M [102].

Regarding a vector field ξ as a map $M \to TM$, Ishihara [62] considered the problem of its harmonicity and established that if M is compact, then ξ is harmonic if and only if ξ is covariantly constant, that is, parallel. Associating this assertion with Walczak's result [102], one can say that $\xi(M)$ is minimal in TM if $\xi(M)$ is totally geodesic.

§6. Certain geometric applications of the Sasaki metric

Besides having independent geometric interest, the Sasaki metrics of the tangent and the normal bundles have important applications.

Weinstein [103] proved a theorem on the volumes of manifolds with closed geodesics. A manifold (M, g) is called a *C_l-manifold* if all the geodesics of the metric g are closed and have the same length l.

Theorem 6.1. *If* (M, g) *is an n-dimensional* C_l*-manifold, then the ratio* $\frac{\mathrm{Vol}(M,g)}{\mathrm{Vol}(S^n)}\left(\frac{2\pi}{l}\right)^n$ *is an integer (the Weinstein number).*

A crucial step in the proof of this theorem is the possibility of computing the volume of T_1M when T_1M has the Sasaki metric. Namely, the following formula applies:

$$\mathrm{Vol}\,(T_1M,\, T_1g) = \mathrm{Vol}\,(M,\, g)\cdot\mathrm{Vol}\,(S^{n-1}).$$

The Sasaki metric on the normal bundle of a submanifold is applied to study the geometry of the submanifold itself in Riemannian space [89].

Let us consider, for instance, a strongly parabolic surface. Let $F^l \subset M^{l+p}$ be an l-dimensional surface in an $(l+p)$-dimensional Riemannian manifold. The *exterior nullity index* of a point $Q \in F^l$ is defined as the dimension of the maximal linear subspace $L_Q \subset T_QF^l$ such that for every $Y \in L_Q$ and every $X \in T_QF^l$ and $\xi \in N_QF^l$ we have $A_\xi(X, Y) = 0$, where A_ξ is the matrix of the operator of the second quadratic form with respect to the normal ξ.

If $k = \dim L_Q$ does not depend on the point of the surface, then the surface is called *strongly k-parabolic*.

It is known that strongly k-parabolic surfaces in constant curvature spaces are foliated into k-dimensional totally geodesic submanifolds along which the normal space is stationary.

For k-parabolic surfaces (that is, for $k \neq$ const), through each point of the manifold there is a totally geodesic submanifold of dimension k of the ambient space along which the normal is stationary (see [3*] and [4*]).

These assertions also hold for various classes of surfaces in a symmetric space of rank 1 (see [5*] and [6*]).

The theorem on the structure of strongly parabolic surfaces is also valid for surfaces in a Riemannian space M^n if at the points of the surface the Riemannian tensor R of the ambient space satisfies $\langle R(X, Y)\xi, Z\rangle = 0$, where $X, Y, Z \in T_QF^l$ and ξ is an arbitrary normal to the surface [9*].

We shall show how to apply the Sasaki metric of a normal bundle to prove a theorem on the structure of parabolic surfaces in Riemannian space.

To obtain a meaningful result, in addition to the k-parabolicity of the surface one needs to require that the curvature tensor of the ambient space satisfies

$$\text{(A)}\qquad R\,(X,\, Y)\xi = 0$$

at the points of the surface, for every $X, Y \in TF^l$ and $\xi \in NF^l$. In what follows we shall assume that condition (A) is fulfilled.

Observe that a surface F^l in a Riemannian space M^n is k-parabolic if the second quadratic form of the surface with respect to each normal has no more than k zero coefficients after reducing it to the diagonal form. In other words, the rank of the second quadratic form of the surface $r(Q) = \max_{\xi\in N_Q} r(Q, \xi)$, where N_Q is the normal space at Q and $r(Q, \xi)$ is the rank of the second quadratic form of the surface with respect to the normal ξ at Q, satisfies the inequality $r(Q) \leqslant l-k$ at each point [3*].

Let $r^*(Q, \xi)$ be the maximal rank for points close to Q and normals close to ξ. We shall assume that the surface is of class C^3 and the Riemannian space is of class C^4. A normal ξ to the surface $F^l \subset M^n$ is called *stationary* along a submanifold $R^k \subset F^l$ if under a parallel translation in the ambient space along any path in R^k it remains normal to F^l. The following result holds.

***Theorem* 6.2.** *Let F^l be an l-dimensional surface in the Riemannian space M^n, assume that in a neighbourhood of $Q_0 \in F^l$ the rank of the second quadratic form is constant, $r(Q) = r(Q_0) = l-k$, and let ξ be a normal at Q_0 for which $R(Q, \xi) = R(Q_0) = l-k$. If condition* (A) *holds at the points of the surface, then through Q_0 there is a totally geodesic k-dimensional submanifold $R^k(Q_0, \xi)$ of the ambient space that belongs to the surface. Along the submanifold $R^k(Q_0, \xi)$ the normal ξ is stationary, $r(Q, \xi) = r(Q_0, \xi)$ for the points $Q \in R^k(Q_0, \xi)$, and $r^*(Q, \xi) \geqslant r(Q_0, \xi)$ for the boundary points of $R^k(Q_0, \xi)$. If the surface is complete and*

$$r(Q_0, \xi) = r_0 = \max_{Q \in F^l} r(Q),$$

then $R^k(Q_0, \xi)$ is a complete Riemannian manifold.

Let us consider the normal vector bundle to F^l in the Riemannian space M^n and let us introduce a metric on it.

Let us agree that in what follows the indices take the following values: $i, j, k, m, s, t = 1, \ldots, l$; $a, b, c, d = 1, \ldots, n$; $\alpha, \beta, \tau, \mu, \nu, \sigma, \lambda = 1, \ldots, \ldots, p = n-l$.

Let ξ be a normal at Q for which the rank $r(Q, \xi) = r(Q) = l-k$. Then it is constant for normals close to ξ. For ξ let us consider the null subspace of its second quadratic form. It satisfies the equations

$$(*) \qquad \sum_\alpha \xi^\alpha A^\alpha_{ij} X^i = 0,$$

where A^α_{ij} are the coefficients of the second quadratic forms with respect to basis normals n_α. Since the rank of $(*)$ is constant, it follows that the solution space $L^k(Q, \xi)$ depends regularly on the point and on the normal. Let us perform a horizontal lift of the k-dimensional planes $L^k(Q, \xi)$ into the points $\tilde{Q} = (Q, \xi)$ of the normal bundle. The horizontal lift of the planes $L^k(Q, \xi)$ in a neighbourhood of $\tilde{Q}$ is a differentiable distribution $\mathscr{L}^k(\tilde{Q})$.

The proof of Theorem 6.2 reduces to the study of $\mathscr{L}^k(\tilde{Q})$. The following result holds.

***Theorem* 6.2′.** *A differentiable horizontal distribution $\mathscr{L}^k(\tilde{Q})$ on the normal bundle of a k-parabolic surface F^l in a Riemannian space M^n is holonomic if at the points of the surface the curvature tensor of M^n satisfies condition* (A).

The fibre $\widetilde{R}^k(Q_0)$ tangent to $\mathscr{L}^k(\widetilde{Q}_0)$ is a totally geodesic submanifold of the normal bundle with the Sasaki metric. If $\widetilde{Q}$ is a boundary point of $\widetilde{R}^k(\widetilde{Q}_0)$, then $r^*(Q, \xi) \geqslant r(Q_0, \xi)$. If the surface is complete and $r(Q_0, \xi) = r_0 = \max_{Q \in F^l} r(Q)$, then $\widetilde{R}^k(\widetilde{Q})$ is a complete Riemannian manifold.

Proof of Theorems 6.2 *and* 6.2′. a) Let us prove that the differential distribution $\mathscr{L}^k(\widetilde{Q})$, introduced earlier on the normal bundle NF^l, is holonomic. Let X and Y be regular vector fields on NF^l such that $X(\widetilde{Q})$, $Y(\widetilde{Q}) \subset \mathscr{L}^k(\widetilde{Q})$ and $X(\widetilde{Q})$, $Y(\widetilde{Q}) \subset L^k(\widetilde{Q})$. Since $\mathscr{L}^k(\widetilde{Q})$ is horizonal, the vector fields X and Y are horizontal. Therefore, $X = (\pi_* X)^H$ and $Y = (\pi_* Y)^H$. Since $X = X^i \partial/\partial u^i + X^{l+\alpha}\partial/\partial\xi^\alpha$, then

$$\pi_* X = X^i \frac{\partial}{\partial u^i}. \tag{1}$$

The horizontal lift of a vector $a \in T_Q F^l$ with coordinates a^i is the vector

$$a^H = \{a^1, \ldots, a^l; -\mu_{1\tau|i}\xi^\tau a^i, \ldots, -\mu_{p\tau|i}\xi^\tau a^i\}. \tag{2}$$

From (1) and (2) it follows that

$$\begin{cases} X = \{X^1(u, \xi), \ldots, X^l(u, \xi); & -\mu_{1\tau|i}\xi^\tau X^i, \ldots, -\mu_{p\tau|i}\xi^\tau X^i\}, \\ Y = \{Y^1(u, \xi), \ldots, Y^l(u, \xi); & -\mu_{1\tau|i}\xi^\tau Y^i, \ldots, -\mu_{p\tau|i}\xi^\tau Y^i\}. \end{cases} \tag{3}$$

The Lie bracket of X and Y is

$$[X, Y] = \left(\frac{\partial X^a}{\partial v^b} Y^b - \frac{\partial Y^a}{\partial v^b} X^b\right)\frac{\partial}{\partial v^a}, \tag{4}$$

where $\partial/\partial v^i = \partial/\partial u^i$ and $\partial/\partial v^{l+\alpha} = \partial/\partial\xi^\alpha$. Substituting (3) into (4) we obtain

$$[X, Y]^i = \frac{\partial X^i}{\partial u^j} Y^j - \frac{\partial Y^i}{\partial u^j} X^j - \mu_{\alpha\tau|j}\left(Y^j \frac{\partial X^i}{\partial \xi^\alpha} - X^j \frac{\partial Y^i}{\partial \xi^\alpha}\right), \tag{5}$$

$$\begin{aligned} [X, Y]^{l+\beta} = & -Y^k \frac{\partial}{\partial u^k}(\mu_{\beta\tau|j}\xi^\tau X^j) + X^k \frac{\partial}{\partial u^k}(\mu_{\beta\tau|j}\xi^\tau Y^j) + \\ & + (\mu_{\alpha\tau|k}\xi^\tau Y^k)\frac{\partial}{\partial \xi^\alpha}(\mu_{\beta\tau|j}\xi^\tau X^j) - (\mu_{\alpha\tau|k}\xi^\tau X^k)\frac{\partial}{\partial \xi^\alpha}(\mu_{\beta\tau|j}\xi^\tau Y^j). \end{aligned} \tag{6}$$

Since $\mu_{\alpha\beta|i} = 0$ at Q_0, then at Q_0 (5) and (6) become

$$[X, Y]^i = \frac{\partial X^i}{\partial u^j} Y^j - \frac{\partial Y^i}{\partial u^j} X^j, \tag{7}$$

$$[X, Y]^{l+\beta} = (Y^j X^k - X^j Y^k)\,\xi^\tau \frac{\partial \mu_{\beta\tau|i}}{\partial u^k}. \tag{8}$$

On F^l the local coordinates are chosen so that at Q_0 the subspace $\mathscr{L}^k(Q_0, \xi)$ is spanned by the first k basis vectors and the vectors

$$\pi_*(X(\widetilde{Q}_0)) = (1, 0, \ldots, 0), \quad \pi_*(Y(\widetilde{Q}_0)) = (0, 1, 0, \ldots, 0).$$

From (2) it follows that $X(\widetilde{Q}_0) = \{1, 0, \ldots, 0\}$, and $Y(\widetilde{Q}_0) = \{0, 1, 0, \ldots, 0\}$. Assume that the normal vector ξ coincides with the basis vector $n_{1|}$. Then

$\xi = \{1, 0, ..., 0\}$. Using the concrete definition of the coordinates, (8) becomes

(9) $$[X, Y]^{l+\beta} = \frac{\partial \mu_{\beta 1|2}}{\partial u^1} - \frac{\partial \mu_{\beta 1|1}}{\partial u^2}.$$

The Ricci equations for the immersion of a surface in the Riemannian space M^n in local coordinates $v^\alpha = v^\alpha(u^1, ..., u^l)$ have the form

(10) $$\mu_{\tau\sigma|j,k} - \mu_{\tau\sigma|k,j} + \sum_\rho (\mu_{\rho\tau|j}\mu_{\rho\sigma|k} - \mu_{\rho\tau|k}\mu_{\rho\sigma|j}) + \\ + g^{lh}(A^\tau_{lj}A^\sigma_{hk} - A^\tau_{lk}A^\sigma_{hj}) + R^d_{abc}v^b_j v^c_k n^a_{\sigma|} n_{\tau|d}.$$

Since $\mu_{\alpha\beta|i} = 0$, $\Gamma^k_{ij} = 0$ along the surface $g^\perp_{\alpha\beta} = \delta_{\alpha\beta}$ at Q_0, and by assumption condition (A) holds at the points of the surface, then for $\sigma = 1$, $\tau = \beta$, $j = 2$, $k = 1$ the Ricci equations at Q_0 can be rewritten as

(11) $$\frac{\partial \mu_{\beta 1|2}}{\partial u^1} - \frac{\partial \mu_{\beta 1|1}}{\partial u^2} + g^{sh}(A^\beta_{s2}A^1_{h1} - A^\beta_{s1}A^1_{h2}).$$

By the special choice of the coordinate system at Q_0 and the fact that the surface is parabolic, we have $A^1_{s1} = A^1_{s2} = 0$. Therefore, equations (11) reduce to

(12) $$\frac{\partial \mu_{\beta 1|2}}{\partial u^1} - \frac{\partial \mu_{\beta 1|1}}{\partial u^2} = 0.$$

From (9) and (12) it follows that

(13) $$[X, Y]^{l+\beta} = 0.$$

The Codazzi equations for surfaces in a Riemannian space have the form

(14) $$A^\sigma_{ij,k} - A^\sigma_{ik,j} = \sum_\tau (\mu_{\tau\sigma|k}A^\tau_{ij} - \mu_{\tau\sigma|j}A^\tau_{ik}) + R_{abcd}v^a_i v^c_j v^d_k n^b_{\sigma|}.$$

At Q_0, by condition (A) and the fact that $\mu_{\alpha\beta|i} = 0$, the Codazzi equations can be rewritten as

(15) $$\frac{\partial A^\alpha_{ij}}{\partial u^k} - \frac{\partial A^\alpha_{ik}}{\partial u^j} = 0.$$

Condition (A) yields the same form of the Ricci equations and the Codazzi equations as those for surfaces in Euclidean space.

By definition of the distribution $\mathscr{L}^k(\tilde{Q})$, the projection of the vector fields X and Y onto the base F^l satisfies the condition

(16) $$\sum_\alpha \xi^\alpha A^\alpha_{ij}X^i = 0; \quad \sum_\alpha \xi^\alpha A^\alpha_{ij}Y^i = 0.$$

Differentiating (16) with respect to u^k, multiplying by Y^k, and summing over k we deduce that at Q_0

(17) $$\sum_\alpha \xi^\alpha A^\alpha_{ij}\frac{\partial X^i}{\partial u^k}Y^k = -\sum_\alpha \xi^\alpha \frac{\partial A^\alpha_{ij}}{\partial u^k}X^iY^k.$$

From (15), (16), and (17) it follows that

(18) $$\sum_\alpha \xi^\alpha A^\alpha_{ij} \frac{\partial X^i}{\partial u^k} Y^k = \sum_\alpha \xi^\alpha A^\alpha_{ik} Y^k \frac{\partial X^i}{\partial u^j} = 0.$$

Similarly,

(19) $$\sum_\alpha \xi^\alpha A^\alpha_{ij} \frac{\partial Y^i}{\partial u^k} X^k = 0.$$

Therefore,

(20) $$\sum_\alpha \xi^\alpha A^\alpha_{ij} \left(\frac{\partial X^i}{\partial u^k} Y^k - \frac{\partial Y^i}{\partial u^k} X^k \right) = 0.$$

This means that $\pi_*[X, Y] \subset L^k(Q_0, \xi)$. From (13) it follows that $[X, Y]$ is a horizontal vector. Consequently, (7) implies that $[X, Y] \subset \mathscr{L}^k(\widetilde{Q}_0)$. But since Q_0 is an arbitrary point of the surface, it follows that $[X, Y] \subset \mathscr{L}^k(\widetilde{Q})$. Since the assumptions of the Frobenius theorem hold, the distribution $\mathscr{L}^k(\widetilde{Q})$ is holonomic, that is, through each point $\widetilde{Q} \in NF^l$ there is a unique k-dimensional manifold $R^k(\widetilde{Q}_0)$ whose tangent spaces coincide with the planes of the distribution $\mathscr{L}^k(\widetilde{Q})$.

b) The projection of the fibre $R^k(\widetilde{Q}_0)$ onto the base F^l is a regular submanifold $R^k(Q_0, \xi)$ of the surface F^l. We claim that the fibre $R^k(\widetilde{Q}_0)$ is totally geodesic in the normal bundle with the Sasaki metric, and that so is $R^k(Q_0, \xi)$ in the ambient Riemannian space M^n. Let us choose local coordinates on F^l so that $R^k(Q_0, \xi)$ is the coordinate surface $u^1, \ldots, u^k$. Then the fibre $R^k(\widetilde{Q}_0)$ has the following parametric expression:

$$v^1 = u^1, \ldots, v^k = u^k; \; v^{k+1} = 0, \ldots, v^l = 0, \; v^{l+\alpha} = \xi^\alpha(u^1, \ldots, u^k).$$

The tangent space to $R^k(\widetilde{Q}_0)$ is spanned by the vectors

(21) $$V_s = \Big(0, \ldots, \underset{s}{1}, \ldots, 0, \frac{\partial \xi^\alpha}{\partial u^s}\Big) \quad (s = 1, \ldots, k).$$

On the other hand, these vectors are horizontal lifts of the tangent vectors $\partial/\partial u^s$ to the submanifold $R^k(Q_0, \xi)$:

$$\frac{\partial}{\partial u^s} = \Big\{0, \ldots, \underset{s}{1}, 0, \ldots, 0\Big\}.$$

From (2) it follows that

(22) $$\left(\frac{\partial}{\partial u^s}\right)^H = \Big\{0, \ldots, \underset{s}{1}, \ldots, 0; -\mu_{1\tau|s}\xi^\tau, \ldots, -\mu_{p\tau|s}\xi^\tau\Big\}.$$

From (21) and (22) it follows that along the fibre $R^k(\widetilde{Q}_0)$

(23) $$\frac{\partial \xi^\alpha}{\partial u^s} + \mu_{\alpha\tau|s}\xi^\tau = 0.$$

In other words, the normal vector field $\xi(u^1, \ldots, u^k)$ that corresponds to the fibre $R^k(\tilde{Q}_0)$ is parallel in the normal connection of F^l along the submanifold $R^k(Q_0, \xi)$. Let us choose a normal vector field $n_{1|}$ so that at the points of $R^k(Q_0, \xi)$ it coincides with $\xi(u^1, \ldots, u^k)$. Then along $R^k(Q_0, \xi)$ we have

$$\mu_{\alpha 1|s} = 0 \quad (s = 1, \ldots, k). \tag{24}$$

In the new coordinates, $R^k(\tilde{Q}_0)$ has the following parametric expression:

$$v^s = u^s, \; v^{k+1} = \ldots = v^l = 0; \; \xi^1 = 1, \ldots, \xi^p = 0.$$

Moreover, only the first k coordinates of the vector fields X and Y are nonzero. Let $\tilde{\nabla}$ be the covariant derivative in the metric of the normal bundle. Then at $\tilde{Q}_0$ we have

$$[\tilde{\nabla}_X Y]^i = \frac{\partial Y^i}{\partial u^k} X^k + \tilde{\Gamma}^i_{js} X^j Y^s;$$
$$[\tilde{\nabla}_X Y]^{l+\alpha} = \tilde{\Gamma}^{l+\alpha}_{js} X^j Y^s.$$

Lemma 1.6 implies that

$$\begin{cases} [\tilde{\nabla}_X Y]^i = \dfrac{\partial Y^i}{\partial u^k} X^k, \\ [\tilde{\nabla}_X Y]^{l+\alpha} = \dfrac{1}{2}\left(\dfrac{\partial \mu_{\alpha 1|j}}{\partial u^s} + \dfrac{\partial \mu_{\alpha 1|s}}{\partial u^j}\right) X^j Y^s. \end{cases} \tag{25}$$

From (24) and (25) it follows that

$$[\tilde{\nabla}_X Y]^{l+\alpha} = 0, \tag{26}$$

and the vector field $\tilde{\nabla}_X Y$ is horizontal. From (19), (25), and (26) we obtain

$$\tilde{\nabla}_X Y \subset \mathcal{L}^k(\tilde{Q}_0). \tag{27}$$

But $\tilde{\nabla}_X Y = \nabla_X Y + A(X, Y)$, where ∇ is the covariant derivative in the metric of the fibre $R^k(\tilde{Q}_0)$, and $A(X, Y)$ is the vector of the second quadratic form of the fibre. From (27) it follows that $\tilde{\nabla}_X Y$ is a vector tangent to the fibre. Hence, $A(X, Y) = 0$ and the fibre $R^k(\tilde{Q}_0)$ is a totally geodesic submanifold.

We prove similarly that the fibre $R^k(Q_0, \xi)$ is totally geodesic in the metric of F^l.

It is known that if F^l is a k-parabolic surface in Riemannian space and ξ is a normal at a point Q for which the rank of the second quadratic form is maximal, $r(Q, \xi) = l-k$, then the restriction of the second quadratic forms of the surface to the plane $L^k(Q, \xi) \subset T_Q F^l$ are zero forms [3*].

Consequently, the fibre $R^k(Q_0, \xi)$ is a totally geodesic surface in the ambient Riemannian manifold M^n.

c) Let Q be a boundary point of $R^k(Q_0, \xi)$. Join Q to an interior point of the fibre, by means of a geodesic $\gamma \subset R^k(Q_0, \xi)$. With no loss of generality, we shall assume that this is precisely the point Q_0. Let us direct the coordinate line u^1 along γ, introduce coordinates on the surface so as to have $\Gamma_{ij}^k = 0$ along γ, and choose normal vector fields $n_{\alpha|}$ so that $\mu_{\alpha\beta|i} = 0$. Then the fibre is still the coordinate surface $u^1, \dots, u^k$. Let A_{ij}^1 be the second quadratic form of F^l corresponding to the normal $n_{\alpha|} = \xi$. Then the parabolicity of the surface implies that $A_{1j}^1 = 0$. Hence, from (14) and (18) it follows that along the geodesic the Codazzi equations for the form A_{1j}^1 are $\partial A_{1j}^1/\partial u^1 = 0$. Let X be a vector at Q such that $X \subset L(Q, \xi)$, that is, $A_{ij}^1 X^i = 0$. Let us perform a parallel translation of X along the geodesic at Q_0; then along the geodesic we get a vector field $X(u^1)$. Then

$$\frac{\partial}{\partial u^1}(A_{ij}^1 X^i(u^1)) = \frac{\partial A_{ij}^1}{\partial u^1} X^i(u^1) = 0.$$

In other words, $X(Q_0) \subset L^k(Q_0, \xi)$. This implies that the rank of the second quadratic form of the surface with respect to the normal field ξ at the points of the boundary fibre $R^k(Q_0, \xi)$ cannot decrease, that is, $r^*(Q, \xi) \geqslant r(Q_0, \xi)$. Let $r(Q_0, \xi) = r_0 = \max_{Q \in F^l} r(Q)$ and assume that the surface F^l is complete. Then $r(Q, \xi) = r_0$. Through Q there also passes a fibre for which Q is an interior point. By the uniqueness of the fibre, it is an extension of $R^k(Q_0, \xi)$. In other words, the geodesics of the fibre $R^k(Q_0, \xi)$ can be extended indefinitely. From the Hopf–Rinow theorem it follows that the fibre is complete as a Riemannian manifold.

From the definition of the metric Ng on the normal bundle and the fact that the normal vector field ξ is parallel along $R^k(Q_0, \xi)$ in the normal connection of the surface, it follows that the projection map $\pi : R^k(\tilde{Q}_0) \to R^k(Q_0, \xi)$ is an isometry. Consequently, the boundary points of $R^k(\tilde{Q}_0)$ and those of $R^k(Q_0, \xi)$ correspond to each other and the completeness of $R^k(\tilde{Q}_0, \xi)$ implies that of $R^k(\tilde{Q}_0)$. The proof is complete.

Remark. In a metric space there arises a difficulty when determining the bundles between directions and tangent elements at different points of the space. This difficulty is overcome by means of a non-regular analogue of the Sasaki metric [8*].

§7. Questions and problems

In this section we shall state several unsolved problems that, in our opinion, are not devoid of interest.

1. In [95] and [96] a classification is given of the geodesics in TM and T_1M when M is a manifold of constant curvature $+1$, -1, or 0. TM and T_1M are spaces of a Riemannian submersion with totally geodesic fibres. The fibres contain geodesics of vertical type.

The base M, embedded in TM via the zero section, is also totally geodesic.

Problem (Borisenko). *Give a complete classification of totally geodesic submanifolds in TM and T_1M when M has constant sectional curvature.*

2. Let (M, g) be a Riemannian manifold. A metric g is called *strongly q-spherical* if at each point $Q \in M$ there is a linear subspace $L_Q \subset T_Q M$ of dimension q such that if $Y \in L_Q$, then for every $X, Z \in T_Q M$ the curvature tensor of M satisfies

$$R(X, Y)Z = K(\langle Y, Z\rangle X - \langle X, Z\rangle Y),$$

q is called the *sphericity index*, and K the *sphericity magnitude*. If $K \equiv 0$, then this is the definition of a strongly q-parabolic metric M.

A description of a strongly parabolic Sasaki metric on TM is given in [4].

If $\dim M = 2$, the following cases are possible:

a) if M is not a constant curvature manifold, then $q = 0$;

b) if M has consant curvature $K_0 \neq 1$, then $q = 1$ and $K = K_0^2/4$;

c) if M is a standard two-dimensional sphere, then $q = 3$ and $K = 1/4$.

For higher dimensions it is only known that for $T_1 S^n$ $(n \geqslant 3)$ we have $q = 1$ and $K = 1/4$.

Problem (Borisenko and Yampol'skii). *Prove that the sphericity index q for $T_1 M$, if $\dim M \geqslant 3$, is equal to:*

a) $q = 1$ *if* $M = S^n$; *here* $K = 1/4$;

b) $q = 0$ *in the remaining cases.*

3. Let F^l be a surface in Euclidean space E^{l+p}. Then TF^l is a surface in TE^{l+p}. On TF^l we can consider the Sasaki metric (that is, an internally defined metric) and the metric induced from $TE^{l+p} = E^{2(l+p)}$. These metrics are not isometric.

Problem (Borisenko). *What is the minimal dimension of a Euclidean space in which the Sasaki metric can be embedded (immersed) isometrically?*

4. The curvature of the Sasaki metric $T_\rho(M^n, K)$ $(n \geqslant 3)$ is non-negative if $0 \leqslant \rho^2 K \leqslant 4/3$, that is, for $\rho^2 \leqslant 4/(3K)$. This estimate is sharp. If M^n is locally isometric and has positive curvature, then for sufficiently small ρ the curvature of the Sasaki metric of $T_\rho M^n$ is non-negative.

Problem (Borisenko). *Find a sharp estimate for ρ for which the sectional curvature of the Sasaki metric of $T_\rho \mathbb{C}P^n$ and $T_\rho \mathbb{H}P^n$ is non-negative.*

5. Let M^n be a pseudo-Riemannian manifold with metric of type (p, q). On TM^n we can define a Sasaki metric of type $(2p, 2q)$. Let $T_\rho M^n$ be the subbundle in TM^n consisting of the tangent vectors of constant length ρ (>0 or <0). Let us consider a pseudo-Riemannian sphere $S^{1,1}$ of curvature -1. Then for $\rho = -1$ the Sasaki metric of $T_\rho S^{1,1}$ has constant curvature equal to $1/4$. (This is an analogue of a theorem from [64]).

***Problem* 1** (Yampol'skii). *Find the range of variation of the sectional curvature of the Sasaki metric of $T_\rho S^{1,1}$.*

***Problem* 2** (Yampol'skii). *Give a classification of the geodesics on $TS^{p,q}$ and $T_\rho S^{p,q}$. Consider the problem on totally geodesic manifolds in these spaces.*

6. In a large class of problems of Riemannian geometry one uses assertions concerning a certain k-dimensional distribution on a given Riemannian manifold.

Problem (Borisenko). *Carry over the definition of Sasaki metric to the Grassmanian bundle of a given Riemannian manifold and study the properties of the resulting metric for various types of Grassmanian bundles.*

7. It is interesting to study the geometries of the normal and the spherical normal bundles with Sasaki metric for various classes of surfaces. Thus, if $V^2 \subset S^4$ is the two-dimensional Veronese surface, then for certain ρ the manifold $N_\rho V^2$ has constant sectional curvature.

***Problem* 1** (Borisenko). *Prove that V^2 is the only compact surface in S^4 for which there is a ρ such that $N_\rho V^2$ is a constant curvature manifold.*

***Problem* 2** (Borisenko). *Study the spherical normal bundle of a multi-dimensional Veronese surface.*

8. In a way similar to the definition of strong k-parabolicity, on can define the strong k-defectivity of the normal connection of a manifold.

***Problem* 1** (Borisenko). *Study the structure of the normal bundle of a submanifold with Sasaki metric if the submanifold has a constant index of normal defectivity.*

***Problem* 2** (Borisenko). *Assume that the Sasaki metric of NF^l is strongly k-parabolic. What can be said about F^l and its embedding (immersion) in a given Riemannian manifold?*

9. The following problem also arises in connection with the curvature tensor of the Sasaki metric of NF^l.

Problem (Yampol'skii). *Describe the manifolds (for instance, in E^n) having a locally symmetric curvature tensor of normal connection.*

Surfaces with parallel second quadratic form in E^n are of this type. Are there other examples?

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