Riemann-Hilbert problems and integrable nonlinear partial differential equations, IV

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From RHP to NLS, I

If $M(x,t,k)$ solves RHP:

- $M_+(x,t,k) = M_-(x,t,k)J(x,t,k)$, $k \in \mathbb{R} \cup \Sigma$ (with $\bar{\Sigma} = \Sigma$) with

$$J(x,t,k) = e^{(-ikx-2ik^2t)\sigma_3} J_0(k) e^{(ikx+2ik^2t)\sigma_3}$$

where $\overline{J_0(\bar{k})} = \begin{cases} \sigma J_0^{-1}(k)\sigma, & k \in \mathbb{R} \\ \sigma J_0(k)\sigma, & k \in \Sigma \end{cases}$ $\quad \sigma = \begin{cases} \sigma_1, & \lambda = 1 \\ \sigma_2, & \lambda = -1 \end{cases}$

- $M(x,t,k) \to I$ as $k \to \infty$

then $q(x,t) := 2i \lim_{k\to\infty} kM_{12}(x,t,k)$ solves (at least, locally) NLS equation $iq_t + q_{xx} - 2\lambda|q|^2 q = 0$.

From RHP to NLS, II

Idea of proof. Starting from $M(x,t,k)$, re-derive the Lax pair equations. Given M, define $\Phi := Me^{(-ikx-2ik^2t)\sigma_3}$. Then

1. noticing that Φ, Φ_x, Φ_t have same jumps (independent of x and t), conclude that $\Phi_x\Phi^{-1}$ and $\Phi_t\Phi^{-1}$ are entire functions of k;
2. calculate large-k asymptotics of $U := \Phi_x\Phi^{-1}$ and $V := \Phi_t\Phi^{-1}$ up to $\mathrm{O}(k^{-1})$ and apply Liouville Theorem to conclude that U and V are polynomials of k, of respective orders (1 and 2);
3. show that compatibility condition $U_t - V_x = [V, U]$ reduces to NLS.

From RHP to NLS, III

Let $M = I + \frac{M_1(x,t)}{ik} + \frac{M_2(x,t)}{(ik)^2} + \dots,\ k \to \infty$. Then

$$\Phi_x \Phi^{-1} = M_x M^{-1} - ikM\sigma_3 M^{-1} = -ik\sigma_3 + [\sigma_3, M_1] + \mathrm{O}(k^{-1})$$

Then, by Liouville Th., $\Phi_x \Phi^{-1} = -ik\sigma_3 + [\sigma_3, M_1]$ and thus we retrieve x-equation of the Lax pair $\Phi_x = (-ik\sigma_3 + U_0)\Phi$ with $U_0 := [\sigma_3, M_1] = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix} =$ [**symmetry**] $= \begin{pmatrix} 0 & q \\ \lambda\bar{q} & 0 \end{pmatrix}$.

Similarly,

$$\Phi_t \Phi^{-1} = M_x M^{-1} - 2ik^2 M\sigma_3 M^{-1} = -2ik^2\sigma_3 + 2kU_0 + V_0 + \mathrm{O}(k^{-1})$$

with $V_0 = 2i\left([\sigma_3, M_1]M_1 - [\sigma_3, M_2]\right)$ and, by Liouville, $\Phi_t = (-2ik^2\sigma_3 + 2kU_0 + V_0)\Phi$. Separating diagonal and off-diagonal parts when expressing $U_t - V_x = [V, U]$ in terms of M_1 and M_2, one obtains that U and V are exactly as they are in the Lax pair for NLS w.r.t. q.

Riemann-Hilbert problems and singular integral equations.

J. Lenells, Matrix Riemann-Hilbert problems with jumps across Carleson contours, Monatsh. Math. 186 (2018)

Classical RHP: smooth contour Σ; smooth jump matrix.
Problem: generalize the RHP formalism to jumps from $L^p(\Sigma)$ $(1 < p < \infty)$; more general contours.
Key tool: Cauchy operators. Principal value integrals

$$S_\Sigma(s^*) = \frac{1}{\pi i} \fint_\Sigma \frac{f(s)}{s - s^*} ds := \frac{1}{\pi i} \lim_{\delta \downarrow 0} \int_{\Sigma \setminus B(s^*,\delta)} \frac{f(s)}{s - s^*} ds,$$

where $B(s^*, \delta) = \{z : |z - s^*| \leq \delta\}$.
Singular Cauchy operators

$$\mathcal{C}_\Sigma^\pm f(s^*) := \lim_{\substack{z \to s^* \\ z \in \Omega_\pm}} \mathcal{C}_\Sigma f(z),$$

where the Cauchy integral is defined by

$$\mathcal{C}_\Sigma f(z) := \frac{1}{2\pi i} \int_\Sigma \frac{f(s)}{s - z} ds, \quad z \in \mathbb{C} \setminus \Sigma.$$

Key property to hold: boundedness of S_Σ and $\mathcal{C}_\Sigma^\pm$ as operators: $L^p(\Sigma) \mapsto L^p(\Sigma)$.

Contours for RHP, I

Definition: Carleson curve

Union of finite number of arcs (homeomorphic to intervals of $\mathbb{R}$; rectifiable); each arc Σ satisfies

$$\sup_{z\in\Sigma}\sup_{r>0}\frac{|\Sigma\cap B(z,r)|}{r}<\infty.$$

Proposition

$S_\Sigma : L^p(\Sigma) \mapsto L^p(\Sigma)$ is a bounded operator $\Leftrightarrow$ Σ is a Carleson curve.

Contours for RHP, II

Definition: Carleson jump contour Σ

Union of finite number of Carleson arcs (with only common points,if intersect, being the end points) s.t.

- $\mathbb{C} \setminus \Sigma = D_+ \cup D_-$; $D_\pm = \cup_j D_{\pm,j}$ finite number of connected components;
- Orientation: $\Sigma = \partial D_+ = -\partial D_-$;
- $\partial D_{\pm,j}$ is a Carleson curve for all j.

Question

Assume that $v \in L^p(\Sigma)$ is given. How to understand the jump condition $m_+ = m_- v$? How to understand $m_\pm$?

Classes for solution of RHP

Definition: Smirnoff classes of analytic functions

(i) Let D be a bounded component of $\mathbb{C} \setminus \Sigma$. We say that $f(z) \in E^p(D)$, if

- $f(z)$ is analytic in D;
- there exist $\{C_n\}_1^\infty \subset D$ rectifiable Jordan curves s.t. for any compact $D_c \subset D$ there exists N: (a) for all $n > N$, C_n surrounds D_c; (b) $\sup_{n \geq 1} \int_{C_n} |f(z)|^p |dz| < \infty$.

(ii) If D is unbounded, then $f \in E^p(D)$, if $f \circ \varphi^{-1} \in E^p(\varphi(D))$, where $\varphi(z) := \frac{1}{z - z_0}$ with some $z_0 \in \mathbb{C} \setminus \bar{D}$.

(iii) $\dot{E}^p(D) := \{f \in E^p(D) : zf(z) \in E^p(D)\}$ (i.e., $f(z) \to 0$ as $z \to \infty$).

(iv) If $D = D_1 \cup \cdots \cup D_n$, then $f \in E^p(D)$ if $f|_{D_j} \in E^p(D_j)$ for all j.

Proposition

- If $f \in \dot{E}^2(D_\pm)$, then there exist (almost everywhere) non-tangential limits $f_\pm(s) = \lim_{\substack{z \to s \\ z \in D_\pm}} f(z)$, $s \in \Sigma$; moreover, $f_\pm - I \in L^2(\Sigma)$.
- if $h \in L^2(\Sigma)$, then $\mathcal{C}h \in \dot{E}^2(D_+ \cup D_-)$, where $\mathcal{C}h(z) := \frac{1}{2\pi i} \int_\Sigma \frac{h(s)}{s-z} ds$, $z \in \mathbb{C} \setminus \Sigma$ (Cauchy integral).
- $\mathcal{C}_+$ and $\mathcal{C}_-$ are bounded operators $L^2(\Sigma) \mapsto L^2(\Sigma)$; moreover, $\mathcal{C}_+ - \mathcal{C}_- = I$, and Sokhotski-Plemelj formulas hold:

$$\mathcal{C}_+ = \frac{1}{2}(I + S_\Sigma), \qquad \mathcal{C}_- = \frac{1}{2}(-I + S_\Sigma),$$

where S_Σ is the principal value Cauchy integral operator.

Riemann-Hilbert problem in L^2 sense

The properties of E^2 and L^2 justify the correctness of the formulation of the RHP:

L^2 – RHP

Let Σ be a Carleson jump contour, and let a matrix-valued function $v : \Sigma \mapsto GL(n, \mathbb{C})$ is given on Σ. Then $m(z)$, $z \in \mathbb{C} \setminus \Sigma$ is called the solution of the RHP $\{\Sigma, v\}$, if

- $m \in I + \dot{E}^2(\mathbb{C} \setminus \Sigma)$;
- $m_+(s) = m_-(s)v(s)$ almost everywhere on Σ, where $m_\pm$ are non-tangential limits of m.

Questions

What conditions should be imposed on v in order to have solvability/uniqueness of the RHP? How to solve the RHP?

Reducing RHP to integral equation, I

Assumption on v

We assume that $v(s)$ can be (algebraically!) factorized as $v(s) = (v_-(s))^{-1} v_+(s)$ s.t.

$$v_\pm, (v_\pm)^{-1} \in I + L^2(\Sigma) \cap L^\infty(\Sigma)$$

(it can be many factorizations like this!).
Introduce $w_+ = v_+ - I$, $w_- = I - v_-$ (so that $v_\pm = I \pm w_\pm$ with $w_\pm \in L^2(\Sigma) \cap L^\infty(\Sigma)$).
Introduce the integral operator $\mathcal{C}_w$ by

$$\mathcal{C}_w h(z) = \mathcal{C}_+(h w_-)(z) + \mathcal{C}_-(h w_+)(z), \quad z \in \Sigma.$$

Proposition

$\mathcal{C}_w$ is bounded operator $L^2(\Sigma) \mapsto L^2(\Sigma)$.

Proof

$$\begin{aligned} \|\mathcal{C}_w h(z)\|_{L^2(\Sigma)} &\leq \|\mathcal{C}_+(hw_-)(z)\|_{L^2(\Sigma)} + \|\mathcal{C}_-(hw_+)(z)\|_{L^2(\Sigma)} \\ &\leq C\left(\|hw_-\|_{L^2} + \|hw_+\|_{L^2}\right) \\ &\leq C\|h\|_{L^2}\left(\|w_-\|_{L^\infty} + \|w_+\|_{L^\infty}\right), \end{aligned}$$

where $C = \max\{\|\mathcal{C}_+\|_{\mathcal{L}(L^2(\Sigma))}, \|\mathcal{C}_-\|_{\mathcal{L}(L^2(\Sigma))}\}$. Therefore,

$$\|\mathcal{C}_w\|_{\mathcal{L}(L^2(\Sigma))} \leq 2C \max\{\|w_-\|_{L^\infty}, \|w_+\|_{L^\infty}\}.$$

Moreover, if $h \in L^\infty(\Sigma)$, then
$\|\mathcal{C}_w h(z)\|_{L^2(\Sigma)} \leq C\left(\|w_-\|_{L^2} + \|w_+\|_{L^2}\right)$; therefore,

$$\mathcal{C}_w : L^2(\Sigma) + L^\infty(\Sigma) \mapsto L^2(\Sigma).$$

Reducing RHP to integral equation, III

Consider the singular integral equation

$$(I - \mathcal{C}_w)\mu = I, \qquad \mu \in I + L^2(\Sigma)$$

(or, equivalently,

$$(I - \mathcal{C}_w)(\mu - I) = \mathcal{C}_w I,$$

where the r.h.s. $\mathcal{C}_w I \in L^2(\Sigma)$, and the solution $\mu - I$ is also sought in $L^2(\Sigma)$).

Theorem

If there exists $(I - \mathcal{C}_w)^{-1}$ as a bounded operator $L^2(\Sigma) \mapsto L^2(\Sigma)$ and $\mu \in I + L^2(\Sigma)$ is the unique solution of $(I - \mathcal{C}_w)\mu = I$, then $m(z) := I + (\mathcal{C}(\mu(w_+ + w_-)))(z)$ is a solution of the L^2–RHP: $m \in I + \dot{E}^2(\mathbb{C} \setminus \Sigma)$ and $m_+ = m_- v$ a.e. on Σ.

Proof (i)

$$\begin{aligned} m_+ &= I + \mathcal{C}_+(\mu(w_+ + w_-)) = I + \mathcal{C}_+(\mu w_-) + \mathcal{C}_+(\mu w_+) \\ &= [\mathcal{C}_+ - \mathcal{C}_- = I] = I + \mathcal{C}_+(\mu w_-) + \mathcal{C}_-(\mu w_+) + \mu w_+ \\ &= I + \mathcal{C}_w \mu + \mu w_+ = [I + \mathcal{C}_w \mu = \mu] = \mu + \mu w_+ \\ &= \mu(I + w_+) = \mu v_+. \end{aligned}$$

Similarly, $m_- = \mu v_- \Rightarrow \mu = m_- v_-^{-1}$ and thus

$$m_+ = \mu v_+ = m_- v_-^{-1} v_+ = m_- v.$$

(ii) Since $\|w_\pm\|_{L^2} < \infty$, we have $\mu(w_+ + w_-) \in L^2(\Sigma)$. Then, by Proposition, $m \in I + \dot{E}^2(\mathbb{C} \setminus \Sigma)$.

Having $\mu(x,t,s)$, $s \in \Sigma$, solution of the NLS equation is given by

$$q(x,t) = 2i \lim_{z\to\infty} z m_{12}(z) = -\frac{1}{\pi} \int_\Sigma \left(\mu(s)(w_+(s) + w_-(s))\right)_{12} ds.$$

Theorem

If $I-\mathcal{C}_w$ is a Fredholm operator, then the existence of $(I-\mathcal{C}_w)^{-1}$ is equivalent to the (unique) solvability of the RHP.

Fredholm property: $\exists(I-\mathcal{C}_w)^{-1} \Leftrightarrow \{(I-\mathcal{C}_w)\mu=0$ has only trivial solution $\mu\equiv 0\}$.

Proof Let RHP $\{\Sigma, v\}$ has a unique solution. Let $(I-\mathcal{C}_w)\mu=0$. Determine $\hat{m}:=(\mathcal{C}(\mu w))(z)$ with $w=w_-+w_+$. Then

$$\begin{aligned}\hat{m}_+ &= \mathcal{C}_+(\mu(w_++w_-)) = \mathcal{C}_+(\mu w_-)+\mathcal{C}_+(\mu w_+)\\ &= [\mathcal{C}_+-\mathcal{C}_-=I] = \mathcal{C}_+(\mu w_-)+\mathcal{C}_-(\mu w_+)+\mu w_+\\ &= \mathcal{C}_w\mu+\mu w_+ = [\mathcal{C}_w\mu=\mu] = \mu+\mu w_+ = \mu(I+w_+) = \mu v_+.\end{aligned}$$

Similarly, $\hat{m}_-=\mu v_-$ and thus $\hat{m}_+=\hat{m}_-v$. Also $\hat{m}\to 0$ as $z\to\infty$. Therefore, $m_\gamma := m+\gamma\hat{m}$ is a solution of the RHP for any γ. But by assumption, $\exists!$ solution of RHP; hence, $\hat{m}=0 \Rightarrow \mathcal{C}_\pm(\mu w)=0$. Then

$$\begin{aligned}0 &= \mathcal{C}_+(\mu w) = \mathcal{C}_+(\mu w_-)+\mathcal{C}_+(\mu w_+) = \mathcal{C}_+(\mu w_-)+\mathcal{C}_-(\mu w_+)+\mu w_+\\ &= \mathcal{C}_w\mu+\mu w_+ = \mu+\mu w_+ = \mu(I+w_+) = \mu v_+.\end{aligned}$$

Since $\exists v_+^{-1}$, it follows that $\mu\equiv 0$.

Theorem

Let $v^{(n)} = (v_-^{(n)})^{-1} v_+^{(n)} = (I - w_-^{(n)})^{-1}(I + w_+^{(n)})$, $n = 1, 2, \dots$ and $v^\infty = (v_-^\infty)^{-1} v_+^\infty = (I - w_-^\infty)^{-1}(I + w_+^\infty)$.
Let $\exists (I - \mathcal{C}_{w^\infty})^{-1}$. Assume that

$$\|w_\pm^{(n)} - w_\pm^\infty\|_{L^\infty(\Sigma) \cap L^2(\Sigma)} \to 0, \qquad n \to \infty.$$

Then:

- $\exists$ solutions of RHP m^∞ and $m^{(n)}$ for all $n > N$.
- $\|m_\pm^{(n)} - m_\pm^\infty\|_{L^2(\Sigma)} \to 0$ as $n \to \infty$.

Proof (i) First, discuss (and estimate) $(I - \mathcal{C}_{w^{(n)}})^{-1}$. As $n \to \infty$,

$$\|\mathcal{C}_{w^\infty} f - \mathcal{C}_{w^{(n)}} f\|_{L^2(\Sigma)} = \|\mathcal{C}_+(f(w_-^{(n)} - w_-^\infty)) + \mathcal{C}_-(f(w_-^{(n)} - w_-^\infty))\|$$
$$\leq C\|f\|_{L^2(\Sigma)} \left(\|w_-^{(n)} - w_-^\infty\|_{L^\infty(\Sigma)} + \|w_+^{(n)} - w_+^\infty\|_{L^\infty(\Sigma)}\right) \to 0.$$

Thus $\|\mathcal{C}_{w^\infty} - \mathcal{C}_{w^{(n)}}\|_{L^2(\Sigma) \mapsto L^2(\Sigma)} \to 0$. Now apply the Second Resolvent Identity: if $R(z, A) := (zI - A)^{-1}$, then

$$R(z, A) - R(z, B) = R(z, A)(B - A)R(z, B)$$

and thus $R(z, A) = (I - (A - B)R(z, B))^{-1} R(z, B)$. Hence

1. $\exists (I - \mathcal{C}_{w^{(n)}})^{-1}$, $n > N$;
2. $\|(I - \mathcal{C}_{w^{(n)}})^{-1} - (I - \mathcal{C}_{w^\infty})^{-1}\|_{L^2(\Sigma) \mapsto L^2(\Sigma)} \to 0$

(ii) Next, estimate $\|\mu^{(n)} - \mu^{\infty}\|$.

$$\|\mu^{(n)} - \mu^{\infty}\|_{L^2(\Sigma)} = \|(I - \mathcal{C}_{w^{(n)}})^{-1} I - (I - \mathcal{C}_{w^{\infty}})^{-1} I\|_{L^2}$$
$$= \|(I - \mathcal{C}_{w^{(n)}})^{-1} (\mathcal{C}_{w^{(n)}} - \mathcal{C}_{w^{\infty}}) (I - \mathcal{C}_{w^{\infty}})^{-1} I\|_{L^2}$$
$$= \|(I - \mathcal{C}_{w^{(n)}})^{-1} (\mathcal{C}_{w^{(n)}} - \mathcal{C}_{w^{\infty}}) (I + (I - \mathcal{C}_{w^{\infty}})^{-1} \mathcal{C}_{w^{\infty}} I\|_{L^2}$$
$$\leq \|(I - \mathcal{C}_{w^{(n)}})^{-1}\|_{L^2 \mapsto L^2} \|\mathcal{C}_{+}(w_{-}^{(n)} - w_{-}^{\infty}) + \mathcal{C}_{-}(w_{+}^{(n)} - w_{+}^{\infty})\|_{L^2}$$
$$+ \|(I - \mathcal{C}_{w^{(n)}})^{-1}\|_{L^2 \mapsto L^2} \|\mathcal{C}_{w^{(n)}} - \mathcal{C}_{w^{\infty}}\|_{L^2 \mapsto L^2} \|(I - \mathcal{C}_{w^{\infty}})^{-1}\| \|\mathcal{C}_{+} w_{-}^{\infty} + \mathcal{C}_{-} w_{+}^{\infty}\|_{L^2}$$
$$\leq C \left(\|w_{-}^{(n)} - w_{-}^{\infty}\|_{L^2 \cap L^{\infty}} + \|w_{+}^{(n)} - w_{+}^{\infty}\|_{L^2 \cap L^{\infty}} \right) \to 0.$$

(iii) Finally,

$$\|m_{\pm}^{(n)} - m_{\pm}^{\infty}\|_{L^2} = \|\mu^{(n)} v_{\pm}^{(n)} - \mu^{\infty} v_{\pm}^{\infty}\|_{L^2}$$
$$\leq \|\mu^{\infty}\|_{L^2} \|v_{\pm}^{(n)} - v_{\pm}^{\infty}\|_{L^{\infty}} + \|v_{\pm}^{(n)}\|_{L^{\infty}} \|\mu^{(n)} - \mu^{\infty}\|_{L^2} \to 0.$$

Large-t asymptotics for regular RHP: nonlinear steepest descent, I

Considerations when reducing a non-regular RHP to a regular one have already demonstrated the flexibility of the RHP approach: one can change the contour (without changing the quantity that we want to extract from the solution of the RHP), in view of making the RHP with "better" jumps (e.g., approaching I for large values of the parameter).

- One of the contour deformations most useful for the asymptotic analysis is called

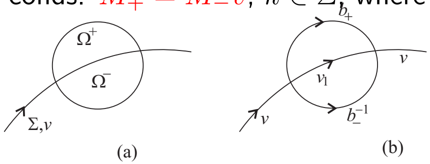
"making lenses"

Let the original RHP be $M_+ = M_- v$, $k \in \Sigma$. Assume that the jump matrix v admits factorization $v = v_-^{-1} v_1 v_+$ on a part of Σ s.t. $v_\pm$ is analytic in $\Omega_\pm$ (analytically continued from Σ into the left/right domain). Define $\hat{M}$:

$$\hat{M} = \begin{cases} M v_+, & k \in \Omega_+, \\ M v_-, & k \in \Omega_-, \\ M, & \text{otherwise} \end{cases}$$

Then $\hat{M}$ satisfies the jump conds. $\hat{M}_+ = \hat{M}_- \hat{v}$, $k \in \hat{\Sigma}$, where

$$\hat{v} = \begin{cases} v_+^{-1}, & k \in \Sigma_+, \\ v_-, & k \in \Sigma_-, \\ v_1, & k \in \Sigma \cap \Omega, \\ v, & \text{otherwise} \end{cases}$$



Particularly, if $v_1 \equiv I$, then a part of Σ is erased from the contour.

- Making lenses is useful if v is oscillatory w.r.t. t, but the entries of $v_\pm$ are exponentially decaying (analogue of (linear) steepest descent method).

Idea: use the lenses mechanism in order to make the original RHP for the NLS more treatable asymptotically (large t).
Recall the jump matrix of original RHP: $J(x,t,k) \mapsto J(\xi,t,k)$, where

$$J(\xi,t,k) = \begin{pmatrix} 1-\lambda|r(k)|^2 & \bar{r}(k)e^{-2it\theta(\xi,k)} \\ -\lambda r(k)e^{2it\theta(\xi,k)} & 1 \end{pmatrix}, \quad k \in \mathbb{R} \quad \xi = \frac{x}{4t}.$$

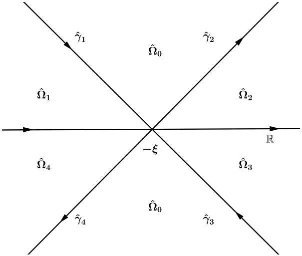
Notice that J admits the triangular factorization

$$J(\xi,t,k) = \begin{pmatrix} 1 & \bar{r}(k)e^{-2it\theta(\xi,k)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\lambda r(k)e^{2it\theta(\xi,k)} & 1 \end{pmatrix}$$

Recall "signature table": for $\operatorname{Re} k > k_0 = -\xi$, $e^{-2it\theta(\xi,k)}$ decays to 0 (as $t \to \infty$), if $\operatorname{Im} k < 0$ and $e^{2it\theta(\xi,k)}$ decays to 0, if $\operatorname{Im} k > 0$.

We assume that $r(k)$ can be analytically continued into $\mathbb{C}_+$ (otherwise, use rational approximations for $r(k)$).

- Good factorization for $k > -\xi$: introducing



$$\hat{M} = \begin{cases} M \begin{pmatrix} 1 & \bar{r}(k)e^{-2it\theta(\xi,k)} \\ 0 & 1 \end{pmatrix}, & k \in \hat{\Omega}_3 \\ M \begin{pmatrix} 1 & 0 \\ \lambda r(k)e^{2it\theta(\xi,k)} & 1 \end{pmatrix}, & k \in \hat{\Omega}_2 \end{cases}$$

  1. erases $(-\xi, \infty)$ from the contour;
  2. introduces $\hat{\gamma}_2$ and $\hat{\gamma}_3$ as new parts of the contour, where the jump matrices decay to I (as $t \to \infty$) exponentially fast.

- But bad factorization for $k < -\xi$: introducing $\hat{M}$ as above in $\hat{\Omega}_1$ and $\hat{\Omega}_4$ would lead to jumps exponentially increasing on $\hat{\gamma}_1$ and $\hat{\gamma}_4$!

- For $k < -\xi$, it would be nice to have an opposite, "left-triangular" – "right-triangular" factorization; then, due to signature table, the new jump matrices would decay to I as well.
- But we have problem: the "left-triangular" – "right-triangular" factorization involves also a diagonal factor:

$$J = \begin{pmatrix} 1 & 0 \\ -\frac{\lambda r e^{2it\theta(\xi,k)}}{1-\lambda|r|^2} & 1 \end{pmatrix} \begin{pmatrix} 1-\lambda|r|^2 & 0 \\ 0 & \frac{1}{1-\lambda|r|^2} \end{pmatrix} \begin{pmatrix} 1 & \frac{\bar{r} e^{-2it\theta(\xi,k)}}{1-\lambda|r|^2} \\ 0 & 1 \end{pmatrix}$$

- In fact, the triangular factorizations used above are of the form:

$$\begin{pmatrix} 1+AB & A \\ B & 1 \end{pmatrix} = \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} =$$
$$= \begin{pmatrix} 1 & 0 \\ \frac{B}{1+AB} & 1 \end{pmatrix} \begin{pmatrix} 1+AB & 0 \\ 0 & \frac{1}{1+AB} \end{pmatrix} \begin{pmatrix} 1 & \frac{A}{1+AB} \\ 0 & 1 \end{pmatrix}$$

- Remedy: remove the diagonal factor, multiplying by an appropriate diagonal matrix having appropriate jump across $k < -\xi$.

In order to construct the appropriate diagonal factor, introduce $\delta(\xi; k)$ (ξ is a parameter) as the solution of the scalar Riemann-Hilbert problem: find a function analytic in $\mathbb{C} \setminus (-\infty, -\xi)$ s.t.

$$\begin{cases} \delta_+(k) = \delta_-(k)(1 - \lambda|r(k)|^2), & k \in (-\infty, -\xi) \\ \delta(k) \to 1, & k \to \infty \end{cases}$$

(notice that $1 - \lambda|r(k)|^2 > 0$). The solution of this RHP is given explicitly:

$$\delta(\xi; k) = \exp\left\{ \frac{1}{2\pi i} \int_{-\infty}^{-\xi} \frac{\log(1 - \lambda|r(s)|^2)}{s - k} ds \right\}.$$

- For k near $-\xi$: $\delta(\xi; k) = (k + \xi)^{i\nu(\xi)} e^{\chi(\xi;k)}$, where $\nu(\xi) = -\frac{1}{2\pi} \log(1 - \lambda|r(-\xi)|^2) \in \mathbb{R}$ and $\chi(\xi; k)$ is bounded and has a limit as $k \to -\xi$:

$$\chi(\xi; k) = -\frac{1}{2\pi i} \int_{-\infty}^{-\xi} \log(k - s) d_s \log(1 - \lambda|r(s)|^2)$$

With $\delta(\xi;k)$ defined as above, the contour/jump deformation process involves 2 steps: (i) $M \mapsto \overset{(1)}{M} := M \left(\begin{smallmatrix} \delta^{-1} & 0 \\ 0 & \delta \end{smallmatrix}\right)$;
(ii) $\overset{(1)}{M} \mapsto \overset{(2)}{M} := \overset{(1)}{M} \cdot$ (triangular factors).

First step: the jump conditions are $\overset{(1)}{M}_+ = \overset{(1)}{M}_- \overset{(1)}{J}$, $k \in \mathbb{R}$, where

$$\overset{(1)}{J} = \begin{cases} \begin{pmatrix} 1 & \bar{r}\delta^2 e^{-2it\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\lambda r \delta^{-2} e^{2it\theta} & 1 \end{pmatrix}, & k \in (-\xi, \infty) \\ \begin{pmatrix} 1 & 0 \\ -\frac{\lambda r \delta_-^{-2} e^{2it\theta}}{1-\lambda|r|^2} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{\bar{r}\delta_+^2 e^{-2it\theta}}{1-\lambda|r|^2} \\ 0 & 1 \end{pmatrix}, & k \in (-\infty, -\xi) \end{cases}$$

Second step: use triangular factors from above.

Second step: Introduce $\overset{(2)}{M}$ by

$$\overset{(2)}{M}(\xi,t,k)=\overset{(1)}{M}(\xi,t,k)\begin{cases}\begin{pmatrix}1 & 0\\ \lambda r(k)\delta^{-2}(k)e^{2it\theta(\xi,k)} & 1\end{pmatrix}, & k\in\hat{\Omega}_2\\ \begin{pmatrix}1 & \bar{r}(k)\delta^{2}(k)e^{-2it\theta(\xi,k)}\\ 0 & 1\end{pmatrix}, & k\in\hat{\Omega}_3\\ \begin{pmatrix}1 & -\frac{\bar{r}(k)\delta_+^{2}(k)e^{-2it\theta(\xi,k)}}{1-\lambda|r|^2}\\ 0 & 1\end{pmatrix}, & k\in\hat{\Omega}_1\\ \begin{pmatrix}1 & 0\\ -\frac{\lambda r(k)\delta_-^{-2}(k)e^{2it\theta(\xi,k)}}{1-\lambda|r|^2} & 1\end{pmatrix}, & k\in\hat{\Omega}_4\end{cases}$$

Then $\overset{(2)}{M}(k)$ satisfies the jump conditions $\overset{(2)}{M}_+(k)=\overset{(2)}{M}_-(k)\overset{(2)}{J}(k)$ across the "cross" $k\in\hat{\Sigma}:=\cup_{j=1}^4\hat{\gamma}_j$, where $\overset{(2)}{J}(k)$ is as the triangular matrices above respectively on $\hat{\gamma}_2$, $\hat{\gamma}_3$, $\hat{\gamma}_1$, and $\hat{\gamma}_4$.

- $\overset{(2)}{J}(k)$ decays to I as $t\to\infty$ for any $k\in\hat{\Sigma}\setminus\{-\xi\}$. Moreover, it decays uniformly on $\hat{\Sigma}$ outside any small vicinity of $k=-\xi$!

We face a problem: as $t \to \infty$, the jump matrix $\overset{(2)}{J}(\xi, t, k)$ does not approach I in L^∞; thus we can't even state that $M(\xi, t, k)$ approaches I as $t \to \infty$ (which correspond to $q(x, t) \to 0$)!
But we want even more: to obtain (explicitly!) the main (decaying!) term of $q(x, t)$ as $t \to \infty$ explicitly. How can this be achieved?

Idea (of local parametrix)

In a small vicinity $D_\varepsilon(\xi)$ of $k = -\xi$, "replace" $\overset{(2)}{M}(k)$ by $\tilde{m}_0(k)$ s.t.

- $\tilde{m}_0(k)$ can be constructed explicitly, via the solution of a (model) RH problem with constant jump matrix;
- $\hat{m} := \begin{cases} \overset{(2)}{M}(k)\tilde{m}_0^{-1}(k), & |k+\xi| < \varepsilon \\ \overset{(2)}{M}(k), & |k+\xi| > \varepsilon \end{cases}$ satisfies a RH problem with jump matrix $\hat{v}(k)$ close to I in $L^2 \cap L^\infty$ (as $t \to \infty$), with an error estimate.

Then:

- estimate the solution μ of the respective integral equation $\mu - \mathcal{C}_{\hat{w}}\mu = I$ with $\hat{w} = \hat{v} - I$;
- show that in the expression $(\Sigma_\varepsilon(\xi) = \hat{\Sigma} \setminus D_\varepsilon(\xi))$

$$2i \lim_{k\to\infty} k\left(\hat{m}(\xi,t,k) - I\right) =$$
$$-\frac{1}{\pi}\int_{\partial D_\varepsilon(\xi)} \mu(s)\hat{w}(s)ds - \frac{1}{\pi}\int_{\Sigma_\varepsilon(\xi)} \mu(s)\hat{w}(s)ds$$

  the first term is dominating as $t \to \infty$;
- calculate explicitly this term and thus the main term of the large-t asymptotics of $q(x,t)$.

The (explicit) construction of $\tilde{m}_0$ is based on:

- rescaling the vicinity $D_\varepsilon(\xi)$ of $k=-\xi$ by introducing new spectral variable $z=(k+\xi)\sqrt{8t}$; then $2it\theta(\xi;k)=\frac{iz^2}{2}-4it\xi^2$.
- approximating $r(k)\approx r(-\xi)$ and $\delta(\xi;k(z))\approx\left(\frac{z}{\sqrt{8t}}\right)^{i\nu(\xi)}e^{\chi(-\xi)}$

This suggests introducing the RH problem in the complex z-plane:

$m^p_+(\xi,z)=m^p_-(\xi,z)J^p(\xi,z)$, $z\in C$ (the cross centered at 0), with

$$J^p(\xi,z)=\begin{cases}\begin{pmatrix}1 & 0\\ -\lambda r(-\xi)e^{\frac{iz^2}{2}}z^{-2i\nu(\xi)} & 1\end{pmatrix}, & z\in C_2\\ \begin{pmatrix}1 & -\bar{r}(-\xi)e^{\frac{-iz^2}{2}}z^{2i\nu(\xi)}\\ 0 & 1\end{pmatrix}, & z\in C_3\\ \begin{pmatrix}1 & \frac{\bar{r}(-\xi)e^{\frac{-iz^2}{2}}z^{2i\nu(\xi)}}{1-\lambda|r(-\xi)|^2}\\ 0 & 1\end{pmatrix}, & z\in C_1\\ \begin{pmatrix}1 & 0\\ \frac{\lambda r(-\xi)e^{\frac{iz^2}{2}}z^{-2i\nu(\xi)}}{1-\lambda|r(-\xi)|^2} & 1\end{pmatrix}, & z\in C_4\end{cases}$$

with standard normalization: $m^p(\xi,z)\to I$ as $z\to\infty$.

Importance of $m^p(\xi,z)$ (parametrix):

- $\tilde{m}_0(x,t,k)$ can be given explicitly in terms of $m^p(\xi,z)$;
- the RHP for $m^p(\xi,z)$ can be solved explicitly.

Proposition 1

$$\tilde{m}_0(x,t,k)=\Delta(\xi,t)m^p(\xi,z(k))\Delta^{-1}(\xi,t),$$

where $\Delta(\xi,t)=e^{(2it\xi^2+\chi(-\xi))\sigma_3}(8t)^{-\frac{i\nu(\xi)}{2}\sigma_3}$.

Proposition 2

$$m^p(\xi,z)=m_0(\xi,z)D_j^{-1}(\xi,z),\quad z\in\Omega_j, j=0,1,...,4$$

where $D_0=e^{-\frac{iz^2}{4}\sigma_3}z^{i\nu(\xi)\sigma_3}$, $D_1=D_0\begin{pmatrix}1 & \frac{\bar{r}(-\xi)}{1-\lambda|r(-\xi)|^2}\\ 0 & 1\end{pmatrix}$, $D_2=D_0\begin{pmatrix}1 & 0\\ -\lambda r(-\xi) & 1\end{pmatrix}$, $D_3=D_0\begin{pmatrix}1 & -\bar{r}(-\xi)\\ 0 & 1\end{pmatrix}$, $D_4=D_0\begin{pmatrix}1 & 0\\ \frac{\lambda r(-\xi)}{1-\lambda|r(-\xi)|^2} & 1\end{pmatrix}$.

and $m_0(\xi, z)$ is the solution of the RHP with constant jump condition (independent of z; across $\mathbb{R}$!)

$$m_{0+}(\xi, z) = m_{0-}(\xi, z) j_0(\xi), \quad z \in \mathbb{R},$$

where $j_0(\xi) = \begin{pmatrix} 1 - \lambda |r(-\xi)|^2 & \bar{r}(-\xi) \\ -\lambda r(-\xi) & 1 \end{pmatrix}$, and with normalization condition $m_0(\xi, z) = \left(I + O(\frac{1}{z})\right) e^{-\frac{iz^2}{4}\sigma_3} z^{i\nu(\xi)\sigma_3}$, $z \to \infty$.

The fact that the jump matrix is constant w.r.t. z suggests that this RHP can be solved explicitly. And indeed, this is the case!

Proposition 3

$m_0(\xi, z) = \begin{pmatrix} (m_0)_{11} & -\frac{(\frac{d}{dz}-\frac{iz}{2})(m_0)_{22}}{\gamma(\xi)} \\ -\frac{(\frac{d}{dz}+\frac{iz}{2})(m_0)_{11}}{\beta(\xi)} & (m_0)_{22} \end{pmatrix}$ where $(m_0)_{11}$ and $(m_0)_{22}$ are solutions of Bessel-type equations (other names: parabolic cylinder eq.; Weber eq.; Weber-Hermite eq.):

$$\frac{d^2}{dz^2}(m_0)_{11}(\xi, z) + \left(\frac{i}{2} - \nu(\xi) + \frac{z^2}{4}\right)(m_0)_{11}(\xi, z) = 0.$$

with $\nu(\xi) = -\frac{1}{2\pi}\log(1 - \lambda|r(-\xi)|^2$.

Hint for proof: since j_0 does not depends on z, it follows that $m_0(z)$ and $\frac{d}{dz}m_0(z)$ satisfy the same jump conds.; consequently, $\frac{dm_0}{dz}m_0^{-1}(z)$ is entire function, whose large-z asymptotics is $-\frac{iz}{2}\sigma_3 - \begin{pmatrix} 0 & \beta_1 \\ \beta_2 & 0 \end{pmatrix}$ with some β_1 and β_2 s.t. $\beta_1(\xi)\beta_2(\xi) = \nu(\xi)$. Then, by Liouville theorem,

$$\frac{dm_0}{dz}m_0^{-1}(\xi, z) = -\frac{iz}{2}\sigma_3 - \begin{pmatrix} 0 & \beta_1(\xi) \\ \beta_2(\xi) & 0 \end{pmatrix}.$$

- Thus we have explicit expressions (the prescribed large-z asymptotics specifies the solution of ODE) for entries of m_0: $(m_0)_{11}(\xi,z) = e^{-\frac{3\pi}{4}} D_{i\nu(\xi)}\left(e^{-\frac{3\pi i}{4}} z\right)$ for $z \in \mathbb{C}_+$, etc.

  $D_a(z)$ parabolic cylinder function
- but what we actually need is the large-z asymptotics of m_0 and, subsequently, of m^p:

$$m^p(\xi,z) = I + \frac{i}{z}\begin{pmatrix} 0 & \beta_1(\xi) \\ -\beta_2) & 0 \end{pmatrix} + O(z^{-2})$$

with $\beta_1(\xi) = \frac{2\pi e^{-\frac{\pi\nu(\xi)}{2}} e^{\frac{-3\pi i}{4}}}{\lambda r(-\xi)\Gamma(-i\nu(\xi))}$.
Indeed, from this asymptotics it follows that

$$\tilde{m}_0^{-1}(x,t,k) = I + \frac{B(\xi,t)}{\sqrt{8t}(k+\xi)} + O(t^{-1}),$$

where $B_{12}(\xi,t) = -i\beta_1(\xi)e^{4it\xi^2+2\chi(-\xi))}(8t)^{-i\nu(\xi)}$ (notice $|B_{12}| = \sqrt{|\nu(\xi)|}$).

Now recall our consecutive transformations:

$$
\begin{aligned}
q(x,t) =& 2i \lim_{k\to\infty} k(\overset{(2)}{M} - I)_{12} = 2i \lim_{k\to\infty} k(\hat{m} - I)_{12} = \\
=& -\frac{1}{\pi} \int\limits_{|k+\xi|=\varepsilon} (\tilde{m}_0^{-1} - I)_{12} dk - \frac{1}{\pi} \int\limits_{|k+\xi|=\varepsilon} (\mu - I)(\tilde{m}_0^{-1} - I)_{12} dk \\
& - \frac{1}{\pi} \int\limits_{\Sigma_\varepsilon} (\mu\hat{w})_{12} dk - \frac{1}{\pi} \int\limits_{\Sigma_{ext}} (\mu\hat{w})_{12} dk
\end{aligned}
$$

Estimates for $\hat{w}$ and μ imply that the dominating term is the first one (in red). But it can be computed by Residue Theorem:

$$
q(x,t) = -\frac{2iB_{12}(\xi,t)}{\sqrt{8t}} + O(t^{-1}\log t)
$$

Large-t asymptotics for regular RHP. Final result

Recalling the expression for B_{12}, we finally arrive at the asymptotic formula for $q(x,t)$: $q(x,t) = q_{as}(\xi,t) + O(t^{-1}\log t)$, where

$$q_{as}(\xi,t) = \frac{\sqrt{|\nu(\xi)|}}{\sqrt{2t}} e^{i(4\xi^2 t - \nu(\xi)\log t + \varphi(\xi))}$$

with

$$\begin{aligned}\varphi(\xi) =& \frac{1}{\pi i}\int_{-\infty}^{-\xi} \log|\xi + s| d_s \log(1-\lambda|r(s)|^2) - 3\nu(\xi)\log 2 - \frac{3\pi}{4} \\ & - \arg r(-\xi) + \arg\Gamma(i\nu(\xi))\end{aligned}$$

(Γ is the Gamma function).
Recall that $\xi = \frac{x}{4t}$ and $\nu(\xi) = -\frac{1}{2\pi}\log(1-\lambda|r(-\xi)|^2)$.

Asymptotics associated with non-regular RHP

Recall that we have reduced non-reg. RHP to a regular RHP s.t.

- On $\mathbb{R}$, the jump matrix keeps its original structure, with r replaced by $\tilde{r}(k) := r(k) \prod_{j \in K(\xi)} \left(\frac{k - \bar{k}_j}{k - k_j} \right)^2$ (reflecting the influence of discrete spectrum on the continuous one);
- On all ∂D_j and $\partial \bar{D}_j$, jump matrices decay to I (as $t \to \infty$).

Thus for any direction $x/4t = const$ that does not coincide with $x/4t = -\operatorname{Re} k_j$ for all $j \in \{1, \ldots, N\}$, the large-t asymptotics have the same structure as in the case without solitons (of order $t^{-\frac{1}{2}}$), but with parameters influenced by discrete spectrum.

Along $x/4t = -\operatorname{Re} k_j$, the asymptotics is non-decaying; it is dominated by the corresponding soliton.

- The first way (of reducing non-reg. RHP to reg. RHP) is more convenient for calculating the soliton asymptotics along $x/4t = -\operatorname{Re} k_j$; the second way is more convenient for calculating the parameters of decaying asymptotics.