Розглянемо перехід від сферичних коорд. до стереогр. (u, v) на S^2:

$$\varphi_N \circ \Phi^{-1}(\varphi, \psi) = \varphi_N(\cos\varphi\cos\psi, \cos\varphi\sin\psi, \sin\varphi) = \left(\frac{\cos\varphi\cos\psi}{1-\sin\varphi}, \frac{\cos\varphi\sin\psi}{1-\sin\varphi}\right).$$

М-ця Якобі:

$$\begin{pmatrix} \frac{\partial u}{\partial\varphi} & \frac{\partial u}{\partial\psi} \\ \frac{\partial v}{\partial\varphi} & \frac{\partial v}{\partial\psi}\end{pmatrix} = \begin{pmatrix} \frac{-\sin\varphi\cos\psi(1-\sin\varphi)-\cos\varphi\cos\psi(-\cos\varphi)}{(1-\sin\varphi)^2} & -\frac{\cos\varphi\sin\psi}{1-\sin\varphi} \\ \frac{-\sin\varphi\sin\psi(1-\sin\varphi)-\cos\varphi\sin\psi(-\cos\varphi)}{(1-\sin\varphi)^2} & \frac{\cos\varphi\cos\psi}{1-\sin\varphi}\end{pmatrix} =$$

$$= \frac{1}{1-\sin\varphi}\begin{pmatrix}\cos\psi & -\cos\varphi\sin\psi \\ \sin\psi & \cos\varphi\cos\psi\end{pmatrix}$$

Тому $\frac{\partial}{\partial\varphi} = \frac{\partial u}{\partial\varphi}\frac{\partial}{\partial u} + \frac{\partial v}{\partial\varphi}\frac{\partial}{\partial v} = \frac{1}{1-\sin\varphi}\left(\cos\psi\frac{\partial}{\partial u} + \sin\psi\frac{\partial}{\partial v}\right)$

$\frac{\partial}{\partial\psi} = \frac{\partial u}{\partial\psi}\frac{\partial}{\partial u} + \frac{\partial v}{\partial\psi}\frac{\partial}{\partial v} = \frac{\cos\varphi}{1-\sin\varphi}\left(-\sin\psi\frac{\partial}{\partial u} + \cos\psi\frac{\partial}{\partial v}\right).$

Якщо $a = \lambda\frac{\partial}{\partial\varphi} + \mu\frac{\partial}{\partial\psi} = \tilde\lambda\frac{\partial}{\partial u} + \tilde\mu\frac{\partial}{\partial v} \in T_pS^2$, то

$\tilde\lambda = \lambda\frac{\partial u}{\partial\varphi} + \mu\frac{\partial u}{\partial\psi} = \frac{1}{1-\sin\varphi}(\lambda\cos\psi - \mu\cos\varphi\sin\psi)$

$\tilde\mu = \lambda\frac{\partial v}{\partial\varphi} + \mu\frac{\partial v}{\partial\psi} = \frac{1}{1-\sin\varphi}(\lambda\sin\psi + \mu\cos\varphi\cos\psi)$

Нарешті,

$$\begin{pmatrix} \frac{\partial \varphi}{\partial u} & \frac{\partial \varphi}{\partial v} \\ \frac{\partial \psi}{\partial u} & \frac{\partial \psi}{\partial v}\end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial\varphi} & \frac{\partial u}{\partial\psi} \\ \frac{\partial v}{\partial\varphi} & \frac{\partial v}{\partial\psi}\end{pmatrix}^{-1} = \frac{1-\sin\varphi}{\cos\varphi}\begin{pmatrix}\cos\varphi\cos\psi & \cos\varphi\sin\psi \\ -\sin\psi & \cos\psi\end{pmatrix}$$

Д/З Записати у термінах (u,v)

~~знайдемо довжини вектора до кривої~~

Наприклад, нехай $p = \left(\frac{\sqrt2}{2}, -\frac{\sqrt2}{2}, 0\right)$, $a = (1, 1, \sqrt2) \in T_pS^2$.

Для p $\varphi = 0$, $\psi = \frac{7\pi}{4}$:

$$a = \lambda\underset{\substack{\uparrow \\ d_p\Psi\left(\frac{\partial}{\partial\varphi}\right) \\ \| \\ \Psi_\varphi}}{\frac{\partial}{\partial\varphi}} + \mu\underset{\substack{\uparrow \\ d_p\Psi\left(\frac{\partial}{\partial\psi}\right) \\ \| \\ \Psi_\psi}}{\frac{\partial}{\partial\psi}} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \sqrt2\frac{\partial}{\partial z}$$

Точніше,

$a = \lambda\left(\frac{\partial x}{\partial\varphi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\varphi}\frac{\partial}{\partial y} + \frac{\partial z}{\partial\varphi}\frac{\partial}{\partial z}\right) + \mu\left(\frac{\partial x}{\partial\psi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\psi}\frac{\partial}{\partial y} + \frac{\partial z}{\partial\psi}\frac{\partial}{\partial z}\right) =$

$= \lambda\left(-\sin\varphi\cos\psi\frac{\partial}{\partial x} - \sin\varphi\sin\psi\frac{\partial}{\partial y} + \cos\varphi\frac{\partial}{\partial z}\right) + \mu\left(-\cos\varphi\sin\psi\frac{\partial}{\partial x} + \cos\varphi\cos\psi\frac{\partial}{\partial y}\right) = \lambda\frac{\partial}{\partial z} + \frac{\sqrt2}{2}\mu\frac{\partial}{\partial x} + \frac{\sqrt2}{2}\mu\frac{\partial}{\partial y}$

Тобто $\lambda = \mu = \sqrt2$: $a = \sqrt2\left(\frac{\partial}{\partial\varphi} + \frac{\partial}{\partial\psi}\right)$.

Якщо $a = \tilde\lambda\frac{\partial}{\partial u} + \tilde\mu\frac{\partial}{\partial v}$, то $\tilde\lambda = \frac{\sqrt2}{2}\lambda + \frac{\sqrt2}{2}\mu = 2$, $\tilde\mu = -\frac{\sqrt2}{2}\lambda + \frac{\sqrt2}{2}\mu = 0$,

тобто $a = 2\frac{\partial}{\partial u}$ (точніше, $2\Psi_u$)

Дійсно, у p $(u,v) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right) = \left(\frac{\sqrt2}{2}, -\frac{\sqrt2}{2}\right)$, тому за ф-лами вище (тільки для Φ_N замість Φ_S) $\Psi_u = \Big(\frac{2(1-u^2+v^2)}{(1+u^2+v^2)^2}$.

$\frac{\partial}{\partial x} - \frac{4uv}{(1+u^2+v^2)^2}\frac{\partial}{\partial y} + \frac{4u}{(1+u^2+v^2)^2}\frac{\partial}{\partial z} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial}{\partial y} + \frac{\sqrt{2}}{2}\frac{\partial}{\partial z} = \frac{1}{2}a.$

—

Знайдемо у $\mathbb{R}P^2$ дотичний вектор $\gamma'(\frac{\pi}{4})$ до $\gamma(t) = (\cos t : \sin t : t)$.

$\gamma(\frac{\pi}{4}) = (\frac{\sqrt{2}}{2} : \frac{\sqrt{2}}{2} : \frac{\pi}{4}) \in U_3$, тому використаємо (U_3, φ_3):

лок. задання γ:

$\varphi_3 \circ \gamma(t) = \left(\frac{\cos t}{t}, \frac{\sin t}{t}\right)$

Отже, $\gamma'(t) = (\gamma^1)'(t)\frac{\partial}{\partial u} + (\gamma^2)'(t)\frac{\partial}{\partial v} = \frac{-t\sin t - \cos t}{t^2}\frac{\partial}{\partial u} +$
$+ \frac{t\cos t - \sin t}{t^2}\frac{\partial}{\partial v}$

Отже, $\gamma'(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}\cdot\frac{16}{\pi^2}\left(-(\frac{\pi}{4}+1)\frac{\partial}{\partial u} + (\frac{\pi}{4}-1)\frac{\partial}{\partial v}\right)$

—

Чи буде відображення, що кожній лок. системі коорд. в околі $p \in M$ ставить у відп. одиничну м-цю $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ тензором валентності $(2,0)$, $(1,1)$ або $(0,2)$?

—

Тобто для $(2,0)$: $T = \delta_{ij}\, dx^i \otimes dx^j = \sum_{i=1}^{n} dx^i \otimes dx^i$.

При переході до $(\tilde{x}^1, \dots, \tilde{x}^n)$: $\tilde{T}_{ij} = \delta_{kl}\frac{\partial x^k}{\partial \tilde{x}^i}\frac{\partial x^l}{\partial \tilde{x}^j} = \sum_{k=1}^{n}\frac{\partial x^k}{\partial \tilde{x}^i}\frac{\partial x^k}{\partial \tilde{x}^j}$.

Це повинно дорівнювати δ_{ij}.

Це не так, бо для $(\tilde{x}^1, \dots, \tilde{x}^n) = (2x^1, \dots, x^n)$ $\tilde{T}_{11} = \sum_k \left(\frac{\partial x^k}{\partial \tilde{x}^1}\right)^2 =$
$= \left(\frac{\partial x^1}{\partial \tilde{x}^1}\right)^2 = \frac{1}{4} \neq 1$.

Для $(0,2)$ аналогічно: ні.

Для $(1,1)$: $T = \delta^i_j\, dx^j \otimes \frac{\partial}{\partial x^i}$, $\tilde{T}^i_j = \delta^k_l \frac{\partial \tilde{x}^i}{\partial x^k}\frac{\partial x^l}{\partial \tilde{x}^j} =$
$= \frac{\partial \tilde{x}^i}{\partial x^k}\frac{\partial x^k}{\partial \tilde{x}^j} = \frac{\partial \tilde{x}^i}{\partial \tilde{x}^j} = \delta^i_j$. Тобто так (це тотожній оператор).

—

Нехай $f \in C^k(M)$, $k \geq 2$. Коли $(x^1, \dots, x^n) \mapsto \left\{\frac{\partial^2 f}{\partial x^i \partial x^j}\right\}_{i,j=1}^{n}$ задає $(2,0)$-тензор у p?

Тобто $T = \frac{\partial^2 f}{\partial x^i \partial x^j}\, dx^i \otimes dx^j$.

$\frac{\partial^2 f}{\partial \tilde{x}^i \partial \tilde{x}^j} = \frac{\partial}{\partial \tilde{x}^i}\left(\frac{\partial f}{\partial \tilde{x}^j}\right) = \frac{\partial x^k}{\partial \tilde{x}^i}\frac{\partial}{\partial x^k}\left(\frac{\partial x^l}{\partial \tilde{x}^j}\frac{\partial f}{\partial x^l}\right) = \boxed{\frac{\partial x^k}{\partial \tilde{x}^i}\frac{\partial^2 x^l}{\partial x^k \partial \tilde{x}^j}}\frac{\partial f}{\partial x^l} +$
$+ \frac{\partial x^k}{\partial \tilde{x}^i}\frac{\partial x^l}{\partial \tilde{x}^j}\frac{\partial^2 f}{\partial x^k \partial x^l}$.

$\frac{\partial^2 x^l}{\partial \tilde{x}^i \partial \tilde{x}^j}$

Це так при $\frac{\partial f}{\partial x^l}=0\ \forall l$. У навпаки: нехай $\exists\, l_0: \frac{\partial f}{\partial x^{l_0}}\neq 0$.
Перейдемо до $(\tilde{x}^1,\dots,\tilde{x}^n)=(x^1,\dots,\sqrt{x^{l_0}},\dots,x^n)$ в околі точки,
де $x^{l_0}>0$, тоді $x^{l_0}=(\tilde{x}^{l_0})^2$. Тоді при $i=j=l_0$:

$$\underbrace{\frac{\partial x^k}{\partial \tilde{x}^i}\frac{\partial^2 x^l}{\partial x^k\partial \tilde{x}^j}}_{=\frac{\partial^2 x^l}{\partial\tilde{x}^i\partial\tilde{x}^j}}\frac{\partial f}{\partial x^l}=\begin{bmatrix}\text{доданок } \neq 0 \text{ лише} \\ \text{при } i=j=k=l=l_0\end{bmatrix}=\underbrace{\frac{\partial^2 x^{l_0}}{(\partial \tilde{x}^{l_0})^2}}_{=2}\cdot\frac{\partial f}{\partial x^{l_0}}\neq 0,$$

тобто тенз. закон не виконується.

Т.ч. це $(2,0)$-тензор $\Leftrightarrow \forall\, l\ \frac{\partial f}{\partial x^l}(p)=0$ (p — крит. точка f).

Д/з. Показати, що для рім. м-ки $g=g_{ij}dx^i dx^j$ кометрика $g^{ij}\frac{\partial}{\partial x^i}\otimes\frac{\partial}{\partial x^j}$, де $(g^{ij})=(g_{ij})^{-1}$ — обернена м-ця, є $(0,2)$-тенз. полем.

Нехай $\dim M=2$, T — $(2,1)$-тензор у $p\in M$, у коорд. (x^1,x^2) всі його компоненти $T^k_{ij}=1$.
– чи буде T симетричним або кососиметричним?
Тут маємо на увазі $T(v,w)=T(w,v)$ або $-T(w,v)$ для $T: T_pM\times T_pM\to T_pM \Leftrightarrow T^k_{ij}=T^k_{ji}\ (=-T^k_{ji})\ \forall\, i,j,k=1,2$
(бо $T(v^i\frac{\partial}{\partial x^i}, w^j\frac{\partial}{\partial x^j})=v^iw^jT^k_{ij}\frac{\partial}{\partial x^k}$).
Тобто симетричний, не кососиметричний.

– Нехай $(\tilde{x}^1,\tilde{x}^2)$ — лок. коорд. в околі p: $\begin{pmatrix}\frac{\partial\tilde{x}^1}{\partial x^1} & \frac{\partial\tilde{x}^1}{\partial x^2}\\ \frac{\partial\tilde{x}^2}{\partial x^1} & \frac{\partial\tilde{x}^2}{\partial x^2}\end{pmatrix}(p)=\begin{pmatrix}2&1\\5&3\end{pmatrix}$

Знайти $\tilde{T}^1_{12}$.

$$\begin{pmatrix}\frac{\partial x^1}{\partial\tilde{x}^1} & \frac{\partial x^1}{\partial\tilde{x}^2}\\ \frac{\partial x^2}{\partial\tilde{x}^1} & \frac{\partial x^2}{\partial\tilde{x}^2}\end{pmatrix}(p)=\begin{pmatrix}2&1\\5&3\end{pmatrix}^{-1}=\frac{1}{2\cdot3-5\cdot1}\begin{pmatrix}3&-1\\-5&2\end{pmatrix}=\begin{pmatrix}3&-1\\-5&2\end{pmatrix}.$$

$$\tilde{T}^1_{12}=T^k_{ij}\frac{\partial\tilde{x}^1}{\partial x^k}\frac{\partial x^i}{\partial\tilde{x}^1}\frac{\partial x^j}{\partial\tilde{x}^2}=T^1_{11}\frac{\partial\tilde{x}^1}{\partial x^1}\frac{\partial x^1}{\partial\tilde{x}^1}\frac{\partial x^1}{\partial\tilde{x}^2}+T^1_{12}\frac{\partial\tilde{x}^1}{\partial x^1}\frac{\partial x^1}{\partial\tilde{x}^1}\frac{\partial x^2}{\partial\tilde{x}^2}+$$
$$+T^1_{21}\frac{\partial\tilde{x}^1}{\partial x^1}\frac{\partial x^2}{\partial\tilde{x}^1}\frac{\partial x^1}{\partial\tilde{x}^2}+T^1_{22}\frac{\partial\tilde{x}^1}{\partial x^1}\frac{\partial x^2}{\partial\tilde{x}^1}\frac{\partial x^2}{\partial\tilde{x}^2}+T^2_{11}\frac{\partial\tilde{x}^1}{\partial x^2}\frac{\partial x^1}{\partial\tilde{x}^1}\frac{\partial x^1}{\partial\tilde{x}^2}+T^2_{12}\frac{\partial\tilde{x}^1}{\partial x^2}\frac{\partial x^1}{\partial\tilde{x}^1}\frac{\partial x^2}{\partial\tilde{x}^2}+$$
$$+T^2_{21}\frac{\partial\tilde{x}^1}{\partial x^2}\frac{\partial x^2}{\partial\tilde{x}^1}\frac{\partial x^1}{\partial\tilde{x}^2}+T^2_{22}\frac{\partial\tilde{x}^1}{\partial x^2}\frac{\partial x^2}{\partial\tilde{x}^1}\frac{\partial x^2}{\partial\tilde{x}^2}=1\cdot2\cdot3\cdot(-1)+1\cdot2\cdot3\cdot2+1\cdot2\cdot(-5)\cdot$$
$$(-1)+1\cdot2\cdot(-5)\cdot2+1\cdot1\cdot3\cdot(-1)+1\cdot1\cdot3\cdot2+1\cdot1\cdot(-5)\cdot(-1)+1\cdot1\cdot(-5)\cdot2=-6.$$

Д/з. Знайти $\tilde{T}^1_{21}$ і пересвідчитися, що $\tilde{T}^1_{21}=\tilde{T}^1_{12}$.

Нехай A — $(1,1)$-тенз. поле на M, і $\forall$ в. полів X, Y

$$N(X,Y) = A^2([X,Y]) - A([A(X),Y]) - A([X,A(Y)]) + [A(X), A(Y)]$$

– Показати, що N – $(2,1)$-тенз. поле.

Дійсно, $\forall$ полів X, Y, Z і ф-цій f, g на M:

$$N(fX+gY, Z) = A^2([fX+gY, Z]) - A([A(fX+gY), Z]) -$$
$$- A([fX+gY, A(Z)]) + [A(fX+gY), A(Z)] = \left[\begin{matrix}\text{власт.}\\ [\cdot,\cdot],\ A - (1,1)\text{-т.п.}\end{matrix}\right] =$$
$$= A^2(f[X,Z] - Z(f)X + g[Y,Z] - Z(g)Y) - A([fA(X) +$$
$$+ gA(Y), Z]) - A(f[X, A(Z)] - A(Z)(f)X + g[Y, A(Z)] -$$
$$- A(Z)(g)Y) + [fA(X) + gA(Y), A(Z)] = \underline{fA^2([X,Z])} - \cancel{Z(f)A^2(X)} +$$
$$+ \underline{\underline{gA^2([Y,Z])}} - \cancel{Z(g)A^2(Y)} - \underline{fA([A(X),Z])} + \cancel{Z(f)A^2(X)} -$$
$$- \underline{\underline{gA([A(Y),Z])}} + \cancel{Z(g)A^2(Y)} - \underline{fA([X,A(Z)])} + \cancel{A(Z)(f)A(X)}$$
$$\underline{\underline{- gA([Y,A(Z)])}} + \cancel{A(Z)(g)A(Y)} + \underline{f[A(X),A(Z)]} - \cancel{A(Z)(f)A(X)} +$$
$$+ \underline{\underline{g[A(Y),A(Z)]}} - \cancel{A(Z)(g)A(Y)} = \underline{fN(X,Z)} + \underline{\underline{gN(Y,Z)}},$$

і ан-но для другого аргумента.

– Чи є N симетричним або кососиметричним?

$$\forall X, Y \quad N(Y,X) = A^2([Y,X]) - A([A(Y),X]) - A([Y,A(X)]) +$$
$$+ [A(Y),A(X)] = -A^2([X,Y]) + A([X,A(Y)]) + A([A(X),Y]) -$$
$$- [A(X),A(Y)] = -N(X,Y).$$ Кососиметричний.