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Stabilization of von Kármán plate in the presence of thermal effects in a subsonic potential flow of gas

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Abstract

We discuss the problem of nonlinear oscillations of a clamped plate in the presence of thermal effects in a subsonic gas flow. The dynamics of the plate is described by von Kármán system in the presence of thermal effects, in which rotational inertia is accounted for. To describe influence of the gas flow we apply the linearized theory of potential flows. Our main result states that each weak solution of the problem considered tends to the set of the stationary points of the problem.

Keywords: Stabilization; Subsonic gas flow; Nonlinear thermoelastic plate

1. Introduction

In the present paper we study stabilization of a coupled system of partial differential equations, consisting of an undamped wave equation, defined on the half-space $\mathbb{R}^3_+$, and a nonlinear thermoelastic plate equation, defined on a two-dimensional bounded smooth domain $\Omega \subset \{x = (x_1, x_2, x_3)\colon\ x_3 = 0\}$.

Nonlinear oscillations of a clamped plate in the presence of thermal effects are described by the following equations:

$$P_\alpha u_{tt} + \Delta^2 u - [u, v + \eta] + \Delta\theta = p(x', t), \quad x' \in \Omega, \tag{1}$$

$$\theta_t - \Delta\theta - \Delta u_t = 0, \tag{2}$$

$$u|_{\partial\Omega} = \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = \theta|_{\partial\Omega} = 0, \tag{3}$$

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$$u(0) = u_0, \quad u_t(0) = u_1, \quad \theta(0) = \theta_0, \tag{4}$$

where $v = v(u)$ is Airy's stress function defined by

$$\Delta^2 v = -[u, u], \qquad v|_{\partial\Omega} = \left.\frac{\partial v}{\partial n}\right|_{\partial\Omega} = 0. \tag{5}$$

Ω is a smooth bounded domain in $\mathbb{R}^2$, n is the outward unit normal vector to $\partial\Omega$, Δ is the Laplace operator, $P_\alpha = (1 - \alpha\Delta)$. The von Kármán brackets are defined by $[u, v] = \partial^2_{x_1} u \cdot \partial^2_{x_2} v + \partial^2_{x_2} u \cdot \partial^2_{x_1} v - 2\partial^2_{x_1 x_2} u \cdot \partial^2_{x_1 x_2} v$. The function $u = u(x', t)$ describes transverse displacement of the plate, the function $\theta = \theta(x', t)$ denotes the temperature; $\eta(x') \in H^4(\Omega)$ is a given function determined by mechanical loads. The parameter $\alpha > 0$ accounts for rotational inertia.

In this paper we consider interaction of the plate with the linearized flow of gas. If the gas moves over the plate in the direction of x_1-axis, the aerodynamic pressure on the plate is given by the formula (see, e.g., [1])

$$p(x', t) = p_0(x') + \nu(\partial_t + U\partial_{x_1}) r_\Omega \gamma[\phi], \quad x' \in \Omega, \tag{6}$$

where $p_0 \in L^2(\Omega)$. Here and below $\gamma[\phi]$ denotes a trace of ϕ onto the plane $\{x\colon x_3 = 0\}$, r_Ω is the operator of restriction from $\mathbb{R}^2$ onto Ω. The parameter $\nu > 0$ is proportional to the intensity of interaction between the gas and the plate, $U > 0$ ($U \neq 1$) is the velocity of the unperturbed flow and $\phi(x, t)$ is the potential of the velocity of the perturbed flow. It satisfies the following equations:

$$(\partial_t + U\partial_{x_1})^2 \phi = \Delta\phi, \quad x = (x', x_3) \in \mathbb{R}^3_+ = \{x\colon x_3 > 0\}, \tag{7}$$

$$\left.\frac{\partial\phi}{\partial x_3}\right|_{x_3=0} = \begin{cases} (\partial_t + U\partial_{x_1}) u(x', t), & x' \in \Omega, \\ 0, & x' \notin \Omega, \end{cases} \tag{8}$$

$$\phi(0) = \phi_0, \qquad \phi_t(0) = \phi_1. \tag{9}$$

In recent years problems related to the stability of thermoelastic plates without transversal loads (i.e., with $p(x', t) = 0$) were studied by many authors. In particular, G. Avalos and I. Lasiecka in [2,3] showed exponential stability of linear thermoelastic systems with various boundary conditions. Uniform decay of solutions to the nonlinear thermoelastic systems of the type (1)–(5) with various boundary conditions and $\eta = 0$, $p(x', t) = 0$ was established in [4]. For a survey of other thermoelastic models we refer to [2,3].

Only recently several authors have addressed problems of stability of interactive models consisting of wave and plate equations coupled at the interface. In particular, hybrid PDE systems that arise from structural acoustic models were studied by I. Lasiecka and C. Lebiedzik in [5–7]. In this case the undamped wave equation of type (7) with $U = 0$ is defined on a bounded three-dimensional domain $\mathcal{O}$ and the thermoelastic plate equation is defined on an interface Ω, the flat part of $\partial\mathcal{O}$. The domain $\mathcal{O}$ represents an acoustic chamber and Ω represents a vibrating wall. The coupling between the acoustic and the structural medium takes place on Ω. Asymptotic behaviour of solutions of such systems was studied in [5–7]. The first paper is devoted to a linear model with free boundary conditions, the others deal with nonlinear models. It was shown in [5–7] that such systems are uniformly stable if some additional boundary dissipation is placed on a suitable portion of $\partial\mathcal{O} \setminus \Omega$. Moreover, in the case of a nonlinear thermoelastic system with free boundary conditions (see [7]) some additional mechanical damping acting on $\partial\Omega$ is assumed, but there is no need for such damping in the case of clamped or hinged boundary conditions (see [6]).

The rigorous mathematical study of the PDE system that describes nonlinear oscillations of an isothermal plate in a subsonic gas flow first appears in [8]. This system corresponds to the problem of aeroelasticity. Further it was addressed in [9,10]. In [10] another approach to the problem was suggested, that enables to treat both subsonic and supersonic flows simultaneously. In this work it is also shown that, provided initial data have compact supports, the problem can be reduces to a retarded PDE. The technique of considering retarded PDEs is another approach to the problem of aeroelasticity, with the aid of which existence of a global attractor for the plate can be achieved, for both subsonic and supersonic flows (see, e.g., [11,12]), but no information can be obtained about the behaviour of the gas flow. The result concerning stabilization of entire structure was presented by I.D. Chueshov in [13]. The problem of type (1)–(9) with additional damping term $\epsilon P_\alpha \partial_t u$ and without thermal effects was considered there and for generic η, p_0 it was proved that for any weak solution of the problem there exists stationary point $(\bar{u}, 0, \bar{\phi}, 0)$ such that

$$\lim_{t\to+\infty}\left\{\left\|u(t)-\bar{u}\right\|_{2,\Omega}^2+\left\|u_t(t)\right\|_{1,\Omega}^2+\left\|\nabla\big(\phi(t)-\bar{\phi}\big)\right\|_{B_R^+}^2+\left\|\phi_t(t)\right\|_{B_R^+}^2\right\}=0 \qquad (10)$$

for any $R>0$, where $B_R^+=\{x\in\mathbb{R}_+^3\colon |x|<R\}$, $\|\cdot\|_{\mathcal{O}}$ is the norm in $L^2(\mathcal{O})$ and $\|\cdot\|_{i,\Omega}$ is the Sobolev norm of order i on Ω.

The main novelty of the present paper is that no mechanical damping (interior or boundary) is included in the model. The stabilization obtained is of the same character as in [13]. In contrast to [13], structural damping (that is described by $\epsilon P_\alpha \partial_t u$) is replaced with “less strong” thermal damping. The main result of the present paper can also be regarded as a complement to the results of [5–7] for the case of unbounded domain $\mathcal{O}=\mathbb{R}_+^3$. In our case the stabilization obtained is not uniform, but there is no need for damping acting on $\partial\mathcal{O}$.

This work relies on some results and ideas from [2,9,10]. To achieve our goal we decompose the solution $W(t)=(u,u_t,\theta,\phi,\phi_t)(t)$ to (1)–(9) into the sum $W(t)=W^1(t)+W^2(t)$ such that $W^1(t)\to 0$, $t\to+\infty$, and $W^2(t_k)$ is compact for every initial data and $t_k\to+\infty$. We use the following decomposition: $W^1(t)=(u^1,u_t^1,\theta^1,\phi^*,\phi_t^*)(t)$, $W^2(t)=(u^2,u_t^2,\theta^2,\phi^{**},\phi_t^{**})(t)$, where components of W^1, W^2 solve the problems

$$\text{(E1):}\quad \begin{cases} P_\alpha u_{tt}^1+\Delta^2 u^1+\Delta\theta^1=0,\\ \theta_t^1-\Delta\theta^1-\Delta u_t^1=0,\\ u^1(0)=u_0,\quad u_t^1(0)=u_1,\quad \theta^1(0)=\theta_0,\end{cases}$$

$$\text{(E2):}\quad \begin{cases} P_\alpha u_{tt}^2+\Delta^2 u^2+\Delta\theta^2=p_0+[u,v+\eta]+\nu(\partial_t+U\partial_{x_1})r_\Omega\gamma[\phi^*+\phi^{**}],\\ \theta_t^2-\Delta\theta^2-\Delta u_t^2=0,\\ u^2(0)=u_t^2(0)=\theta^2(0)=0,\end{cases}$$

where v solves (5). The functions u^j,θ^j, $j=1,2$ satisfy boundary conditions (3);

$$\text{(E3):}\quad \begin{cases} (\partial_t+U\partial_{x_1})^2\phi^*=\Delta\phi^*,\\ \frac{\partial\phi^*}{\partial x_3}\Big|_{x_3=0}=\begin{cases}(\partial_t+U\partial_{x_1})u^1, & x'\in\Omega,\\ 0, & x'\notin\Omega,\end{cases}\\ \phi^*(0)=\phi_0,\qquad \phi_t^*(0)=\phi_1,\end{cases}$$

$$\text{(E4):}\quad \begin{cases} (\partial_t + U\partial_{x_1})^2\phi^{**} = \Delta\phi^{**}, \\ \frac{\partial\phi^{**}}{\partial x_3}\Big|_{x_3=0} = \begin{cases} (\partial_t + U\partial_{x_1})u^2, & x' \in \Omega, \\ 0, & x' \notin \Omega, \end{cases} \\ \phi^{**}(0) = \phi^{**}_t(0) = 0. \end{cases}$$

This decomposition enables us to prove our main result on stabilization which states that every weak solution to (1)–(9) tends to the set of stationary points of this problem. That is, in addition to convergence of type (10), the temperature $\theta(t)$ tends to zero in $L^2(\Omega)$-norm.

The paper is organized as follows. In Section 2 we introduce notations we need and state our main results. In Section 3 we establish results concerning the potential ϕ that satisfies (7)–(9) with a given function $u(t)$. In Section 4 we prove theorem of existence, uniqueness and continuity of solutions to the problem (1)–(9) and in Section 5 we prove our stabilization theorem.

2. Notations and main results

Before formulating our main results we introduce the following notations. In addition to the classical notations and the norms used for the Sobolev spaces we define an equivalent norm and inner product in $H_0^1(\Omega)$: $(u,v)_{1,\alpha} = (u,v)_\Omega + \alpha(\nabla u, \nabla v)_\Omega$, $\|u\|^2_{1,\alpha} = \|u\|^2_\Omega + \alpha\|\nabla u\|^2_\Omega$.

To describe behaviour of the plate, we will use the space $\mathcal{X} = H_0^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$ with the norm $\|(u,u_t,\theta)\|^2_{\mathcal{X}} = \|\Delta u\|^2_\Omega + \|u_t\|^2_{1,\alpha} + \|\theta\|^2_\Omega$, where $u(\cdot,t) \in H_0^2(\Omega)$, $u_t(\cdot,t) \in H_0^1(\Omega)$, $\theta(\cdot,t) \in L^2(\Omega)$ for almost all t.

We define a homogeneous Sobolev space $\mathcal{H}^1(\mathbb{R}^3)$ (see, e.g., [14]) as a closure of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|u\|_{\mathcal{H}^1(\mathbb{R}^3)} = \|\nabla u\|_{\mathbb{R}^3}$. For $\mathcal{H}^1(\mathbb{R}^3_+)$ defined as a space of restrictions of functions from $\mathcal{H}^1(\mathbb{R}^3)$ onto $\mathbb{R}^3_+$ we will use the equivalent norm $\|\nabla\phi\|_{\mathbb{R}^3_+}$.

We use two spaces to describe behaviour of the gas flow. The space $\mathcal{Y} = \mathcal{H}^1(\mathbb{R}^3_+) \times L^2(\mathbb{R}^3_+)$ with the norm $\|(\phi_0,\phi_1)\|^2_{\mathcal{Y}} = \|\nabla\phi_0\|^2_{\mathbb{R}^3_+} + \|\phi_1\|^2_{\mathbb{R}^3_+}$, where $\phi(\cdot,t) \in \mathcal{H}^1(\mathbb{R}^3_+)$ and $\phi_t(\cdot,t) \in L^2(\mathbb{R}^3_+)$.

For $(\phi_0,\phi_1) \in \mathcal{Y}$ we define the local energy by

$$\mathcal{E}_R(\phi_0,\phi_1) = \int_{B_R^+} \left|\nabla\phi_0(x)\right|^2 dx + \int_{B_R^+} \left|\phi_1(x)\right|^2 dx, \tag{11}$$

where $B_R^+ = \{x\colon |x| < R,\ x_3 > 0\}$. We define the space $\tilde{\mathcal{Y}}$ as the set $\mathcal{Y}$ with the following convergence, which will be referred to as a local energy convergence:

$$\left(\phi_0^n, \phi_1^n\right) \xrightarrow{\tilde{\mathcal{Y}}} (\phi_0,\phi_1) \quad \text{if and only if} \quad \mathcal{E}_R\left(\phi_0^n - \phi_0, \phi_1^n - \phi_1\right) \to 0 \quad \forall R > 0.$$

We also introduce the energy functional

$$\mathcal{E}^{(1)}(t) = E_{\mathrm{pl}}\big(u(t), u_t(t), \theta(t)\big) + \nu E^{(1)}_{\mathrm{fl}}\big(\phi(t), \phi_t(t)\big) + E_{\mathrm{int}}\big(u(t), \phi(t)\big),$$

where $E_{\mathrm{pl}}(u(t), u_t(t), \theta(t))$ is the energy of the thermoelastic plate given by

$$E_{\mathrm{pl}}\big(u(t),u_t(t),\theta(t)\big)=\frac{1}{2}\bigg(\big\|u_t(t)\big\|_{1,\alpha}^2+\big\|\Delta u(t)\big\|_{\Omega}^2+\frac{1}{2}\big\|\Delta v\big(u(t)\big)\big\|_{\Omega}^2 + \big\|\theta(t)\big\|_{\Omega}^2-\big(\big[u(t),u(t)\big],\eta\big)_{\Omega}-2\big(p_0,u(t)\big)_{\Omega}\bigg),$$

the symbol $E_{\mathrm{fl}}^{(1)}(\phi(t),\phi_t(t))$ denotes the energy of the gas flow and is defined by

$$E_{\mathrm{fl}}^{(1)}\big(\phi(t),\phi_t(t)\big)=\frac{1}{2}\big(\big\|\phi_t(t)\big\|_{\mathbb{R}_+^3}^2+\big\|\nabla\phi(t)\big\|_{\mathbb{R}_+^3}^2-U^2\big\|\partial_{x_1}\phi(t)\big\|_{\mathbb{R}_+^3}^2\big)$$

and the energy $E_{\mathrm{int}}(u(t),\phi(t))$ of interaction of the plate and the flow is given by

$$E_{\mathrm{int}}\big(u(t),\phi(t)\big)=\nu U\big(r_\Omega\gamma[\phi](t),\partial_{x_1}u(t)\big)_\Omega.$$

Our energy functional should be compared to the energy functional $\mathcal{E}^\alpha(t)$ used in [13]. Note that in the absence of the mechanical damping we need to incorporate the thermal energy term $\|\theta(t)\|_\Omega^2$.

Our first result is the following theorem.

Theorem 1 (Existence and uniqueness of weak solution). *For every* $W_0=(u_0,u_1,\theta_0,\phi_0,\phi_1)\in\mathcal{X}\times\mathcal{Y}$ *and* $T>0$ *there exists precisely one weak solution* $W(t)=(u(t),u_t(t),\theta(t),\phi(t),\phi_t(t))$ *to* (1)–(9).

(i) *The solution* $W(t)$ *possesses the properties*

$$u(t)\in C\big(0,T;H_0^2(\Omega)\big),\qquad u_t(t)\in C\big(0,T;H_0^1(\Omega)\big),$$
$$\theta(t)\in C\big(0,T;L^2(\Omega)\big)\cap L^2\big(0,T;H_0^1(\Omega)\big),$$
$$\phi(t)\in C\big(0,T;\mathcal{H}^1\big(\mathbb{R}_+^3\big)\big),\qquad \phi_t(t)\in C\big(0,T;L^2\big(\mathbb{R}_+^3\big)\big).$$

(ii) *The following energy relation holds*:

$$\mathcal{E}^{(1)}(t)=\mathcal{E}^{(1)}(0)-\int_0^t\big\|\nabla\theta(\tau)\big\|_\Omega^2\,d\tau. \tag{12}$$

(iii) *The problem* (1)–(9) *generates the evolution operator* S_t*, defined by the formula* $S_tW_0=W(t)$*, where* $W(t)$ *is the weak solution to* (1)–(9) *with the initial value* $W_0\in\mathcal{X}\times\mathcal{Y}$*. The operator* S_t *is continuous in* $\mathcal{X}\times\mathcal{Y}$ *and in* $\mathcal{X}\times\tilde{\mathcal{Y}}$ *in the following sense. Let* $W^j(t)$*,* $j=1,2$*, be two weak solutions to* (1)–(9) *with the initial data* W_0^j*, respectively, such that* $\|W_0^j\|_{\mathcal{X}\times\mathcal{Y}}^2\leqslant Q^2$*. Then the following estimates are valid for all* $t<T$:

$$\big\|W^1(t)-W^2(t)\big\|_{\mathcal{X}\times\mathcal{Y}}^2\leqslant C(T,Q)\big\|W_0^1-W_0^2\big\|_{\mathcal{X}\times\mathcal{Y}}^2,$$
$$\big\|\big(u^1(t),u_t^1(t),\theta^1(t)\big)-\big(u^2(t),u_t^2(t),\theta^2(t)\big)\big\|_{\mathcal{X}}^2 +\mathcal{E}_R\big(\phi^1(t)-\phi^2(t),\phi_t^1(t)-\phi_t^2(t)\big)$$
$$\leqslant C(T,Q,R)\big(\big\|\big(u_0^1,u_1^1,\theta_0^1\big)-\big(u_0^2,u_1^2,\theta_0^2\big)\big\|_{\mathcal{X}}^2+\mathcal{E}_{R_1}\big(\phi_0^1-\phi_0^2,\phi_1^1-\phi_1^2\big)\big),$$

where R_1 *depends only on* R,T,U,Ω.

Remark 2. It is easy to see, that stationary solutions to the problem (1)–(9) have the form $(u(x'), 0, 0, \phi(x), 0)$, where $u(x')$ and $\phi(x)$ solve the problem

$$\begin{aligned}
&\Delta^2 u - [u, v + \eta] = p_0 + \nu U \partial_{x_1} r_\Omega \gamma[\phi], \quad x' \in \Omega,\\
&\Delta\phi - U^2 \partial_{x_1}^2 \phi = 0, \quad x = (x', x_3) \in \mathbb{R}^3_+,\\
&u|_{\partial\Omega} = \frac{\partial u}{\partial n}\bigg|_{\partial\Omega} = 0,\\
&\frac{\partial \phi}{\partial x_3}\bigg|_{x_3=0} = \begin{cases} U\partial_{x_1} u(x'), & x' \in \Omega,\\ 0, & x' \notin \Omega,\end{cases}
\end{aligned} \tag{13}$$

where v is defined by (5). Solutions to the problem (13) were studied in [9]. Using the Sard–Smale theorem (see, e.g., [15]) it is possible to prove that for generic p_0, η the set of the stationary solutions to (1)–(9) is finite.

Our main result is the stabilization theorem for subsonic flows.

Theorem 3 (Stabilization). *Let $0 < U < 1$. Then for every $W_0 = (u_0, u_1, \theta_0, \phi_0, \phi_1) \in H_0^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \times \mathcal{H}^1(\mathbb{R}^3_+) \times L^2(\mathbb{R}^3_+)$ the solution $S_t W_0$ tends to the set $\mathcal{M}$ of the stationary solutions to the problem* (1)–(9) *in the local energy topology, i.e.,*

$$\inf_{(u^*,\phi^*)\in\mathcal{M}} \Bigg(\big\|\Delta\big(u(t) - u^*\big)\big\|_\Omega^2 + \alpha\big\|\nabla u_t(t)\big\|_\Omega^2 + \big\|u_t(t)\big\|_\Omega^2 + \big\|\theta(t)\big\|_\Omega^2 + \int_B \Big(\big|\nabla\big(\phi(x,t) - \phi^*(x)\big)\big|^2 + \big|\phi_t(x,t)\big|^2\Big)\, dx \Bigg) \to 0, \quad t \to +\infty,$$

for every bounded set $B \subset \mathbb{R}^3_+$. Here $\mathcal{M}$ is a set of solutions to (13).

Corollary 4. *If the set $\mathcal{M}$ is finite, then for every $W_0 = (u_0, u_1, \theta_0, \phi_0, \phi_1) \in H_0^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \times \mathcal{H}^1(\mathbb{R}^3_+) \times L^2(\mathbb{R}^3_+)$ there exists a unique point $(u^*, \phi^*) \in \mathcal{M}$, such that $S_t W_0 \to (u^*, 0, 0, \phi^*, 0)$ in the local energy topology, i.e.,*

$$\lim_{t\to+\infty} \Bigg(\big\|\Delta\big(u(t) - u^*\big)\big\|_\Omega^2 + \alpha\big\|\nabla u_t(t)\big\|_\Omega^2 + \big\|u_t(t)\big\|_\Omega^2 + \big\|\theta(t)\big\|_\Omega^2 + \int_B \Big(\big|\nabla\big(\phi(x,t) - \phi^*(x)\big)\big|^2 + \big|\phi_t(x,t)\big|^2\Big)\, dx \Bigg) = 0$$

for every bounded set $B \subset \mathbb{R}^3_+$.

Remark 5. Theorem 1 is valid for all $U \neq 1$, but stabilization is established only for subsonic flows. The proof of stabilization is heavily dependent on the boundedness of solutions to the problem (1)–(9), but in the case of supersonic flow boundedness is not guaranteed, as the energy $\mathcal{E}^{(1)}(\cdot)$ is not bounded below. We also note that the main mechanisms responsible for the stabilization of the entire structure are thermal effects on the plate and local energy decay for the wave equation. Whether the structure stabilizes without thermal effects, is still an open question.

3. Preliminary results

The main goal of this section is description of properties of the problem (7)–(9) with a given $u(t)$. More precisely, we consider the potential gas flow equation in $\mathbb{R}^3_+$,

$$(\partial_t + U\partial_{x_1})^2\phi = \Delta\phi, \quad x \in \mathbb{R}^3_+, \tag{14}$$

$$\phi(0) = \phi_0, \quad \phi_t(0) = \phi_1, \quad \left.\frac{\partial\phi}{\partial x_3}\right|_{x_3=0} = h(x', t), \quad x' \in \mathbb{R}^2, \tag{15}$$

where $\phi_0 \in \mathcal{H}^1(\mathbb{R}^3_+)$, $\phi_1 \in L^2(\mathbb{R}^3_+)$, $h \in L^p_{\mathrm{loc}}(\mathbb{R}_+; H^1(\mathbb{R}^2))$ for some $p > 2$. The results of this section are close to the ones of [10].

In the case $h \equiv 0$ this equation has precisely one solution for every $(\phi_0, \phi_1) \in \mathcal{Y}$ (the proof is similar to the one for the case $U = 0$, see, e.g., [16]), for which the energy conservation law is valid:

$$E^{(2)}\big(\phi(t), \phi_t(t)\big) = E^{(2)}(\phi_0, \phi_1) = \frac{1}{2}\big(\|\phi_1 + U\partial_{x_1}\phi_0\|^2_{\mathbb{R}^3_+} + \|\nabla\phi_0\|^2_{\mathbb{R}^3_+}\big).$$

The problem (14)–(15) with $h \equiv 0$ generates the evolution semigroup $G_t : \tilde{\mathcal{Y}} \to \tilde{\mathcal{Y}}$ in the following way: $G_t(\phi_0, \phi_1) = (\phi(t), \phi_t(t))$, where $(\phi(t), \phi_t(t))$ is the solution to (14)–(15) with the initial values (ϕ_0, ϕ_1) and the boundary conditions $h \equiv 0$ at the moment t. For the proof of continuity of S_t on $\mathcal{X} \times \tilde{\mathcal{Y}}$ we need continuity of G_t on $\tilde{\mathcal{Y}}$.

Lemma 6. *The evolution semigroup G_t generated by the problem* (14)–(15) *with $h \equiv 0$ is continuous on $\tilde{\mathcal{Y}}$.*

The proof is based on the following representation of the solution to (14)–(15) (see [10]):

$$\phi(x, t) = \frac{1}{4\pi}\int_S dS_\xi \left[\bar{\phi}_0 - t_0(\xi - Ue_1, \nabla\bar{\phi}_0) + t_0\bar{\phi}_1\right]\big(x - (\xi + Ue_1)t\big), \tag{16}$$

where S is a unit sphere in $\mathbb{R}^3$, $e_1 = (1, 0, 0)$ and $\bar{\phi}_j$ are even extensions of ϕ_j on $\mathbb{R}^3$, $j = 0, 1$. It is easy to see that, for $x \in B^+_R$ and $t \leqslant T$, values of $\phi(x, t)$ depend only on values of (ϕ_0, ϕ_1) in $B^+_{R_1}$, where $R_1 = R + (1 + U)T$.

The following result on the decay of the local energy is well known for the case $U = 0$. To study problem (E3) we need similar result for the case $U > 0$ $(U \neq 1)$. The following lemma can be proved by the methods presented in [16], for instance.

Lemma 7. *For every solution $(\phi(t), \phi_t(t))$ to the problem* (14)–(15) *with the initial data $(\phi_0, \phi_1) \in \mathcal{Y}$ and $h \equiv 0$ the local energy $\mathcal{E}_R(\phi(t), \phi_t(t)) \to 0$ when $t \to +\infty$ for all $R > 0$.*

As it was shown in [10], the solution to (14)–(15) with $\phi_0 = \phi_1 = 0$ is given by

$$\phi(x,t) = -\frac{\chi(t-x_3)}{2\pi}\int_{x_3}^{t} ds \int_0^{2\pi} d\theta\, h\big(x_1 - k_1(\theta,s,x_3), x_2 - k_2(\theta,s,x_3), t-s\big), \quad (17)$$

where

$$k_1(\theta,s,x_3) = Us + \sqrt{s^2 - x_3^2}\sin\theta, \qquad k_2(\theta,s,x_3) = \sqrt{s^2 - x_3^2}\cos\theta.$$

If h has compact support in $\bar{\Omega}$, it vanishes for $s \geqslant t^*(U,\Omega,x_3)$. An argument similar to the one given in [10] shows that we can assume $t^*(U,\Omega,x_3) = \max(\bar{t}(U,\Omega), \omega(U)x_3)$, where $\omega = 1/\sqrt{1-(U+1)^2/4}$ for $U < 1$ and $\omega = 1$ for $U > 1$. We now have the following result which will be widely used in Sections 4 and 5.

Lemma 8. *For the solution to the problem* (14)–(15) *with* $\phi_0 = \phi_1 = 0$ *and provided* h *has compact support in* $\bar{\Omega}$ *the following estimates are valid for every* $R > 0$ *and* $t \geqslant t^* = t^*(U,\Omega,R)$:

(i) *if* $h(x',\tau) \in C(t-t^*, t; H_0^1(\Omega))$, *then*

$$\mathcal{E}_R\big(\phi(t), \phi_t(t)\big) \leqslant C(R)\Big\{\Big(\max_{\tau\in[0,t^*]}\big\|\nabla h(t-\tau)\big\|_\Omega\Big)^2 + \Big(\max_{\tau\in[0,t^*]}\big\|h(t-\tau)\big\|_\Omega\Big)^2\Big\}; \quad (18)$$

(ii) *if* $h(x',\tau) \in H^s(t-t^*, t; H_0^1(\Omega))$, $0 < s < 1/2$, *then*

$$\big\|\nabla\phi(t)\big\|_{B_R^+}^2 + \big\|\phi_t(t)\big\|_{B_R^+}^2 \leqslant C(R)\|h\|^2_{H^s(t-t^*,t;H_0^1(\Omega))}; \quad (19)$$

(iii) *if* $h(x',\tau) \in H^s(t-t^*, t; H_0^2(\Omega))$, $h_\tau(x',\tau) \in H^s(t-t^*, t; H_0^1(\Omega))$, $0 < s < 1/2$, *then*

$$\big\|\nabla\phi(t)\big\|_{1,B_R^+}^2 + \big\|\phi_t(t)\big\|_{1,B_R^+}^2 \leqslant C(R)\big\{\|h\|^2_{H^s(t-t^*,t;H_0^2(\Omega))} + \|h_\tau\|^2_{H^s(t-t^*,t;H_0^1(\Omega))}\big\}; \quad (20)$$

(iv) *if* $h(x',\tau) \in C(t-t^{**}, t; H_0^1(\Omega))$, *where* $t^{**} = \inf\{t\colon (x_1 - (U+\sin\theta)s, x_2 - s\cos\theta) \notin \Omega$ *for all* $(x_1,x_2) \in \Omega$, $\theta \in [0,2\pi]$, $s > t\}$, *then*

$$\big\|(\partial_t + U\partial_{x_1})\gamma[\phi](t)\big\|_\Omega \leqslant \big\|h(t)\big\|_\Omega + \int_0^{\min\{t,t^{**}\}} \big\|\nabla h(t-\tau)\big\|_\Omega\, d\tau. \quad (21)$$

Proof. In the proof of each inequality we fix R and assume $t \geqslant t^* = t^*(U,\Omega,R)$, so we can replace the upper integration limit in (17) with t^*.

Inequality (i). Similarly as in [10] we obtain

$$\big\|\partial_{x_j}\phi(\cdot,t)\big\|_{B_R^+}^2 \leqslant C(R)\Big(\max_{\tau\in[0,t^*]}\big\|\nabla h(t-\tau)\big\|_\Omega\Big)^2 \quad (22)$$

for $j = 1, 2$. Further we denote

$$h^*(x,t,s,\theta) = h\bigl(x_1 - Us - \sqrt{s^2 - x_3^2}\sin\theta, x_2 - \sqrt{s^2 - x_3^2}\cos\theta, t - s\bigr).$$

Using (17) and the formula

$$\partial_t h^*(x,t,s,\theta) = -\frac{d}{ds}h^*(x,t,s,\theta) - U\partial_{x_1}h^*(x,t,s,\theta) - \frac{s}{\sqrt{s^2 - x_3^2}}[M_\theta h^*](x,t,s,\theta), \tag{23}$$

where $M_\theta = \sin\theta\,\partial_{x_1} + \cos\theta\,\partial_{x_2}$, we obtain

$$\partial_{x_3}\phi(x,t) = h(x_1 - Ux_3, x_2, t - x_3) - \frac{1}{2\pi}\int_{x_3}^{t^*}\frac{x_3\,ds}{\sqrt{s^2 - x_3^2}}\int_0^{2\pi} d\theta\,[M_\theta h^*](x,t,s,\theta) \tag{24}$$

and

$$\partial_t\phi(x,t) = -h(x_1 - Ux_3, x_2, t - x_3) + \frac{1}{2\pi}\Biggl\{U\int_{x_3}^{t^*} ds\int_0^{2\pi} d\theta\,\partial_{x_1}h^*(x,t,s,\theta) + \int_{x_3}^{t^*}\frac{s\,ds}{\sqrt{s^2 - x_3^2}}\int_0^{2\pi} d\theta\,[M_\theta h^*](x,t,s,\theta)\Biggr\}.$$

It is easy to see, that for $\partial_{x_3}\phi(t)$ and $\partial_t\phi(t)$ estimates similar to (22) are valid. Hence (i) is proved.

Inequality (ii). In what follows we assume that $h(x', \tau) \in C^\infty(t - t^*, t; H_0^1(\Omega))$. To prove (ii), we need another estimate for norms of ϕ_t and $\partial_{x_3}\phi$. Using Hölder's inequality with $p > 2$, $1/p + 1/q = 1$, similarly as in [10] we obtain

$$\bigl\|\partial_{x_3}\phi(\cdot, x_3, t)\bigr\|_{\mathbb{R}^2} + \bigl\|\phi_t(\cdot, x_3, t)\bigr\|_{\mathbb{R}^2} \leqslant C(R,p)Q(x_3)\Biggl(\int_{x_3}^{t^*}\bigl\|h(t-\tau)\bigr\|_{1,\Omega}^p\Biggr)^{1/p} + C(R)\bigl\|h(t - x_3)\bigr\|_\Omega,$$

where $Q(x_3)$ is a square-integrable on $(0, R)$ function. Squaring this inequality and integrating with respect to x_3 along $(0, R)$, we get

$$\bigl\|\nabla\phi(\cdot,t)\bigr\|_{B_R^+}^2 + \bigl\|\phi_t(\cdot,t)\bigr\|_{B_R^+}^2 \leqslant C(R,p)\bigl(\|h\|^2_{L^p(t-t^*,t;H_0^1(\Omega))} + \|h\|^2_{L^2(t-t^*,t;H_0^1(\Omega))}\bigr). \tag{25}$$

Due to the Sobolev embedding theorem $H^s(t-t^*,t;\,H_0^1(\Omega)) \subset L^p(t-t^*,t;\,H_0^1(\Omega))$ for $p \leqslant 2/(1-2s)$ (see, e.g., [14]). As $C^\infty(t-t^*,t;\,H_0^1(\Omega))$ is dense in $H^s(t-t^*,t;\,H_0^1(\Omega))$, (ii) is proved.

Inequality (iii). In what follows we assume $h(x',\tau) \in C^\infty(t-t^*,t;\,H_0^2(\Omega))$. Differentiating (24) with respect to x_j, $j=1,2$, and repeating previous argumentation, we obtain that

$$\left\|\partial_{x_i x_j}\phi(\cdot,t)\right\|^2_{B_R^+} \leqslant C(R,p)\|h\|^2_{H^s(t-t^*,t;\,H_0^2(\Omega))}$$

for $i=1,2,3$ and $j=1,2$.

After a simple calculation we get that

$$\begin{aligned}
\partial^2_{x_3}\phi(x,t) &= -U\partial_{x_1}h(x_1-Ux_3,x_2,t-x_3) - \partial_t h(x_1-Ux_3,x_2,t-x_3)\\
&\quad + \frac{1}{2\pi}\Biggl\{-\int_0^{t^*-x_3}\frac{\tau+x_3}{\sqrt{\tau}(\tau+2x_3)^{3/2}}\,d\tau\int_0^{2\pi}d\theta\,[M_\theta\bar{h}](x,t,\tau,\theta)\\
&\quad + \int_0^{t^*-x_3}\frac{x_3}{\sqrt{\tau}(\tau+2x_3)^{1/2}}\,d\tau\int_0^{2\pi}d\theta\,\bigl[(U\partial_{x_1}+\partial_t)M_\theta\bar{h}\bigr](x,t,\tau,\theta)\\
&\quad + \int_0^{t^*-x_3}\frac{x_3}{(\tau+2x_3)}\,d\tau\int_0^{2\pi}d\theta\,\bigl[M_\theta^2\bar{h}\bigr](x,t,\tau,\theta)\Biggr\},
\end{aligned}\tag{26}$$

where $\bar{h}(x,t,\tau,\theta) = h^*(x,t,\tau+x_3,\theta)$. Then we apply to (26) Hölder's inequality with $p>2$, $1/q+1/p=1$ and integrate the inequality obtained along $(0,R)$ with respect to x_3. Using Sobolev's embedding theorem, we get the estimate

$$\left\|\partial^2_{x_3}\phi(\cdot,t)\right\|^2_{B_R^+} \leqslant C(R,p)\bigl\{\|h\|^2_{H^s(t-t^*,t;\,H_0^2(\Omega))} + \|h_\tau\|^2_{H^s(t-t^*,t;\,H_0^1(\Omega))}\bigr\}.$$

It is easy to see that $\phi_t(x,t)$ is a solution to (14)–(15) with $\phi_0=\phi_1=0$ and the boundary conditions $h_\tau(x',\tau)$. As the part (ii) of the lemma is proved, using (19) we get

$$\left\|\nabla\phi_t(t)\right\|^2_{B_R^+} \leqslant \|h_\tau\|^2_{H^s(t-T,t;\,H_0^1(\Omega))}.$$

We finish the proof of (iii) by applying the density argument.

Proof of (iv) can be found in [10]. □

In the proof of Theorem 1 we will use the following property of the operator $(\partial_t + U\partial_{x_1})r_\Omega\gamma[\phi]$, which occurs in the aerodynamic pressure $p(x',t)$.

Lemma 9. *Let $\Omega \subset \{x'\colon |x'|<R\} = B_R$ and*

$$\phi \in \mathcal{W} = \left\{\phi(t) \in C\bigl(0,T;\mathcal{H}^1\bigl(\mathbb{R}^3_+\bigr)\bigr),\ \frac{d}{dt}\phi(t) \in C\bigl(0,T;L^2\bigl(\mathbb{R}^3_+\bigr)\bigr)\right\}.$$

Then $(\partial_t + U\partial_{x_1})r_\Omega\gamma[\phi](t) \in C(0,T;\,H^{-1/2-\delta}(\Omega))$ for every $\delta>0$, $T>0$ and

$$\left\|(\partial_t + U\partial_{x_1})r_\Omega\gamma[\phi](t)\right\|^2_{-1/2-\delta,\Omega} \leqslant \mathcal{E}_R\bigl(\phi(t),\partial_t\phi(t)\bigr).\tag{27}$$

Proof. First we prove, that $\partial_t r_\Omega \gamma[\phi](t)$ exists and belongs to $L^\infty(0,T;H^{-1/2-\delta}(\Omega))$ for all $\delta > 0$. Let $w \in (H^{-1/2-\delta}(\Omega))^* = H_0^{1/2+\delta}(\Omega)$. Then the extension by zero $l_0 : H_0^{1/2+\delta}(\Omega) \to H^{1/2+\delta}(\mathbb{R}^2)$ and the lifting operator $l : H^{1/2+\delta}(\mathbb{R}^2) \to H^{1+\delta}(\mathbb{R}^3_+)$ are continuous operators. If $\alpha(x) \in C_0^\infty(\mathbb{R}^3)$ and $\alpha(x',0) = 1$ for $x' \in \Omega$, $\alpha(x) = 0$ for $x \notin B_R^+$, we obtain the operator $L = \alpha(x)\cdot l \circ l_0$, such that $r_\Omega \gamma[Lw] = w$, $\|Lw\|_{1+\delta,\mathbb{R}^3_+} \leqslant C\|w\|_{1/2+\delta,\Omega}$ and $\operatorname{supp} Lw \subset B_R^+$. Using integration by parts, we obtain the equality

$$(\partial_{x_3}\phi, Lw)_{B_R^+} + (\phi, \partial_{x_3}Lw)_{B_R^+} = -\big(\gamma[\phi], w\big)_\Omega \tag{28}$$

for some $w \in H_0^{1/2+\delta}(\Omega)$ and $\phi \in \mathcal{H}^1(\mathbb{R}^3_+)$. Let the sequences $t_m, s_n \to 0$ as $m,n \to +\infty$. Using (28), we get

$$\begin{aligned}
&\left|\left(\frac{\gamma[\phi](t+t_m)-\gamma[\phi](t)}{t_m} - \frac{\gamma[\phi](t+s_n)-\gamma[\phi](t)}{s_n}, w\right)_\Omega\right| \\
&\quad \leqslant C\left\|\frac{\phi(t+t_m)-\phi(t)}{t_m} - \frac{\phi(t+s_n)-\phi(t)}{s_n}\right\|_{-\delta,B_R^+}\|w\|_{1/2+\delta,\Omega} \to 0, \\
&\quad n,m \to +\infty.
\end{aligned}$$

This implies that $r_\Omega\gamma[\phi](t)$ is differentiable with respect to t in $H^{-1/2-\delta}(\Omega)$. Similarly we obtain the inequality

$$\big\|\partial_t r_\Omega \gamma[\phi](t)\big\|_{-1/2-\delta,\Omega} \leqslant C\big\|\phi_t(t)\big\|_{B_R^+}. \tag{29}$$

To estimate $\partial_{x_1} r_\Omega \gamma[\phi](t)$, we consider the function $\bar\phi(t)$ defined by

$$\begin{cases} \nabla\bar\phi(t) = \nabla\phi(t), & x \in B_R^+, \\ \nabla\bar\phi(t) = 0, & x \notin B_R^+, \end{cases}$$

such that $\bar\phi(t) = 0$, $x \notin B_R^+$. Thus, $\bar\phi(t)$ has compact support in $\bar B_R^+$ and $\phi(t) - \bar\phi(t) = C$. Then

$$\big\|\partial_{x_1} r_\Omega\gamma[\phi](t)\big\|_{-1/2,\Omega} = \big\|\partial_{x_1} r_\Omega\gamma[\bar\phi](t)\big\|_{-1/2,\Omega} \leqslant C\big\|\nabla\phi(t)\big\|_{B_R^+}. \tag{30}$$

Combining (29) and (30) we obtain the inequality (27). Continuity with respect to t easily follows from (29) and (30). □

For the proof of stabilization we need the following criterion of compactness in $\tilde{\mathcal{Y}}$.

Lemma 10. *Let $\{(\phi_0^m, \phi_1^m)\}_{m=1}^\infty$ be a bounded sequence in $\mathcal{Y}$ and let the constant $\beta > 0$. If for every $R > 0$ there exists $N(R) \in \mathbb{N}$ and $C(R) > 0$ such that*

$$\big\|\nabla\phi_0^m\big\|^2_{\beta,B_R^+} + \big\|\phi_1^m\big\|^2_{\beta,B_R^+} \leqslant C(R) \quad \textit{for } m \geqslant N(R), \tag{31}$$

then $\{(\phi_0^m, \phi_1^m)\}_{m=1}^\infty$ is compact in $\tilde{\mathcal{Y}}$.

Proof. As the sequence is bounded we can extract a subsequence that converges to $(\bar\phi_0, \bar\phi_1)$ weakly in $\mathcal{Y}$. Let $r_{B_R^+}$ be the operator of restriction from $L^2(\mathbb{R}^3_+)$ to $L^2(B_R^+)$.

The sequences $\{r_{B_R^+}\nabla\phi_0^m\}_{m\geqslant N(R)}$ and $\{r_{B_R^+}\phi_1^m\}_{m\geqslant N(R)}$ are compact in $L^2(B_R^+)$, therefore $(r_{B_R^+}\nabla\phi_0^m, r_{B_R^+}\phi_1^m)) \to (r_{B_R^+}\nabla\bar\phi_0, r_{B_R^+}\bar\phi_1)$ by norm in $L^2(B_R^+)\times L^2(B_R^+)$ for every fixed $R>0$. Thus we have that $(\phi_0^m,\phi_1^m)\to(\bar\phi_0,\bar\phi_1)$ in $\tilde{\mathcal{Y}}$. □

The next lemma will be used in our proof of stabilization.

Lemma 11. *Let $f(t)\geqslant 0$, $f(t)\in AC[0,+\infty)$, $f'(t)\leqslant C$ or $f'(t)\geqslant -C$ almost everywhere and $\int_0^\infty f(t)\,dt<\infty$. Then $f(t)\to 0$ when $t\to+\infty$.*

Proof. We consider the case $f'(t)\leqslant C$ only. Let the statement be false, i.e., assume there exist $a>0$ and the sequence $x_n\to+\infty$ such that $f(x_n)\geqslant a$. We introduce the sets $B_n=\{x\in[x_n-1,x_n+1]:\ f(x)>a/2\}$. Their measures $\mu_n=\mu(B_n)\to 0$, as far as $\int_0^\infty f(t)\,dt<\infty$. We fix $\epsilon>0$ and N such that $\mu_n<\epsilon$ for $n>N$ and choose $y_n<x_n$ such that $x_n-y_n<2\epsilon$ and $f(y_n)\leqslant a/2$ for $n>N$. Since $f(t)$ is absolutely continuous,

$$f(x_n)=f(y_n)+\int_{y_n}^{x_n} f'(t)\,dt\leqslant f(y_n)+2\epsilon C\leqslant\frac{a}{2}+2\epsilon C.$$

Choosing $\epsilon<a/8C$ we get that $f(x_n)<a$. This contradicts the assumption $f(x_n)\geqslant a$. The lemma is proved. □

4. Existence, uniqueness and continuity

This section is devoted to a proof of Theorem 1. To prove existence, uniqueness and continuity of solutions to problem (1)–(9) we use the same method as in [10]. It uses the regularized variant of Galerkin's method for finding $u(x',t)$.

Let $\{e_k\}$ be eigenvectors of the positive self-adjoint operator A in $H_0^1(\Omega)$ with the domain $H^3(\Omega)\cap H_0^2(\Omega)$ defined by $(Au,v)_{1,\alpha}=(\Delta u,\Delta v)$. Let $\{\bar e_k\}$ be eigenvectors of $-\Delta$ with the Dirichlet boundary conditions, that is a positive self-adjoint operator in $L^2(\Omega)$. In what follows P_N and $\bar P_N$ are orthogonal projections onto $\mathrm{Lin}(\{e_k\}_{k=1}^N)$ and $\mathrm{Lin}(\{\bar e_k\}_{k=1}^N)$, respectively, J is the operator from $H^{-1}(\Omega)$ to $H_0^1(\Omega)$ such that $(Ju,v)_{1,\alpha}=(u,v)$.

Similarly as in [10] we define an approximate solution of order N to the problem (1)–(9) as a triple of the functions $\{u_N(t),\theta_N(t),\phi_N(t)\}$,

$$u_N(t)=\sum_{k=1}^N g_k(t)e_k\in\mathcal{L}_T^N\equiv C^1\bigl(0,T;P_NH_0^1(\Omega)\bigr),$$

$$\theta_N(t)=\sum_{k=1}^N \bar g_k(t)\bar e_k\in\bar{\mathcal{L}}_T^N\equiv C^1\bigl(0,T;\bar P_NH_0^1(\Omega)\bigr),$$

which satisfy the following relations in $H_0^1(\Omega)$:

$$u_N(t) = u_N(0) + \int_0^t \partial_t u_N(\tau)\,d\tau, \tag{32}$$

$$\partial_t u_N(t) = \partial_t u_N(0) + \int_0^t \bigl\{-Au_N(\tau) + P_N J\bigl(\bigl[u_N(\tau), v\bigl(u_N(\tau)\bigr) + \eta\bigr] - \Delta\theta(\tau) + p_0\bigr)\bigr\}\,d\tau + \nu P_N J r_\Omega\bigl(\gamma\bigl[\phi_N(t)\bigr] - \gamma[\phi_0]\bigr) + \nu U \int_0^t P_N J r_\Omega \partial_{x_1}\gamma\bigl[\phi_N(\tau)\bigr]\,d\tau, \tag{33}$$

$$\theta_N(t) = \theta_N(0) + \int_0^t \bigl\{\Delta\theta_N(\tau) + \bar{P}_N \Delta\partial_t u_N(\tau)\bigr\}\,d\tau \tag{34}$$

for all $0 \leqslant t < T$, where $u_N(0) = P_N u_0$, $\partial_t u_N(0) = P_N u_1$, $\theta_N(0) = \bar{P}_N\theta_0$; $v(u_N)$ is defined in terms of u_N by (5); ϕ_N is a solution to (14)–(15) with the initial data $(\phi_0, \phi_1) \in \mathcal{Y}$ and the boundary conditions

$$\left.\frac{\partial\phi_N}{\partial x_3}\right|_{x_3=0} = h_N(x', t) = \begin{cases} (\partial_t + U\chi_N(x')\partial_{x_1})u_N(x', t), & x' \in \Omega, \\ 0, & x' \notin \Omega, \end{cases} \tag{35}$$

where $\chi_N(x') \in C_0^\infty(\Omega)$ is chosen so that $0 \leqslant \chi_N(x') \leqslant 1$, $\chi_N(x') \to 1$ almost everywhere and $|\nabla\chi_N(x')|\operatorname{dist}(x', \partial\Omega) \leqslant C$ for $x' \in \Omega$ with the constant C independent on N.

Theorem 12 (Existence and uniqueness of approximate solutions). *For every* $(u_0, u_1, \theta_0) \in \mathcal{X}$, $(\phi_0, \phi_1) \in \mathcal{Y}$ *there exists precisely one approximate solution of order* N *to the problem* (1)–(9). *If* $(u_N(t), \theta_N(t), \phi_N(t))$ *is an approximate solutions such that* $\|(u_N(0), \partial_t u_N(0), \theta_N(0), \phi_0, \phi_1)\|^2_{\mathcal{X}\times\mathcal{Y}} \leqslant Q^2$, *then for* $t < T$,

$$\bigl\|\bigl(u_N(t), \partial_t u_N(t), \theta_N(t)\bigr)\bigr\|^2_{\mathcal{X}} + \bigl\|\bigl(\phi_N(t), \partial_t\phi_N(t)\bigr)\bigr\|^2_{\mathcal{Y}} \leqslant C(T, Q). \tag{36}$$

Approximate solutions depend continuously on initial data in $\mathcal{X}\times\mathcal{Y}$. *The following energy relation is valid*:

$$\mathcal{E}_N^{(1)}(t) = \mathcal{E}_N^{(1)}(0) - \int_0^t \bigl\|\nabla\theta_N(\tau)\bigr\|^2_\Omega\,d\tau - \nu U \int_0^t d\tau \int_\Omega (1 - \chi_N)\partial_{x_1}\partial_t u_N(\tau)\gamma[\phi_N](\tau)\,dx', \tag{37}$$

where

$$\mathcal{E}_N^{(1)}(t) = E_{\mathrm{pl}}\bigl(u_N(t), \partial_t u_N(t), \theta_N(t)\bigr) + E_{\mathrm{fl}}\bigl(\phi_N(t), \partial_t\phi_N(t)\bigr) + \nu U\bigl(\gamma[\phi_N](t), \chi_N\partial_{x_1}u_N(t)\bigr)_\Omega.$$

The proof of this theorem is similar to the one in [10] and therefore it is omitted. To obtain existence, uniqueness and continuity of weak solutions to the problem (1)–(9) we pass to the limit in the same way as in [10]. Letting $N \to +\infty$ in (37), similarly as in [10] we obtain (12).

Theorem 13 (Continuity of approximate solution in $\mathcal{X} \times \tilde{\mathcal{Y}}$). *Let* $(u_{j,N}(t), \theta_{j,N}(t), \phi_{j,N}(t))$, $j = 1, 2$, *be two approximate solutions such that* $\|(u_{j,N}(0), \partial_t u_{j,N}(0), \theta_{j,N}(0), \phi_{j,0}, \phi_{j,1})\|^2_{\mathcal{X}\times\mathcal{Y}} \leqslant Q^2$. *Then for all* $t < T$ *and* $R > 0$,

$$\begin{aligned}
&\bigl\|\partial_t\bigl(u_{1,N}(t) - u_{2,N}(t)\bigr)\bigr\|^2_{1,\Omega} + \bigl\|\Delta\bigl(u_{1,N}(t) - u_{2,N}(t)\bigr)\bigr\|^2_{\Omega} + \bigl\|\theta_{1,N}(t) - \theta_{2,N}(t)\bigr\|^2_{\Omega} \\
&\quad\leqslant C(T,Q)\bigl\{\bigl\|\partial_t u_{1,N}(0) - \partial_t u_{2,N}(0)\bigr\|^2_{1,\Omega} + \bigl\|\Delta\bigl(u_{1,N}(0) - u_{2,N}(0)\bigr)\bigr\|^2_{\Omega} \\
&\qquad + \bigl\|\theta_{1,N}(0) - \theta_{2,N}(0)\bigr\|^2_{\Omega} + \mathcal{E}_{R_1}(\phi_{1,0} - \phi_{2,0}, \phi_{1,1} - \phi_{2,1})\bigr\},
\end{aligned} \tag{38}$$

$$\begin{aligned}
&\mathcal{E}_R\bigl(\phi_{1,N}(t) - \phi_{2,N}(t), \partial_t\phi_{1,N}(t) - \partial_t\phi_{2,N}(t)\bigr) \\
&\quad\leqslant C(T,R,Q)\bigl\{\bigl\|\partial_t u_{1,N}(0) - \partial_t u_{2,N}(0)\bigr\|^2_{1,\Omega} + \bigl\|\Delta\bigl(u_{1,N}(0) - u_{2,N}(0)\bigr)\bigr\|^2_{\Omega} \\
&\qquad + \bigl\|\theta_{1,N}(0) - \theta_{2,N}(0)\bigr\|^2_{\Omega} + \mathcal{E}_{R_1}\bigl(\phi_{1,0} - \phi_{2,0}, \phi_{1,1} - \phi_{2,1}\bigr)\bigr\},
\end{aligned} \tag{39}$$

where $C(T,R,Q)$ *and* $C(T,Q)$ *do not depend on* N *and* R_1 *depends only on* R, U, T, Ω.

Proof. We denote $w_N = u_{1,N} - u_{2,N}$, $\zeta_N = \theta_{1,N} - \theta_{2,N}$, $\varphi_N = \phi_{1,N} - \phi_{2,N}$. The functions w_N, ζ_N satisfy the relations

$$\begin{aligned}
\partial_t w_N(t) = \partial_t w_N(0) + \int_0^t &\bigl\{-Aw_N(\tau) + P_N J\bigl(\bigl[u_{1,N}(\tau), v\bigl(u_{1,N}(\tau)\bigr) + \eta\bigr] \\
&- \bigl[u_{2,N}(\tau), v\bigl(u_{2,N}(\tau)\bigr) + \eta\bigr] - \Delta\zeta_N(\tau)\bigr)\bigr\}\,d\tau \\
&+ \nu\int_0^t P_N J r_\Omega(\partial_t + U\partial_{x_1})\gamma[\varphi_N](\tau)\,d\tau,
\end{aligned} \tag{40}$$

$$\zeta_N(t) = \zeta_N(0) + \int_0^t \bigl\{\Delta\zeta_N(\tau) + \bar{P}_N\Delta\partial_t w_N(\tau)\bigr\}\,d\tau. \tag{41}$$

Taking in (40) the scalar product with $\partial_t w_N$ in $H^1_0(\Omega)$ and in (41) with ζ_N in $L^2(\Omega)$, we get

$$\begin{aligned}
&\bigl\|\partial_t w_N(t)\bigr\|^2_{1,\alpha} + \bigl\|\zeta_N(t)\bigr\|^2_{\Omega} + \frac{1}{2}\bigl\|\Delta w_N(t)\bigr\|^2_{\Omega} \\
&\quad= \bigl(\partial_t w_N(0), \partial_t w_N(t)\bigr)_{1,\alpha} + \frac{1}{2}\bigl\|\Delta w_N(0)\bigr\|^2_{\Omega} \\
&\qquad+ \int_0^t \bigl(\bigl[u_{1,N}(\tau), v\bigl(u_{1,N}(\tau)\bigr) + \eta\bigr] - \bigl[u_{2,N}(\tau), v\bigl(u_{2,N}(\tau)\bigr) + \eta\bigr], \partial_t w_N(\tau)\bigr)_{\Omega}\,d\tau
\end{aligned}$$

$$+\big(\zeta_N(0),\zeta_N(t)\big)_\Omega-\int_0^t\big\|\nabla\zeta_N(\tau)\big\|_\Omega^2\,d\tau$$
$$+\nu\int_0^t\big(r_\Omega(\partial_t+U\partial_{x_1})\gamma[\varphi_N](\tau),\partial_t w_N(\tau)\big)_\Omega\,d\tau. \tag{42}$$

For the components of this expression the following estimates are valid:

$$\big|\big(\partial_t w_N(0),\partial_t w_N(t)\big)_{1,\alpha}\big|\leqslant\delta\big\|\partial_t w_N(t)\big\|_{1,\alpha}^2+C_\delta\big\|\partial_t w_N(0)\big\|_{1,\alpha}^2, \tag{43}$$
$$\big|\big(\zeta_N(0),\zeta_N(t)\big)_\Omega\big|\leqslant\delta\big\|\zeta_N(t)\big\|_\Omega^2+C_\delta\big\|\zeta_N(0)\big\|_\Omega^2. \tag{44}$$

Using Lemma 2.2 from [10] and the estimate (36), we obtain

$$\begin{aligned}&\big|\big(\big[u_{1,N}(\tau),v\big(u_{1,N}(\tau)\big)+\eta\big]-\big[u_{2,N}(\tau),v\big(u_{2,N}(\tau)\big)+\eta\big],\partial_t w_N(\tau)\big)_\Omega\big|\\&\quad\leqslant C(Q,T)\big(\big\|\Delta w_N(\tau)\big\|_\Omega^2+\big\|\partial_t w_N(\tau)\big\|_{1,\alpha}^2\big).\end{aligned} \tag{45}$$

To estimate the last term in (42), we represent φ_N as $\varphi_N^*+\varphi_N^{**}$, where φ_N^* is a solution to (14)–(15) with $h\equiv 0$ and the initial data $\varphi_N^*(0)=\phi_{1,0}-\phi_{2,0}$, $\partial_t\varphi_N^*(0)=\phi_{1,1}-\phi_{2,1}$ and φ_N^{**} is a solution of (14)–(15) with zero initial values and the boundary conditions (35), where u_N is replaced with w_N. Due to Lemma 9 and the energy conservation law (3) we have

$$\begin{aligned}&\big|\big(r_\Omega(\partial_t+U\partial_{x_1})\gamma\big[\varphi_N^*\big](\tau),\partial_t w_N(\tau)\big)_\Omega\big|\\&\quad\leqslant C\big(\mathcal{E}_{R_1}(\phi_{1,0}-\phi_{2,0},\phi_{1,1}-\phi_{2,1})+\big\|\partial_t w_N(\tau)\big\|_\Omega^2\big).\end{aligned} \tag{46}$$

Due to Lemma 8 and Theorem 12 we have the following estimate for the term including ϕ_N^{**}:

$$\begin{aligned}&\int_0^t\big|\big(r_\Omega(\partial_t+U\partial_{x_1})\gamma\big[\varphi_N^{**}\big](\tau),\partial_t w_N(\tau)\big)_\Omega\big|\,d\tau\\&\quad\leqslant C(T,Q)\int_0^t\big(\big\|\partial_t w_N(\tau)\big\|_\Omega^2+\big\|\chi_N(x')\partial_{x_1}w_N(\tau)\big\|_\Omega^2\big)\,d\tau\\&\qquad+C(T,Q)\int_0^t d\tau\int_0^{\min\{\tau,t^{**}\}}ds\big(\big\|\nabla\partial_t w_N(\tau-s)\big\|_\Omega^2+\big\|\nabla(\chi_N\partial_{x_1}w_N)(\tau-s)\big\|_\Omega^2\big).\end{aligned}$$

Since for $v\in H_0^1(\Omega)$ the estimate $\|\chi_N v\|_{1,\Omega}\leqslant\|v\|_{1,\Omega}$ is valid (see Theorem 11.8 in [19]), after a simple calculation we obtain

$$\int_0^t\big|\big(r_\Omega(\partial_t+U\partial_{x_1})\gamma\big[\varphi_N^{**}\big](\tau),\partial_t w_N(\tau)\big)_\Omega\big|\,d\tau$$

$$\leqslant C(T, Q)\int_0^t \left(\left\|\Delta w_N(\tau)\right\|_\Omega^2 + \left\|\partial_t w_N(\tau)\right\|_{1,\Omega}^2\right) d\tau. \tag{47}$$

Applying the estimates (43)–(47) to (42) and using Gronwall's lemma, we obtain (38). Taking into account Lemmas 6, 8 and (38), we have proved (39). □

This theorem and the properties of weak convergence imply that solutions to (1)–(9) depend continuously on initial data in $\mathcal{X} \times \tilde{\mathcal{Y}}$. Thus, the problem (1)–(9) generates the continuous evolution operator S_t on $\mathcal{X} \times \tilde{\mathcal{Y}}$ described in (iii) of Theorem 1. The proof of Theorem 1 is now complete.

5. Stabilization in the case of subsonic flow

In this section we prove Theorem 3. As we consider only subsonic flows ($0 < U < 1$), the energy $\mathcal{E}^{(1)}(\cdot)$ is bounded below by $C_1(E_0(\cdot) + \|\cdot\|_{\mathcal{Y}}^2) - C_2$, where $E_0(u_0, u_1, \theta_0) = 1/2(\|u_1\|_{1,\alpha}^2 + \|\Delta u_0\|_\Omega^2 + 1/2\|\Delta v(u_0)\|_\Omega^2 + \|\theta_0\|_\Omega^2)$ (see Lemma 3.2 from [17]). Thus, the energy of a solution to (1)–(9) is bounded by the initial data energy

$$\begin{aligned} &E_0\big(u(t), u_t(t), \theta(t)\big) + \left\|\big(\phi(t), \phi_t(t)\big)\right\|_{\mathcal{Y}}^2 \\ &\quad \leqslant C_1\big(E_0(u_0, u_1, \theta_0) + \left\|(\phi_0, \phi_1)\right\|_{\mathcal{Y}}^2\big) + C_2 \end{aligned} \tag{48}$$

and the energy equality (12) implies that

$$\int_0^{+\infty} \left\|\nabla\theta(t)\right\|^2 dt < +\infty. \tag{49}$$

Now we study problems (E1)–(E4) in detail.

Exponential stability of solutions to problem (E1) was shown in [2]. In particular we have that there exist $\delta > 0$, $M_\delta \geqslant 1$ such that $\|(u^1, u_t^1, \theta^1)(t)\|_{\mathcal{X}} \leqslant M_\delta e^{-\delta t}\|(u_0, u_1, \theta_0)\|_{\mathcal{X}}$ for every $(u_0, u_1, \theta_0) \in \mathcal{X}$. To study problem (E2) we need a result on exponential stability in stronger norms.

Lemma 14. *Let* $\mathcal{X}^\beta = H_0^2(\Omega) \cap H^{2+\beta}(\Omega) \times H_0^{1+\beta}(\Omega) \times H_0^{2\beta}(\Omega)$, $0 \leqslant \beta \leqslant 1$ (*note, that* $\mathcal{X}^0 = \mathcal{X}$), *and let* $(u(t), u_t(t), \theta(t))$ *be a solution to* (E1) *with the initial data* $(u_0, u_1, \theta_0) \in \mathcal{X}^\beta$. *Then the following estimate is valid*:

$$\left\|\big(u(t), u_t(t), \theta(t)\big)\right\|_{\mathcal{X}^\beta}^2 \leqslant C M_\delta e^{-\delta t}\left\|(u_0, u_1, \theta_0)\right\|_{\mathcal{X}^\beta}^2. \tag{50}$$

Proof. The idea presented in [18] is used here. We define an approximate solution to the problem (E1) as the functions $u^m(t) = \sum_{k=1}^m f_k(t)e_k$, $\theta^m(t) = \sum_{k=1}^m g_k(t)\bar{e}_k$ (for notations see Section 4) satisfying the relations

$$\big(P_\alpha u_{tt}^m, e_k\big)_\Omega + (\Delta u^m, \Delta e_k)_\Omega - (\nabla\theta^m, \nabla e_k)_\Omega = 0, \quad k = 1, \ldots, m, \tag{51}$$

$$\big(\theta_t^m, \bar{e}_k\big)_\Omega + (\nabla\theta^m, \nabla\bar{e}_k)_\Omega + \big(\nabla u_t^m, \nabla\bar{e}_k\big)_\Omega = 0 \tag{52}$$

with the initial values $u^m(0) = P_m u_0$, $u_t^m(0) = P_m u_1$, $\theta^m(0) = \bar{P}_m \theta_0$. Obviously, $f_k(t)$, $g_k(t)$ are infinitely differentiable. Differentiating (51)–(52) with respect to t and denoting $w^m = u_t^m$, $\zeta^m = \theta_t^m$ we obtain that $(w^m(t), w_t^m(t), \zeta^m(t))$ satisfy system (51)–(52) with the initial values $w^m(0) = u_t^m(0) = P_m u_1$, $w_t^m(0) = u_{tt}^m(0)$, $\zeta^m(0) = \theta_t^m(0)$. Let $(u_0, u_1, \theta_0) \in \mathcal{X}^1$. Then

$$u_{tt}^m(0) = -AP_m u_0 - P_m J\Delta\theta_0 \to -Au_0 - J\Delta\theta_0 = u_{tt}(0) \quad \text{in } H_0^1(\Omega),$$
$$\theta_t^m(0) = \Delta\bar{P}_m\theta_0 + \Delta\bar{P}_m u_1 \to \Delta\theta_0 + \Delta u_1 = \theta_t(0) \quad \text{in } L^2(\Omega).$$

Similarly to [18] we obtain

$$\left\|u^m(t)\right\|_{3,\Omega}^2 + \left\|\Delta u_t^m(t)\right\|_{\Omega}^2 + \left\|\Delta\theta^m(t)\right\|_{\Omega}^2 \leqslant CM_\delta e^{-\delta t}\left\|(u_0, u_1, \theta_0)\right\|_{\mathcal{X}^1}^2.$$

Hence, we can extract the subsequence $(u^m(t), u_t^m(t), \theta^m(t)) \to (u(t), u_t(t), \theta(t))$ $*$-weakly in $L^\infty(0, T; \mathcal{X}^1)$, where $(u(t), u_t(t), \theta(t))$ is a solution to (E1) with initial data $(u_0, u_1, \theta_0) \in \mathcal{X}^1$. Thus, the problem (E1) generates the linear evolution operator $S_t^1 \in \mathcal{L}(\mathcal{X}, \mathcal{X}) \cap \mathcal{L}(\mathcal{X}^1, \mathcal{X}^1)$ such that $\|S_t^1\|_{\mathcal{X}} \leqslant M_\delta e^{-\delta t}$, $\|S_t^1\|_{\mathcal{X}^1} \leqslant CM_\delta e^{-\delta t}$. Due to the interpolation Theorem 5.1 from [19], $S_t^1 \in \mathcal{L}([\mathcal{X}, \mathcal{X}^1]_\beta, [\mathcal{X}, \mathcal{X}^1]_\beta) = \mathcal{L}(\mathcal{X}^\beta, \mathcal{X}^\beta)$ and $\|S_t^1\|_{\mathcal{X}^\beta} \leqslant CM_\delta e^{-\delta t}$, $0 \leqslant \beta \leqslant 1$, for some $\delta > 0$. Inequality (50) is proved. □

For problem (E2) the following result is true.

Lemma 15. *The trajectory* $(u^2(t), u_t^2(t), \theta^2(t))$ *is compact and Lipschitz in* $\mathcal{X}^\beta$ *for* $0 \leqslant \beta < 1/2$.

Proof. Obviously, every solution to (E2) can be written by means of Duhamel's principle, i.e.,

$$\begin{aligned}\left(u^2(t), u_t^2(t), \theta^2(t)\right) = &\int_0^t S_{t-\tau}^1\left(0, P_\gamma^{-1}\left(\left[u(\tau), v\left(u(\tau)\right) + \eta\right] + p_0\right), 0\right) d\tau \\ &+ \int_0^t S_{t-\tau}^1\left(0, P_\gamma^{-1}\left(\nu r_\Omega(\partial_t + U\partial_{x_1})\gamma[\phi](\tau)\right), 0\right) d\tau, \end{aligned} \tag{53}$$

where S_t^1 is the evolution operator generated by (E1). Using Lemma 2.1 from [10], we obtain the estimate for von Kármán brackets,

$$\left\|\left[u(t), v\left(u(t)\right) + \eta\right]\right\|_{-\epsilon,\Omega}^2 \leqslant C\left(E_0(u_0, u_1, \theta_0) + \left\|(\phi_0, \phi_1)\right\|_{\mathcal{Y}}^2\right), \quad \epsilon > 0.$$

Since P_α^{-1} is a bounded linear operator from $H^s(\Omega)$ to $H^{s+2}(\Omega) \cap H_0^1(\Omega)$ for $s \geqslant -1$, using (53) and Lemma 9, we obtain

$$\begin{aligned}&\left\|\left(u^2(t), u_t^2(t), \theta^2(t)\right)\right\|_{\mathcal{X}^\beta} \\ &\quad \leqslant \int_0^t M_\delta e^{-\delta(t-\tau)}\left\|\left(\nu(\partial_t + U\partial_{x_1})\gamma[\phi]\right)(\tau)\right.\end{aligned}$$

$$+ \big[u(\tau), v\big(u(\tau)\big) + \eta\big] + p_0\big\|_{-1+\beta,\Omega}\, d\tau$$
$$\leqslant C\big(E_0(u_0, u_1, \theta_0) + \big\|(\phi_0, \phi_1)\big\|^2_{\mathcal{Y}}\big)^{1/2}, \quad 0 \leqslant \beta < 1/2. \tag{54}$$

Hence, the trajectory $(u^2(t), u^2_t(t), \theta^2(t))$ is compact in $\mathcal{X}^\beta$. Similarly we obtain that it is also Lipschitz in $\mathcal{X}^\beta$ and

$$\big\|\big(u^2(t), u^2_t(t), \theta^2(t)\big)\big\|^2_{C^\mu(0,T;\mathcal{X}^\beta)} \leqslant C\big(E_0(u_0, u_1, \theta_0) + \big\|(\phi_0, \phi_1)\big\|^2_{\mathcal{Y}}\big), \tag{55}$$

where $C^\mu(0, T; \mathcal{X}^\beta)$ is a space of μ-Hölder continuous $\mathcal{X}^\beta$-valued functions. □

Due to Lemma 7 and inequality (18) we have the following estimate for the solutions to problem (E3): for every $R > 0$,

$$\mathcal{E}_R\big(\phi^*(t), \phi^*_t(t)\big) \leqslant C(R)\Big(f(t) + \max_{\tau > t-t^*}\big\|u^1_t(\tau)\big\|^2_{1,\Omega} + \max_{\tau > t-t^*}\big\|\Delta u^1(\tau)\big\|^2_{\Omega}\Big),$$

where $f(t) \to 0$, $t \to +\infty$ and $t^* = t^*(U, \Omega, R)$ (see Section 3). Taking into account exponential decay of the solutions to the problem (E1) we obtain that $(\phi^*(t), \phi^*_t(t)) \to 0$ in $\tilde{\mathcal{Y}}$ as $t \to +\infty$.

It is left to show that any sequence of the form $(\phi^{**}(t_k), \phi^{**}_t(t_k))$, $t_k \to +\infty$, is compact. To prove that such sequence satisfies conditions of Lemma 10 we use the estimates (19) and (20), so we need to interpolate functional spaces used there. Applying Theorem 13.1 from [19, Chapter 1] about interpolation of intersections and the standard techniques presented in [20] we obtain that

$$\big[H^s\big((a,b); H^2_0(\Omega)\big) \cap H^{s+1}\big((a,b); H^1_0(\Omega)\big), H^s\big((a,b); H^1_0(\Omega)\big)\big]_\theta$$
$$= H^s\big((a,b); H^{2-\theta}_0(\Omega)\big) \cap H^{s+1-\theta}\big((a,b); H^1_0(\Omega)\big).$$

This result and Lemma 8 imply that

$$\big\|\nabla\phi^{**}(t)\big\|^2_{\beta,B^+_R} + \big\|\phi^{**}_t(t)\big\|^2_{\beta,B^+_R}$$
$$\leqslant C(R)\big(\big\|u^2_t(t)\big\|^2_{H^{s+\beta}(t-t^*,t;H^{1+\beta}_0(\Omega))} + \big\|u^2(t)\big\|^2_{H^{s+\beta}(t-t^*,t;H^{2+\beta}_0(\Omega))}\big) \tag{56}$$

for $s + \beta < 1/2$ and $t > t^* = t^*(U, \Omega, R)$. Due to the embedding $C^\mu(0, T; X) \subset H^s(0, T; X)$, $0 < s < \mu < 1/2$, and estimates (54)–(56) we get

$$\big\|\nabla\phi^{**}(t)\big\|^2_{\beta,B^+_R} + \big\|\phi^{**}_t(t)\big\|^2_{\beta,B^+_R}$$
$$\leqslant C\big(E_0(u_0, u_1, \theta_0) + \big\|(\phi_0, \phi_1)\big\|^2_{\mathcal{Y}}\big), \quad t > t^*, \ \beta < 1/2.$$

Thus $(\phi^{**}(t_k), \phi^{**}(t_k))$, $t_k \to +\infty$ is compact.

The analysis of problems (E1)–(E4) shows that there exists the decomposition of the solution we described in Introduction. Indeed, $S_t W_0 = (u, u_t, \theta, \phi, \phi_t)(t) = (u^1, u^1_t, \theta^1, \phi^*, \phi^*_t)(t) + (u^2, u^2_t, \theta^2, \phi^{**}, \phi^{**}_t)(t)$, where $(u^1, u^1_t, \theta^1, \phi^*, \phi^*_t)(t) \to 0$ in $\mathcal{X} \times \tilde{\mathcal{Y}}$ as $t \to +\infty$ and $(u^2, u^2_t, \theta^2, \phi^{**}, \phi^{**}_t)(t_k)$ is compact in $\mathcal{X} \times \tilde{\mathcal{Y}}$ for any $t_k \to +\infty$. Thus, for an arbitrary $W_0 \in \mathcal{X} \times \tilde{\mathcal{Y}}$ and $t_k \to +\infty$ the sequence $S_{t_k} W_0$ contains convergent subsequence $S_{t_m} W_0 \xrightarrow{\mathcal{X}\times\tilde{\mathcal{Y}}} \bar{W}$. Now we prove that $\|\theta(t)\|^2_\Omega \to 0$ when $t \to +\infty$. Note that

$$\frac{d}{dt}\big\|\theta(t)\big\|^2_\Omega \leqslant -\big\|\nabla\theta(t)\big\|^2_\Omega + \big\|\nabla\theta(t)\big\|_\Omega \cdot \big\|\nabla u_t(t)\big\|_\Omega \leqslant \frac{1}{4}\big\|\nabla u_t(t)\big\|^2_\Omega.$$

Thus by (48) we have $(d/dt)\|\theta(t)\|^2_\Omega \leqslant C(E_0(u_0, u_1, \theta_0) + \|(\phi_0, \phi_1)\|^2_{\mathcal{Y}})$. Therefore the convergence $\|\theta(t)\|^2_\Omega \to 0$ follows from (49) and Lemma 11. Thus, any convergent sequence of the form $S_{t_k} W_0$, $t_k \to +\infty$, tends to a point $\bar{W} = (\bar{u}_0, \bar{u}_1, 0, \bar{\phi}_0, \bar{\phi}_1)$. Since S_t is a continuous operator, $S_\tau \bar{W} = \lim_{t_m\to+\infty} S_{t_k}(S_\tau W_0)$ for every fixed τ. Thus, $S_\tau \bar{W} = (\tilde{u}_0(\tau), \tilde{u}_1(\tau), 0, \tilde{\phi}_0(\tau), \tilde{\phi}_1(\tau))$. This implies that for the trajectory $S_t \bar{W}$ $\theta(t) \equiv 0$. Equation (2) implies, that $u_t(t) \equiv 0$ for this trajectory too. Hence, $\bar{W} = (\bar{u}_0, 0, 0, \bar{\phi}_0, \bar{\phi}_1)$. As far as W_0 was chosen arbitrary, the result obtained means that any convergent sequence $S_{t_m} W_0$, where $t_m \to +\infty$, converges to a point of the form $(\bar{u}_0, 0, 0, \bar{\phi}_0, \bar{\phi}_1)$. Using the standard contradiction argument we can prove that $u_t(t) \to 0$ when $t \to +\infty$.

To prove that $\bar{W}$ is a stationary solution to the problem (1)–(9), it is enough to show that $\phi_t^{**}(t) \to 0$ in $\tilde{\mathcal{Y}}$ along the trajectory. Obviously,

$$\phi_t^{**}(x,t) = -\frac{1}{2\pi}\int\limits_{x_3}^{t} ds \int\limits_0^{2\pi} d\theta\, (\partial_t + U\partial_{x_1}) u_t^*(x,t,s,\theta),$$

where $u_t^*(x,t,s,\theta) = u_t(x_1 - k_1(\theta,s,x_3), x_2 - k_2(\theta,s,x_3), t-s)$. Let $R > 0$ be a fixed value, $x_3 < R$, and $t > t^* = t^*(U,\Omega,R)$. Using formula (23), we obtain

$$\begin{aligned}\phi_t^{**}(x,t) = \frac{1}{2\pi}\Bigg\{ &\int\limits_0^{2\pi} d\theta\, u_t^*(x,t,t^*,\theta) - \int\limits_0^{2\pi} d\theta\, u_t^*(x,t,x_3,\theta) \\ &+ U\int\limits_{x_3}^{t^*} ds \int\limits_0^{2\pi} d\theta\, \big[\partial_{x_1} u_t^*\big](x,t,s,\theta) \\ &+ \int\limits_{x_3}^{t^*} \frac{s\,ds}{\sqrt{s^2 - x_3^2}} \int\limits_0^{2\pi} d\theta\, \big[M_\theta u_t^*\big](x,t,s,\theta)\Bigg\}.\end{aligned}$$

Repeating arguments from the proof of Lemma 8, part (i), we get

$$\big\|\phi_t^{**}(\cdot,t)\big\|^2_{B_R^+} \leqslant C(R) \max_{\tau > t - t^*} \big\|u_t(\tau)\big\|^2_{1,\Omega} \to 0, \quad t \to +\infty.$$

Hence, $\phi_t^{**}(t) \to 0$ in $\tilde{\mathcal{Y}}$ when $t \to +\infty$ and $\bar{W} = (\bar{u}_0, 0, 0, \bar{\phi}_0, 0)$. Thus, we have proved that every convergent sequence $S_{t_k} W_0$ converges to a stationary point $\bar{W}$ as $t_k \to +\infty$.

Now we complete the proof of Theorem 3 by applying the standard contradiction argument.

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References

[1] V.V. Bolotin, Non-Conservative Problems of the Theory of Elastic Stability, Fizmatgiz, Moscow, 1961, English transl.: Pergamon Press, Oxford, 1963.
[2] G. Avalos, I. Lasiecka, Exponential stability of thermoelastic system without mechanical dissipation, Rend. Istit. Mat. Univ. Trieste 28 (1997) 1–28.
[3] G. Avalos, I. Lasiecka, Exponential stability of thermoelastic system without mechanical dissipation II: the case of simply supported boundary conditions, SIAM J. Math. Anal. 29 (1998) 155–182.
[4] G. Avalos, I. Lasiecka, Uniform decays in nonlinear thermoelastic systems, in: Optimal Control: Theory, Algorithms, and Applications, Kluwer Academic, Dordrecht, 1998, Appl. Optim. 15 (1998) 1–23.
[5] I. Lasiecka, C. Lebiedzik, Decay rates of interactive hyperbolic–parabolic PDE models with thermal effects on the interface, Appl. Math. Optim. 42 (2000) 127–167.
[6] I. Lasiecka, C. Lebiedzik, Boundary stabilizability of nonlinear acoustic models with thermal effects on the interface, C. R. Acad. Sci. Paris 328 (2000) 187–192.
[7] I. Lasiecka, C. Lebiedzik, Asymptotic behaviour of nonlinear structural acoustic interactions with thermal effects on the interface, Nonlinear Anal. Ser. A 49 (2002) 703–735.
[8] I.D. Chueshov, Construction of solutions in a problem of the oscillation of a shell in a potential subsonic flow, in: V.A. Marchenco (Ed.), Operator Theory, Subharmonic Functions, Naukova Dumka, Kiev, 1991, pp. 147–154 (in Russian).
[9] A. Boutet de Monvel, I.D. Chueshov, The problem of interaction of von Kármán plate with subsonic flow of gas, Math. Methods Appl. Sci. 22 (1999) 801–810.
[10] L. Boutet de Monvel, I.D. Chueshov, Oscillation of von Kármán's plate in a potential flow of gas, Izv. Ross. Akad. Nauk Ser. Mat. 63 (1999) 219–244.
[11] I.D. Chueshov, On a certain system with delay, occurring in aeroelasticity, J. Soviet Math. 58 (1992) 385–390.
[12] I.D. Chueshov, A.V. Rezounenko, Global attractor for a class of retarded quasilinear partial differential equations, C. R. Acad. Sci. Paris Ser. I 321 (1995) 607–612.
[13] I.D. Chueshov, Dynamics of von Karman plate in a potential flow of gas: rigorous results and unsolved problems, in: Proceedings of 16th IMACS World Congress, 2000, pp. 1–6.
[14] H. Triebel, Theory of Functional Spaces, Birkhäuser, Basel, 1983.
[15] A. Babin, M. Vishik, Attractors of Evolutional Equations, North-Holland, Amsterdam, 1992.
[16] P.D. Lax, R.S. Phillips, Scattering Theory, Academic Press, New York, 1967.
[17] I.D. Chueshov, Finite-dimensionality of the attractor in some problems of the nonlinear theory of shells, Math. USSR Sb. 61 (1988) 411–420.
[18] I.D. Chueshov, Strong solutions and attractor of a system of von Kármán equations, Math. USSR Sb. 69 (1991) 25–36.
[19] J.-L. Lions, E. Magenes, Problèmes aux limites non homogènes et applications, vol. 1, Dunod, Paris, 1968.
[20] J. Bergh, J. Löfström, Interpolation Spaces. An Introduction, Springer-Verlag, Berlin, 1976.