Riemann-Hilbert problems and integrable nonlinear partial differential equations, III

Dmitry Shepelsky

V.N. Karazin Kharkiv National University, Ukraine

Integrability of NLS: Lax pair representation

From now on, the main object of our study is Nonlinear Schrödinger equation (NLS):

$$\mathrm{i}q_t + q_{xx} - 2\lambda|q|^2 q = 0.$$

Actually, two equations: (i) defocusing NLS, with $\lambda = 1$; and focusing NLS, with $\lambda = -1$.

NLS is compatibility condition for 2 linear equations (Lax pair).

They are (i) matrix-valued (2×2) and (ii) involve parameter k:

$$\Phi_x(x,t,k) = U(x,t,k)\Phi(x,t,k), \quad \Phi_t(x,t,k) = V(x,t,k)\Phi(x,t,k)$$

where

- $U(x,t,k) = -\mathrm{i}k\sigma_3 + \begin{pmatrix} 0 & q(x,t) \\ \lambda\bar{q}(x,t) & 0 \end{pmatrix}$ with $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$;
- $V(x,t,k) = -2\mathrm{i}k^2\sigma_3 + 2k\begin{pmatrix} 0 & q(x,t) \\ \lambda\bar{q}(x,t) & 0 \end{pmatrix} + \begin{pmatrix} -\mathrm{i}\lambda|q|^2 & \mathrm{i}q_x \\ -\mathrm{i}\lambda\bar{q}_x & \mathrm{i}\lambda|q|^2 \end{pmatrix}$

$$\{\text{NLS for } q\} \Longleftrightarrow \{\Phi_{xt} = \Phi_{tx} \text{ for all } k\} \Longleftrightarrow \boxed{U_t - V_x = [V,U]}$$

where $[V,U] := VU - UV$ (matrix commutator).

Problems for NLS

We are going to study

Cauchy problem for NLS with decaying boundary conds.

Given $q_0(x)$, $-\infty < x < \infty$ s.t. $q_0(x) \to 0$ as $x \to \infty$, find $q(x,t)$:

- $iq_t(x,t) + q_{xx}(x,t) - 2\lambda|q|^2 q = 0, \quad -\infty < x < \infty,\ t > 0;$
- $q(x,0) = q_0(x), \quad -\infty < x < \infty$ (initial conditions);
- $q(x,t) \to 0$ as $x \to \infty$ for all $t > 0$.

- Goal: obtain a representation of solution $q(x,t)$ of Cauchy problem in terms of solution of a RHP;
- Use this representation in order to obtain (the main term of) large time asymptotics of $q(x,t)$

Jost solutions of Lax pair, I

General Idea:

- Assuming that $q(x,t)$ is known, construct appropriate solutions of the Lax pair equations having good control as functions of k (analyticity; asymptotics);
- Use these solutions for constructing multiplicative RH problem s.t. the data for this problem (jump conds.) can be uniquely determined by the data for our Cauchy problem (i.e., initial data $q_0(x)$).

Noticing that Lax pair can be written as

$$\Phi_x = -ik\sigma_3\Phi + \tilde{U}(x,t)\Phi, \quad \Phi_t = -2ik^2\sigma_3\Phi + \tilde{V}(x,t,k)\Phi,$$

where $\tilde{U}, \tilde{V} \to 0$ as $|x| \to \infty$, one can fix the Jost solutions, Φ_- and Φ_+, by their asymptotics as $x \to -\infty$ and as $x \to +\infty$, reducing differential equation to integral equations:

$$\Phi_\pm(x,t,k) = e^{(-ikx-2ik^2t)\sigma_3} + \int_{\pm\infty}^{x} e^{ik(y-x)\sigma_3}\tilde{U}(y,t)\Phi_\pm(y,t,k)dy$$

- It is easily seen that solution of int. equ. satisfy the x-equation of Lax pair;
- The fact that they satisfy also the t-equation of Lax pair: comes from compatibility condition $U_t - V_x + [U,V] = 0$!

Jost solutions of Lax pair, II

- For $k \in \mathbb{R}$, both Φ_+ and Φ_- are well defined. Indeed, for such k, all exponentials in int. equ. are bounded, and, assuming that $q(x,t) \to 0$ as $|x| \to \infty$ sufficiently fast, these int. equ. (which are Volterra integral equations) can be solved by iterations.
- For non-real k, some of exponentials are growing (as $x \to -\infty$ or as $x \to +\infty$, which requires more careful analysis of the int. equations.

It is convenient to introduce $\Psi_\pm := \Phi_\pm e^{(ikx+2ik^2t)\sigma_3}$. Then, the int. equ. for $\Phi_\pm$ reduce to those for $\Psi_\pm$:

$$\Psi_\pm(x,t,k) = I + \int_{\pm\infty}^{x} e^{ik(y-x)\sigma_3}\tilde{U}(y,t)\Psi_\pm(y,t,k)e^{-ik(y-x)\sigma_3}dy$$

(here and below, I is 2×2 identity matrix. An advantage of $\Psi_\pm$ is that it becomes easier to control them (column-wise!) for non-real k.
Introduce notations for columns: $\Psi \equiv (\Psi^{(1)}, \Psi^{(2)})$ and consider, e.g., the int. equ. for $\Psi_-^{(1)} \equiv \begin{pmatrix}\Psi_{-,11}\\ \Psi_{-,21}\end{pmatrix}$:

$$\begin{pmatrix}\Psi_{-,11}\\ \Psi_{-,21}\end{pmatrix}(x,t,k) = \begin{pmatrix}1\\0\end{pmatrix} + \int_{-\infty}^{x}\begin{pmatrix} q(y,t)\Psi_{-,21}(y,t,k) \\ \lambda\bar{q}(y,t)\Psi_{-,11}e^{2ik(x-y)}\end{pmatrix}dy$$

Jost solutions of Lax pair, III

Since the only exponentials in the int. equ. for $\Psi_-^{(1)}$ decays to 0 as $y \to -\infty$ for all k with $\operatorname{Im} k > 0$, the follows result holds true.

Theorem

$\Psi_-^{(1)}(\cdot,\cdot,k)$ is analytic in $\mathbb{C}_+$ and continuous in $\overline{\mathbb{C}_+}$.

Proof. Rewrite the int. equ. for $h(x) := \Psi_-^{(1)}(x) \equiv \binom{h_1}{h_2}$ in operator form (omitting t and k). Important: variable limit of integration!

$$h(x) = \binom{1}{0} + (Ah)(x) \quad \text{with} \quad A\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}(x) = \int_{-\infty}^{x} \begin{pmatrix} q(y)h_2(y) \\ \lambda\bar{q}(y)h_1(y)e^{2ik(x-y)} \end{pmatrix} dy$$

for $h \in C((-\infty, M], \mathbb{R}^2)$ assuming $q(x) \in L_1(-\infty, M)$.
Introduce $\alpha(x) := \int_{-\infty}^{x} |q(y)|dy$. Then

- $\|(Ah)\|(x) \le \|h\|\alpha(x)$;
- $\|(A^2h)\|(x) \le \|h\| \int\limits_{-\infty}^{x} |q(y)|\alpha(y)dy = \frac{\|h\|}{2} \int\limits_{-\infty}^{x} \frac{d}{dy}(\alpha^2(y))dy = \|h\|\frac{\alpha^2(x)}{2}$
- by induction, $\|(A^n h)\|(x) \le \|h\|\frac{\alpha^n(x)}{n!}$

It follow that $\Psi_-^{(1)}(x,t,k) := \sum_{n=0}^{\infty} A^n \binom{1}{0}$ converges uniformly for $k \in \overline{\mathbb{C}_+}$.
Moreover, integrating by parts, $\Psi_-^{(1)}(x,t,k) = \binom{1}{0} + O(1/k)$ as $k \to \infty$.

Jost solutions of Lax pair, IV

Similarly, the other columns of Ψ_-, Ψ_+ are analytic (as functions of k) in the respective half-planes, so that they can be combined into two matrices: one is analytic in $\mathbb{C}_+$ and continuous in $\overline{\mathbb{C}_+}$, another in $\mathbb{C}_-$ and continuous in $\overline{\mathbb{C}_-}$; moreover, they approach I at infinity:

$$\left(\Psi_-^{(1)}, \Psi_+^{(2)}\right): \quad \text{analytic in } \mathbb{C}_+; \quad = I + O(\frac{1}{k}) \text{ as } k \to \infty;$$

$$\left(\Psi_+^{(1)}, \Psi_-^{(2)}\right): \quad \text{analytic in } \mathbb{C}_-; \quad = I + O(\frac{1}{k}) \text{ as } k \to \infty;$$

Consequently, these matrices are good candidates for the construction of RH problem, in view of (i) analyticity and (ii) large-k asymptotics. Moreover, more detailed analysis of the large-k behavior of $\Psi_\pm$ reveals that, e. g.,

$$\Psi_{+,12}(x,t,k) = \frac{q(x,t)}{2ik} + O(1/k^2), \qquad k \to \infty.$$

Question: how these matrices are related on $\mathbb{R}$ (which would serve as a contour for the RHP)?

Scattering relation

Let's go back to the Jost solutions $\Phi_{\pm}(x,t,k)$ and write down (formally) the relation

$$\Phi_{+}(x,t,k) = \Phi_{-}(x,t,k)S(x,t,k).$$

Now notice the following:

- Since the both Φ_{+} and Φ_{-} are solutions of an ordinary differential equation w.r.t. x (from Lax pair), it follows that matrix S is independent of x.
- Since the both Φ_{+} and Φ_{-} are solutions of an ordinary differential equation w.r.t. t (again from Lax pair), it follows that matrix S is independent of t.

Therefore we have that S in the relation above depends on k only and thus this relation actually reads

$$\Phi_{+}(x,t,k) = \Phi_{-}(x,t,k)S(k), \qquad k \in \mathbb{R}.$$

This relation is called the scattering relation; matrix $S(k)$ is called the scattering matrix.

Properties of scattering matrix, I

Now let's discuss the properties of $S(k)$.

First, we notice that $\det \Phi_+ = \det \Phi_- \equiv 1$.

Indeed, $\Phi_\pm$ are solutions of the differential equation $\frac{d}{dx}\Phi_\pm = U\Phi_\pm$, where the coefficient matrix U is such that $\operatorname{Tr} U \equiv 0$. It follows that $\frac{d}{dx}(\det \Phi_\pm) \equiv 0$. Similarly, since $\operatorname{Tr} V \equiv 0$ for V in $\frac{d}{dt}\Phi_\pm = V\Phi_\pm$, we have $\frac{d}{dt}(\det \Phi_\pm) \equiv 0$. Thus $\det \Phi_\pm$ is a constant. On the other hand, $\det \Phi_\pm|_{x\to\pm\infty} = 1$. Therefore, $\det(\Phi_\pm) \equiv 1$.

Consequently, $\det S(k) \equiv 1$.

Properties of scattering matrix, II

Further, notice that U satisfies the symmetry conditions:

- If $\lambda = 1$ (the parameter in NLS equation), then $U(x,t,k) = \sigma_1 \overline{U(x,t,\bar{k})} \sigma_1$, where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
- If $\lambda = -1$, then $U(x,t,k) = \sigma_2 \overline{U(x,t,\bar{k})} \sigma_2$, where $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

Similarly for V. Since the limiting values of $\Phi_\pm$ as $x \to \pm\infty$ also satisfies these symmetries, it follows (from the ODEs for $\Phi_\pm$) that (for $k \in \mathbb{R}$)

- $\Phi_\pm(x,t,k) = \sigma_1 \overline{\Phi_\pm(x,t,k)} \sigma_1 \quad (\lambda = 1)$;
- $\Phi_\pm(x,t,k) = \sigma_2 \overline{\Phi_\pm(x,t,k)} \sigma_2 \quad (\lambda = -1)$.

It follows that $S(k)$ also has these symmetries: for $k \in \mathbb{R}$,

- $S(k) = \sigma_1 \overline{S(k)} \sigma_1 \quad (\lambda = 1)$;
- $S(k) = \sigma_2 \overline{S(k)} \sigma_2 \quad (\lambda = -1)$.

Properties of scattering matrix, III

The symmetries can be expressed in terms of entries of $S(k)$.
Denote $a(k) := S_{22}(k)$ and $b(k) := S_{12}(k)$. Then

- $S(k) = \begin{pmatrix} \bar{a}(k) & b(k) \\ \bar{b}(k) & a(k) \end{pmatrix} \qquad (\lambda = 1)$;
- $S(k) = \begin{pmatrix} \bar{a}(k) & b(k) \\ -\bar{b}(k) & a(k) \end{pmatrix} \qquad (\lambda = -1)$.

$a(k)$ and $b(k)$ can be expressed in terms of determinants of matrices constructed from columns of $\Phi_\pm$ (linear algebra!):

$$a(k) = \det\left(\Phi_-^{(1)}, \Phi_+^{(2)}\right), \quad b(k) = \det\left(\Phi_+^{(2)}, \Phi_-^{(2)}\right).$$

Since $\Phi_-^{(1)}$ and $\Phi_+^{(2)}$ are analytic in $\mathbb{C}_+$, it follows that $a(k)$ is analytic in $\mathbb{C}_+$; moreover,

$$a(k) \to 1 \qquad k \to \infty,\ k \in \overline{\mathbb{C}_+}.$$

As for $b(k)$, it is (in general) defined for $k \in \mathbb{R}$ only, with $b(k) \to 0$ as $k \to \infty$. But in particular situations (e.g. if the support of $q_0(x)$ is finite), $b(k)$ also can be analytically continued from $\mathbb{R}$ into (a part of) $\mathbb{C}$.

Recall the scattering relation

$$\Phi_+(x,t,k) = \Phi_-(x,t,k)S(k), \qquad k \in \mathbb{R}.$$

This relation already has a form of a "multiplicative jump condition". But this relation cannot be interpreted as a jump condition for a RH problem, because neither Φ_+ nor Φ_- are analytic (as whole matrices!) in any domain of $\mathbb{C}$! On the other hand, particular columns of Φ_+ and Φ_- do analytic in either $\mathbb{C}_+$ or $\mathbb{C}_-$. This suggests to combine the columns analytic in one or another half-plane into the respective matrices:

$$\left(\Phi_-^{(1)}, \Phi_+^{(2)}\right)(x,t,k) = \left(\Phi_+^{(1)}, \Phi_-^{(2)}\right)(x,t,k)\tilde{S}(k), \qquad k \in \mathbb{R}.$$

Linear algebra problem: express $\tilde{S}(k)$ in terms of $S(k)$ (in terms of $a(k)$ and $b(k)$).

Solution of this (algebra) problem:

$$\left(\Phi_-^{(1)}, \Phi_+^{(2)}\right)(x,t,k) = \left(\Phi_+^{(1)}, \Phi_-^{(2)}\right)(x,t,k) \begin{pmatrix} \frac{1}{\bar{a}(k)} & \frac{b(k)}{\bar{a}(k)} \\ -\lambda\frac{\bar{b}(k)}{\bar{a}(k)} & \frac{1}{\bar{a}(k)} \end{pmatrix}, \quad k \in \mathbb{R}.$$

This can already be interpreted as a jump across $\mathbb{R}$!

But — further problems:

- the determinants of $\left(\Phi_-^{(1)}, \Phi_+^{(2)}\right)$ and $\left(\Phi_+^{(1)}, \Phi_-^{(2)}\right)$ may equal 0 at some points in respectively $\mathbb{C}_+$ and $\mathbb{C}_-$, which is not good for uniqueness (of the solution of RHP).
- large-k asymptotics is not I.

Indeed,

- $\det\left(\Phi_-^{(1)}, \Phi_+^{(2)}\right) = a(k)$ and $\det\left(\Phi_+^{(1)}, \Phi_-^{(2)}\right) = \overline{a(\bar{k})}$;
- recall that $\Psi_\pm := \Phi_\pm e^{(ikx+2ik^2t)\sigma_3} \to I$ as $k \to \infty$.

In order to have all determinants to equal 1: introduce

$$\hat{\Phi}(x,t,k) := \begin{cases} \left(\frac{\Phi_-^{(1)}(x,t,k)}{a(k)}, \Phi_+^{(2)}(x,t,k) \right), & k \in \mathbb{C}_+, \\ \left(\Phi_+^{(1)}(x,t,k), \frac{\Phi_-^{(2)}(x,t,k)}{a^*(k)} \right), & k \in \mathbb{C}_- \end{cases}$$

with $a^*(k) := \overline{a(\bar{k})}$. Then we have

$$\hat{\Phi}_+(x,t,k) = \hat{\Phi}_-(x,t,k) J_0(k), \qquad k \in \mathbb{R},$$

or

$$\left(\frac{\Phi_-^{(1)}}{a}, \Phi_+^{(2)} \right) = \left(\Phi_+^{(1)}, \frac{\Phi_-^{(2)}}{a^*} \right) J_0(k), \quad k \in \mathbb{R},$$

where

$$J_0(k) = \begin{pmatrix} 1 - \lambda |r(k)|^2 & \bar{r}(k) \\ -\lambda r(k) & 1 \end{pmatrix} \qquad \text{with } r(k) := \frac{\bar{b}(k)}{a(k)}.$$

Finally, in order to correct large-k behavior, introduce $M(x,t,k) := \hat{\Phi}(x,t,k)e^{(ikx+2ik^2t)\sigma_3}$. Then

$$M(x,t,k) \to I, \qquad k \to \infty \tag{i}$$

whereas the jump condition for $M(x,t,k)$ takes the form:

$$M_+(x,t,k) = M_-(x,t,k)J(x,t,k), \qquad k \in \mathbb{R}, \tag{ii}$$

where

$$\begin{aligned} J(x,t,k) &= e^{(-ikx-2ik^2t)\sigma_3} J_0(k) e^{(ikx+2ik^2t)\sigma_3} \\ &= \begin{pmatrix} 1-\lambda|r(k)|^2 & \bar{r}(k)e^{-2ikx-4ik^2t} \\ -\lambda r(k)e^{2ikx+4ik^2t} & 1 \end{pmatrix} \end{aligned} \tag{iii}$$

(i)+(ii)+(iii) looks like a RH problem: given $r(k), k \in \mathbb{R}$, find 2×2 matrix function $M(\cdot,\cdot,k)$ analytic in $\mathbb{C} \setminus \mathbb{R}$ and satisfying normalization (i) and jump (ii)+(iii).

Problem

If $a(k) = 0$ for some $k \in \overline{\mathbb{C}_+}$, then M has to have singularities.

The RHP formalism for the NLS equation, I

Assume for a moment that $a(k) \neq 0$ for all $k \in \overline{\mathbb{C}_+}$. Then the analysis above leads to the following algorithm for solving the Cauchy problem for the NLS equation.

1. Given $q_0(x)$, construct $a(k)$ and $b(k)$ from
$$\begin{pmatrix} \bar{a}(k) & b(k) \\ \lambda\bar{b}(k) & a(k) \end{pmatrix} = \Phi_-^{-1}(0,0,k)\Phi_+(0,0,k),$$
where $\Phi_\pm(0,0,k)$ are calculated by solving the linear integral equations
$$\Phi_\pm(x,0,k) = e^{-ikx\sigma_3} + \int_{\pm\infty}^{x} e^{ik(y-x)\sigma_3} \begin{pmatrix} 0 & q_0(y) \\ \lambda\bar{q}_0(y) & 0 \end{pmatrix} \Phi_\pm(y,0,k)dy$$
2. Given $r(k) := \bar{b}(k)/a(k)$, $k \in \mathbb{R}$, construct the jump matrix $J(x,t,k)$ by (iii).
3. Solve the RH problem (i)+(ii) for $M(x,t,k)$.
4. Obtain $q(x,t)$ from the large-k behavior of M:
$$q(x,t) = 2i \lim_{k\to\infty} kM_{12}(x,t,k).$$

- Since $\det J(x,t,k) \equiv 1$, it follows that $\det M(x,t,k) \equiv 1$.
- Then, by the Liouville theorem, the solution M of the RHP is unique, if exist.
- Existence of a solution to a matrix RHP: more subtle question. Basically, it can be established in two situations:
  1. "Symmetry case": the jump matrix possesses a particular (Schwartz) symmetry:
     - $J(k) + J^*(k)$ is positive definite for $k \in \mathbb{R}$;
     - $J(k) = J^*(\bar{k})$ for $k \in \Sigma \setminus \mathbb{R}$.

     (* denotes matrix adjoint);
  2. "Small norm case": the jump matrix is "close" to a matrix, for which the RHP has a solution.
     For instance, when $J(k)$ is "close" to I (if $J(k) \equiv I$, then $M(k) \equiv I$).

Cases $\lambda = 1$ and $\lambda = -1$ are different.

Case $\lambda = 1$

If $\lambda = 1$, then $a(k) \neq 0$ for all $k \in \overline{\mathbb{C}_+}$.

Proof. (i) Assume that $a(\hat{k}) = 0$ for some $\hat{k} \in \mathbb{C}_+$ (nonreal). Then, from representation $a(k) = \det(\Phi_-^{(1)}, \Phi_+^{(2)})$ it follows that $\Phi_+^{(2)}(x, 0, \hat{k}) = \Phi_-^{(1)}(x, 0, \hat{k})\hat{b}$ with some $b \in \mathbb{C}$. Now notice that (a) $\Phi_+^{(2)}(x, 0, \hat{k})$ decays exponentially as $x \to +\infty$ and $\Phi_-^{(1)}(x, 0, \hat{k})$ decays exponentially as $x \to -\infty$; (b) both are (vector) solutions of equation $L\Phi = \hat{k}\Phi$, where $L\Phi := i\sigma_3\Phi_x - i\sigma_3 \begin{pmatrix} 0 & q_0(x) \\ \bar{q}_0(x) & 0 \end{pmatrix} \Phi$. But L is self-adjoint operator and, therefore, can't have non-real eigenvalues; this contradicts the existence of $\hat{\Phi}(x) := \Phi_+^{(2)}(x, 0, \hat{k})$.

(ii) For $k \in \mathbb{R}$ we have (determinant relation) $|a(k)|^2 - |b(k)|^2 = 1$. Therefore, $|a(k)|^2 = 1 + |b(k)|^2 > 0$ and thus $a(k) \neq 0$ for $k \in \mathbb{R}$.

Zeros of $a(k)$ for $k \in \overline{\mathbb{C}_+}$, II

Now consider the case $\lambda = -1$.

- The corresponding operator L is not self-adjoint.
- Since $a(k) \to 1$ as $k \to \infty$, it follows that the zeros of $a(k)$ are located in a bounded domain.
- But it is possible that there are infinitely many of them, accumulating to some points on $\mathbb{R}$.
- Determinant relation on $\mathbb{R}$ reads $|a(k)|^2 + |b(k)|^2 = 1$, giving no information about possible real zeros.

From now on, we will assume that in case $\lambda = -1$:

- There are only finite number of zeros of $a(k)$ in $\mathbb{C}_+$ (and thus there are no zeros on $\mathbb{R}$);
- All zeros of $a(k)$ are simple.

Such assumption corresponds to generic situation: the set of initial data for with the assumptions above hold is an open, dense set (in respective topology).

If $a(k)$ has zeros, in $\mathbb{C}_+$, we have to enlarge our notion of Riemann-Hilbert problem, in order to allow singularities of a solution. But then, in order to preserve uniquness of a solution, we need to add more conditions, characterizing possible singularities.

Let $a(k_j) = 0$ for some $k_j \in \mathbb{C}_+$, $j = 1, \dots, N$. Then $\Phi_+^{(2)}(x, 0, k_j) = \Phi_-^{(1)}(x, 0, k_j) b_j$ with some $b_j \in \mathbb{C}$. Introduce $c_j := \frac{1}{\dot{a}(k_j) b_j}$, where dot denotes the derivative w.r.t. k. Then this relation, being considered in terms of M, reads:

$$\operatorname{Res}_{k \to k_j} M^{(1)}(x, t, k) = c_j e^{2ik_j x + 4ik_j^2 t} M^{(2)}(x, t, k_j), j = \overline{1, N}. \tag{iv-a}$$

The corresponding relations in $\mathbb{C}_-$ read

$$\operatorname{Res}_{k \to \bar{k}_j} M^{(2)}(x, t, k) = -\bar{c}_j e^{-2i\bar{k}_j x - 4i\bar{k}_j^2 t} M^{(1)}(x, t, \bar{k}_j), \quad j = \overline{1, N}. \tag{iv-b}$$

Notice that $\{k_j, c_j\}_{j=1}^N$ in these Residue conditions (similarly to $r(k)$, $k \in \mathbb{R}$) are determined by the initial condition $q_0(x)$.

Thus, in the case $a(k)$ has zeros, the RH problem for the NLS equation is as follows:

RH problem for NLS

Given $r(k)$, $k \in \mathbb{R}$ and $\{k_j, c_j\}_{j=1}^{N}$, find a 2×2 function M piece-wise meromorphic relative to $k \in \mathbb{R}$ such that it satisfies the normalization (i), the jump conditions (ii)+(iii), and the residue conditions (iv).

Having this RHP solved, the solution of the Cauchy problem for NLS is given by

$$q(x,t) = 2i \lim_{k\to\infty} kM_{12}(x,t,k).$$

Question

Do conditions (i)-(iv) determine M uniquely?

Proposition

Conditions (i)-(iv) determine M uniquely.

Proof Let $C_j(x,t,k) := c_j e^{2ik_jx+4ik_j^2t}$. (i) First, consider determinant of M.
By Residue conds, as $k \to k_j$, $M = \begin{pmatrix} \frac{C_j\alpha}{k-k_j} + O(1) & \alpha + O(k-k_j) \\ \frac{C_j\beta}{k-k_j} + O(1) & \beta + O(k-k_j) \end{pmatrix}$ with some α and β and thus $\det M = O(1)$ as $k \to k_j$ (non-singular!). Similarly as $k \to \bar{k}_j$. Since $\det J \equiv 1$, by the Liouville theorem we have $\det M \equiv 1$.

(ii) Let M and $\tilde{M}$ are two solutions of RHP. Then
$\tilde{M}^{-1} = \begin{pmatrix} \tilde{\beta} + O(k-k_j) & -\tilde{\alpha} + O(k-k_j) \\ -\frac{C_j\tilde{\beta}}{k-k_j} + O(1) & \frac{C_j\tilde{\alpha}}{k-k_j} + O(1) \end{pmatrix}$ and thus
$M\tilde{M}^{-1} = \frac{C_j}{k-k_j}\begin{pmatrix} \alpha\tilde{\beta} - \alpha\tilde{\beta} & -\alpha\tilde{\alpha} + \alpha\tilde{\alpha} \\ \beta\tilde{\beta} - \beta\tilde{\beta} & \alpha\tilde{\beta} - \alpha\tilde{\beta} \end{pmatrix} + O(1) = O(1)$. Thus $M\tilde{M}^{-1}$ is non-singular; moreover, it has no jump across $\mathbb{R}$ and approaches I as $k \to \infty$; then by the Liouville theorem, $M\tilde{M}^{-1} \equiv I$.

Solitons, I

In general, the RH problem can't be solved explicitly: in what follows, we will discuss how to reduce a RHP to a (singular) integral equation.

Cases when the RHP can be solved explicitly: if the jump conditions are trivial ($J(x,t,k) \equiv I$) and thus the only nontrivial information are Residue conditions.

The simplest case: there is only one pair of Res. conds. ($N = 1$), at $k = k_1 \in \mathbb{C}_+$ and $k = \bar{k}_1$:

$$\operatorname{Res}_{k \to k_1} M^{(1)}(x,t,k) = c_1 e^{2ik_1 x + 4ik_1^2 t} M^{(2)}(x,t,k_1),$$

$$\operatorname{Res}_{k \to \bar{k}_1} M^{(2)}(x,t,k) = -\bar{c}_1 e^{-2i\bar{k}_1 x - 4i\bar{k}_1^2 t} M^{(1)}(x,t,\bar{k}_1).$$

Thus in this case, the RHP is as follows: given $\{k_1, c_1\}$, find $M(x,t,k)$ s.t.: (i) it is meromorphic in $\mathbb{C}$; (ii) $M(x,t,k) \to I$ as $k \to \infty$; (iii) it has simple poles at k_1 and $\bar{k}_1$ according to the jump conditions above.

Solitons, II

Thus the solution of the RHP is a matrix with rational entries. Moreover, the Res. cond. and the normalization at ∞ dictate the form of M:

$$M(x,t,k) = \begin{pmatrix} \frac{k-B_1(x,t)}{k-k_1} & \frac{D_2(x,t)}{k-\bar{k}_1} \\ \frac{B_2(x,t)}{k-k_1} & \frac{k-D_1(x,t)}{k-\bar{k}_1} \end{pmatrix}$$

with some B_j, D_j. Further, the symmetry $M(\cdot,\cdot,k) = \sigma_2 \overline{M(\cdot,\cdot,\bar{k})} \sigma_2$ implies:

$$D_1(x,t) = \bar{B}_1(x,t), \qquad D_2(x,t) = -\bar{B}_2(x,t).$$

Finally, in order to determine $B_1(x,t)$ and $B_2(x,t)$ we use the Res. conds.: introducing $\hat{C}_1(x,t) := c_1 e^{2ik_1 x + 4ik_1^2 t}$, Res. cond. at $k = k_1$ reduce to the system of 2 linear algebraic equations for B_1, B_2:

$$k_1 - B_1 = -\frac{\hat{C}_1 \bar{B}_2}{k_1 - \bar{k}_1}, \qquad B_2 = \frac{\hat{C}_1 (k_1 - \bar{B}_1)}{k_1 - \bar{k}_1}$$

The solution of this algebraic equations:

$$B_1(x,t) = \left(k_1 - \frac{|\hat{C}_1(x,t)|^2}{(k_1-\bar{k}_1)^2}\right)\left(1 - \frac{|\hat{C}_1(x,t)|^2\bar{k}_1}{(k_1-\bar{k}_1)^2}\right)^{-1},$$

$$B_2(x,t) = \frac{\hat{C}_1(x,t)}{1 - \frac{|\hat{C}_1(x,t)|^2}{(k_1-\bar{k}_1)^2}}.$$

Since $q(x,t) = -2i\bar{B}_2(x,t)$, we obtain the corresponding solution q of the NLS:

$$q(x,t) = \frac{-2i\hat{C}_1(x,t)}{1 - \frac{|\hat{C}_1(x,t)|^2}{(k_1-\bar{k}_1)^2}},$$

where $\hat{C}_1(x,t) = c_1 e^{2ik_1x+4ik_1^2t}$.

Denote $k_1 = k_R + ik_I$. One can distinguish two cases:
(i) $k_R = 0$; (ii) $k_R \neq 0$.

Case 1: $k_1 = i\nu$ with $\nu > 0$.
In this case, $\hat{C}_1(x,t) = c_1 e^{-2\nu x - 4i\nu^2 t}$.

$$q(x,t) = -8i\nu^2 \bar{c}_1 \frac{e^{-2\nu x}}{4\nu^2 + |c_1|^2 e^{-4\nu x}} e^{4i\nu^2 t}.$$

Thus $q(x,t) = f(x)g(t)$, where $f(x) > 0$ and $|g(t)| = 1$. Moreover, $f(x) \sim \cdot e^{-2\nu|x|}$ as $x \to \pm\infty$. Such solution is called stationary soliton.

Soliton – from "solitary wave".

Case 2: $k_1 = k_R + ik_I$ with $k_R \neq 0$.

$$q(x,t) = -8ik_I^2\bar{c}_1 \frac{e^{-2k_I(x+4k_Rt)}}{4k_I^2 + |c_1|^2 e^{-4k_I(x+4k_Rt)}} e^{-2ik_R\left(x+\frac{2(k_R^2-k_I^2)}{k_R}t\right)}.$$

Particularly,

$$|q(x,t)| = const \frac{e^{-2k_I(x+4k_Rt+\phi)}}{1+e^{-4k_I(x+4k_Rt+\phi)}}$$

with some $\phi \in \mathbb{R}$, i.e., the "envelope" of q moves with velocity $v^{(1)} = -4k_R$. At the same time, the "inner oscillations" move with velocity $v^{(2)} = -\frac{2(k_R^2-k_I^2)}{k_R}$:

$$q(x,t) = f(x - v^{(1)}t)g(x - v^{(2)}t)$$

with $f > 0$ and $|g| = 1$. This is a moving soliton.

Solitons, VI

In the general case of N pairs of residue conditions with parameters $\{k_j, c_j\}_{j=1}^{N}$: obtaining $q(x,t)$ reduces to solving the system of $2N$ linear algebraic equations.
If all $\mathrm{Re}\, k_j$ are different, then as $t \to -\infty$ or $t \to +\infty$, $q(x,t)$ reduces (asymptotically!) to a sum of one-soliton solutions:

$$q(x,t) \sim \sum_{j=1}^{N} f_j(x - v_j^{(1)} t + \phi_j^{(1,-)}) g_j(x - v_j^{(2)} t + \phi_j^{(2,-)})$$

as $t \to -\infty$ and

$$q(x,t) \sim \sum_{j=1}^{N} f_j(x - v_j^{(1)} t + \phi_j^{(1,+)}) g_j(x - v_j^{(2)} t + \phi_j^{(2,+)})$$

as $t \to +\infty$.
In other words, the interaction between solitons can be viewed as elastic collisions, with only effect being the phase change. It is this property that distinguishes "solitons" among "solitary waves".

Reducing RHP with res. conds. to regular RHP

Question

What to do in the general case when the RHP is non-regular, i.e. it involves the residue conditions?

We consider 2 ways how to reduce a RHP with res. conds. to a regular RHP:

1. Multiplying (basically) from the left, to “kill” the singularities of the solution of the RHP;
2. Multiplying from the right, augment the contour of the RHP replacing the residue conditions by the jump conditions on the additional parts of the contour.

From non-regular to regular RHP: the first way, I

Lemma

Let M satisfies the RHP with the jump cond.

$$M_+(k) = M_-(k)J(k), \qquad k \in \Sigma.$$

Let $d_j(k)$, $j = 1, 2$ be functions analytic (meromorphic) and non-zero on Σ. Let $\tilde{M} := M \begin{pmatrix} d_1(k) & 0 \\ 0 & d_2(k) \end{pmatrix}$.
Then $\tilde{M}$ satisfies the jump cond.

$$\tilde{M}_+(k) = \tilde{M}_-(k)\tilde{J}(k),$$

where the entries of $\tilde{J}$ are:

$$\tilde{J}_{11} = J_{11}, \tilde{J}_{22} = J_{22}, \tilde{J}_{12} = J_{12}\frac{d_2}{d_1}, \tilde{J}_{21} = J_{21}\frac{d_1}{d_2}.$$

From non-regular to regular RHP: the first way, II

Theorem

Let $M(x,t,k)$ be the solution of the RHP corresponding to the focusing NLS, i.e. it satisfies

- $M_+(x,t,k) = M_-(x,t,k)J(x,t,k)$ with
  $J = \begin{pmatrix} 1+|r(k)|^2 & \bar{r}(k)e^{-2ikx-4ik^2t} \\ r(k)e^{2ikx+4ik^2t} & 1 \end{pmatrix}$, $k \in \mathbb{R}$
- $M(x,t,k) \to I$ as $k \to \infty$
- $\operatorname{Res}_{k\to k_j} M^{(1)}(x,t,k) = c_j e^{2ik_jx+4ik_j^2t} M^{(2)}(x,t,k_j), j = \overline{1,N}$
  $\operatorname{Res}_{k\to \bar{k}_j} M^{(2)}(x,t,k) = -\bar{c}_j e^{-2i\bar{k}_jx-4i\bar{k}_j^2t} M^{(1)}(x,t,\bar{k}_j), \quad j = \overline{1,N}.$

Let $\tilde{M}$ be the solution of the regular RHP: (a) without res. conds.; (b) with $r(k)$ replaced by $r(k)\prod\limits_{j=1}^{N}(k-k_j)/(k-\bar{k}_j)$. Then M can be expressed in terms of $\tilde{M}$ as follows:

$$M(x,t,k) = (kI+B_N)(kI+B_{N-1})\ldots(kI+B_1)\tilde{M}(x,t,k)\begin{pmatrix} \prod\limits_{j=1}^{N}\frac{1}{k-k_j} & 0 \\ 0 & \prod\limits_{j=1}^{N}\frac{1}{k-\bar{k}_j} \end{pmatrix}$$

From non-regular to regular RHP: the first way, III

Here the 2×2 matrices $B_j(x,t)$ are independent of k; they can be determined recursively:

$$\tilde{M}_0 \mapsto B_1 \mapsto \tilde{M}_1 \mapsto B_2 \mapsto \tilde{M}_2 \mapsto \cdots \mapsto \tilde{M}_{N-1} \mapsto B_N,$$

where $\tilde{M}_0 \equiv \tilde{M}$, $\tilde{M}_j = (kI + B_j)\tilde{M}_{j-1}$, and B_j is the solution of the linear algebraic equations

$$(k_j I + B_j)\tilde{M}_{j-1}(k_j)\begin{pmatrix} 1 \\ -d_j \end{pmatrix} = 0,$$

$$(\bar{k}_j I + B_j)\tilde{M}_{j-1}(\bar{k}_j)\begin{pmatrix} \bar{d}_j \\ 1 \end{pmatrix} = 0,$$

with $d_j(x,t) = c_j \dfrac{\prod\limits_{l=1, l\neq j}^{N}(k_j - k_l)}{\prod\limits_{l=1}^{N}(\bar{k}_j - k_l)} e^{2ik_j x + 4ik_j^2 t}$.

- If $r(k) \equiv 0$ (pure soliton case), then $\tilde{M} \equiv I$, and we arrive at another way to present a pure N-soliton solution (but still the calculations reduce to solving linear algebraic equations).

Let's discuss the details in the case $N = 1$. In this case, $d_1(x,t) = c_1 \frac{1}{k_1 - \bar{k}_1} e^{2ik_1x + 4ik_1^2 t}$ and

$$M(x,t,k) = (kI + B_1)\tilde{M}(x,t,k) \begin{pmatrix} \frac{1}{k-k_1} & 0 \\ 0 & \frac{1}{k-\bar{k}_1} \end{pmatrix}.$$

Thus $\operatorname{Res}_{k\to k_1} M^{(1)} = (k_1 I + B_1)\tilde{M}^{(1)}(k_1)$ and $M^{(2)}(k_1) = (k_1 I + B_1)\tilde{M}^{(2)}(k_1)\frac{1}{k_1 - \bar{k}_1}$. Let's check the residue condition:

$$\begin{aligned} \operatorname{Res}_{k\to k_1} M^{(1)} &= (k_1 I + B_1)\tilde{M}^{(1)}(k_1) = d_1(k_1 I + B_1)\tilde{M}^{(2)}(k_1) \\ &= d_1(k_1 - \bar{k}_1)M^{(2)}(k_1) = c_1 e^{2ik_1x + 4ik_1^2 t} M^{(2)}(k_1). \end{aligned}$$

The equations for determining B_1 reduce to $B_1 W = -W \left(\begin{smallmatrix} k_1 & 0 \\ 0 & \bar{k}_1 \end{smallmatrix}\right)$, where W is the 2×2 matrix: $W = \left(\tilde{M}(k_1)\begin{pmatrix} 1 \\ -d_1 \end{pmatrix}, \tilde{M}(\bar{k}_1)\begin{pmatrix} \bar{d}_1 \\ 1 \end{pmatrix}\right)$. Due to the symmetry, $\det W > 0$ and thus B_1 is uniquely determined.

The **second way** of reducing non-regular RHP to a regular one is based on adding (small) circles surrounding k_j and $\bar{k}_j$ to the jump contour and respective re-definition of the solution of the RHP in the disks $D_j = \{k : |k - k_j| < \varepsilon\}$ and their complex conjugates $\bar{D}_j$

At the **first step**, define
$$\hat{M} := M \begin{cases} \begin{pmatrix} 1 & 0 \\ -\frac{c_j e^{2ik_j x + 4ik_j^2 t}}{k - k_j} & 1 \end{pmatrix}, & k \in D_j \\ \begin{pmatrix} 1 & \frac{\bar{c}_j e^{-2i\bar{k}_j x - 4i\bar{k}_j^2 t}}{k - \bar{k}_j} \\ 0 & 1 \end{pmatrix}, & k \in \bar{D}_j \\ I, & \text{otherwise} \end{cases}$$

Let's check that $\hat{M}$ is non-singular at $k = k_1$. Indeed, for k near k_1,

$$\hat{M}^{(1)}(k) = M^{(1)}(k) - \frac{c_j e^{2ik_j x + 4ik_j^2 t}}{k - k_j} M^{(2)}(k)$$
$$= \frac{c_j e^{2ik_j x + 4ik_j^2 t}}{k - k_j} M^{(2)}(k_1) + O(1) - \frac{c_j e^{2ik_j x + 4ik_j^2 t}}{k - k_j} M^{(2)}(k_1) + O(1) = O(1).$$

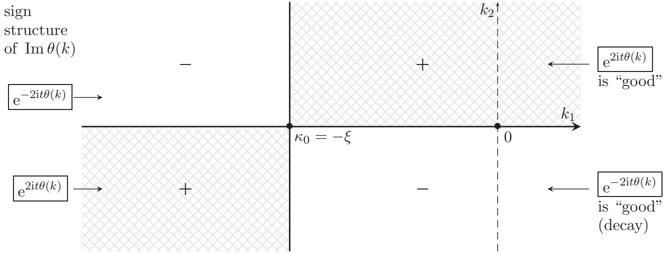
Then $\hat{M}_+ = \hat{M}_- \hat{J}$, $k \in \Sigma = \mathbb{R} \cup_{j=1}^{N} \left(\partial D_j \cap \partial \bar{D}_j\right)$, with $\hat{J} = J$ on $\mathbb{R}$ and the triangular jumps across ∂D_j and $\partial \bar{D}_j$ as above.

From non-regular to regular RHP: the second way, II

In principal, $\hat{M}$ satisfies a regular RHP, whose solution gives the solution of the NLS. But this RHP is not appropriate for studying properties of this solution, particularly, large-t asymptotics, because the triangular jumps may exponentially grow as $t \to \infty$! Indeed, let's rewrite the exponentials above as

$$e^{2ikx+4ik^2t} = e^{2it\theta(\xi,k)}, \quad \xi := \frac{x}{4t}, \quad \theta(\xi,k) := 2k^2 + 4\xi.$$

Consider the Signature Table, i.e., the distribution of signs of $\operatorname{Im}\theta(\xi,k))$ in the k plane, depending on the value of ξ:



From non-regular to regular RHP: the second way, III

Now notice that if, for a given ξ, some k_j is located in the upper-right quarter of the signature table, then the corresponding exponentials in the jump matrix on the circle surrounding k_j decays to 0 as $t \to \infty$ (keeping ξ fixed!); thus one may expect that this part of the jump conditions give exponentially small contribution to $M(x,t,k)$ and thus to $q(x,t)$.

On the other hand, for those k_j located in the upper-left quarter, the exponential is growing. In order to cope with this growth, we need the **second step** in the transformation of the original RHP, based on the matrix identity

$$\begin{pmatrix} 1 & 0 \\ A & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{A} \\ -A & 0 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{A} \\ 0 & 1 \end{pmatrix}.$$

From non-regular to regular RHP: the second way, IV

The **second step**: define $\tilde{M} :=$

$$\hat{M}\begin{cases}\begin{pmatrix}1 & -\frac{k-k_j}{c_j}e^{-2it\theta(k_j)}\\ \frac{c_j e^{2it\theta(k_j)}}{k-k_j} & 0\end{pmatrix}\begin{pmatrix}\prod\limits_{j\in K(\xi)}\frac{k-k_j}{k-\bar{k}_j} & 0\\ 0 & \prod\limits_{j\in K(\xi)}\frac{k-\bar{k}_j}{k-k_j}\end{pmatrix}, & k\in D_j\\ \begin{pmatrix}0 & -\frac{\bar{c}_j e^{-2it\theta(\bar{k}_j)}}{k-\bar{k}_j}\\ \frac{k-\bar{k}_j}{\bar{c}_j}e^{2it\theta(\bar{k}_j)} & 1\end{pmatrix}\begin{pmatrix}\prod\limits_{j\in K(\xi)}\frac{k-k_j}{k-\bar{k}_j} & 0\\ 0 & \prod\limits_{j\in K(\xi)}\frac{k-\bar{k}_j}{k-k_j}\end{pmatrix}, & k\in \bar{D}_j\\ \begin{pmatrix}\prod\limits_{j\in K(\xi)}\frac{k-k_j}{k-\bar{k}_j} & 0\\ 0 & \prod\limits_{j\in K(\xi)}\frac{k-\bar{k}_j}{k-k_j}\end{pmatrix}, & \text{otherwise}\end{cases}$$

where $K(\xi)=\{j : \operatorname{Re} k_j < -\xi\}$. Then $\tilde{M}_+ = \tilde{M}_-\tilde{J}$, where the jump matrix $\tilde{J}$ is:

1. $\tilde{J} = \begin{pmatrix}1 & -\frac{k-k_j}{c_j}\prod\limits_{j\in K(\xi)}\left(\frac{k-\bar{k}_j}{k-k_j}\right)^2 e^{-2it\theta(k_j)}\\ 0 & 1\end{pmatrix}$ for $k\in\partial D_j$ (corr. $\partial\bar{D}_j$);
2. $\tilde{J} = \begin{pmatrix}1+|\tilde{r}(k)|^2 & \bar{\tilde{r}}(k)e^{-2it\theta(k)}\\ \tilde{r}(k)e^{2it\theta(k)} & 1\end{pmatrix}$, $k\in\mathbb{R}$, with

   $$\tilde{r}(k) := r(k)\prod_{j\in K(\xi)}\left(\frac{k-\bar{k}_j}{k-k_j}\right)^2.$$