

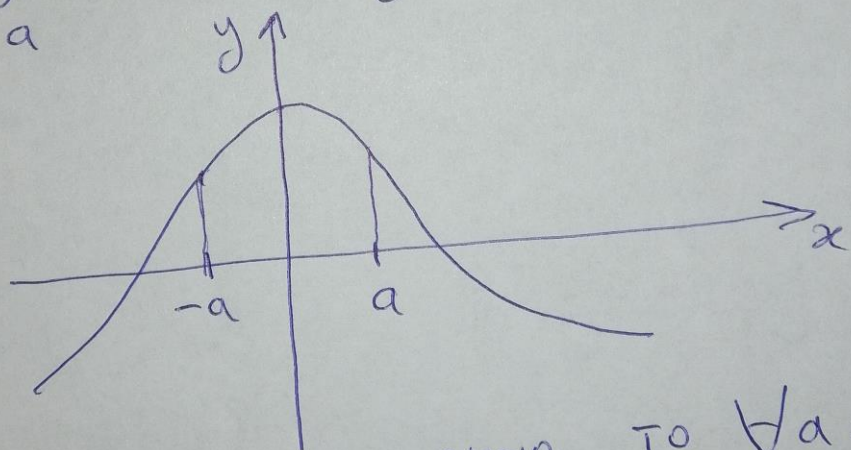
# Властивості інтеграла (продовження)

Теорема (парність, непарність, періодичність).

Нехай функція  $f$  є інтегрованою на кожному скінченному відрізку. Тоді:

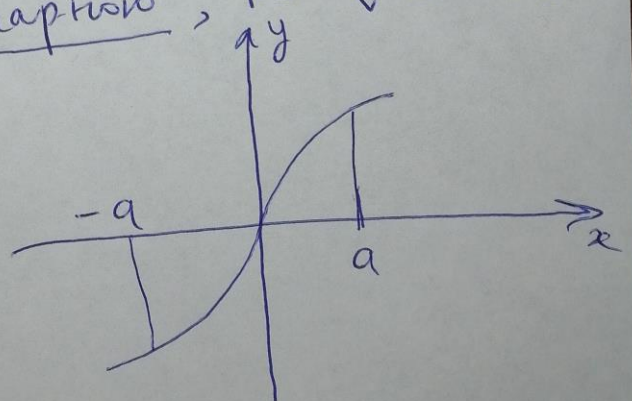
① Якщо  $f \in \underline{\text{парною}}$ , то  $\forall a > 0$ :

$$\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$$

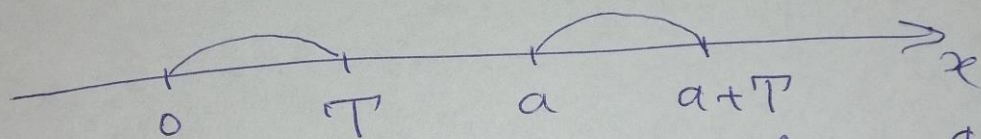


② Якщо  $f \in \underline{\text{непарною}}$ , то  $\forall a > 0$ :

$$\int_{-a}^a f(x) dx = 0.$$



③ اگرچه  $f \in \underline{\text{نپریودیک}} \text{ } f \text{ } \underline{\text{نپریودیک}}$  2  
 $T$ , тодіто  $\forall x \in \mathbb{R} : f(x+T) = f(x)$ , то  
 $\forall a \in \mathbb{R} : \int_a^{a+T} f(x) dx = \int_0^T f(x) dx$ .



Dobegennus. ①  $\int_{-a}^a f(x) dx = \int_{-a}^0 f(x) dx + \int_0^a f(x) dx$

$$\int_{-a}^0 f(x) dx = \left[ \begin{array}{l} x = -t \\ dx = -dt \\ t = -x \end{array} \right] = - \int_a^0 f(-t) dt = \int_a^0 f(t) dt$$

$$= \int_0^a f(t) dt. \text{ Тоді } \int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx.$$

② - теорема загальна.

Пример 1.  $\int_{-\pi}^{\pi} \sqrt[3]{\sin x} dx = 0,$

оскільки  $\sqrt[3]{\sin(-x)} = \sqrt[3]{-\sin x} = -\sqrt[3]{\sin x}.$

$\int \sqrt[3]{\sin x} dx = ?$

Пример 2.  $\int_0^{2\pi} \frac{dx}{1 + \cos^2 x} = ?$

$\int \frac{dx}{1 + \cos^2 x} = \left[ \begin{array}{l} t = \operatorname{tg} x \\ dt = \frac{dx}{\cos^2 x}, dx = \cos^2 x dt \\ dx = \frac{dt}{1+t^2} \end{array} \right]$

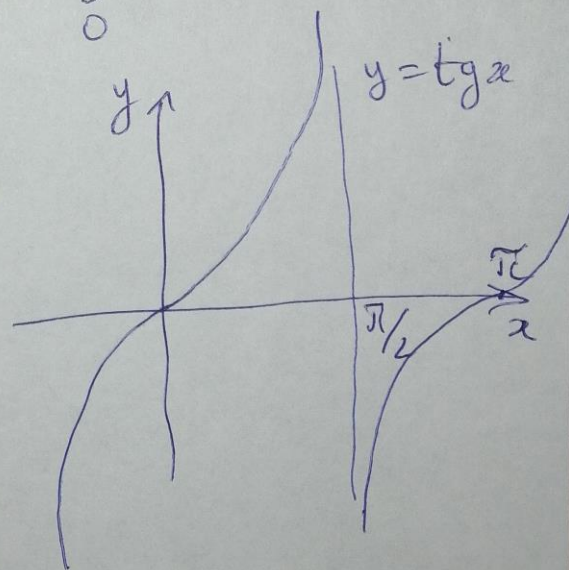
$= \int \frac{\frac{dt}{1+t^2}}{1 + \frac{1}{1+t^2}} = \int \frac{dt}{1+t^2+1} = \int \frac{dt}{t^2+2} =$

$= \frac{1}{\sqrt{2}} \operatorname{arctg} \frac{t}{\sqrt{2}} + C = \frac{1}{\sqrt{2}} \operatorname{arctg} \left( \frac{\operatorname{tg} x}{\sqrt{2}} \right) + C$

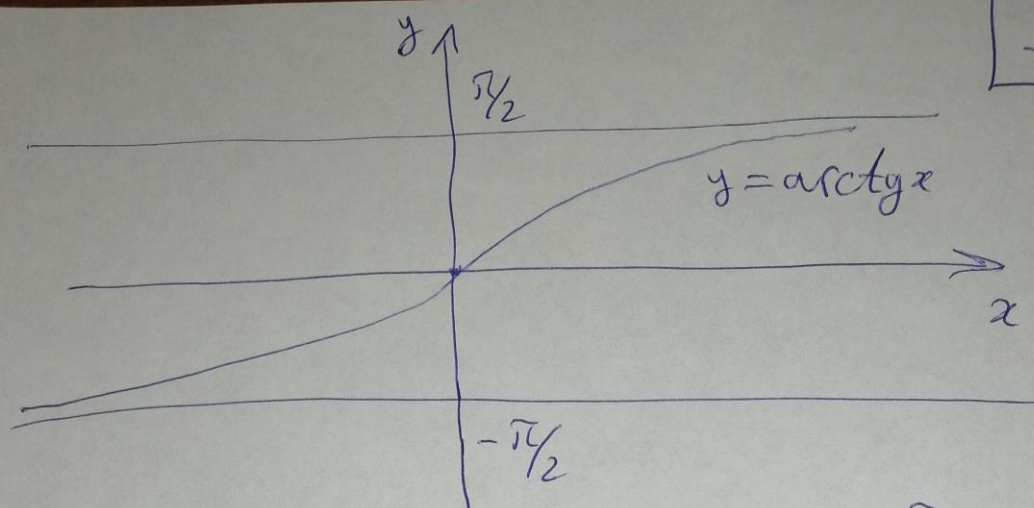
Теоретична задача: знайти первичну для  
 $f(x) = \frac{1}{1+\cos^2 x}$  на  $[0, \pi]$ .

$(\operatorname{tg} x)' = \frac{1}{\cos^2 x} > \frac{1}{\cos^2 x} = 1 + \operatorname{tg}^2 x; \cos^2 x = \frac{1}{1+t^2}$

$$\begin{aligned}
 \int_0^{2\pi} \frac{dx}{1+\cos^2 x} &= \int_0^{\pi} \frac{dx}{1+\cos^2 x} + \int_{\pi}^{2\pi} \frac{dx}{1+\cos^2 x} = \boxed{4} \\
 &= 2 \int_0^{\pi} \frac{dx}{1+\cos^2 x} = 2 \int_{-\pi/2}^{\pi/2} \frac{dx}{1+\cos^2 x} = \\
 &= 4 \int_0^{\pi/2} \frac{dx}{1+\cos^2 x} = \left[ \begin{array}{l} t = \operatorname{tg} x, x = \operatorname{arctg} t \\ dx = \frac{dt}{1+t^2} \end{array} \right] = \\
 &= 4 \int_0^{+\infty} \frac{\frac{dt}{1+t^2}}{1 + \frac{1}{1+t^2}} = 4 \int_0^{+\infty} \frac{dt}{t^2 + (\sqrt{2})^2} = \\
 &= \frac{4}{\sqrt{2}} \operatorname{arctg} \frac{t}{\sqrt{2}} \Big|_0^{+\infty} = \\
 &= \frac{4}{\sqrt{2}} \lim_{t \rightarrow +\infty} \operatorname{arctg} \frac{t}{\sqrt{2}} = \\
 &= \frac{4}{\sqrt{2}} \cdot \frac{\pi}{2} = \frac{2\pi}{\sqrt{2}}.
 \end{aligned}$$



$$\cos^2 x = \frac{1}{2}(1 + \cos 2x), \quad \pi = \pi.$$



$$\lim_{x \rightarrow +\infty} \arctg x = \frac{\pi}{2}, \quad \lim_{x \rightarrow -\infty} \arctg x = -\frac{\pi}{2}.$$

## Інтегрування частинами в інтегралі Ріманна

Теорема. Нехай  $u, v \in C^1[a, b]$ .

Тоді:

$$\int_a^b u(x) v'(x) dx = (u(x) \cdot v(x)) \Big|_a^b - \int_a^b v(x) u'(x) dx,$$

$$\text{або } \int_a^b u dv = (u \cdot v) \Big|_a^b - \int_a^b v du.$$

Доведення.  $\int_a^b (u(x) \cdot v'(x))' dx = (u(x) v'(x)) \Big|_a^b$

$$\int_a^b (u(x) \cdot v'(x))' dx = \int_a^b (u'(x)v(x) + u(x)v'(x)) dx \Big|_6$$

$$= \int_a^b u(x)v'(x) dx + \int_a^b v(x)u'(x) dx.$$

Пример.  $\int_0^{\pi/2} \cos^n x dx = I_n, n \in \mathbb{N}_{\geq 0}$ .

$$I_0 = \int_0^{\pi/2} dx = \frac{\pi}{2}, \quad I_1 = \int_0^{\pi/2} \cos x dx = \sin x \Big|_0^{\pi/2} = 1.$$

Для  $n \geq 2$ . Тогда  $I_n = \int_0^{\pi/2} \cos^n x dx =$

$$= \int_0^{\pi/2} \cos^{n-1} x \cdot \cos x dx = \left[ \begin{array}{l} u = \cos^{n-1} x \\ dv = \cos x dx \\ v = \sin x \end{array} \right] =$$

$$= \left( \cos^{n-1} x \cdot \sin x \right) \Big|_0^{\pi/2} + (n-1) \int_0^{\pi/2} \cos^{n-2} x \cdot \sin^2 x dx =$$

$$= (n-1) \int_0^{\pi/2} \cos^{n-2} x \cdot (1 - \cos^2 x) dx = (n-1) I_{n-2} -$$

$$- (n-1) I_n^0. \text{ Тогда } n I_n = (n-1) I_{n-2},$$

$$\text{To show } I_n = \frac{n-1}{n} I_{n-2}.$$

$$(1) \quad n = 2k. \quad \text{To show } I_{2k} = \frac{2k-1}{2k} \cdot I_{2(k-1)}$$

$$= \frac{2k-1}{2k} \cdot \frac{2k-3}{2(k-1)} I_{2(k-2)} = \dots =$$

$$= \frac{(2k-1)(2k-3)(2k-5) \dots 3 \cdot 1}{2k \cdot 2(k-1) \cdot 2(k-2) \dots 4 \cdot 2} \cdot I_0 =$$

$$= \frac{(2k-1)!!}{(2k)!!} \cdot \frac{\pi}{2}, \quad \text{To show}$$

$$\int_0^{\pi/2} \cos^{2k} x \, dx = \frac{(2k-1)!!}{(2k)!!} \cdot \frac{\pi}{2}, \quad k \geq 1.$$

$$(2) \quad n = 2k+1. \quad \text{To show } I_{2k+1} = \frac{2k}{2k+1} I_{2k-1}$$

$$= \frac{2k \cdot 2(k-1)}{(2k+1)(2k-1)} I_{2k-3} = \dots = \frac{(2k)!!}{(2k+1)!!} I_0$$

$$\text{To show } \int_0^{\pi/2} \cos^{2k+1} x \, dx = \frac{(2k)!!}{(2k+1)!!}.$$