

Riemann-Hilbert problems and integrable nonlinear partial differential equations, V

Dmitry Shepelsky

V.N. Karazin Kharkiv National University, Ukraine

Step-like Cauchy problem for NLS

We consider the Cauchy problem for the **focusing nonlinear Schrödinger (NLS) equation**

$$\begin{aligned}iq_t + q_{xx} + 2|q|^2q &= 0, & x \in \mathbb{R}, \quad t \geq 0, \\q(x, 0) &= q_0(x), & x \in \mathbb{R},\end{aligned}$$

where the initial data are assumed to approach, for large $|x|$, the **non-zero backgrounds**:

$$q_0(x) \sim \begin{cases} A_1 e^{i\phi_1} e^{-2iB_1 x}, & x \rightarrow -\infty \\ A_2 e^{i\phi_2} e^{-2iB_2 x}, & x \rightarrow +\infty, \end{cases}$$

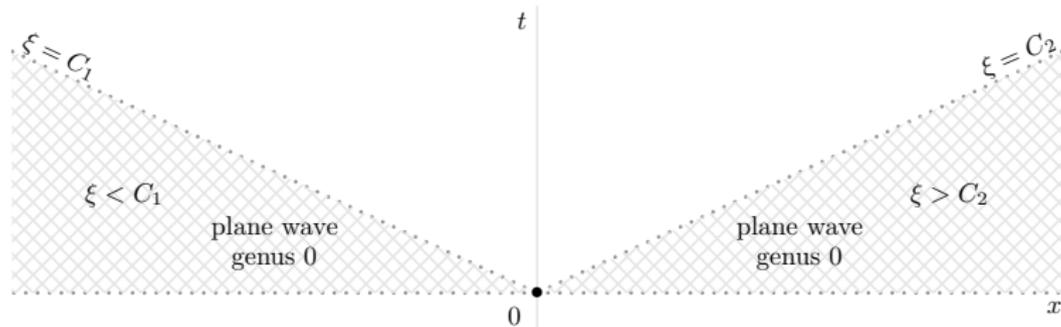
where $\{A_j, B_j, \phi_j\}_1^2$ are real constants; $A_j > 0$. The solution $q(x, t)$ is assumed to approach the **associated plane wave backgrounds for all $t \geq 0$** ("**nontrivial boundary conditions**")

$$q(x, t) = q_{0j}(x, t) + o(1) \quad \text{as } x \rightarrow (-1)^j \infty \text{ for all } t,$$

where

$$q_{0j}(x, t) = A_j e^{i\phi_j} e^{-2iB_j x + 2i\omega_j t} \quad \text{with } \omega_j = A_j^2 - 2B_j^2, \quad j = 1, 2.$$

Asymptotics in the case $B_1 = B_2, A_1 = A_2$



- $|\xi| > C(A, B)$ (here $C_1 = C_2 = C$): **modulated plane waves**

$$q(x, t) = Ae^{2i(\omega t - Bx - \phi(\xi))} + O(t^{-1/2})$$

- $|\xi| < C$: **modulated elliptic wave**

$$q(x, t) = \hat{A} \frac{\Theta(\beta t + \gamma)}{\Theta(\beta t + \tilde{\gamma})} e^{2i(\nu t - \phi)} + O(t^{-1/2}).$$

Here $\hat{A}, \beta, \gamma, \tilde{\gamma}, \nu, \phi$ are functions of $\xi = \frac{x}{t}$;

$\Theta(z) = \sum_{m \in \mathbb{Z}} e^{\pi i \tau m^2 + 2\pi i m z}$ is the **theta function** of invariant $\tau(\xi)$.

Background solutions and Jost solutions

In what follows we are dealing with the case $B_1 \neq B_2$, $A_j \neq 0$, $j = 1, 2$.

- Determine solutions of Lax pair associated with **background solutions** of NLS $q_{0j}(x, t) = A_j e^{-2iB_j x + 2i\omega_j t + i\phi_j}$:

$$\Phi_{0j}(x, t, k) = e^{(-iB_j x + i\omega_j t)\sigma_3} \mathcal{N}_{0j}(k) e^{(-iX_j(k)x - i\Omega_j(k)t)\sigma_3}$$

where $X_j(k) = \sqrt{(k - E_j)(k - \bar{E}_j)}$, $E_j = B_j + iA_j$,
 $\Omega_j(k) = 2(k + B_j)X_j(k)$,

$$\mathcal{E}_j(k) = \frac{1}{2} \begin{pmatrix} \varkappa_j(k) + \varkappa_j^{-1}(k) & \varkappa_j(k) - \varkappa_j^{-1}(k) \\ \varkappa_j(k) - \varkappa_j^{-1}(k) & \varkappa_j(k) + \varkappa_j^{-1}(k) \end{pmatrix}$$

with $\varkappa_j(k) = \left(\frac{k - E_j}{k - \bar{E}_j}\right)^{1/4}$, $\mathcal{N}_{0j}(k) = e^{\frac{i\phi_j}{2}} \mathcal{E}_j(k) e^{-\frac{i\phi_j}{2}}$.

- Let $q(x, t) \rightarrow q_{0j}(x, t)$ as $x \rightarrow (-1)^j \infty$. Then **Jost solutions** $\Phi_j(x, t, k)$ of Lax pair are fixed by ini. conds.:

$$\Phi_j \sim \Phi_{0j}, x \rightarrow (-1)^j \infty, \quad k \in \mathbb{R} \cup (E_j, \bar{E}_j)$$

- **scattering**: $\Phi_2(x, t, k) = \Phi_1(x, t, k)s(k)$, $k \in \mathbb{R}$;
 $s(k) = \begin{pmatrix} a^*(k) & b(k) \\ -b^*(k) & a(k) \end{pmatrix}$, where $f^*(k) := \overline{f(\bar{k})}$.

NLS with non-zero background: RH problem

Let $\phi_2 = 0$, $\phi_1 \equiv \phi$; $\Sigma_j := (E_j, \bar{E}_j)$. Define

$$M(x, t, k) := \begin{cases} \begin{pmatrix} \frac{\Phi_1^{(1)}}{a(k)} & \Phi_2^{(2)} \\ \Phi_2^{(1)} & \frac{\Phi_1^{(2)}}{\bar{a}(k)} \end{pmatrix} e^{(ikx+2ik^2t)\sigma_3}, & k \in \mathbb{C}^+, \\ \begin{pmatrix} \frac{\Phi_1^{(1)}}{a(k)} & \Phi_2^{(2)} \\ \Phi_2^{(1)} & \frac{\Phi_1^{(2)}}{\bar{a}(k)} \end{pmatrix} e^{(ikx+2ik^2t)\sigma_3}, & k \in \mathbb{C}^-. \end{cases}$$

Then M satisfies the RH problem:

- $M_+(x, t, k) = M_-(x, t, k)J(x, t, k)$, $k \in \Sigma = \mathbb{R} \cup \Sigma_1 \cup \Sigma_2$,
- $M(x, t, \infty) = I$,

where $J(x, t, k) = e^{-(ikx+2ik^2t)\sigma_3} J_0(k) e^{(ikx+2ik^2t)\sigma_3}$ with $J_0(k)$ defined by

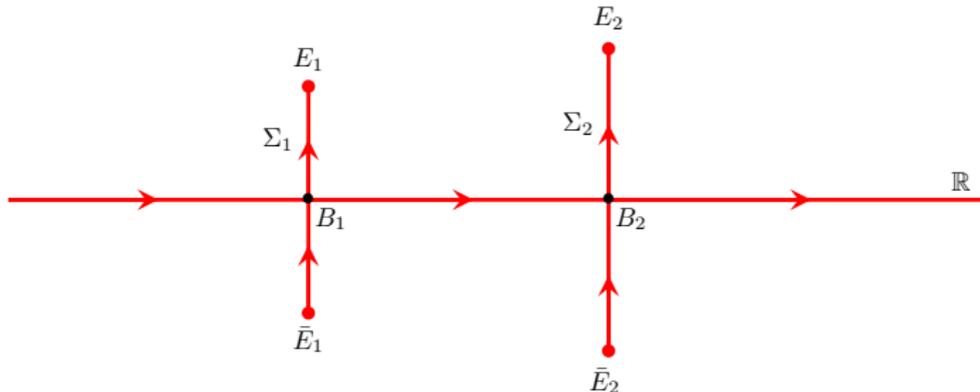
- ① for $k \in \mathbb{R}$,

$$J_0(k) = \begin{pmatrix} 1 + r(k)r^*(k) & r^*(k) \\ r(k) & 1 \end{pmatrix} \quad \text{with } r(k) := \frac{b^*(k)}{a(k)};$$

- ② for $k \in \Sigma_1 \cup \Sigma_2$,

$$J_0(k) = \begin{cases} \begin{pmatrix} 1 & 0 \\ \frac{ie^{-i\phi}}{a_+ a_-} & 1 \end{pmatrix}, & k \in \Sigma_1 \cap \mathbb{C}^+ \\ \begin{pmatrix} \frac{a_-}{a_+} & i \\ 0 & \frac{a_+}{a_-} \end{pmatrix}, & k \in \Sigma_2 \cap \mathbb{C}^+ \end{cases} \quad J_0(k) = \begin{cases} \begin{pmatrix} 1 & \frac{ie^{i\phi}}{a_+^* a_-^*} \\ 0 & 1 \end{pmatrix}, & k \in \Sigma_1 \cap \mathbb{C}^- \\ \begin{pmatrix} \frac{a_+^*}{a_-^*} & 0 \\ i & \frac{a_-^*}{a_+^*} \end{pmatrix}, & k \in \Sigma_2 \cap \mathbb{C}^- \end{cases}$$

RHP: dependence on x and t , I

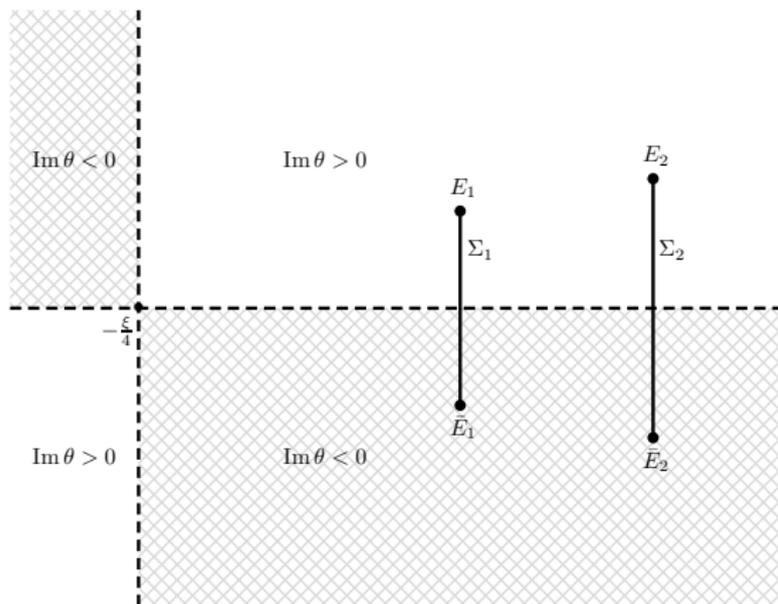


- Dependence on x and t of jump matrix: **same as in the case of decaying ini. conditions**, through $e^{(ikx+2ik^2t)\sigma_3} \equiv e^{it\theta(\xi,k)\sigma_3}$, where

$$\theta(\xi, k) = 2k^2 + \xi k \quad (\xi = x/t).$$

- For **large- t** analysis, it would be nice to have that as $t \rightarrow \infty$, $J(x, t, k) \rightarrow \tilde{J}$ (piecewise) independent of k . Then the limiting RH problem can (hopefully) be solved explicitly, thus giving explicit asymptotics for $q(x, t)$.

RHP: dependence on x and t , II



Considering original RHP, we face **PROBLEM**: at some parts of contour, depending on value of $\text{Im } \theta(\xi, k)$,

$e^{it\theta(\xi, k)}$ or $e^{-it\theta(\xi, k)}$ grows as $t \rightarrow \infty$!

“ g -function mechanism”, I

- **SOLUTION**: deform the contour and replace (in the jump matrix) the original “phase function” $\theta(\xi, k)$ by another phase function $g(\xi, k)$ (“ g -function mechanism”), which has appropriate behavior on the (deformed) contour. $M \mapsto M^{(1)}$:

$$M^{(1)}(x, t, k) := e^{-itg^{(0)}(\xi)\sigma_3} M(x, t, k) e^{it(g(\xi; k) - \theta(\xi; k))\sigma_3}.$$

- in order to keep large- k asymptotics for RHP:

$$g(\xi; k) = 2k^2 + \xi k + g^{(0)}(\xi) + O(k^{-1})$$

- deformed contour: $\hat{\Sigma} = \{k : \text{Im } g(k) = 0\}$.
- **RESULT**: (i) appropriate $g(\xi; k)$ turn to be structurally different for different ranges of ξ ; (ii) asymptotic structure depends on relationship amongst A_j, B_j .

"g-function mechanism", II

- for $|\xi| \gg 1$, the appropriate **g-functions** are (not surprising!) those involved in the construction of background solutions :

$$g(\xi; k) := \begin{cases} \Omega_1(k) + \xi X_1(k), & \xi \ll -1, \\ \Omega_2(k) + \xi X_2(k), & \xi \gg 1, \end{cases}$$

where $X_j(k) = \sqrt{(k - E_j)(k - \bar{E}_j)}$, $\Omega_j(k) = 2(k + B_j)X_j(k)$

- in terms of derivative w.r.t. k : for $\xi \gg 1$,

$$g'(\xi; k) = 4 \frac{(k - \mu_1(\xi))(k - \mu_2(\xi))}{\sqrt{(k - E_2)(k - \bar{E}_2)}}$$

with $\mu_1(\xi) < \mu_2(\xi)$ real (similarly for $\xi \ll -1$, with E_2 replaced by E_1).

- Series of deformations of RH problem** with this phase function lead finally to two **model RH problems** (applicable for $(-1)^j \xi \gg 1$), each with a **single jump arc**; for $\xi \gg 1$:

$$M_+^{(mod)}(k) = M_-^{(mod)}(k) \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad k \in \hat{\Sigma}_2$$

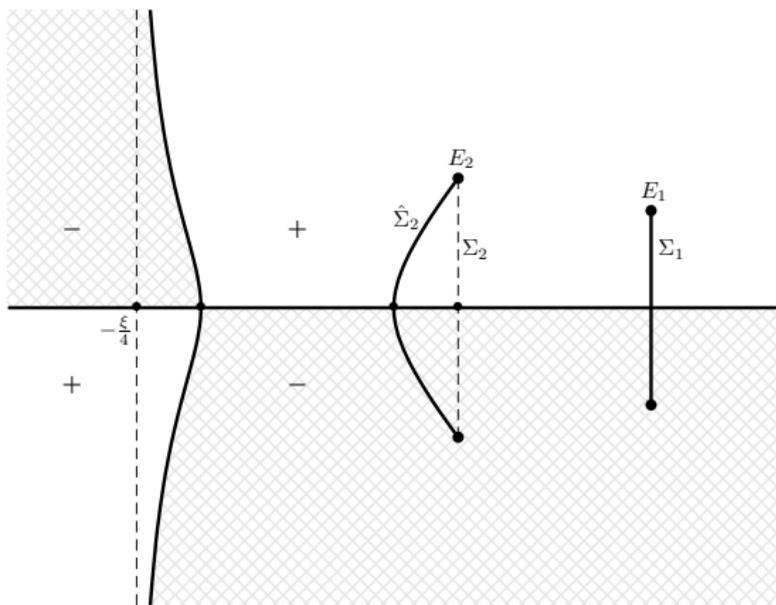
- model RHPs have **explicit solutions**: $M^{(mod)}(k) = \mathcal{E}_j(k)$, leading to

$$q(x, t) = A_j e^{-2iB_j x + 2i\omega_j t - 2i\psi(\xi)} + O(t^{-\frac{1}{2}}), \quad (-1)^j \xi \gg 1,$$

with $\psi(\xi)$ determined by A_j , B_j , with $\psi(-\infty) = \phi_1$ and $\psi(+\infty) = \phi_2$.

Signature table

“Signature table” (distribution of signs of $\text{Im } g(\xi, k)$) for $\xi \gg 1$:



Here the jump matrix on Σ_1 decays exponentially fast to I as $t \rightarrow \infty$ and thus gives negligible contribution to the large- t asymptotics.

- Introducing g -function. $M \mapsto M^{(1)}$:

$$M^{(1)}(x, t, k) := e^{-itg^{(0)}(\xi)\sigma_3} M(x, t, k) e^{it(g(\xi; k) - \theta(\xi; k))\sigma_3}.$$

$$M_+^{(1)}(x, t, k) = M_-^{(1)}(x, t, k) J^{(1)}(x, t, k), \quad k \in \mathbb{R} \cup \Sigma_1 \cup \hat{\Sigma}_2,$$

where

- $J^{(1)}(x, t, k) = \begin{pmatrix} 1 + r(k)r^*(k) & r^*(k)e^{-2itg(\xi, k)} \\ r(k)e^{2itg(\xi, k)} & 1 \end{pmatrix}, \quad k \in \mathbb{R},$

- $J^{(1)}(x, t, k) = \begin{pmatrix} \frac{a_-(k)}{a_+(k)} e^{it(g_+ - g_-)} & i \\ 0 & \frac{a_+(k)}{a_-(k)} e^{-it(g_+ - g_-)} \end{pmatrix}, \quad k \in \hat{\Sigma}_2 \cap \mathbb{C}^+$

(and by symmetry for $\hat{\Sigma}_2 \cap \mathbb{C}^-$). Here we have used (for matrix entry (12)) that $g_+ + g_- = 0$ on $\hat{\Sigma}_2$.

Since $\text{Im } g_{\pm} = 0$ on $\hat{\Sigma}_2$, we have replaced the growth by oscillations (w.r.t t)!

- $J^{(1)}(x, t, k) = \begin{pmatrix} 1 & 0 \\ \frac{ie^{-i\phi}}{a_+(k)a_-(k)} e^{2ig(\xi, k)} & 1 \end{pmatrix}, \quad k \in \Sigma_1 \cap \mathbb{C}^+$

(and by symmetry for $\Sigma_1 \cap \mathbb{C}^-$).

RHP deformations, II

Since $\text{Im } g(\xi, k) = 0$ for $k \in \mathbb{R}$, we “do lenses” along $k \in \mathbb{R}$ as in the decaying case but with $-\xi$ replaced by μ_1 .

- Preparation for lenses. $M^{(1)} \mapsto M^{(2)}$:

$$M^{(2)}(x, t, k) := M^{(1)}(x, t, k) \delta^{-\sigma_3}(\xi, k)$$

$$\text{with } \delta(\xi; k) = \exp \left\{ \frac{1}{2\pi i} \int_{-\infty}^{\mu_1(\xi)} \frac{\log(1-\lambda|r(s)|^2)}{s-k} ds \right\},$$

$$M_+^{(2)}(x, t, k) = M_-^{(2)}(x, t, k) J^{(2)}(x, t, k), \quad k \in \mathbb{R} \cup \hat{\Sigma}_2,$$

$$\bullet J^{(2)} = \begin{cases} \begin{pmatrix} 1 & \bar{r}\delta^2 e^{-2itg} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\lambda r \delta^{-2} e^{2itg} & 1 \end{pmatrix}, & k \in (\mu_1(\xi), \infty) \\ \begin{pmatrix} 1 & 0 \\ -\frac{\lambda r \delta_-^{-2} e^{2itg}}{1-\lambda|r|^2} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{\bar{r}\delta_+^2 e^{-2itg}}{1-\lambda|r|^2} \\ 0 & 1 \end{pmatrix}, & k \in (-\infty, \mu_1(\xi)) \end{cases}$$

$$\bullet J^{(2)}(x, t, k) = \begin{pmatrix} \frac{a_-(k)}{a_+(k)} e^{2itg_+(\xi, k)} & i\delta^2(\xi, k) \\ 0 & \frac{a_+(k)}{a_-(k)} e^{2itg_-(\xi, k)} \end{pmatrix}, \quad k \in \hat{\Sigma}_2 \cap \mathbb{C}^+$$

(and by symmetry for $\hat{\Sigma}_2 \cap \mathbb{C}^-$),

Notice we have also **oscillating jumps across $\hat{\Sigma}_2$** , which means that we need lenses near $\hat{\Sigma}_2$ as well, suggested by the algebraic factorization (for $\hat{\Sigma}_2 \cap \mathbb{C}^+$)

$$\begin{pmatrix} e^{2itg_+(k)} & Y \\ 0 & e^{2itg_-(k)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ Y^{-1}e^{2itg_-(k)} & 1 \end{pmatrix} \begin{pmatrix} 0 & Y \\ -Y^{-1} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -Y^{-1}e^{2itg_+(k)} & 1 \end{pmatrix}$$

But in our case, the “lenses near \mathbb{R} ” serve $\hat{\Sigma}_2$ as well.

- It would be nice to have $\text{Im } g(k) > 0$ for all k near $\hat{\Sigma}_2 \cap \mathbb{C}^+$ (and, by symmetry, $\text{Im } g(k) < 0$ for all k near $\hat{\Sigma}_2 \cap \mathbb{C}^-$).

RHP deformations, IV

- Doing lenses. $M^{(2)} \mapsto M^{(3)}$:

$$M^{(3)}(x, t, k) := M^{(2)}(x, t, k) \begin{cases} \begin{pmatrix} 1 & 0 \\ \lambda r(k) \delta^{-2}(k) e^{2itg(\xi, k)} & 1 \end{pmatrix}, & k \in \hat{\Omega}_2 \\ \begin{pmatrix} 1 & \bar{r}(k) \delta^2(k) e^{-2itg(\xi, k)} \\ 0 & 1 \end{pmatrix}, & k \in \hat{\Omega}_3 \\ \begin{pmatrix} 1 & -\frac{\bar{r}(k) \delta_+^2(k) e^{-2itg(\xi, k)}}{1 - \lambda |r|^2} \\ 0 & 1 \end{pmatrix}, & k \in \hat{\Omega}_1 \\ \begin{pmatrix} 1 & 0 \\ -\frac{\lambda r(k) \delta_-^2(k) e^{2itg(\xi, k)}}{1 - \lambda |r|^2} & 1 \end{pmatrix}, & k \in \hat{\Omega}_4 \end{cases}$$

$$M_+^{(3)}(x, t, k) = M_-^{(3)}(x, t, k) J^{(3)}(x, t, k), \quad k \in \cup_{j=1}^4 \hat{\gamma}_j \cup \hat{\Sigma}_2,$$

- $J^{(3)}$ across $\hat{\gamma}_j$, $j = 1, \dots, 4$ are triangular matrices as above
- important:** $\hat{\Sigma}_2 \in \hat{\Omega}_2 \cup \hat{\Omega}_3$; then all these **triangular jumps decay to I** as $t \rightarrow \infty$
- $J^{(3)} = J^{(3)}(\xi, k) = \begin{pmatrix} 0 & i\delta^2(\xi, k) \\ i\delta^{-2}(\xi, k) & 0 \end{pmatrix}$, $k \in \hat{\Sigma}_2$ (**doesn't involve large parameter!**)

- Getting rid of k dependence of jump. $M^{(3)} \mapsto M^{(mod)}$:

$$M^{(mod)}(x, t, k) := \Delta^{\sigma_3}(\xi, \infty) M^{(3)}(x, t, k) \Delta^{-\sigma_3}(\xi, k),$$

where Δ solves scalar RHP: find $\Delta(\xi, k)$ analytic in $\mathbb{C} \setminus \hat{\Sigma}_2$ and **bounded as $k \rightarrow \infty$** satisfying “jump” cond.

$$\Delta_+(\xi, k) \Delta_-(\xi, k) = \delta^{-2}(\xi, k), \quad k \in \hat{\Sigma}_2.$$

Then

$$M_+^{(mod)}(x, t, k) = M_-^{(mod)}(x, t, k) \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad k \in \hat{\Sigma}_2.$$

- RHP for Δ reduces to: $\log \Delta_+ + \log \Delta_- = \log \delta^{-2}$, or $\left(\frac{\log \Delta}{X}\right)_+ - \left(\frac{\log \Delta}{X}\right)_- = \frac{\log \delta^{-2}}{X_+}$, where $X(k) = \sqrt{(k - E_2)(k - \bar{E}_2)}$.
- Solution by Cauchy int. $\Delta(\xi, k) = \exp \left\{ \frac{X(k)}{2\pi i} \int_{\hat{\Sigma}_2} \frac{\log \delta^{-2}(\xi, s) ds}{X_+(s)(s-k)} \right\}$
- RHP for $M^{(mod)}$ is solved explicitly.
- Following back the transformations $M^{(mod)} \mapsto M^{(3)} \mapsto M^{(2)} \mapsto M^{(1)} \mapsto M \mapsto q$ we obtain

$$q(x, t) = A_j e^{-2iB_j x + 2i\omega_j t - 2i\psi(\xi)} + O(t^{-\frac{1}{2}}), \quad (-1)^j \xi \gg 1.$$

Why g -mechanism works

Properties of the g -function used in the deformations above:

- i $\text{Im } g_{\pm}(k) = 0, k \in \hat{\Sigma}_2.$
- ii $\text{Im } g(k) > 0$ “near $\hat{\Sigma}_2 \cap \mathbb{C}^+$ ”, $\text{Im } g(k) < 0$ “near $\hat{\Sigma}_2 \cap \mathbb{C}^-$ ”.
- iii $\text{Im } g(k) = 0, k \in \mathbb{R}.$
- iv $g_+(k) + g_-(k) = 0, k \in \hat{\Sigma}_2.$
- v $\text{Im } g(k) > 0, k \in \Sigma_1 \cap \mathbb{C}^+$ and $\text{Im } g(k) < 0, k \in \Sigma_1 \cap \mathbb{C}^-.$
- vi $g(\xi, k) = 2k^2 + \xi k + g^0(\xi) + O(k^{-1})$

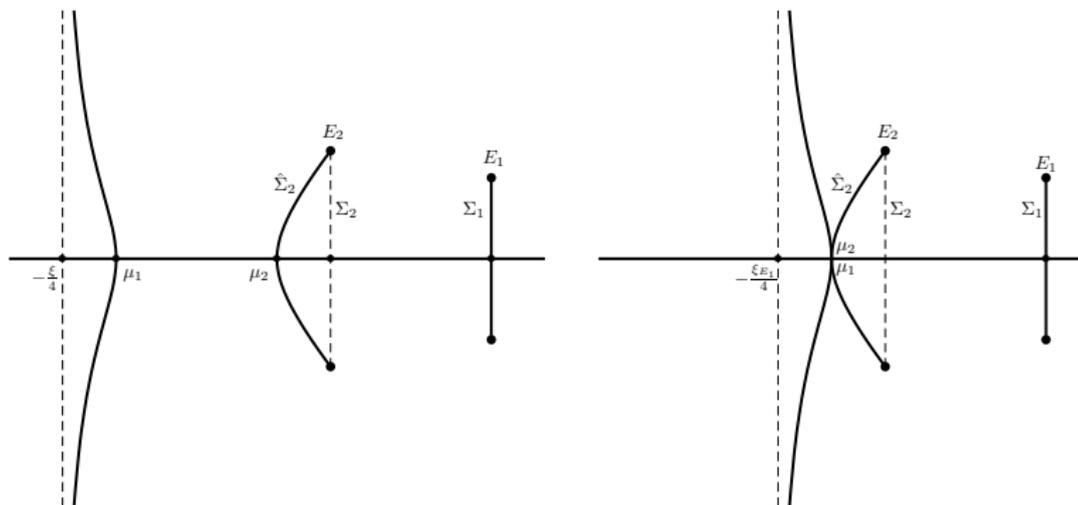
These properties determine $g(k)$ uniquely!

$B_2 < B_1$: rarefaction, I

Decreasing $|\xi|$, this type of g -function stops to work when

- Ⓛ either $\mu_1(\xi)$ and $\mu_2(\xi)$ collides,
- Ⓜ or the infinite branch of $\{k : \text{Im } g(k) = 0\}$ hits E_j .

If $B_2 < B_1$, it is **always (i)** that occurs:



$\mu_1(\xi)$ and $\mu_2(\xi)$ merge at $\xi = C_2 = -4B_2 + 4\sqrt{2}A_2$ and $\xi = C_1 = -4B_1 - 4\sqrt{2}A_2$, which signifies the **ends of the plane wave sectors** and the **necessity of a new g -function**.

$B_2 < B_1$: rarefaction, II

The transition to the new sectors is characterized by the emergence of two complex zeros of $g'(\xi; k)$, $\beta(\xi)$ and $\bar{\beta}(\xi)$, at the place of merging real zeros (keeping one real zero μ):

$$g'(\xi; k) = 4 \frac{(k - \mu(\xi))(k - \beta(\xi))(k - \bar{\beta}(\xi))}{\sqrt{(k - E_2)(k - \bar{E}_2)(k - \beta(\xi))(k - \bar{\beta}(\xi))}}$$

for $-4B_2 < \xi < -4B_2 + 4\sqrt{2}A_2$; similarly for $-4B_1 - 4\sqrt{2}A_1 < \xi < -4B_1$.

- ① The associated model RHP has **jumps across two arcs**: $\Sigma = \tilde{\Sigma}_1 \cup \tilde{\Sigma}_2$,

$$M_+^{(mod)}(x, t, k) = M_-^{(mod)}(x, t, k) \begin{pmatrix} 0 & ie^{ixD_j x + itG_j t + \phi_j} \\ ie^{-ixD_j x - itG_j t - \phi_j} & 0 \end{pmatrix}, k \in \tilde{\Sigma}_j, \\ j = 1, 2 \text{ (on each arc, the jump is independent of } k!).$$

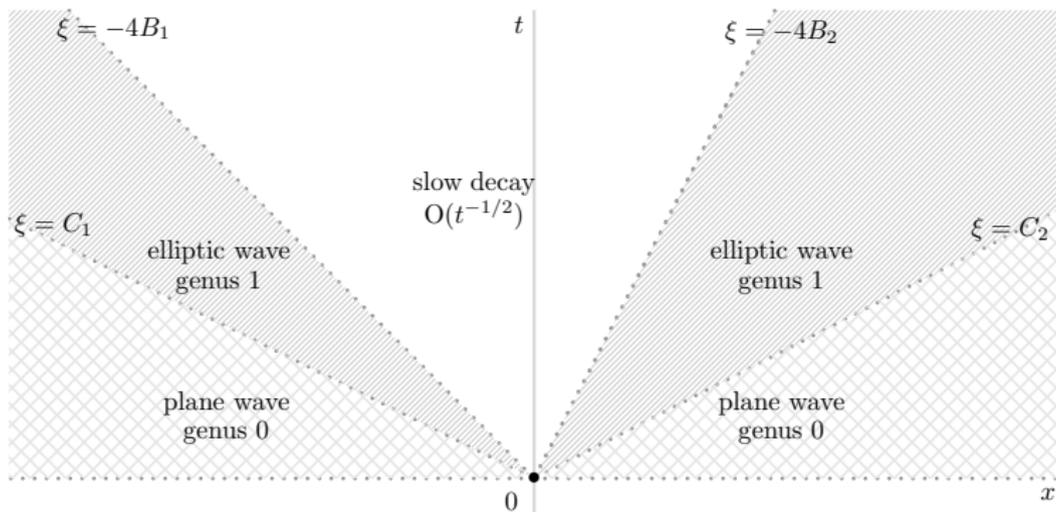
- ② The solution can be given in terms of **Riemann theta functions** of dimension 1 (**genus-1 solution**):

$$q(x, t) = \hat{A}_j \frac{\Theta(\beta_j t + \gamma_j)}{\Theta(\beta_j t + \tilde{\gamma}_j)} e^{i\nu_j t} + O(t^{-1/2}),$$

where all coefficients are functions of $\xi = \frac{x}{t}$.

$B_2 < B_1$: rarefaction, III

For $-4B_1 < \xi < -4B_2$: the original phase function $it\theta(\xi; k)$ is such that the jumps in the original RH problem across both arcs Σ_1 and Σ_2 **decay (exponentially) to the identity matrix** as $t \rightarrow \infty$ and thus one can keep $g(\xi; k) = \theta(\xi; k)$ for this range. It follows that the asymptotic analysis in this sector essentially follows that in the case of the zero background, giving $q(x, t) = O(t^{-1/2})$.



The case $B_2 > B_1$ turns out to be much richer: there are several asymptotic scenarios depending on the values $A_1/(B_2 - B_1)$ and $A_2/(B_2 - B_1)$, each being characterized by a set of appropriate g -functions; but there are always two infinite branches of $\text{Im } g(\xi; k) = 0$: the real axis and a branch approaching the vertical line $\text{Re } k = -\xi/4$.

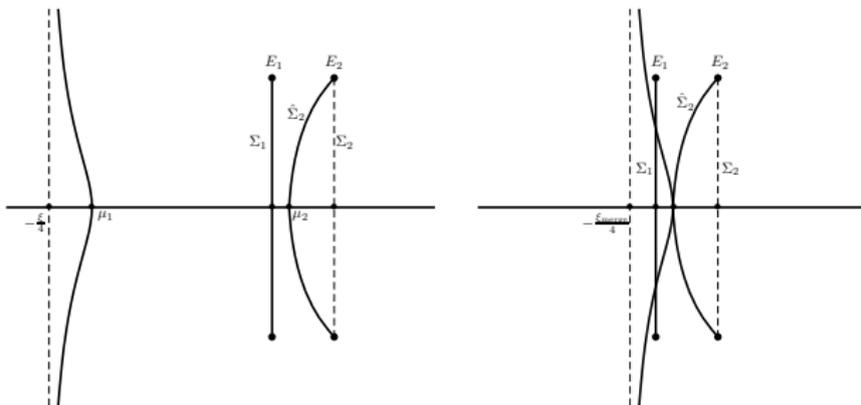
In what follows, for simplicity we consider the symmetric case where $A_1 = A_2 = A$ and $B_2 = -B_1 = B > 0$. In this case, the asymptotic picture is symmetric in ξ and thus it is sufficient to consider ξ ranging from $+\infty$ down to $\xi = 0$.

As ξ decreases from $+\infty$, there are three possibilities for ending the plane wave sector:

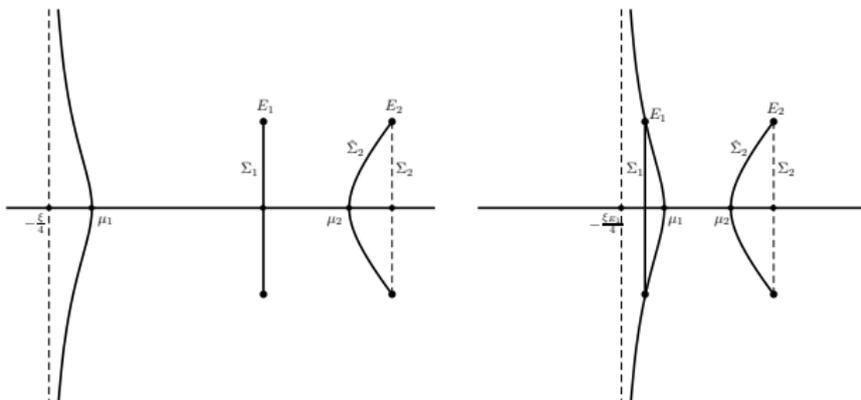
- 1 If $\frac{A}{B} > \frac{2}{7}(2 + 3\sqrt{2})$, then μ_1 and μ_2 merge, at $\xi = \xi_{merge}^{(1)} = -4B + 4\sqrt{2}A$, **before** the infinite branch hits E_1 and \bar{E}_1 .
- 2 If $\frac{A}{B} = \frac{2}{7}(2 + 3\sqrt{2})$, then the infinite branch hits E_1 and \bar{E}_1 **at the same moment** $\xi = \frac{4}{7}(4\sqrt{2} + 5)B$ as μ_1 and μ_2 merge.
- 3 If $\frac{A}{B} < \frac{2}{7}(2 + 3\sqrt{2})$, then the **infinite branch hits E_1 and \bar{E}_1** , at $\xi = \xi_{E_1}^{(1)} = 2(B + \sqrt{A^2 + B^2})$, **before** μ_1 and μ_2 merge.

$B_2 > B_1$: shock, III

Case (i):



Case (iii):



$B_2 > B_1$, case (i): $\frac{A}{B} > \frac{2}{7}(2 + 3\sqrt{2})$. Four types of asymptotics, I

- the asymptotics in the range $\xi_{E_1}^{(2)} < \xi < \xi_{merge}^{(1)}$ is characterized, similarly to the rarefaction case, by the **genus-1 g -function**

$$g'(\xi; k) = 4 \frac{(k - \mu(\xi))(k - \beta(\xi))(k - \bar{\beta}(\xi))}{\sqrt{(k - E_2)(k - \bar{E}_2)(k - \beta(\xi))(k - \bar{\beta}(\xi))}}$$

- The left end $\xi_{E_1}^{(2)}$ of the genus-1 range: when the infinite branch of g function above hits E_1 and \bar{E}_1 : for $\xi < \xi_{E_1}^{(2)}$, a new, **genus-3 g -function**

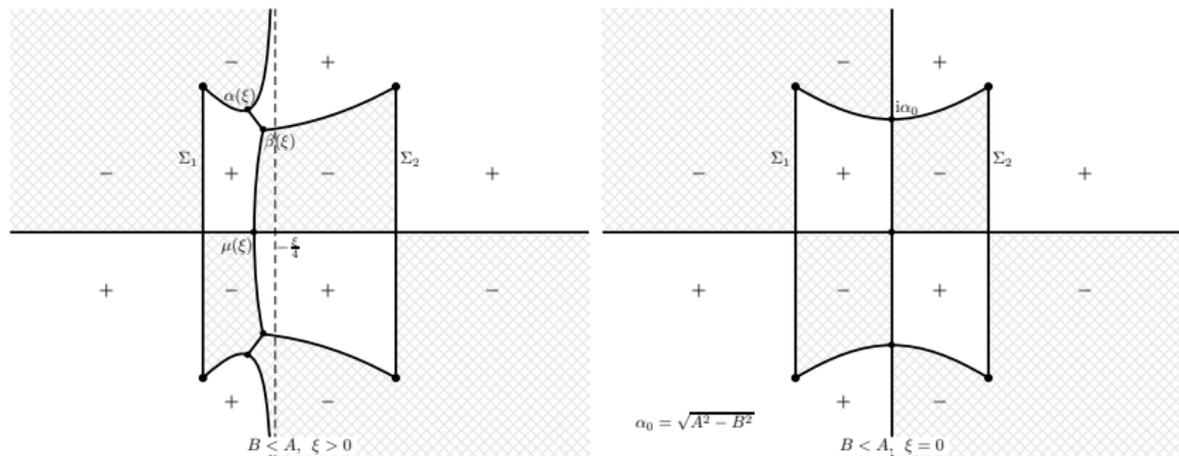
$$g'(\xi; k) = 4 \frac{(k - \mu(\xi))(k - \beta(\xi))(k - \bar{\beta}(\xi))(k - \alpha(\xi))(k - \bar{\alpha}(\xi))}{\sqrt{(k - E_1)(k - \bar{E}_1)(k - E_2)(k - \bar{E}_2)(k - \beta(\xi))(k - \bar{\beta}(\xi))(k - \alpha(\xi))(k - \bar{\alpha}(\xi))}}$$

becomes appropriate, with $\alpha(\xi)$ “emerging from E_1 ”.

- The left end of the genus-3 range is $\xi = 0$: as $\xi \rightarrow 0$, $\alpha(\xi)$ and $\beta(\xi)$ both approach a single point $i\alpha_0 = i\sqrt{A^2 - B^2}$ whereas $\mu(\xi) \rightarrow 0$, and the g -function takes a **genus-1** form:

$$g'(0; k) = 4 \frac{k(k^2 + \alpha_0^2)}{\sqrt{(k - E_1)(k - \bar{E}_1)(k - E_2)(k - \bar{E}_2)}}$$

$B_2 > B_1$, case (i): $\frac{A}{B} > \frac{2}{7}(2 + 3\sqrt{2})$. Four types of asymptotics, II



$$\frac{A}{B} > \frac{2}{7}(2 + 3\sqrt{2})$$

$\xi = 0$	$0 < \xi < \xi_{E_1}^{(2)}$	$\xi = \xi_{E_1}^{(2)}$	$\xi_{E_1}^{(2)} < \xi < \xi_{merge}^{(1)}$	$\xi = \xi_{merge}^{(1)}$	$\xi > \xi_{merge}^{(1)}$
genus 1	genus 3		genus 1		genus 0
α, β merge		the infinite branch hits E_1, \bar{E}_1		the real zeros μ_1, μ_2 merge	

RHP for genus- N solutions

- The model RHP associated with a genus- N solution has **jumps across**

$$N + 1 \text{ arcs: } \Sigma = \cup_{j=1}^{N+1} \tilde{\Sigma}_j,$$

$$M_+^{(mod)}(x, t, k) =$$

$$M_-^{(mod)}(x, t, k) \begin{pmatrix} 0 & ie^{ixD_jx+itG_jt+\phi_j} \\ ie^{-ixD_jx-itG_jt-\phi_j} & 0 \end{pmatrix}, k \in \tilde{\Sigma}_j,$$

$j = 1, \dots, N + 1$ (on each arc, the jump is independent of $k!$); D_j, G_j are determined by the arc ends.

- The solution $M^{(mod)}$ (and, consequently, $q^{(ass)}(x, t)$) can be given in terms of Riemann theta functions of dimension N (**genus- N solution**):

$$q^{(ass)}(x, t) = \alpha \frac{\Theta(\mathbf{B}t + \mathbf{F})}{\Theta(\mathbf{B}t + \tilde{\mathbf{F}})} e^{i\nu t},$$

\mathbf{B} and \mathbf{F} are N -component vectors; all coefficients are functions of ξ .

- The Riemann theta function $\Theta(u_1, \dots, u_N)$ associated with τ (matrix of periods) is defined for $\mathbf{u} \in \mathbb{C}^N$ by the Fourier series

$$\Theta(u_1, \dots, u_N) = \sum_{\mathbf{l} \in \mathbb{Z}^N} \exp \{ \pi i (\tau \mathbf{l}, \mathbf{l}) + 2\pi i (\mathbf{l}, \mathbf{u}) \},$$

where $(\mathbf{l}, \mathbf{u}) = l_1 u_1 + \dots + l_n u_N$.

$B_2 > B_1$, case (ii): $\frac{A}{B} = \frac{2}{7}(2 + 3\sqrt{2})$. Three types of asymptotics

This case, comparing to Case (i), is characterized by the equality $\xi_{E_1}^{(2)} = \xi_{merge}^{(1)}$; thus, the genus-3 range $0 < \xi < \xi_{merge}^{(1)}$ is directly adjacent to the plane wave range (no genus-1 range).

$$\frac{A}{B} = \frac{2}{7}(2 + 3\sqrt{2})$$

$\xi = 0$	$0 < \xi < \xi_{E_1}^{(2)}$	$\xi = \xi_{E_1}^{(2)} = \xi_{merge}^{(1)}$	$\xi > \xi_{merge}^{(1)}$
genus 1	genus 3		genus 0
α, β merge		the infinite branch hits E_1, \bar{E}_1 and the real zeros μ_1, μ_2 merge	

In this case, the infinite branch of the plane wave g -function hits E_1 and \bar{E}_1 , leading, for ξ just to the left of $\xi_{E_1}^{(1)}$, to a **genus-2 g -function** with the complex zeros at $\alpha(\xi)$ and $\bar{\alpha}(\xi)$ (“emerging” from E_1 and \bar{E}_1):

$$g'(\xi; k) = 4 \frac{(k - \mu_1(\xi))(k - \mu_2(\xi))(k - \alpha(\xi))(k - \bar{\alpha}(\xi))}{\sqrt{(k - E_1)(k - \bar{E}_1)(k - E_2)(k - \bar{E}_2)}(k - \alpha(\xi))(k - \bar{\alpha}(\xi))}$$

- There is the **second “bifurcation value”** of A/B : $A/B = 1$, separating, when ξ is decreasing, **three different scenarios** of appearance of further asymptotic ranges.

- Case (iii-a): $\frac{A}{B} < 1$. In this case, the left end $\xi_{merge}^{(2)}$ of the genus-2 sector corresponds to merging of $\alpha(\xi)$ and $\bar{\alpha}(\xi)$ into a third real zero, μ_0 , leading to a genus-1 ansatz (different from above!) for the g -function in $-\xi_{merge}^{(2)} < \xi < \xi_{merge}^{(2)}$ (thus including $\xi = 0$).

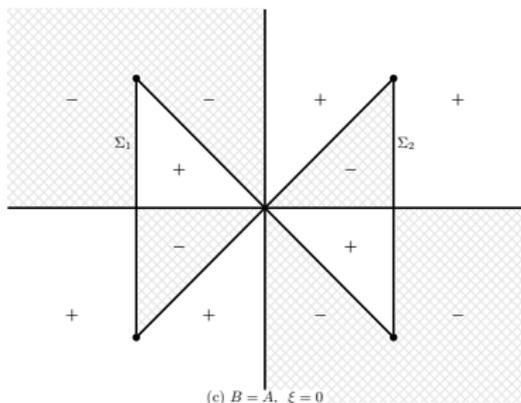
$$g'(\xi; k) = 4 \frac{(k - \mu_1(\xi))(k - \mu_2(\xi))(k - \mu_0(\xi))}{\sqrt{(k - E_1)(k - \bar{E}_1)(k - E_2)(k - \bar{E}_2)}}$$

$$0 < \frac{A}{B} < 1$$

$0 \leq \xi < \xi_{merge}^{(2)}$	$\xi = \xi_{merge}^{(2)}$	$\xi_{merge}^{(2)} < \xi < \xi_{E_1}$	$\xi = \xi_{E_1}$	$\xi > \xi_{E_1}$
genus 1		genus 2		genus 0
residual region	$\alpha, \bar{\alpha}$ merge into a third real zero	transition region	infinite branch hits E_1, \bar{E}_1	wave plane region

$B_2 > B_1$, case (iii): $\frac{A}{B} < \frac{2}{7}(2 + 3\sqrt{2})$. Three different asymptotic scenarios, III

- Case (iii-b): $\frac{A}{B} = 1$. In this case, $\xi_{merge}^{(2)}$ becomes 0 and thus the genus-1 range from Case (iii-a) shrinks to a single value $\xi = 0$, with $\alpha_0 = 0$.



$$\frac{A}{B} = 1$$

$\xi = 0$	$0 < \xi < \xi_{E_1}$	$\xi = \xi_{E_1}$	$\xi > \xi_{E_1}$
genus 1	genus 2		genus 0
$\alpha, \bar{\alpha}, \mu_1$ all merge at the origin		the infinite branch hits E_1, \bar{E}_1	

$B_2 > B_1$, case (iii): $\frac{A}{B} < \frac{2}{7}(2 + 3\sqrt{2})$. Three different asymptotic scenarios, IV

- Case (iii-c): $1 < \frac{A}{B} < \frac{2}{7}(2 + 3\sqrt{2})$. In this case, the left end $\xi_{merge}^{(2)}$ of the genus-2 sector $\xi_{merge}^{(2)} < \xi < \xi_{E_1}^{(1)}$ corresponds to merging of the real zeros $\mu_1(\xi)$ and $\mu_2(\xi)$ and emerging a pair of complex zeros $\beta(\xi)$ and $\bar{\beta}(\xi)$ thus leading to the genus-3 g -function for the range $0 < \xi < \xi_{merge}^{(2)}$.

$$1 < \frac{A}{B} < \frac{2}{7}(2 + 3\sqrt{2})$$

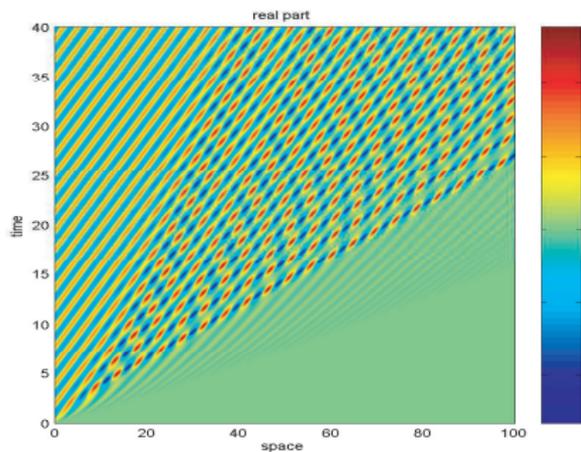
$\xi = 0$	$0 < \xi < \xi_{merge}^{(2)}$	$\xi = \xi_{merge}^{(2)}$	$\xi_{merge}^{(2)} < \xi < \xi_{E_1}^{(1)}$	$\xi = \xi_{E_1}^{(1)}$	$\xi > \xi_{E_1}^{(1)}$
genus 1	genus 3		genus 2		genus 0
α, β merge		the real zeros μ_1, μ_2 merge		the infinite branch hits E_1, \bar{E}_1	

Initial boundary value (IBV) problem for focusing NLS

Let $q(x, t)$ be the solution of the IBV problem for focusing NLS:

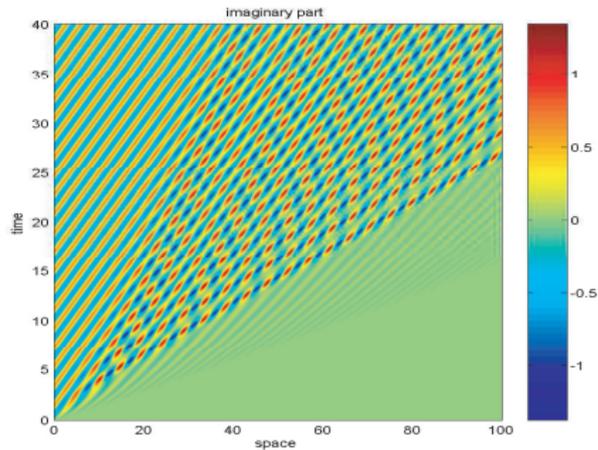
- $iq_t + q_{xx} + 2|q|^2q = 0, \quad x > 0, t > 0,$
- $q(x, 0) = q_0(x)$ fast decaying as $x \rightarrow +\infty$
- $q(0, t) = g_0(t)$ time-periodic $\boxed{g_0(t) = \alpha e^{2i\omega t}}$ $\alpha > 0, \omega \in \mathbb{R}$
($q(0, t) - \alpha e^{2i\omega t} \rightarrow 0$ as $t \rightarrow +\infty$)
- ▶ **Question:** How does $q(x, t)$ behave for large t ?
- ▶ **Numerics:** Qualitatively different pictures for parameter ranges:
 - (i) $\omega < -3\alpha^2$
 - (ii) $-3\alpha^2 < \omega < \frac{\alpha^2}{2}$
 - (iii) $\omega > \frac{\alpha^2}{2}$

Numerics for $\omega < -3\alpha^2$, I



Real part $\text{Re } q(x, t)$

$$\alpha = \sqrt{3/8}, \quad \omega = -13/8$$



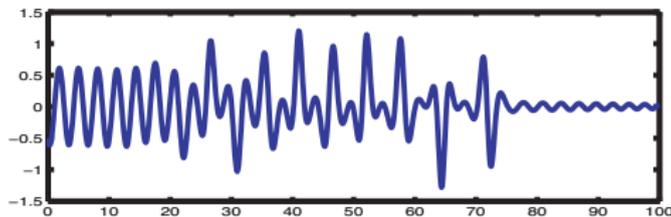
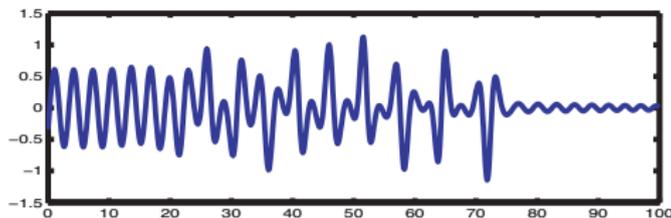
Imaginary part $\text{Im } q(x, t)$

$$q_0(x) \equiv 0, \quad g_0(t) = \alpha e^{2i\omega t} + O(e^{-10t^2})$$

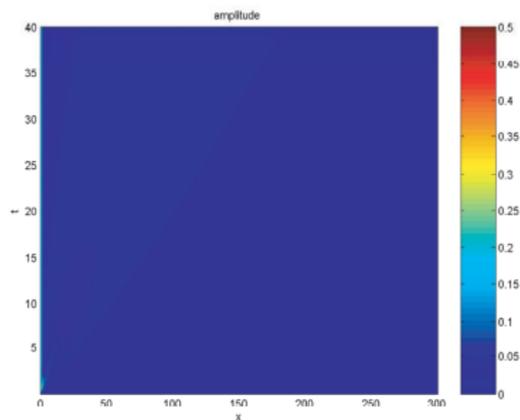
Numerics for $\omega < -3\alpha^2$, II

Numerical solution for $t = 20$, $0 < x < 100$.

Upper: real part $\text{Re } q(x, 20)$. Lower: imaginary part $\text{Im } q(x, 20)$.

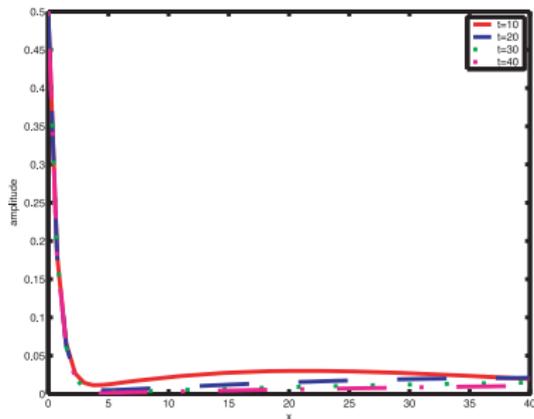


Numerics for $\omega \geq \alpha^2/2$



Amplitude of $q(x, t)$

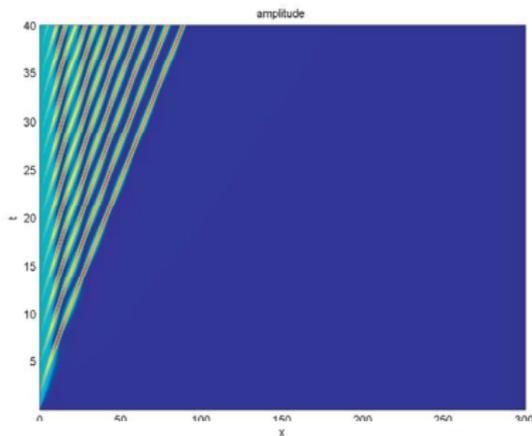
$$\alpha = 0.5, \quad \omega = 1, \quad \omega \geq \alpha^2/2,$$



Amplitude for $t = 10, \dots$

$$q_0(x) \equiv 0, \quad g_0(t) = \alpha e^{2i\omega t} + O(e^{-10t^2})$$

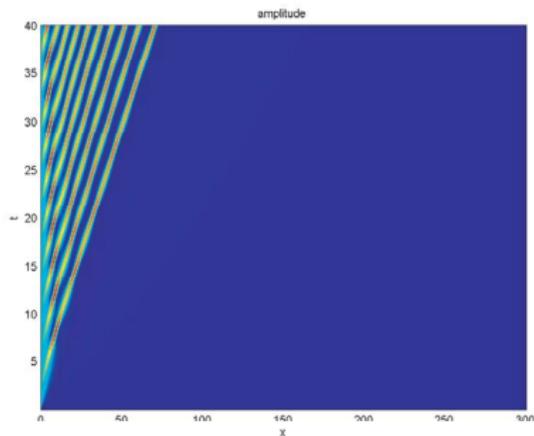
Amplitude of $q(x, t)$



$$\alpha = 0.5$$

$$\omega = -2\alpha^2 = -0.5$$

$$g_0(x) \equiv 0,$$



$$\alpha = 0.5$$

$$\omega = -\alpha^2 = -0.25$$

$$g_0(t) = \alpha e^{2i\omega t} + O(e^{-10t^2})$$

General scheme for boundary value problems via IST

Goal: adapt (generalize) the RHP approach to boundary-value (initial-boundary value) problems for integrable equations.

Data for an IBV problem (e.g. in domain $x > 0, t > 0$):

(i) Initial data: $q(x, 0) = q_0(x), x > 0$

(ii) Boundary data: $q(0, t) = g_0(t), q_x(0, t) = g_1(t), \dots$

Question: How many boundary values?

For a well-posed problem: roughly "half" the number of x -derivatives.

For NLS: one b.c. (e.g., $q(0, t) = g_0(t)$).

General idea for IBV: use **both** equations of the Lax pair as **spectral problems**.

Common difficulty: spectral analysis of the t -equation on the boundary ($x = 0$) involves **more functions** (boundary values $q(0, t), q_x(0, t), \dots$) than possible data for a **well-posed problem**.

Half-line problem for NLS

For NLS: t -equation

$$\Phi_t = \left(-2ik^2\sigma_3 + 2k \begin{pmatrix} 0 & q(x,t) \\ \lambda\bar{q}(x,t) & 0 \end{pmatrix} + \begin{pmatrix} -i\lambda|q|^2 & iq_x \\ -i\lambda\bar{q}_x & i\lambda|q|^2 \end{pmatrix} \right) \Phi$$

involves q and q_x ; hence for the (direct) spectral analysis at $x = 0$ one needs $q(0,t)$ **and** $q_x(0,t)$. Assume that we are given the both. Then one can define two sets of spectral functions coming from the spectral analysis of x -equation and t -equation.

- ① $q(x,0) \mapsto \{a(k), b(k)\}$ (direct problem for x -equ); $s \equiv \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$
- ② $\{q(0,t), q_x(0,t)\} \mapsto \{A(k), B(k)\}$ (direct problem for t -equ)
- ③ From the spectral functions $\{a(k), b(k), A(k), B(k)\}$, the jump matrix $J(x,t,k)$ for the Riemann-Hilbert problem is constructed:

$$\begin{aligned} \{a(k), b(k), A(k), B(k)\} &\mapsto J_0(k) \\ J(x,t,k) &= e^{-i(2k^2t+kx)\sigma_3} J_0(k) e^{i(2k^2t+kx)\sigma_3} \end{aligned}$$

(notice the same explicit dependence on (x,t) !) Jump conditions are across a **contour** Σ determined by the asymptotics of $g_0(t)$ **and** $g_1(t)$

- ④ Similarly to the Cauchy (whole-line) problem, the solution of the IBV (half-line) problem is given in terms of the solution of the RHP:
 $q(x,t) = 2i \lim_{k \rightarrow \infty} (kM_{12}(x,t,k)).$

Given $q(x, t)$, how to construct $M(x, t, k)$?

Define $\Psi_j(x, t, k)$, $j = 1, 2, 3$ solutions (2×2) of the Lax pair equations specified at all “corners” of the (x, t) -domain where the IBV problem is formulated:

- ① $\Psi_1(0, T, k) = e^{-2ik^2 T \sigma_3}$ ($\Psi_1(0, t, k) \simeq e^{-2ik^2 t \sigma_3}$ as $t \rightarrow \infty$)
- ② $\Psi_2(0, 0, k) = I$
- ③ $\Psi_3(x, 0, k) \simeq e^{-ikx \sigma_3}$ as $x \rightarrow \infty$

Being **simultaneous** solutions of x - and t -equation, they are related by two scattering relations:

- ① $\Psi_3(x, t, k) = \Psi_2(x, t, k)s(k) \quad s = \begin{pmatrix} \bar{a} & b \\ -\bar{b} & a \end{pmatrix}$
- ② $\Psi_1(x, t, k) = \Psi_2(x, t, k)S(k; T) \quad S = \begin{pmatrix} \bar{A} & B \\ -\bar{B} & A \end{pmatrix}$

Then M is constructed from columns of Ψ_1 , Ψ_2 and Ψ_3 following their **analyticity and boundedness** properties w.r.t k , and the jump relation for RHP is re-written scattering relations (i)+(ii) for Ψ_j .

For NLS in half-strip ($T < \infty$) or in quarter plane ($T = \infty$) with $g_j(t) \rightarrow 0$ as $t \rightarrow \infty$: first column of $\Psi_1(x, t, k)e^{(-ikx - 2ik^2 t) \sigma_3}$ is bounded in $\{k : \text{Im } k \geq 0, \text{Im } k^2 \leq 0\}$, etc., which leads to $\Sigma = \mathbb{R} \cup i\mathbb{R}$.

- Given $q_0(x)$, determine $a(k)$, $b(k)$: $a(k) = \Phi_2(0, k)$, $b(k) = \Phi_1(0, k)$, where vector $\Phi(x, k)$ is the solution of the x -equation evaluated at $t = 0$:

$$\Phi_x + ik\sigma_3\Phi = Q(x, 0, k)\Phi, \quad 0 < x < \infty, \operatorname{Im} k \geq 0$$

$$\Phi(x, k) = e^{ikx} \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + o(1) \right) \text{ as } x \rightarrow \infty,$$

$$Q(x, 0, k) = \begin{pmatrix} 0 & q_0(x) \\ -\bar{q}_0(x) & 0 \end{pmatrix}$$

- Given $\{g_0(t), g_1(t)\}$, determine $A(k; T)$, $B(k; T)$:

$$A(k; T) = e^{2ik^2T} \bar{\tilde{\Phi}}_1(T, \bar{k}), \quad B(k; T) = -e^{2ik^2T} \tilde{\Phi}_2(T, k),$$

where vector $\tilde{\Phi}(x, k)$ is the solution of the t -equation evaluated at $x = 0$:

$$\tilde{\Phi}_t + 2ik^2\sigma_3\tilde{\Phi} = \tilde{Q}(0, t, k)\tilde{\Phi}, \quad 0 < t < T,$$

$$\tilde{\Phi}(0, k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\tilde{Q}(0, t, k) = \begin{pmatrix} -|g_0(t)|^2 & 2kg_0(t) - ig_1(t) \\ 2k\bar{g}_0(t) + i\bar{g}_1(t) & |g_0(t)|^2 \end{pmatrix}$$

- Contour: $\Sigma = \mathbb{R} \cup i\mathbb{R}$
- Jump matrix: $J(x, t, k) = e^{-(ikx+2ik^2t)\sigma_3} J_0(k) e^{(ikx+2ik^2t)\sigma_3}$ with

$$J_0(k) = \begin{cases} \begin{pmatrix} 1 + |r(k)|^2 & \bar{r}(k) \\ r(k) & 1 \end{pmatrix}, & k > 0, \\ \begin{pmatrix} 1 & 0 \\ C(k; T) & 1 \end{pmatrix}, & k \in i\mathbb{R}_+, \\ \begin{pmatrix} 1 & \bar{C}(\bar{k}; T) \\ 0 & 1 \end{pmatrix}, & k \in i\mathbb{R}_-, \\ \begin{pmatrix} 1 + |r(k) + C(k; T)|^2 & \bar{r}(k) + \bar{C}(\bar{k}; T) \\ r(k) + C(k; T) & 1 \end{pmatrix}, & k < 0, \end{cases}$$

where $r(k) = \frac{\bar{b}(k)}{a(k)}$, $C(k; T) = -\frac{\overline{B(\bar{k}; T)}}{a(k)d(k; T)}$ with $d = a\bar{A} + b\bar{B}$

(also works for $T = +\infty$ if $g_0(t), g_1(t) \rightarrow 0, t \rightarrow \infty$)

Compatibility of boundary values: Global Relation

- The fact that the set of initial and boundary values $\{q_0(x), g_0(t), g_1(t)\}$ cannot be prescribed arbitrarily (as data for IBVP) must be reflected in spectral terms.

Indeed, from scattering relations (i)+(ii):

$S^{-1}(k; T)s(k) = \Psi^{-1}(x, t, k)\Psi_3(x, t, k)$. Evaluating this at $x = 0, t = T$ and using analyticity and boundedness properties of Ψ_j , one deduces for the (12) entry of $S^{-1}s$:

$$A(k; T)b(k) - a(k)B(k; T) = O\left(\frac{e^{4ik^2T}}{k}\right), \quad k \rightarrow \infty$$

$$k \in D = \{\operatorname{Im} k \geq 0, \operatorname{Re} k \geq 0\}$$

- This relation is called **Global Relation (GR)**: it characterizes the compatibility of $\{q_0(x), g_0(t), g_1(t)\}$ in spectral terms.

Typical theorem: Consider the IBVP with given $q_0(x)$ and $g_0(t)$. Assume $g_1(t)$ is such that the spectral functions $\{a(k), b(k), A(k), B(k)\}$ calculated from $\{q_0(x), g_0(t), g_1(t)\}$ satisfy Global Relation. Then the solution of the IBVP is given in terms of the solution of the RHP above. Moreover, it satisfies also the b.c. $q_x(0, t) = g_1(t)$.

- ① construct the **Dirichlet-to-Neumann** map

$$\{q_0(x), g_0(t)\} \mapsto g_1(t):$$

$$g_1(t) = -\frac{i}{\pi} \int_{\partial D} dk e^{-ik^2 t} k \left(\int_0^\infty q_0(x) e^{ikx} dx \right) + \frac{1}{\pi} \int_{\partial D} dk \left\{ ik^2 \int_0^t e^{ik^2(\tau-t)} g_0(\tau) d\tau - g_0(t) \right\}$$

- ② solve the IBVP:

$$q(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx - ik^2 t} (\hat{q}_0(k) - \hat{q}_0(-k)) dk - \frac{1}{\pi} \int_{-\infty}^{\infty} dk e^{-ikx - ik^2 t} k \left(\int_0^t e^{ik^2 \tau} g_0(\tau) d\tau \right)$$

where $\hat{q}_0(k) = \int_0^\infty e^{ikx} q_0(x) dx$.

Using Global Relation for NLS

GR can also be used to describe the **Dirichlet-to-Neumann** map:

$$g_1(t) = \frac{g_0(t)}{\pi} \int e^{-2ik^2t} (\tilde{\Phi}_2(t, k) - \tilde{\Phi}_2(t, -k)) dk + \frac{4i}{\pi} \int e^{-2ik^2t} kr(k) \overline{\tilde{\Phi}_2(t, \bar{k})} dk \\ + \frac{2i}{\pi} \int e^{-2ik^2t} (k[\tilde{\Phi}_1(t, k) - \tilde{\Phi}_1(t, -k)] + ig_0(t)) dk \quad \left(\int = \int_{\partial D} \right)$$

But: nonlinear! (g_1 is involved in the construction of $\tilde{\Phi}_j$)

- In the small-amplitude limit, this reduces to a **formula** giving $g_1(t)$ in terms of $g_0(t)$ and $q_0(x)$ (via $r(k)$); here NLS reduces to a **linear** equation $iq_t + q_{xx} = 0$.
- This suggests **perturbative** approach: given $g_0(t)$ say periodic with small amplitude, derive a perturbation series for $g_1(t)$, with periodic terms.

For $T = \infty$: the approach can be implemented for boundary values **non-decaying as $t \rightarrow \infty$** . But for this: one needs **correct large-time behavior of $g_1(t)$** complying with that of the given $g_0(t)$; this is because **both $g_0(t)$ and $g_1(t)$** determine the spectral problem for t -equation and thus the structure of associated spectral functions $A(k), B(k)$.

Dirichlet-to-Neumann map

Let $q(0, t) = \alpha e^{2i\omega t}$ ($q(0, t) - \alpha e^{2i\omega t} \rightarrow 0, t \rightarrow \infty$)

Neumann values ($q_x(0, t)$):

- from numerics:

$$q_x(0, t) \simeq c e^{2i\omega t} \quad c = \begin{cases} 2i\alpha \sqrt{\frac{\alpha^2 - \omega}{2}}, & \omega \leq -3\alpha^2 \\ \alpha \sqrt{2\omega - \alpha^2}, & \omega \geq \frac{\alpha^2}{2} \end{cases}$$

- theoretical (asymptotic) results: agree with numerics (for all $x > 0, t > 0$) provided c as above.

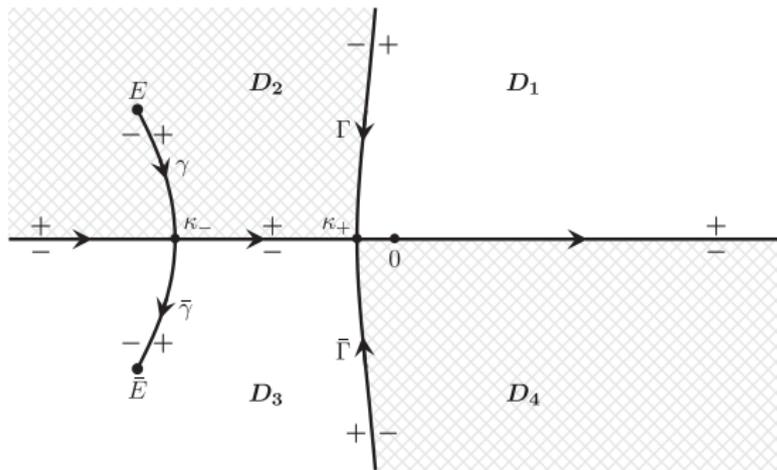
Question: Why these particular values of c ?

(the spectral mapping $\{g_0, g_1\} \mapsto \{A(k), B(k)\}$ is well-defined for any $c \in \mathbb{C}$!)

Idea: Use the global relation (its impact on analytic properties of $A(k), B(k)$) to specify admissible values of parameters α, ω, c .

The RHP for NLS: the contour

for $\omega < -3\alpha^2$, assuming $q_x(0, t) \sim 2i\alpha\beta e^{2i\omega t}$



$$\Sigma = \mathbb{R} \cup \gamma \cup \bar{\gamma} \cup \Gamma \cup \bar{\Gamma} \text{ with } E = -\beta + i\alpha.$$

The RHP for NLS: the jump matrix

$$J(x, t; k) = \begin{cases} \begin{pmatrix} 1 & -\bar{\rho}(k)e^{-2it\theta(k)} \\ -\rho(k)e^{2it\theta(k)} & 1 + |\rho(k)|^2 \end{pmatrix} & k \in (-\infty, \kappa_+), \\ \begin{pmatrix} 1 & -\bar{r}(k)e^{-2it\theta(k)} \\ -r(k)e^{2it\theta(k)} & 1 + |r(k)|^2 \end{pmatrix} & k \in (\kappa_+, \infty), \\ \begin{pmatrix} 1 & 0 \\ c(k)e^{2it\theta(k)} & 1 \end{pmatrix} & k \in \Gamma, \\ \begin{pmatrix} 1 & \bar{c}(\bar{k})e^{-2it\theta(k)} \\ 0 & 1 \end{pmatrix} & k \in \bar{\Gamma}, \\ \begin{pmatrix} 1 & 0 \\ f(k)e^{2it\theta(k)} & 1 \end{pmatrix} & k \in \gamma, \\ \begin{pmatrix} 1 & -\bar{f}(\bar{k})e^{-2it\theta(k)} \\ 0 & 1 \end{pmatrix} & k \in \bar{\gamma}. \end{cases}$$

where

$$\theta(k) = \theta(k, \xi) = 2k^2 + 4\xi k$$

with

$$\xi = \frac{x}{4t}$$

- I. Integrable nonlinear PDE; [Riemann-Hilbert problem as a tool](#)
- T.Trogdon and S.Olver, Riemann-Hilbert problems, their numerical solutions, and the computation of nonlinear special functions, SIAM, Philadelphia, 2012.
 - P.Deift, Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach. Courant Lecture Notes in Mathematics, New York, AMS, 2000.
 - L.D.Faddeev and L.Takhtajan, Hamiltonian Methods in the Theory of Solitons, Springer, 1987.
 - M.J.Ablowitz and H.Segur, Solitons and the Inverse Scattering Transform, Philadelphia, PA, SIAM, 1981.

II. Problems with step-like initial data

- A. Boutet de Monvel, V. Kotlyarov and D. Shepelsky, Focusing NLS equation: long-time dynamics of step-like initial data, International Mathematics Research Notices (2011), no. 7, 1613-1653.
- G. Biondini, S. Li and D. Mantzavinos, Soliton trapping, transmission and wake in modulationally unstable media, Phys. Rev. E (2018), arXiv:1810.00388.
- G. Biondini and D. Mantzavinos, Long-time asymptotics for the focusing nonlinear Schrödinger equation with nonzero boundary conditions at infinity and asymptotic stage of modulational instability, arXiv:1512.06095
- I. Egorova, Z. Gladka, V. Kotlyarov, and G. Teschl, Long-Time Asymptotics for the Korteweg-de Vries Equation with Steplike Initial Data, Nonlinearity 26, 1839-1864 (2013).
- K. Andreiev, I. Egorova, T. L. Lange, and G. Teschl, Rarefaction Waves of the Korteweg-de Vries Equation via Nonlinear Steepest Descent, J. Differential Equations 261, 5371-5410 (2016).