Transverse totally geodesic submanifolds of the tangent bundle.*

Mohamed Tahar Kadaoui ABBASSI
Alexander YAMPOLSKY

July 14, 2002

Abstract

It is well-known that if ξ is a smooth vector field on a given Riemannian manifold M^n then ξ naturally defines a submanifold $\xi(M^n)$ transverse to the fibers of the tangent bundle TM^n with Sasaki metric. In this paper, we are interested in transverse totally geodesic submanifolds of the tangent bundle. We show that a transverse submanifold N^l of TM^n $(1 \leq l \leq n)$ can be realized locally as the image of a submanifold F^l of M^n under some vector field ξ defined along F^l. For such images $\xi(F^l)$, the conditions to be totally geodesic are presented. We show that these conditions are not so rigid as in the case of $l = n$, and we treat several special cases (ξ of constant length, ξ normal to F^l, M^n of constant curvature, M^n a Lie group and ξ a left invariant vector field).

Keywords: Sasaki metric, vector field along submanifolds, totally geodesic submanifolds in the tangent bundle.
AMS subject class: Primary 53B25, 53C42; Secondary 22E15, 53B20, 53B21, 53C12, 53C25, 53M10.

Introduction.

Let (M^n, g) be a Riemannian manifold and (TM^n, g_s) its tangent bundle equipped with the Sasaki metric [12]. Let ξ be a given smooth vector field on M^n. Then ξ naturally defines a mapping $\xi : M^n \to TM^n$ such that the submanifold $\xi(M^n) \subset TM^n$ is transverse to the fibers. This fact allows to ascribe to the vector field ξ some geometrical characteristics from the geometry of submanifolds. We say that the vector field ξ is *minimal, totally umbilic* or *totally geodesic* if $\xi(M^n)$ possesses the same property. In a similar way we can say about the *sectional, Ricci* or *scalar curvature* of a vector field. For the case of a *unit* vector field this approach has been proposed

*Math. Publ. Debrecen 64/1-2 (2004), 129-154

by H.Gluck and W.Ziller [6]. They proved that the Hopf vector field h on three-sphere S^3 is one with globally minimal volume, i.e. $h(S^3)$ is a globally minimal submanifold in the unit tangent bundle T_1S^3. Corresponding local consideration leads to the notion of the *mean curvature of a unit vector field* and a number of examples of locally minimal unit vector fields were found based on a preprint version of [5] (see [1, 2, 7] and references). In a different way, the second author found examples of unit vector fields of *constant mean curvature* [18] and completely described the *totally geodesic* unit vector fields on 2-dimensional manifolds of constant curvature [19]. The energy of a mapping $\xi : M^n \to T_1M^n$ can also be ascribed to the vector field ξ and we can say about the *energy* of a unit vector field (see [17, 4, 15] and references).

In contrast to unit vector fields, there are few results (both of local or global aspects) on the geometry of general vector fields treated as submanifolds in the *tangent bundle*. It is known [10] that if ξ is the zero vector field, then $\xi(M^n)$ is totally geodesic in TM^n. Walczak P. [14] treated the case when ξ is a non-zero vector field on M^n and proved that if ξ is a parallel vector field on M^n, then $\xi(M^n)$ is totally geodesic in TM^n. Moreover, if ξ is of constant length, then $\xi(M^n)$ is totally geodesic in TM^n if and only if ξ is a parallel vector field on M^n. The latter condition is rather burdensome. The basic manifold M^n should be a metrical product $M^{n-k} \times E^k$ $(k \geq 1)$, where E^k is a Euclidean (flat) factor.

Remark that $\xi(M^n)$ has maximal dimension among submanifolds in the tangent bundle, transverse to the fibers. In this paper, we study submanifolds N^l of TM^n with $l \leq n$ which are transverse to the fibers. We show in section 2 that any transverse submanifold N^l of TM^n can be realized locally as the image of a submanifold F^l of M^n under some vector field ξ defined along F^l. We also investigate some cases when the image can be globally realized. Mainly, we are interested in submanifolds among this class which are totally geodesic. In this way, we get a chain of inclusions:

$$\xi(F^l) \subset \xi(M^n) \subset TM^n.$$

In comparison with the case when ξ is defined over the whole M^n or, at least, over a domain $D^n \subset M^n$ as in [14], the picture becomes different, because $\xi(F^l)$ can be totally geodesic in TM^n while $\xi(M^n)$ is not. Our considerations include also the case when the vector field is defined only on F^l, so that ξ defines a "direct" embedding $\xi : F^l \to TM^n$.

For $l = 1$ we get nothing else but a vector field along a curve in M^n which generates a geodesic in TM^n. Sasaki S. [12] described geodesic lines in TM^n in terms of vector fields along curves in M^n and found the differential equations on the curve and the corresponding vector field. Moreover, in the case when M^n is of constant curvature, Sato K. [13] explicitly described the curves and the vector fields.

Evidently, our approach takes an intermediate position between the above mentioned considerations for $l = 1$ and $l = n$.

Necessary and sufficient conditions on $\xi(F^l)$ to be totally geodesic, that we make explicit in section 3 (Proposition 3.1), have a clearer geometrical meaning if we suppose that ξ is of constant length along F^l (Theorem 3.2) or is a normal vector field along F^l (Theorem 3.3). Indeed, an application of Theorem 3.3 to the specific case of foliated Riemannian manifolds allows us to clarify the geometrical structure of $\xi(M^n)$ (Corollary 3.5).

The case of a base space M^n of constant curvature is discussed in detail in section 4. An application to the case of a Riemannian manifold of constant curvature enlightens us as to the non rigidity of the totally geodesic property of $\xi(F^l)$, $l < n$, contrary to the case $l = n$.

Finally, an application of our results to Lie groups endowed with bi-invariant metrics gives a clear geometrical picture of our problem.

Remark. Throughout the paper

- M^n is a given Riemannian manifold with metric $\bar{g}$, F^l is a submanifold of M^n with the induced metric g, TM^n is the tangent bundle of M^n equipped with the Sasaki metric g_s;
- $\bar{\nabla}$, ∇, $\tilde{\nabla}$ are the Levi-Civita connections with respect to $\bar{g}$, g, g_s respectively;
- the indices range is fixed as $a, b, c = 1 \dots n$; $i, j, k = 1 \dots l$;
- all the vector fields are supposed sufficiently smooth, say of class C^∞.

1 Local geometry of $\xi(F^l)$.

1.1 Tangent bundle of $\xi(F^l)$.

Let $(M^n, \bar{g})$ be an n-dimensional Riemannian manifold with metric $\bar{g}$. Denote by $\bar{g}(\cdot\,,\cdot)$ the scalar product with respect to $\bar{g}$. The *Sasaki metric* g_s on TM^n is defined by the following scalar product: if $\tilde{X}, \tilde{Y}$ are tangent vector fields on TM^n, then

$$g_s(\tilde{X}, \tilde{Y}) = \bar{g}(\pi_* \tilde{X}, \pi_* \tilde{Y}) + \bar{g}(K\tilde{X}, K\tilde{Y}) \tag{1}$$

where $\pi_* : TTM^n \to TM^n$ is the differential of the projection $\pi : TM^n \to M^n$ and $K : TTM^n \to TM^n$ is the *connection map* [3]. The local representations for π_* and K are the following ones. Let $(x^1, \dots, x^n)$ be a local coordinate system on M^n. Denote by $\partial/\partial x^a$ the natural tangent coordinate frame. Then, at each point $x \in M^n$, any tangent vector ξ can be decomposed as $\xi = \xi^a \frac{\partial}{\partial x^a}(x)$. The set of parameters $\{x^1, \dots, x^n;\ \xi^1, \dots, \xi^n\}$ forms the natural induced coordinate system in TM^n, i.e. for a point $z = (x, \xi) \in TM^n$, with $x \in M^n$, $\xi \in T_xM^n$, we have $x=(x^1,\dots,x^n)$, $\xi=\xi^a\frac{\partial}{\partial x^a}(x)$. The natural frame in T_zTM^n is formed by $\left\{\frac{\partial}{\partial x^a}(z), \frac{\partial}{\partial \xi^a}(z)\right\}$ and for any $\tilde X \in T_zTM^n$ we have the decomposition $\tilde X = \tilde X^a \frac{\partial}{\partial x^a}(z) + \tilde X^{n+a}\frac{\partial}{\partial \xi^a}(z)$. Now locally, the *horizontal* and *vertical* projections of $\tilde X$ are given by

$$\begin{aligned}\pi_*\tilde X &= \tilde X^a \tfrac{\partial}{\partial x^a}(\pi(z)),\\ K\tilde X &= (\tilde X^{n+a} + \bar\Gamma^a_{bc}(\pi(z))\,\xi^b \tilde X^c)\,\tfrac{\partial}{\partial x^a}(\pi(z)),\end{aligned} \tag{2}$$

where $\bar\Gamma^a_{bc}$ are the Christoffel symbols of the metric $\bar g$. The inverse operations are called *lifts* . If $\bar X = \bar X^a\,\partial/\partial x^a$ is a vector field on M^n then the vector fields on TM given by

$$\begin{aligned}\bar X^h &= \bar X^a \partial/\partial x^a - \bar\Gamma^a_{bc}\,\xi^b \bar X^c\,\partial/\partial \xi^a,\\ \bar X^v &= \bar X^a \partial/\partial \xi^a\end{aligned}$$

are called the *horizontal* and *vertical* lifts of X respectively. Remark that for any vector field $\bar X$ on M^n it holds

$$\begin{aligned}\pi_*\bar X^h &= \bar X, \quad K\bar X^h = 0,\\ \pi_*\bar X^v &= 0, \quad\;\; K\bar X^v = \bar X.\end{aligned} \tag{3}$$

Let F^l be an l-dimensional submanifold in M^n with a local representation given by

$$x^a = x^a(u^1,\dots,u^l).$$

Let ξ be a vector field on M^n defined in some neighborhood of (or only on) the submanifold F^l. Then the restriction of ξ to the submanifold F^l, called *a vector field on M^n along F^l*, generates a submanifold $\xi(F^l) \subset TM^n$ with a local representation of the form

$$\xi(F^l): \left\{\begin{array}{ll} x^a = & x^a(u^1,\dots,u^l),\\ \xi^a = & \xi^a(x^1(u^1,\dots,u^l),\dots,x^n(u^1,\dots,u^l)).\end{array}\right. \tag{4}$$

In what follows we will refer to the submanifold (4) as to one *generated by a vector field on M^n along F^l.*

The following Proposition describes the tangent space of $\xi(F^l)$.

Proposition 1.1 *A vector field $\tilde X$ on TM^n is tangent to $\xi(F^l)$ along $\xi(F^l)$ if and only if its horizontal-vertical decomposition is of the form*

$$\tilde X = X^h + (\bar\nabla_X\,\xi)^v,$$

where X is a tangent vector field on F^l, $\bar\nabla_X\,\xi$ is the covariant derivative of ξ in the direction of X with respect to the Levi-Civita connection of M^n and the lifts are considered as those on TM^n.

Proof. Let us denote by $\tilde{e}_i$ the vectors of the coordinate frame of $\xi(F^l)$. Then, evidently,

$$\tilde{e}_i = \left\{ \frac{\partial x^1}{\partial u^i}, \dots, \frac{\partial x^n}{\partial u^i}; \ \frac{\partial \xi^1}{\partial u^i}, \dots, \frac{\partial \xi^n}{\partial u^i} \right\}.$$

Applying (2), we have

$$\begin{aligned} \pi_* \tilde{e}_i = & \frac{\partial x^a}{\partial u^i} \frac{\partial}{\partial x^a} = \frac{\partial}{\partial u^i}, \\ K \tilde{e}_i = & (\frac{\partial \xi^a}{\partial u^i} + \bar{\Gamma}^a_{bc} \xi^b \frac{\partial x^c}{\partial u^i}) \frac{\partial}{\partial x^a} = (\frac{\partial \xi^a}{\partial x^c} \frac{\partial x^c}{\partial u^i} + \bar{\Gamma}^a_{bc} \xi^b \frac{\partial x^c}{\partial u^i}) \frac{\partial}{\partial x^a} \\ = & \frac{\partial x^c}{\partial u^i} (\frac{\partial \xi^a}{\partial x^c} + \bar{\Gamma}^a_{bc} \xi^b) \frac{\partial}{\partial x^a} = \bar{\nabla}_i \xi, \end{aligned}$$

where $\bar{\Gamma}^a_{bc}$ are the Christoffel symbols of the metric $\bar{g}$ taken along F^l and $\bar{\nabla}_i$ means the covariant derivative of a vector field on M^n with respect to the Levi-Civita connection of $\bar{g}$ along the i-th coordinate curve of the submanifold $F^l \subset M^n$. Summing up, we have

$$\tilde{e}_i = \left(\frac{\partial}{\partial u^i} \right)^h + (\bar{\nabla}_i \xi)^v. \tag{5}$$

Let $\tilde{X}$ be a vector field on TM^n tangent to $\xi(F^l)$ along $\xi(F^l)$. Then the following decomposition holds $\tilde{X} = \tilde{X}^i \tilde{e}_i$. Set $X = \tilde{X}^i \partial/\partial u^i$. The vector field X is tangent to F^l and, taking into account (5), the decomposition of $\tilde{X}$ can be represented as $\tilde{X} = X^h + (\bar{\nabla}_X \xi)^v$, which completes the proof.

■

Corollary 1.1 *Let (F^l, g) be a submanifold of a Riemannian manifold $(M^n, \bar{g})$ with the induced metric. Let ξ be a vector field on M^n along F^l. Then the metric on $\xi(F^l)$, induced by the Sasaki metric of TM^n, is defined by the following scalar product*

$$g_s(\tilde{X}, \tilde{Y}) = g\,(X, Y) + \bar{g}\,(\bar{\nabla}_X \xi, \bar{\nabla}_Y \xi),$$

for all vector fields $\tilde{X} = X^h + (\bar{\nabla}_X \xi)^v$ and $\tilde{Y} = Y^h + (\bar{\nabla}_Y \xi)^v$ on $\xi(F^l)$, where X, Y are vector fields on F^l.

1.2 Normal bundle of $\xi(F^l)$.

To describe the normal bundle of $\xi(F^l)$, we need one auxiliary notion. Let ξ be a given vector field on a submanifold $F^l \subset M^n$. Then $\bar{\nabla}$ enables us to define a point-wise linear mapping $\bar{\nabla}\xi : T_xF^l \to T_xM^n$, $X \to \bar{\nabla}_X \xi$, for all $x \in M^n$. Its dual mapping, with respect to the corresponding scalar products induced by g and $\bar{g}$, gives rise to the linear mapping $(\bar{\nabla}\xi)^* : T_xM^n \to T_xF^l$ defined by the formula

$$g\,((\bar{\nabla}\xi)^* W, X) = \bar{g}\,(\bar{\nabla}_X \xi, W) \text{ for all } W \in T_xM^n \text{ and } X \in T_xF^l. \tag{6}$$

We call the mapping $(\bar{\nabla}\xi)^* : T_xM^n \to T_xF^l$ the *conjugate derivative mapping*, or simply *conjugate derivative*. Remark, that if W is a vector field on M^n, then the application of $(\bar{\nabla}\xi)^*$ gives rise to a vector field $(\bar{\nabla}\xi)^*W$ on F^l by $[(\bar{\nabla}\xi)^*W]_x = (\bar{\nabla}\xi)^*W_x \in T_xF^l$ for all $x \in F^l$.

Now we can prove

Proposition 1.2 *Let η and Z be normal and tangent vector fields on F^l respectively. Then the lifts*

$$\eta^h,\ \eta^v - ((\bar{\nabla}\xi)^*\eta)^h,\ Z^v - ((\bar{\nabla}\xi)^*Z)^h$$

to the points of $\xi(F^l)$ span the normal bundle of $\xi(F^l)$ in TM^n.

Proof. Let $\tilde{X} = X^h + (\bar{\nabla}_X\xi)^v$ be a vector field on $\xi(F^l)$. Let η and Z be vector fields on F^l which are normal and tangent to F^l respectively. Taking into account (1), (3) and (6), we have

$$g_s(\tilde{X}, \eta^h) = \bar{g}\,(X, \eta) = 0$$

$$\begin{aligned} g_s(\tilde{X}, \eta^v - [(\bar{\nabla}\xi)^*\eta]^h) &= -\bar{g}\,(X, (\bar{\nabla}\xi)^*\eta) + \bar{g}\,(\bar{\nabla}_X\xi, \eta) \\ &= -\bar{g}\,(\bar{\nabla}_X\xi, \eta) + \bar{g}\,(\bar{\nabla}_X\xi, \eta) = 0 \\ g_s(\tilde{X}, Z^v - [(\bar{\nabla}\xi)^*Z]^h) &= -\bar{g}\,(X, (\bar{\nabla}\xi)^*Z) + \bar{g}\,(\bar{\nabla}_X\xi, Z) \\ &= -\bar{g}\,(\bar{\nabla}_X\xi, Z) + \bar{g}\,(\bar{\nabla}_X\xi, Z) = 0 \end{aligned}$$

Let $\eta_1, \dots, \eta_p$ $(p = 1, \dots, n-l)$ be a normal frame of F^l while $f_1, \dots, f_l$ span T_xF^l at each point $x \in F^l$. Consider the vector fields

$$N_\alpha = \eta_\alpha^h, \quad P_\alpha = \eta_\alpha^v - ((\bar{\nabla}\xi)^*\eta_\alpha)^h, \quad F_i = f_i^v - ((\bar{\nabla}\xi)^*e_i)^h,$$

where $\alpha = 1, \dots, n-l;\ i = 1, \dots, n$. Let us show that these are linearly independent. Indeed, suppose that

$$\lambda^\alpha N_\alpha + \mu^\alpha P_\alpha + \nu^i F_i = \{\lambda^\alpha\eta_\alpha - \mu^\alpha(\bar{\nabla}\xi)^*\eta_\alpha - \nu^i(\bar{\nabla}\xi)^*e_i\}^h + \{\mu^\alpha\eta_\alpha + \nu^i f_i\}^v = 0.$$

Because of the fact that the horizontal and vertical components are linearly independent, we see that $\mu^\alpha\eta_\alpha + \nu^i f_i = 0$ which is possible iff $\mu^\alpha = 0, \nu^i = 0$. Then, from the horizontal part of the decomposition above we see that $\lambda^\alpha = 0$. So, $N_\alpha,\ P_\alpha$ and F_i are linearly independent, which completes the proof.

■

Remark. In the case when ξ is a normal vector field, the images $(\bar{\nabla}\xi)^*\eta$ and $(\bar{\nabla}\xi)^*Z$ have a simple and natural meaning, namely

$$(\bar{\nabla}\xi)^*\eta = g^{ik}\bar{g}\,(\nabla^\perp_k\xi, \eta)\tfrac{\partial}{\partial u^i}, \quad (\bar{\nabla}\xi)^*Z = -A_\xi Z,$$

where $\nabla^\perp$ is the normal bundle connection of F^l and A_ξ is the shape operator of F^l with respect to the normal vector field ξ. In fact, $(\bar{\nabla}\xi)^*\eta$ is the vector field on F^l dual to the 1-form $\bar{g}\,(\nabla^\perp_k\xi, \eta)\,du^k$.

2 Characterization of submanifolds of TM^n transverse to fibers.

It is clear that all totally geodesic vector fields along submanifolds of M^n generate submanifolds in TM^n which are transverse to the fibers of TM^n. We study in this section the converse question. We start with the local case.

Proposition 2.1 *Let N^l be an embedded submanifold in the tangent bundle of a Riemannian manifold M^n, which is transverse to the fiber at a point $z \in N^l$, then there is a submanifold F^l of M^n containing $x = \pi(z)$, a neighborhood U of x in M^n, a neighborhood V of z in TM^n and a vector field ξ on M^n along $F^l \cap U$ such that $N^l \cap V = \xi(F^l \cap U)$.*

Proof. Since T_zN^l is transverse to the vertical subspace V_zTM^n of TTM^n at z, $\pi_* \restriction T_zN^l : T_zN^l \to T_xM^n$ is injective, and so there is an open neighborhood W of z in TM^n such that $\pi_* \restriction T_{z'}N^l : T_{z'}N^l \to T_{\pi(z')}M^n$ is injective for all $z' \in W \cap N^l$. Hence $\pi \restriction W \cap N^l : W \cap N^l \to M^n$ is an immersion, and thus there exist a cubic centered coordinate system (U, φ) about $x = \pi(z)$ and a neighborhood V of z in W such that $\pi \restriction V \cap N^l$ is 1:1 and $\pi(V \cap N^l)$ is a part of a slice F^l of (U, φ) ([16], p. 28). The slice F^l is a submanifold of M^n and we have $\pi \restriction V \cap N^l : V \cap N^l \to U \cap F^l$ is an imbedding onto, and so there is a C^∞-mapping $\xi : F^l \cap U \to N^l \cap V$ such that $\pi \circ \xi = Id_{F^l \cap U}$. In other words, ξ is a vector field on M^n along $F^l \cap U$ such that $N^l \cap V = \xi(F^l \cap U)$.

∎

The global version of the last result requires further conditions.

Theorem 2.1 *Let N^nbe a connected compact n-dimensional submanifold of the tangent bundle of a connected simply connected Riemannian manifold M^n, which is everywhere transverse to the fibers of TM^n. Then M^n is also compact, and there is a vector field ξ on M^n such that $\xi(M^n) = N^n$.*

Proof. The fact that N^n is everywhere transverse to the fibers of TM^n implies that $\pi \restriction N^n : N^n \to M^n$ is an immersion. Since M^n and N^n are connected of the same dimension and N^n is compact, then M^n is compact and $\pi \restriction N^n$ is a covering projection (cf. [8], Vol. 1, p.178). Now, M^n is simply connected and so $\pi \restriction N^n$ is a diffeomorphism. Let $\xi : M^n \to N^n$ be the inverse of $\pi \restriction N^n$. Then ξ is a vector field on M^n and $\xi(M^n) = N^n$.

∎

In a similar way, we can show the following:

Theorem 2.2 *Let N^l be a connected compact submanifold of the tangent bundle of a connected simply connected manifold M^n, which is transverse to the fibers it meets and projects onto a simply connected submanifold F^l of M^n. Then F^l is compact and there is a vector field ξ on M^n along F^l such that $\xi(F^l) = N^l$.*

In the particular case of horizontal totally geodesic submanifolds of TM^n, i.e. whose tangent space at any point is horizontal, we can state the following:

Theorem 2.3 *Let N^l be a connected complete totally geodesic horizontal submanifold of the tangent bundle of a connected Riemannian manifold M^n which projects into a simply connected Riemannian submanifold F^l of M^n. Then F^l is also complete and totally geodesic in M^n and there is a parallel vector field ξ on M^n along F^l such that $\xi(F^l) = N^l$.*

Proof. By hypothesis, for all $z \in N^l$, T_zN^l is a horizontal subspace of T_zTM^n with respect to the Levi-Civita connection of $\bar{g}$. Hence $\pi \upharpoonright N^l : N^l \to F^l$ is an isometric submersion of N^l into F^l, with N^l and F^l connected and of the same dimension. Since N^l is complete, also F^l is complete and N^l is a covering space of F^l (cf. [8], Vol.1, p.176). The fact that F^l is simply connected implies that $\pi \upharpoonright N^l : N^l \to F^l$ is an isometry, and there is an isometry $\xi : F^l \to N^l$ such that $\pi \upharpoonright N^l \circ \xi = Id_{F^l}$, i.e. ξ is a vector field on M^n along F^l.

Now, F^l is totally geodesic. Indeed, let X and Y be vector fields on F^l, and denote by the same letters some of their extensions to M^n. If we denote by X^h and Y^h their horizontal lifts to TM^n, then $X^h \upharpoonright N^l$ and $Y^h \upharpoonright N^l$ are vector fields on TM^n along N^l. For all $z \in N^l$, T_zN^l being horizontal, $\pi_* \upharpoonright T_zN^l : T_zN^l \to T_xM^n$ is bijective. Since $\pi_*(X^h(z)) = X(\pi(z))$ and $\pi_*(Y^h(z)) = Y(\pi(z))$, we have that $X^h(z)$ and $Y^h(z)$ are tangent to N^l. Thus $(\tilde{\nabla}_{X^h}Y^h) \upharpoonright N^l$ is tangent to N^l and hence horizontal. Consequently $(\tilde{\nabla}_{X^h}Y^h) \upharpoonright N^l = (\bar{\nabla}_X Y)^h \upharpoonright N^l$ and is tangent to N^l. Hence $\bar{\nabla}_X Y = \pi_* \circ (\bar{\nabla}_X Y)^h$ is tangent to F^l and so F^l is totally geodesic. It remains to prove that ξ is parallel along F^l. In fact, for all $x \in F^l$ and $X \in T_xF^l$, the vector $X^h + (\bar{\nabla}_X \xi)^v$ is tangent to $\xi(F^l) = N^l$ at $\xi(x)$ and is mapped onto X. Since $T_{\xi(x)}N^l$ is a horizontal space, $\bar{\nabla}_X \xi = 0$. Therefore, ξ is parallel along F^l.

■

Corollary 2.1 *Let N^n be a connected complete totally geodesic horizontal n-dimensional submanifold of the tangent bundle of a connected simply connected Riemannian manifold M^n. Then M^n is also complete and there is a parallel vector field ξ on M^n such that $\xi(M^n) = N^n$.*

3 The conditions on $\xi(F^l)$ to be totally geodesic.

Evidently, geometrical properties of the submanifold $\xi(F^l)$ depend on the submanifold F^l and the vector field ξ. If one does not pose any restrictions on them, the geometry of $\xi(F^l)$ becomes rather intricate. Nevertheless, it is possible to formulate the conditions on $\xi(F^l)$ to be totally geodesic in more or less geometrical terms.

To do this, we introduce the notion of a ξ-connection on the Riemannian manifold M^n.

Definition 3.1 *Let M^n be a Riemannian manifold with Riemannian connection $\bar\nabla$ and curvature tensor $\bar R$. Let ξ be a fixed smooth vector field on M^n. Denote by $\mathfrak{X}(M^n)$ the set of all smooth vector fields on M^n. The mapping $\overset{*}{\nabla}\colon \mathfrak{X}(M^n)\times\mathfrak{X}(M^n)\to\mathfrak{X}(M^n)$ defined by*

$$\overset{*}{\nabla}_{\bar X}\,\bar Y=\bar\nabla_{\bar X}\bar Y+\frac{1}{2}\Big[\bar R(\xi,\bar\nabla_{\bar X}\xi)\bar Y+\bar R(\xi,\bar\nabla_{\bar Y}\xi)\bar X\Big] \tag{7}$$

is a torsion-free affine connection on M^n. It is called the ξ-connection.

Remark that if ξ is a parallel vector field or the manifold M^n is flat, then the ξ-connection is the same as the Levi-Civita connection of M^n.

It is easy to check that (7) indeed defines a torsion-free affine connection. Now we can state the main technical tool for the further considerations.

Proposition 3.1 *Let F^l be a submanifold in a Riemannian manifold M^n. Let ξ be a vector field on M^n along F^l. Then $\xi(F^l)$ is totally geodesic in TM^n if and only if*

(a) F^l is totally geodesic with respect to the ξ-connection (7);

(b) for any vector fields X,Y on F^l

$$\bar\nabla_X\bar\nabla_Y\xi=\bar\nabla_{\overset{*}{\nabla}_XY}\xi+\frac{1}{2}\bar R(X,Y)\xi.$$

Proof. By definition, the submanifold $\xi(F^l)$ is totally geodesic in TM^n if and only if $g_s\,(\,\tilde\nabla_{\tilde X}\tilde Y,\tilde N)\;=\;0$ for any vector fields $\tilde X,\tilde Y$ tangent to $\xi(F^l)$ along $\xi(F^l)$ and $\tilde N$ normal to $\xi(F^l)$. To calculate $\tilde\nabla_{\tilde X}\tilde Y$, we use the Kowalski formulas [9].

For any vector fields $\bar X,\bar Y$ on M^n, the covariant derivatives of various combinations of lifts to the point $(x,\xi)\in TM^n$ can be found as follows

$$\begin{aligned}&\tilde\nabla_{\bar X^h}\bar Y^h=(\bar\nabla_{\bar X}\bar Y)^h-\tfrac12(\bar R(\bar X,\bar Y)\xi)^v, &&\tilde\nabla_{\bar X^v}\bar Y^h=\tfrac12(\bar R(\xi,\bar X)\bar Y)^h,\\ &\tilde\nabla_{\bar X^h}\bar Y^v=(\bar\nabla_{\bar X}\bar Y)^v+\tfrac12(\bar R(\xi,\bar Y)\bar X)^h, &&\tilde\nabla_{\bar X^v}\bar Y^v=0.\end{aligned} \tag{8}$$

where $\bar\nabla$ and $\bar R$ are the Levi-Civita connection and the curvature tensor of M^n respectively.

Let $\tilde X=X^h+(\bar\nabla_X\xi)^v$ and $\tilde Y=(Y)^h+(\bar\nabla_Y\xi)^v$ be vector fields tangent to $\xi(F^l)$. Then, applying (8), we easily find

$$\tilde\nabla_{\tilde X}\tilde Y=(\bar\nabla_XY+\frac12\bar R(\xi,\bar\nabla_X\xi)Y+\frac12\bar R(\xi,\bar\nabla_Y\xi)X)^h+(\bar\nabla_X\bar\nabla_Y\,\xi-\frac12\bar R(X,Y)\xi)^v$$

or

$$\tilde{\nabla}_{\tilde{X}}\tilde{Y} = (\overset{*}{\nabla}_X\, Y)^h + (\bar{\nabla}_X\bar{\nabla}_Y\,\xi - \frac{1}{2}\bar{R}(X,Y)\xi)^v.$$

Using Proposition 1.2, we see that the totally geodesic property of $\xi(F^l)$ is equivalent to

$$\begin{cases} \bar{g}\,(\overset{*}{\nabla}_X\, Y, \eta) &= 0, \\ \bar{g}\,(\overset{*}{\nabla}_X\, Y, (\nabla\xi)^*\eta) &= \bar{g}\,(\bar{\nabla}_X\bar{\nabla}_Y\xi - \frac{1}{2}\bar{R}(X,Y)\xi, \eta), \\ \bar{g}\,(\overset{*}{\nabla}_X\, Y, (\nabla\xi)^*Z) &= \bar{g}\,(\bar{\nabla}_X\bar{\nabla}_Y\xi - \frac{1}{2}\bar{R}(X,Y)\xi, Z), \end{cases} \tag{9}$$

for any vector fields X, Y, Z tangent to F^l and any vector field η orthogonal to F^l.

From $(9)_1$ we see that F^l must be autoparallel with respect to $\overset{*}{\nabla}$ and hence totally geodesic [8]. Thus, $\overset{*}{\nabla}_X\, Y$ is tangent to F^l and it is possible to apply (6). Therefore, we can rewrite the equations $(9)_2$ and $(9)_3$ as

$$\begin{cases} \bar{g}\,(\bar{\nabla}_{\overset{*}{\nabla}_X Y}\xi - \bar{\nabla}_X\bar{\nabla}_Y\xi + \frac{1}{2}\bar{R}(X,Y)\xi, \eta) = 0, \\ \bar{g}\,(\bar{\nabla}_{\overset{*}{\nabla}_X Y}\xi - \bar{\nabla}_X\bar{\nabla}_Y\xi + \frac{1}{2}\bar{R}(X,Y)\xi, Z) = 0 \end{cases}$$

for any vector fields η normal and Z tangent to F^l along F^l. Thus, we conclude

$$\bar{\nabla}_X\bar{\nabla}_Y\xi = \bar{\nabla}_{\overset{*}{\nabla}_X Y}\xi + \frac{1}{2}\bar{R}(X,Y)\xi,$$

which completes the proof.

■

For the cases when $l = 1$ and $l = n$, we get the known conditions for the totally geodesic property of $\xi(F^l)$.

Corollary 3.1 *If $l = 1$ and $\xi(F^l)$ is a curve Γ in TM^n then this curve is a geodesic if and only if*

$$\begin{cases} x'' + \bar{R}(\xi, \xi')x' = 0, \\ \xi'' = 0, \end{cases}$$

where ($'$) means the covariant derivative with respect to the natural parameter of Γ and $x(\sigma) = (\pi \circ \Gamma)(\sigma)$ (cf. [12])*;*

Proof. Indeed, in this case $\tilde{X} = \tilde{Y} = \Gamma' = (x')^h + (\xi')^v$, $\bar{X} = \bar{Y} = x'$ and $\overset{*}{\nabla}_{\bar{X}}\, \bar{Y} = x'' + \bar{R}(\xi, \xi')x'$. Thus, $x(\sigma)$ is geodesic with respect to the ξ-connection iff $x'' + \bar{R}(\xi, \xi')x' = 0$ and the rest of the proof is evident.

■

Corollary 3.2 *If $l = n$ and $F^l = M^n$, then $\xi(M^n)$ is totally geodesic in TM^n if and only if for any vector fields $\bar{X}, \bar{Y}$ on M^n* (cf. [14])

$$\bar{\nabla}_{\bar{X}} \bar{\nabla}_{\bar{Y}} \xi = \bar{\nabla}_{\overset{*}{\nabla}_{\bar{X}} \bar{Y}} \xi + \frac{1}{2} \bar{R}(\bar{X}, \bar{Y})\xi.$$

Proof. In this case, only (b) of Proposition 3.1 should be checked, which completes the proof.

∎

The result of Corollary 3.2 can be expressed in more geometrical terms. To do this, introduce a symmetric bilinear mapping $h_\xi : \mathfrak{X}(M^n) \times \mathfrak{X}(M^n) \to \mathfrak{X}(M^n)$ by

$$h_\xi(\bar{X}, \bar{Y}) = \frac{1}{2} \Big[\bar{R}(\xi, \nabla_{\bar{X}} \xi)\bar{Y} + \bar{R}(\xi, \nabla_{\bar{Y}} \xi)\bar{X} \Big], \tag{10}$$

for all $\bar{X}$, $\bar{Y} \in \mathfrak{X}(M^n)$. Then the definition of the ξ-connection takes as similar form as the Gauss decomposition

$$\overset{*}{\nabla}_{\bar{X}} \bar{Y} = \bar{\nabla}_{\bar{X}} \bar{Y} + h_\xi(\bar{X}, \bar{Y}). \tag{11}$$

Define a *"shape operator"* A_ξ for the field ξ by

$$A_\xi \bar{Y} = -\bar{\nabla}_{\bar{Y}} \xi, \text{ for all } \bar{Y} \in \mathfrak{X}(M^n). \tag{12}$$

Then the covariant derivative of the $(1,1)$-tensor field A_ξ is given by

$$(\bar{\nabla}_{\bar{X}} A_\xi)\bar{Y} = -\bar{\nabla}_{\bar{X}} \bar{\nabla}_{\bar{Y}} \xi + \bar{\nabla}_{\bar{\nabla}_{\bar{X}} \bar{Y}} \xi.$$

Hence we see that the Codazzi-type equation $\bar{R}(\bar{X}, \bar{Y})\xi = (\bar{\nabla}_{\bar{Y}} A_\xi)\bar{X} - (\bar{\nabla}_{\bar{X}} A_\xi)\bar{Y}$ holds. In these notations

$$\bar{\nabla}_{\overset{*}{\nabla}_{\bar{X}} \bar{Y}} \xi + \frac{1}{2} \bar{R}(\bar{X}, \bar{Y})\xi - \bar{\nabla}_{\bar{X}} \bar{\nabla}_{\bar{Y}} \xi = \bar{\nabla}_{h_\xi(\bar{X}, \bar{Y})} \xi + \frac{1}{2} \Big[(\bar{\nabla}_{\bar{X}} A_\xi)\bar{Y} + (\bar{\nabla}_{\bar{Y}} A_\xi)\bar{X} \Big].$$

If we introduce a symmetric bilinear mapping $\Omega_\xi : \mathfrak{X}(M^n) \times \mathfrak{X}(M^n) \to \mathfrak{X}(M^n)$ defined by

$$\Omega_\xi(\bar{X}, \bar{Y}) = \bar{\nabla}_{h_\xi(\bar{X}, \bar{Y})} \xi + \frac{1}{2} \Big[(\bar{\nabla}_{\bar{X}} A_\xi)\bar{Y} + (\bar{\nabla}_{\bar{Y}} A_\xi)\bar{X} \Big],$$

then Corollary 3.2 can be reformulated as

Corollary 3.3 *If ξ is a smooth vector field on a Riemannian manifold M^n then $\xi(M^n)$ is totally geodesic in TM^n if and only if for any vector fields $\bar{X}, \bar{Y}$ on M^n*

$$\Omega_\xi(\bar{X}, \bar{Y}) = \bar{\nabla}_{h_\xi(\bar{X}, \bar{Y})} \xi + \frac{1}{2} \Big[(\bar{\nabla}_{\bar{X}} A_\xi)\bar{Y} + (\bar{\nabla}_{\bar{Y}} A_\xi)\bar{X} \Big] \equiv 0, \tag{13}$$

where h_ξ and A_ξ are defined by (10) and (12) respectively.

Remark. The statement of Proposition 3.1 can also be reformulated in these terms, namely, *let F^l be a submanifold in a Riemannian manifold M^n and ξ be a vector field on M^n along F^l. Then $\xi(F^l)$ is totally geodesic in TM^n if and only if F^l is totally geodesic with respect to the ξ-connection (7) and Ω_ξ vanishes on the tangent bundle of F^l*

Now, combining Theorem 2.1 with Proposition 3.1, we obtain

Corollary 3.4 *On a connected simply connected compact $n-$dimensional Riemannian manifold, vector fields satisfying (b) of Proposition 3.1 generate the only connected compact totally geodesic n-dimensional submanifolds of the tangent bundle which are transverse to fibers.*

As has been shown in [20], for the case of the unit tangent bundle, the Hopf vector fields on odd dimensional spheres generate totally geodesic submanifolds in T_1S^n. For the tangent bundle the situation is different.

Theorem 3.1 *A non-zero Killing vector field on a space of non-zero constant curvature (M^n, c) never generates a totally geodesic submanifold in TM^n. Moreover, a manifold with positive sectional curvature does not admit a non-zero Killing vector field with totally geodesic property.*

Proof. Let ξ be a Killing vector field on a space M^n of constant curvature c. Then A_ξ is a skew-symmetric linear operator, i.e.

$$\bar g(A_\xi \bar X, \bar Y) + \bar g(\bar X, A_\xi \bar Y) = 0, \tag{14}$$

and moreover,

$$(\bar\nabla_{\bar X} A_\xi)\bar Y = \bar R(\xi, \bar X)\bar Y \tag{15}$$

for all vector fields $\bar X, \bar Y$ on M^n (cf. [8]). Since M^n is of non-zero constant curvature, the equation (13) can be simplified in the following way.

$$\begin{aligned}(\bar\nabla_{\bar X} A_\xi)\bar Y + (\bar\nabla_{\bar Y} A_\xi)\bar X = \ & \bar R(\xi, \bar X)\bar Y + \bar R(\xi, \bar Y)\bar X = \\ & c\Big[2\bar g(\bar X, \bar Y)\,\xi - \bar g(\xi, \bar X)\bar Y - \bar g(\xi, \bar Y)\bar X\Big]\end{aligned}$$

$$\begin{aligned}&\bar R(\xi, \bar\nabla_{\bar X}\xi)\bar Y + \bar R(\xi, \bar\nabla_{\bar Y}\xi)\bar X = c\Big[\bar g(\bar\nabla_{\bar X}\xi, \bar Y) + \bar g(\bar X, \bar\nabla_{\bar Y}\xi)\bar X)\Big]\xi - \\ &c\Big[(\bar g(\xi, \bar X)\bar\nabla_{\bar Y}\xi + \bar g(\xi, \bar Y)\bar\nabla_{\bar X}\xi)\Big] = c\Big[\bar g(\xi, \bar X)A_\xi \bar Y + \bar g(\xi, \bar Y)A_\xi \bar X\Big].\end{aligned}$$

So, ξ is totally geodesic if

$$\bar g(\xi, \bar X)\bar Y + \bar g(\xi, \bar Y)\bar X - \bar\nabla_{\bar g(\xi, \bar X)A_\xi \bar Y + \bar g(\xi, \bar Y)A_\xi \bar X}\xi = 2\bar g(\bar X, \bar Y)\,\xi,$$

or

$$\bar g(\xi, \bar X)\Big[\bar Y + A_\xi(A_\xi \bar Y)\Big] + \bar g(\xi, \bar Y)\Big[\bar X + A_\xi(A_\xi \bar X)\Big] = 2\bar g(\bar X, \bar Y)\,\xi,$$

for all vector fields $\bar{X}, \bar{Y}$ on M^n. Choosing $\bar{X}, \bar{Y}$ such that $\bar{X}_x \neq 0$ and $\bar{X}_x = \bar{Y}_x \perp \xi_x$, we get $2|\bar{X}_x|^2 \xi_x = 0$. Therefore, $\xi = 0$ for all $x \in M^n$.

Let ξ be a non-zero Killing vector field on a manifold with *positive* (non-constant) sectional curvature. From (14) it follows that $A_\xi \xi \perp \xi$. If $A_\xi \xi = 0$, then, after setting $Y = \xi$ in (14), we conclude that ξ has a constant length and therefore can be totally geodesic if it is a parallel vector field [14]. In this case, $M^n = M^{n-1} \times E^1$ and we come to a contradiction. Suppose that $A_\xi \xi \neq 0$. Then $\xi \wedge A_\xi \xi$ is a non-zero bivector field. Setting $\bar{Y} = \bar{X}$ in (13) and using (15), we have

$$A_\xi \Big[\bar{R}(\xi, A_\xi \bar{X}) \bar{X} \Big] + \bar{R}(\xi, \bar{X}) \bar{X} = 0.$$

Taking a scalar product in both sides with ξ and applying (14), we get

$$-\bar{g}(\bar{R}(\xi, A_\xi \bar{X}) \bar{X}, A_\xi \xi) + K_{\xi \wedge \bar{X}} |\xi \wedge \bar{X}|^2 = 0.$$

Finally, setting $\bar{X} = A_\xi \xi$, we have $K_{\xi \wedge \bar{X}} = 0$ and come to a contradiction. ■

The next Theorem is analogous to the one proved by Walczak P. [14], but does not have similar rigid consequences for the structure of M^n.

Theorem 3.2 *Let ξ be a vector field of constant length along a submanifold $F^l \subset M^n$. Then $\xi(F^l)$ is a totally geodesic submanifold in TM^n if and only if F^l is totally geodesic in M^n and ξ is a parallel vector field on M^n along F^l.*

Proof. The condition $|\,\xi\,| = const$ implies $\bar{g}\,(\bar{\nabla}_X \xi, \xi) = 0$ for any vector field X tangent to F^l . As $\xi(F^l)$ is supposed to be totally geodesic, it follows from the second condition of Proposition 3.1 that $\bar{g}\,(\bar{\nabla}_X \bar{\nabla}_Y \xi, \xi) = 0$. Hence $\bar{g}\,(\bar{\nabla}_X \xi, \bar{\nabla}_Y \xi) = 0$ for any $X, Y \in T_x F^l$, $x \in F^l$. Supposing $X = Y$, we see that $\bar{\nabla}_X \xi = 0$, i.e. ξ is parallel along F^l in the ambient space and the second condition of Proposition 3.1 is fulfilled. Moreover, the condition $\bar{\nabla}_X \xi = 0$ means that the ξ-connection (7) coincides with the Levi-Civita connection of M^n, so that by Proposition 3.1 F^l is totally geodesic in M^n.

On the other hand, if F^l is totally geodesic in M^n and $\bar{\nabla}_X \xi = 0$ for any tangent vector field X on F^l, then both conditions from Proposition 3.1 are satisfied evidently.

■

Giving more restrictions on the vector field, we can a more geometrical result.

Theorem 3.3 *Let ξ be a normal vector field on a submanifold $F^l \subset M^n$, which is parallel in the normal bundle. Then $\xi(F^l)$ is totally geodesic in TM^n if and only if F^l is totally geodesic in M^n.*

Proof. If ξ is a normal vector field to F^l and parallel in the normal bundle, then $\bar{\nabla}_X \xi = -A_\xi X$ for each vector field X on F^l, where A_ξ is the shape operator of F^l with respect to ξ, and hence $\bar{g}\,(\bar{\nabla}_X \xi, \xi) = 0$. This means that $|\xi|$=const along F^l.

Let $\xi(F^l)$ be totally geodesic in TM^n. Then from (b) of Proposition 3.1 we see that $\bar{g}\,(\bar{\nabla}_X \bar{\nabla}_Y \xi, \xi) = 0$, which implies $|\bar{\nabla}_X \xi| = 0$ for each X tangent to F^l. In this case, along F^l the ξ-connection (7) coincides with the Levi-Civita connection of M^n and (a) of Proposition 3.1 implies the totally geodesic property of F^l.

Conversely, if ξ is a normal vector field which is parallel in the normal bundle of F^l and F^l is totally geodesic, then $\bar{\nabla}_X \xi = 0$ for any vector field X tangent to F^l. Evidently, both conditions of Proposition 3.1 are fulfilled.

■

The application of Theorem 3.3 to the specific case of a foliated Riemannian manifold allows to clarify the geometrical structure of $\xi(M^n)$. The manifold M^n is said to be *ν-foliated* if it admits a family $\mathcal{F}$ of connected ν-dimensional submanifolds $\{\mathcal{F}_\alpha; \alpha \in A\}$ called *leaves* such that (i) $M^n = \bigcup\limits_{\alpha \in A} \mathcal{F}_\alpha$; (ii) $\mathcal{F}_\alpha \cap \mathcal{F}_\beta = \emptyset$ for $\alpha \neq \beta$; (iii) there exists a coordinate covering $\mathcal{U}$ of M^n such that in each local chart $U \in \mathcal{U}$ the leaves can be expressed locally as level submanifolds, i.e. $u^{\nu+1} = c_{\nu+1}, \dots, u^n = c_n$.

The family $\mathcal{F}$ is called a *ν-foliation* and *hyperfoliation* for $\nu = n-1$. The hyperfoliation is said to be *transversally orientable* if M^n admits a vector field ξ transversal to the leaves. Moreover, with respect to the Riemannian metric on M^n, this vector field can be chosen as a field of unit normals for each leaf.

A submanifold $F^{k+\nu} \subset M^n$ is called *ν-ruled* if $F^{k+\nu}$ admits a ν-foliation $\{\mathcal{F}_\alpha;\ \alpha \in A\}$ such that each leaf $\mathcal{F}_\alpha$ is totally geodesic in M^n. The leaves $\mathcal{F}_\alpha$ are called *elements* or *generators* [11].

Corollary 3.5 *Let M^n be a Riemannian manifold admitting a totally geodesic transversally orientable hyperfoliation $\mathcal{F}$. Let ξ be a field of normals of the foliation having constant length. Then $\xi(M^n)$ is an $(n-1)$-ruled submanifold in TM^n with the elements $\xi(\mathcal{F}_\alpha)$.*

Proof. Indeed, let $\mathcal{F}_\alpha$ be a leaf of the hyperfoliation and ξ be a vector field of constant length on M^n which is a field of normals along each leaf. Applying Theorem 3.3, we get that $\xi(\mathcal{F}_\alpha)$ is totally geodesic in TM^n for each α. Since $\xi : M^n \to \xi(M^n)$ is a homeomorphism, $\xi(\mathcal{F}_\alpha) \cap \xi(\mathcal{F}_\beta) = \emptyset$ for $\alpha \neq \beta$ and $\xi(M^n) = \bigcup\limits_{\alpha \in A} \xi(\mathcal{F}_\alpha)$. Finally, if $\mathcal{F}_\alpha$ is given by $u^n = c_n$ within a local chart U then from (4) we see that $\xi(\mathcal{F}_\alpha)$ is given by the same equalities within the local chart $\xi(U)$. So, $\xi(\mathcal{F}) = \{\xi(\mathcal{F}_\alpha); \alpha \in A\}$ form a hyperfoliation on $\xi(M^n)$ with totally geodesic leaves in TM^n.

■

4 The case of a base space of constant curvature.

If the ambient space is of constant curvature $c \neq 0$ and ξ is a normal vector field on a submanifold $F^l \subset M^n$, then the necessary and sufficient condition on ξ to generate a totally geodesic submanifold in TM^n takes a rather simple form.

Theorem 4.1 *Let F^l be a submanifold of a space $M^n(c)$ of constant curvature $c \neq 0$. Let ξ be a normal vector field on F^l. Then $\xi(F^l)$ is totally geodesic in TM^n if and only if F^l is totally geodesic in $M^n(c)$ and ξ is parallel in the normal bundle.*

Proof. The curvature tensor of $M^n(c)$ is of the form

$$\bar R(\bar X, \bar Y)\bar Z = c\,(\bar g\,(\bar Y, \bar Z)\bar X - \bar g\,(\bar X, \bar Z)\bar Y\,). \tag{16}$$

If ξ is a normal vector field on F^l then $\bar\nabla_X \xi = -A_\xi X + \nabla^\perp_X \xi$. As $A_\xi X$ is tangent and $\nabla^\perp_X \xi$ is normal to F^l, from (16) we find

$$\bar R(\xi, \bar\nabla_X \xi)Y = -c\, g\,(A_\xi X, Y)\,\xi$$

for any vector fields X, Y on F^l. Thus, the conditions from Proposition 3.1 mean that

$$\begin{cases} \bar\nabla_X Y - c\, g\,(A_\xi X, Y)\xi \quad \text{is tangent to } \; F^l, \\ \bar\nabla_{\bar\nabla_X Y - c\, g\,(A_\xi X, Y)\xi}\,\xi = \bar\nabla_X \bar\nabla_Y \xi. \end{cases} \tag{17}$$

Multiplying $(17)_1$ by ξ and by normal vector field η orthogonal to ξ, we have

$$\begin{cases} g\,(A_\xi X, Y)(1 - c\,|\xi|^2) = 0, \\ \qquad\qquad g\,(A_\eta X, Y) = 0. \end{cases}$$

If ξ is of constant length $|\xi|^2 = \frac{1}{c}$ $(c > 0)$ then by Theorem 3.2, F^l is totally geodesic in M^n, otherwise F^l is totally geodesic immediately.

So, F^l is totally geodesic and therefore $\bar\nabla_X \xi = \nabla^\perp_X \xi$, $\bar\nabla_X Y = \nabla_X Y$. The condition $(17)_2$ now takes the form

$$\nabla^\perp_{\nabla_X Y}\xi = \nabla^\perp_X \nabla^\perp_Y \xi. \tag{18}$$

Set $Y = \nabla_V Z$, where V and Z are arbitrary vector fields tangent to F^l. Then from (18), we get

$$\nabla^\perp_{\nabla_X \nabla_V Z}\xi = \nabla^\perp_X \nabla^\perp_{\nabla_V Z}\xi.$$

Applying (18) to $\nabla^\perp_{\nabla_V Z}\xi$ in the right-hand side of the above equation, we see that $\nabla^\perp_{\nabla_V Z}\xi = \nabla^\perp_V \nabla^\perp_Z \xi$ and therefore,

$$\nabla^\perp_{\nabla_X \nabla_V Z}\xi = \nabla^\perp_X \nabla^\perp_V \nabla^\perp_Z \xi. \tag{19}$$

Interchanging the roles of X and V, we get

$$\nabla^{\perp}_{\nabla_V \nabla_X Z}\xi = \nabla^{\perp}_V \nabla^{\perp}_X \nabla^{\perp}_Z \xi. \tag{20}$$

Finally, applying again (18) to the bracket $[X, V]$ and Z, we get

$$\nabla^{\perp}_{\nabla_{[X,V]} Z}\xi = \nabla^{\perp}_{[X,V]} \nabla^{\perp}_Z \xi. \tag{21}$$

Combining (19),(20) and (21), we obtain

$$\nabla^{\perp}_{R(X,V)Z}\xi = R^{\perp}(X, V)\nabla^{\perp}_Z \xi$$

where R is the curvature tensor of F^l and $R^{\perp}$ is the normal curvature tensor. Since F^l is totally geodesic and $M^n(c)$ is of constant curvature, $R^{\perp}(X, Y)\eta \equiv 0$ for any normal vector field η and, moreover,

$$R(X, Y)Z = c\,(g\,(Y, Z)X - g\,(X, Z)Y).$$

So, we have

$$c\,\nabla^{\perp}_{g\,(Y,Z)X - g\,(X,Z)Y}\xi = 0.$$

Setting X orthogonal to Y and $Y = Z$ we get $\nabla^{\perp}_X \xi = 0$ for any vector field X on F^l, which completes the necessary part of the proof. The sufficient part is trivial.

∎

The application of Theorem 4.1 to the case of a space of constant curvature shows the difference between our considerations and Walczak's [14]. Let S^n be the unit sphere and S^{n-1} be the unit totally geodesic great sphere in S^n. Denote by D^n an open equatorial zone around S^{n-1} where the unit geodesic vector field orthogonal to S^{n-1} is regularly defined. Then D^n is a Riemannian manifold of constant positive curvature and S^{n-1} is a totally geodesic submanifold in D^n.

Let ξ be a unit (or of constant length) geodesic vector field on $D^n \subset S^n$ which is normal to the totally geodesic great sphere S^{n-1}. Then $\xi(D^n)$ is not totally geodesic in TD^n while the restriction of ξ to S^{n-1} generates the totally geodesic submanifold $\xi(S^{n-1})$ in TD^n.

Indeed, ξ is of constant length and by Walczak's result, $\xi(D^n)$ can be totally geodesic in TD^n only if ξ is a parallel vector field on D^n [14], which is impossible due to positive curvature of D^n. On the other hand, ξ is parallel in the normal bundle of $S^{n-1} \subset D^n$ and we can apply Theorem 4.1 to see that $\xi(S^{n-1})$ is totally geodesic in TD^n.

As concerns flat Riemannian manifolds, Walczak has shown that every totally geodesic vector field on a flat Riemannian manifold is harmonic (cf. [14]) and that, consequently, on a compact flat Riemannian manifold, a vector field is totally geodesic if and only if it is parallel. We shall give a similar result for vector fields along submanifolds.

Theorem 4.2 *Let F^l be a compact oriented submanifold in a flat Riemannian manifold M^n. Let ξ be a vector field on F^l. Then $\xi(F^l)$ is totally geodesic in TM^n if and only if F^l is totally geodesic in M^n and ξ is parallel along F^l.*

Proof. Since M^n is flat, the ξ-connection is the same as the Levi-Civita connection on M^n. So, by Proposition 3.1, $\xi(F^l)$ is totally geodesic if and only if F^l is totally geodesic and

$$\bar{\nabla}_X \bar{\nabla}_Y \xi = \bar{\nabla}_{\bar{\nabla}_X Y} \xi \tag{22}$$

for all vector fields X and Y on F^l.

Suppose now that $\xi(F^l)$ is totally geodesic. Then F^l is totally geodesic and is thus flat. Hence locally we can choose vector fields X_1, X_2,...,X_l tangent to F^l such that $\bar{\nabla}_{X_i} X_j = \nabla_{X_i} X_j = 0$, and $\bar{g}(X_i, X_j) = g(X_i, X_j) = \delta_{ij}$, for all $i, j = 1, ..., l$. Putting $X = Y = X_i$ in the identity (22), we obtain $\bar{\nabla}_{X_i} \bar{\nabla}_{X_i} \xi = 0$. Hence, $\sum_{i=1}^{l} \bar{g}(\bar{\nabla}_{X_i} \bar{\nabla}_{X_i} \xi, \xi) = 0$, i.e.

$$\sum_{i=1}^{l} X_i.\bar{g}(\bar{\nabla}_{X_i} \xi, \xi) = \sum_{i=1}^{l} |\bar{\nabla}_{X_i} \xi|^2. \tag{23}$$

If we consider the function f defined by $f(x) = \frac{1}{2}\bar{g}_x(\xi, \xi)$, for all $x \in F^l$, then we can define a global vector field X_f on F^l by the local formula $X_f = g(\bar{\nabla}_{X_i} \xi, \xi) X_i$. Formula (23) can thus be written locally as $\mathrm{div} X_f = \sum_{i=1}^{l} |\bar{\nabla}_{X_i} \xi|^2$.

Integrating both sides of the last equality and applying Green's theorem, we obtain $\sum_{i=1}^{l} \int_{F^l} |\bar{\nabla}_{X_i} \xi|^2 dv = 0$, and hence $\bar{\nabla}_{X_i} \xi = 0$, for all $i = 1, ..., l$. Therefore ξ is parallel along F^l.

The sufficient part of the theorem is trivial.

■

Remarks.

1. If in Theorem 4.2 the field ξ is a normal vector field along F^l, then $\bar{\nabla}_X \xi$ is also normal for each vector field X on F^l. Indeed, for the X_i's constructed in the proof of the theorem, we have $\bar{g}(\bar{\nabla}_{X_i} \xi, X_j) = X_i.\bar{g}(\xi, X_j) = 0$, and so $\bar{\nabla}_{X_i} \xi$ is normal to F^l. Hence the identity (22) can be written as

$$\bar{\nabla}^{\perp}_X \bar{\nabla}^{\perp}_Y \xi = \bar{\nabla}_{\nabla_X Y} \xi. \tag{24}$$

Also, ξ is parallel if and only if it is parallel in the normal bundle. Hence $\xi(F^l)$ is totally geodesic if and only if F^l is totally geodesic and ξ is parallel in the normal bundle.

2. The condition of compactness is necessary. Indeed, if we consider $\mathbb{R}^n$ with its canonical coordinates $(x_1, x_2, ..., x_n)$ and its canonical Euclidean metric, and the hypersurface $\mathbb{R}^{n-1}$ which is identified with the subspace

given by: $x_n = 0$, then $\mathbb{R}^{n-1}$ is an oriented totally geodesic submanifold of $\mathbb{R}^n$. We have $\bar{\nabla}_{\partial/\partial x_i}\partial/\partial x_j = 0$ for all $i, j = 1, ..., n$. We consider the vector field ξ on $\mathbb{R}^n$ along $\mathbb{R}^{n-1}$ defined by $\xi(x) = x_1\partial/\partial x_n(x)$, where x_1 is the first component of x. Now, to show that $\xi(\mathbb{R}^{n-1})$ is totally geodesic in $T\mathbb{R}^n$, it suffices to check that (22) is verified. In fact,$\bar{\nabla}_{\partial/\partial x_i}\bar{\nabla}_{\partial/\partial x_j}\xi = \bar{\nabla}_{\partial/\partial x_i}\delta_{1j}\partial/\partial x_n = 0$. But $\bar{\nabla}_{\partial/\partial x_1}\xi = \partial/\partial x_n$, and so ξ is not parallel.

5 The case of Lie groups with bi-invariant metrics

Let us consider a connected Lie group G^n equipped with a bi-invariant metric $\bar{g}$, i.e. invariant by both left and right translations. We shall generalize the results of Walczak P. [14] on totally geodesic left invariant vector fields on G^n to left invariant vector fields along Lie subgroups.

Let H^l be a Lie subgroup of G^n. The metric g induced from $\bar{g}$ on H^l is a bi-invariant metric. If we denote by $\bar{\nabla}$ and ∇ the Levi-Civita connections on G^n and H^l respectively, then we have $\bar{\nabla}_X Y = \frac{1}{2}[X, Y]$, for all X,Y of $\mathfrak{g}$, the Lie algebra of G^n, and $\nabla_X Y = \frac{1}{2}[X, Y]$, for all X,Y of $\mathfrak{h}$, the Lie algebra of H^l.

Lemma 5.1 *A connected complete submanifold F^l of G^n containing the identity element e of G^n, such that T_eF^l is a subalgebra of $\mathfrak{g}$, is totally geodesic if and only if F^l is a Lie subgroup H^l of G^n.*

Proof.

If we denote by exp the exponential mapping $\exp : \mathfrak{g} \to G^n$ of the Lie group G^n, and by $\exp_x : T_xG^n \to G^n$ the exponential map at a point x of G^n with respect to the Levi-Civita connection of the metric g, then for all $x \in G^n$, $\exp_x = \exp \circ (L_{x^{-1}})_*$, where L_x is the left translation of G^n by x. Indeed, we show firstly that $\exp_e = \exp$. Let $X \in \mathfrak{g} \equiv T_eG^n$ and $\gamma(t) = \exp tX$. It suffices to check that γ is a geodesic. We have $\dot{\gamma}(t) = (L_{\gamma(t)})_*(\dot{\gamma}(0)) = (L_{\gamma(t)})_*(X)$, and thus $\bar{\nabla}_{\dot{\gamma}(t)}\dot{\gamma}(t) = \bar{\nabla}_{X(\gamma(t))}X(\gamma(t))$, where X denotes also the left invariant vector field on G^n corresponding to X. Hence $\bar{\nabla}_{\dot{\gamma}(t)}\dot{\gamma}(t) = \frac{1}{2}[X, X](\gamma(t)) = 0$, and so $\exp_e = \exp$. Now, our assertion follows from the fact that left translations are isometries.

We consider a Lie subgroup H^l of G^n and $\mathfrak{h} = T_eH^l$ its Lie algebra. If $X \in \mathfrak{h}$, then $\exp_e tX = \exp tX \in H^l$, for all t in a neighborhood of 0, i.e. H^l contains the geodesic starting from e and with initial condition X, and by the left translations, H^l contains all geodesics starting from points of H^l with initial vectors tangent to H^l at these points. Thus H^l is totally geodesic.

Conversely, suppose that F^l is a connected complete submanifold of G^n such that $e \in F^l$ and $T_eF^l =: \mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. Let H^l be the connected subgroup of G^n with Lie algebra $\mathfrak{h}$. H^l is then a connected totally geodesic submanifold of G^n with $T_eH^l = T_eF^l$. Therefore $H^l = F^l$.

■

Proposition 5.1 *A left invariant vector field on G^n along a submanifold F^l generates a totally geodesic submanifold of TG^n if and only if it is parallel along F^l and F^l is totally geodesic.*

Proof. A left invariant vector field on G^n is necessarily of constant length, and we apply Theorem 3.2.

■

Corollary 5.1 *A left invariant vector field ξ on G^n along a Lie subgroup H^l is totally geodesic if and only if it is an element of the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$.*

Proof. By Lemma 5.1, H^l is a totally geodesic submanifold in G^n. Thus, by virtue of Proposition 5.1, ξ is totally geodesic if and only if ξ is parallel along H^l.

Suppose that ξ is totally geodesic. Then $\bar{\nabla}_X\xi = 0$, for all $X \in \mathfrak{h}$; i.e. ξ is in the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$.

Conversely, if ξ is in the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$, then $\bar{\nabla}_X\xi = 0$, for all $X \in \mathfrak{h}$. Let $x \in H^l$ and $z \in T_xH^l$. It suffices to prove that $\bar{\nabla}_z\xi = 0$. But $X := (L_{x^{-1}})_*(z) \in T_eH^l \equiv \mathfrak{h}$, and consequently $\bar{\nabla}_z\xi = (\bar{\nabla}_X\xi)(x) = 0$.

■

Corollary 5.2 *(a) There are no non-zero left invariant totally geodesic vector fields on a semi-simple Lie subgroup of a Lie group with a bi-invariant Riemannian metric.*

(b) Every left invariant vector field along a subgroup of an abelian Lie group with a bi-invariant Riemannian metric generates a totally geodesic submanifold of the tangent bundle.

Theorem 5.1 *Let N^l be a connected complete totally geodesic embedded submanifold of the tangent bundle of a connected Lie group G^n equipped with a bi-invariant Riemannian metric such that $H^l = \pi(N^l)$ is a Lie subgroup of G^n. Suppose that N^l is horizontal at a point z of T_eG^n.*

(a) If $z \in T_eH^l$, then N^l is the image of H^l by a left invariant vector field on H^l which belongs to the center of $\mathfrak{h}$. In particular, if H^l is semi-simple, then H^l is the only connected totally geodesic embedded submanifold of TG^n which is tangent to H^l at e and orthogonal to the fiber at a point of T_eG^n.

(b) If H^l is simple, then N^l is the image of H^l by a left invariant vector field on G^n along H^l which belongs to the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$.

Proof. Using Proposition 2.1, there is a neighborhood U of e in G^n, a neighborhood V of z in TG^n and a vector field Y on M^n along $H^l \cap U$ such that $N^l \cap V = Y(H^l \cap U), Y(e) = z$. We have $T_zN^l = T_z(N^l \cap V) =$

$T_zY(H^l \cap U)$. Then each vector of T_zN^l can be written as $X^h + (\bar{\nabla}_X Y)^v$, for some $X \in \mathfrak{h}$. But T_zN^l is a subset of the horizontal subspace of TTG^n at z, so at e we have $\bar{\nabla}_X Y = 0$ for all $X \in \mathfrak{h}$. On the other hand, since $N^l \cap V = Y(H^l \cap U)$ is totally geodesic, the second assertion of Proposition 3.1 reduces at e to the identity

$$\bar{\nabla}_{X_1} \bar{\nabla}_{X_2} Y = \frac{1}{2} \bar{R}(X_1, X_2)Y, \text{ for all vector fields } X_1, X_2 \text{ on } H^l.$$

Then for all $W \in \mathfrak{g} = T_eG^n$, we have

$$\bar{g}(\bar{\nabla}_{X_1(e)} \bar{\nabla}_{X_2} Y, W) = \frac{1}{2} \bar{g}(\bar{R}(X_1(e), X_2(e))Y(e), W).$$

If we extend W to a vector field X_3 along H^l, which is orthogonal to $\bar{\nabla}_{X_2} Y$ in a neighborhood of e in H^l, then we can write

$$\bar{g}(\bar{\nabla}_{X_1(e)} \bar{\nabla}_{X_2} Y, W) = -\bar{g}(\bar{\nabla}_{X_2(e)} Y, \bar{\nabla}_{X_1(e)} X_3) = 0,$$

and consequently, $\bar{g}(\bar{R}(X_1(e), X_2(e))Y(e), W) = 0$, for all $X_1(e), X_2(e) \in \mathfrak{h} = T_eH^l$ and $W \in \mathfrak{g} = T_eG^n$. Therefore we have

$$R(\cdot, \cdot)Y(e) = 0, \text{ when applied to vectors in } T_eH^l.$$

Let us denote by ξ the left invariant vector field on G^n along H^l such that $Y(e) = \xi(e)$. Then $\bar{R}(\cdot, \cdot)\xi(e) = 0$ when applied to vectors in T_eH^l, and hence

$$\bar{R}(\cdot, \cdot)\, \xi = 0, \text{ when applied to elements of } \mathfrak{h}. \tag{25}$$

Consider now two cases.

(a) If $\xi(e) = z \in T_eH^l$, then $\xi \in \mathfrak{h}$, and we have, by virtue of (25), $\bar{R}(X, \xi)\xi = 0$, for all $X \in \mathfrak{h}$. Thus $|\,[\xi, X]\,|^2 = 4\bar{g}(\bar{R}(\xi, X)X, \xi) = 0$ for all $X \in \mathfrak{h}$. It follows that ξ belongs to the center of $\mathfrak{h}$.

(b) If H^l is simple, then $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}$. But $\bar{\nabla}_{[X_1, X_2]} \xi = \frac{1}{2}[[X_1, X_2], \xi] = -2R(X_1, X_2)\xi = 0$, for all X_1, $X_2 \in \mathfrak{h}$, by virtue of (25). Since $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}$, we deduce easily that $\bar{\nabla}_X \xi = 0$, for all $X \in \mathfrak{h}$, or equivalently $[X, \xi] = 0$, for all $X \in \mathfrak{h}$. It follows that ξ belongs to the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$.

In both cases, ξ belongs to the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$. Hence, by Lemma 5.1, H^l is totally geodesic in G^n, and Proposition 5.1 implies then that $\xi(H^l)$ is a complete totally geodesic submanifold of TG^n. Therefore $\xi(H^l) = N^l$, because $\xi_*(T_eH^l) = T_zN^l$ and N^l and H^l are connected.

■

Corollary 5.3 *Let N^l be a connected complete horizontal totally geodesic submanifold of the tangent bundle of a connected Lie group G^n equipped with a bi-invariant Riemannian metric such that $H^l = \pi(N^l)$ is a simply connected submanifold of G^n containing the identity element. Suppose that*

$\mathfrak{h} := \pi_*(T_z N^l)$ is a Lie subalgebra of $\mathfrak{g}$ for a point z of $T_e G^n \cap N^l$. If $Z \in T_e H^l$ (resp. $\mathfrak{h}$ is simple), then H^l is a Lie subgroup of G^n and N^l is the image of H^l by a left invariant vector field on H^l (resp. on G^n along H^l) which belongs to the center of $\mathfrak{h}$ (resp. centralizer of $\mathfrak{h}$ in $\mathfrak{g}$).

Proof. By Theorem 2.3, H^l is complete and totally geodesic. It follows from Lemma 5.1 that H^l is a Lie subgroup of G^n. Now, our corollary follows from Theorem 5.1.

■

References

[1] Boeckx E., Vanhecke L. Harmonic and minimal radial vector fields, Acta Math. Hungar. 90(2001), 317-331.

[2] Boeckx E., Vanhecke L. Harmonic and minimal vector fields on tangent and unit tangent bundles, Differential Geom. Appl.13 (2000), 77-93.

[3] Dombrowski P. On the geometry of tangent bundle, J. Reine Angew. Math., 210 (1962), N 1-2, 73-88.

[4] Gil-Medrano O., Llinares-Fuster E. Second variation of volume and energy of vector fields. Stability of Hopf vector fields, Math. Ann. 320 (2001), 531-545.

[5] Gil-Medrano O., Llinares-Fuster E. Minimal unit vector fields, Tôhoku Math. J., 54(2002), 71-84.

[6] Gluck H., Ziller W. On the volume of a unit vector field on the three-sphere, Comm. Math. Helv. 61 (1986), 177-192.

[7] González-Dávila J.C., Vanhecke L. Examples of minimal unit vector fields, Ann. Global Anal. Geom. 18 (2000), 385-404.

[8] Kobayashi S., Nomizu K. *Foundations of differential geometry*, Interscience Publ., Vol.1 (1967) and Vol.1,2 (1969).

[9] Kowalski O. Curvature of the induced Riemannian metric on the tangent bundle of a Riemannian manifold, J. Reine Angew. Math. 250 (1971), 124-129.

[10] Liu M.-S. Affine maps of tangent bundles with Sasaki metric, Tensor, N.S., 28 (1974), 34-42.

[11] Rovenskii V. *Foliations on Riemannian Manifolds and Submanifolds.* Birkhäuser, 1997.

[12] Sasaki S. On the differential geometry of tangent bundles of Riemannian manifolds, Tôhoku Math. J. 10 (1958), 338-354.

[13] Sato K. Geodesics on the tangent bundles over space forms, Tensor, 32 (1978), 5-10.

[14] Walczak P. On totally geodesic submanifolds of tangent bundles with Sasaki metric, Bull. Acad. Pol. Sci., ser. Sci. Math., 28 (1980), N 3-4, 161-165.

[15] Walczak P. On the energy of unit vector fields with isolated singularities, Ann. Pol. Math., LXXII.3 (2000), 269-274.

[16] Warner, F.W. *Foundations of differentiable manifolds and Lie groups*, Academic Press, New York, 1971.

[17] Wood C.M. On the energy of a unit vector field, Geom. Dedicata 64 (1997), 319-330.

[18] Yampolsky A. On the mean curvature of a unit vector field, Math. Publ. Debrecen, 60/1-2 (2002), 131 – 155.

[19] Yampolsky A. On the intrinsic geometry of a unit vector field, Comment. Math. Univ. Carolinae 43, 2 (2002), 299-317.

[20] Yampolsky A. Totally geodesic property of the Hopf vector field, Acta Math. Hungarica, Acta Math. Hungar. 101, 1-2 (2003), 73-92.

Mohamed Tahar Kadaoui ABBASSI,
Département des Mathématiques,
Faculté des sciences Dhar El Mahraz,
Université Sidi Mohamed Ben Abdallah,
B.P. 1796, Fes-Atlas,
Fes, Morocco
e.mail: mtk_abbassi@Yahoo.fr

Alexander YAMPOLSKY,
Department of Geometry,
Faculty of Mechanics and Mathematics,
Kharkiv National University,
Svobody Sq. 4,
61077, Kharkiv,
Ukraine.
e-mail: yamp@univer.kharkov.ua