Riemann-Hilbert problems and integrable nonlinear partial differential equations, I

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Literature

I. Complex analysis: boundary value problems

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II. Integrable nonlinear PDE; Riemann-Hilbert problem as a tool

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Case study:

Cauchy problem for nonlinear Schrödinger (NLS) equation in dim. $1+1$, with decaying boundary conds.

Given $q_0(x)$, $-\infty < x < \infty$ s.t. $q_0(x) \to 0$ as $x \to \infty$, find $q(x,t)$:

- $iq_t(x,t) + q_{xx}(x,t) - 2\lambda|q|^2 q = 0, \quad -\infty < x < \infty,\ t > 0;$
- $q(x,0) = q_0(x), \quad -\infty < x < \infty$ (initial conditions);
- $q(x,t) \to 0$ as $x \to \infty$ for all $t > 0$.

- Tool: a representation of solution $q(x,t)$ of Cauchy problem in terms of solution of a Riemann–Hilbert problem (RHP): a version of the Inverse Scattering Transform (IST) method for integrable nonlinear PDE
- Goal: use this representation in order to obtain (the main term of) large time asymptotics of $q(x,t)$

I. Introduction: Cauchy integrals; RH problems (additive/multiplicative; scalar/matrix)

Riemann–Hilbert problem: boundary value problem in complex analysis

Given (i) contour $\Sigma \in \bar{\mathbb{C}}$, (ii) "jump across Σ", (iii) normalization cond., find a function:

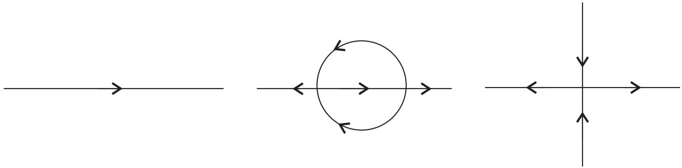
1. is analytic in $\mathbb{C} \setminus \Sigma$;
2. satisfies "jump conditions across Σ";
3. satisfies normalization cond. at some $z_0 \in \bar{\mathbb{C}}$.

Questions

- what are contours, jump conditions, normalization?
- existence of a solution
- uniqueness

Contours

Oriented contours:



- bounded / unbounded
- simple / composed (with self-intersections)
- closed / open (e.g., finite arcs)

Orientation: $\mathbb{C} = \Sigma \cup \Omega_+ \cup \Omega_-$, where $\Omega_\pm$ are (multicomponent) domains s.t. Σ is the counterclockwise boundary of (components of) Ω_+ and Σ is the clockwise boundary of (components of) Ω_-.
Boundary values: $\Phi_\pm(s) := \lim_{\substack{z \to s \\ z \in \Omega_\pm}} \Phi(z)$ (non-tangent limits).

RH problems (scalar)

- Additive RH problem: given a smooth, bounded and closed contour Σ and a function $f(s) \in C^{0,\alpha}(\Sigma)$ (α-Hölder continuous), find $\phi(z)$ piecewise analytic in $\mathbb{C} \setminus \Sigma$ s.t.
  - $\phi_+(s) - \phi_-(s) = f(s)$, $s \in \Sigma$ ("jump cond.");
  - $\phi(\infty) = 0$ (normalization).
- Multiplicative RH problem (**homogeneous**): given contour Σ and a function $g(s) \in C^{0,\alpha}(\Sigma)$ s.t. $g(s) \neq 0$, find $\phi(z)$ piecewise analytic in $\mathbb{C} \setminus \Sigma$ s.t.
  - $\phi_+(s) = \phi_-(s)g(s)$, $s \in \Sigma$ ("jump cond.");
  - $\phi(\infty) = 1$ (normalization).
- Multiplicative RH problem (**inhomogeneous**): given contour Σ and functions $g(s), f(s) \in C^{0,\alpha}(\Sigma)$ s.t. $g(s) \neq 0$, find $\phi(z)$ piecewise analytic in $\mathbb{C} \setminus \Sigma$ s.t.
  - $\phi_+(s) = \phi_-(s)g(s) + f(s)$, $s \in \Sigma$ ("jump cond.");
  - $\phi(\infty) = 0$ (normalization).
- Generalizations: more complicated contours (unbounded; open arcs; composed); less smooth jump functions.

The Cauchy integral is the **fundamental object (tool) in the theory of RH problems**. Given an oriented contour Σ and a function $f : \Sigma \to \mathbb{C}$, the Cauchy integral is defined by

$$\mathcal{C}_\Sigma f(z) := \frac{1}{2\pi i} \int_\Sigma \frac{f(s)}{s-z} ds, \quad z \in \mathbb{C} \setminus \Sigma.$$

We are interested in the boundary values of $\mathcal{C}_\Sigma f(z)$, when z approach a point on Σ:

$$\mathcal{C}_\Sigma^\pm f(s^*) := \lim_{\substack{z \to s^* \\ z \in \Omega_\pm}} \mathcal{C}_\Sigma f(z).$$

For $s^* \in \Sigma$, define also the Cauchy principal value integral:

$$⨍_\Sigma \frac{f(s)}{s-s^*} ds := \lim_{\delta \downarrow 0} \int_{\Sigma \setminus B(s^*, \delta)} \frac{f(s)}{s-s^*} ds,$$

where $B(s^*, \delta) = \{z : |z - s^*| \leq \delta\}$.

Boundary values of Cauchy integrals, I

Sokhotski-Plemelj formulas

Let Σ be a smooth arc from a to b and let $f \in C^{0,\alpha}(\Sigma)$. Then for $s^* \in \Sigma$, $s^* \neq a, b$:

$$(\mathcal{C}_\Sigma^+ f)(s^*) = \frac{1}{2} f(s^*) + \frac{1}{2\pi i} \fint_\Sigma \frac{f(s)}{s - s^*} ds,$$

$$(\mathcal{C}_\Sigma^- f)(s^*) = -\frac{1}{2} f(s^*) + \frac{1}{2\pi i} \fint_\Sigma \frac{f(s)}{s - s^*} ds,$$

Notice the local character of these formulas: f has to be "smooth" only is a vicinity of s^*.

On the other hand, the global result holds true:

Cauchy integrals as operators

If Σ is smooth, bounded, and closed, then the Cauchy integral operators $\mathcal{C}_\Sigma^\pm$ are bounded linear operators on $C^{0,\alpha}(\Sigma)$.

Sokhotski-Plemelj formulas: **proof**.

(i) Let $f(s) \equiv 1$. Then $⨍_{\Sigma} \frac{ds}{s-s^*} = i\pi$.

Indeed, let $l := (s_1, s_2) \subset \Sigma$ s.t. $s^* \in\ l$ and $|s_1 - s^*| = \delta$, $|s_2 - s^*| = \delta$. Let (a, b) be a part of Σ (arc) s.t. $l \subset (a, b)$. Let $\log(z - s^*)$ be defined with a cut from s^* to ∞ outside Σ. Then

$$\int_{(a,b)\setminus l} \frac{ds}{s - s^*} = \log(s - s^*)|_a^{s_1} + \log(s - s^*)|_{s_2}^{b} = \log \frac{b - s^*}{a - s^*} + \log \frac{s_1 - s^*}{s_2 - s^*}.$$

Second term as $\delta \to 0$:

$$\log \left| \frac{s_1 - s^*}{s_2 - s^*} \right| + i\left(\arg(s_1 - s^*) - \arg(s_2 - s^*)\right) \to \log 1 + i\pi = i\pi.$$

Setting $b = a$, the first term vanishes.

Boundary values of Cauchy integrals, III

(ii) For arbitrary smooth $f(s)$, introduce $\phi(s^*) := \frac{1}{2\pi i}\int\limits_{\Sigma} \frac{f(s)-f(s^*)}{s-s^*}ds$

(non-singular integral). Then

$$\frac{1}{2\pi i}\fint\limits_{\Sigma} \frac{f(s)}{s-s^*}ds = \frac{1}{2\pi i}\int\limits_{\Sigma} \frac{f(s)-f(s^*)}{s-s^*}ds + \frac{f(s^*)}{2\pi i}\fint\limits_{\Sigma} \frac{ds}{s-s^*} = \phi(s^*) + \frac{f(s^*)}{2}.$$

(iii) Let $z \in \mathbb{C} \setminus \Sigma$. Then

$$\mathcal{C}_{\Sigma} f(z) = \frac{1}{2\pi i}\int\limits_{\Sigma} \frac{f(s)-f(s^*)}{s-z}ds + \frac{f(s^*)}{2\pi i}\int\limits_{\Sigma} \frac{ds}{s-z}.$$

First term: bounded for z near s^*; has limit $\phi(s^*)$ as $z \to s^*$ along any path!
Second term – by Cauchy Theorem: $\frac{1}{2\pi i}\int\limits_{\Sigma} \frac{ds}{s-z} = 1$ for $z \in \Omega_+$ (inside Σ) and

$\frac{1}{2\pi i}\int\limits_{\Sigma} \frac{ds}{s-z} = 0$ for $z \in \Omega_-$. Then

$$(\mathcal{C}_{\Sigma}^{+} f)(s^*) = \phi(s^*) + f(s^*) = \frac{1}{2\pi i}\fint\limits_{\Sigma} \frac{f(s)}{s-s^*}ds - \frac{f(s^*)}{2} + f(s^*),$$

$$(\mathcal{C}_{\Sigma}^{-} f)(s^*) = \phi(s^*) + 0 = \frac{1}{2\pi i}\fint\limits_{\Sigma} \frac{f(s)}{s-s^*}ds - \frac{f(s^*)}{2}.$$

Boundary values of Cauchy integrals, IV

Cauchy integrals near endpoints

Let Σ be a smooth arc from a to b and let $f \in C^{0,\alpha}(\Sigma)$. The following holds for any endpoint $c = a$ or b:

- for z near c, $z \in \mathbb{C} \setminus \Sigma$,

$$(\mathcal{C}_\Sigma f)(z) = \sigma \frac{f(c)}{2\pi i} \log \frac{1}{z-c} + F_0(z;\sigma),$$

- for s near c, $s \in \Sigma$,

$$(\mathcal{C}_\Sigma^\pm f)(s) = \sigma \frac{f(c)}{2\pi i} \log \frac{1}{|z-c|} + H_0(s;\sigma),$$

  where $\sigma = -1$ if $c = b$ and $\sigma = 1$ if $c = a$; F_0 and H_0 both tend to definite limits as the argument approaches c.

If Σ is smooth, bounded, and closed, then the RH problem $\phi_+(s) - \phi_-(s) = f(s)$, $s \in \Sigma$, $f \in C^{0,\alpha}(\Sigma)$; $\phi(\infty) = 0$ is solved by the Cauchy integral:

$$\phi(z) = \frac{1}{2\pi i} \int_\Sigma \frac{f(s)}{s - z} ds, \quad z \in \mathbb{C} \setminus \Sigma.$$

- Existence: by construction.
- Uniqueness: Assume $\phi(z)$, $\tilde{\phi}(z)$ are solutions. Define $D(z) := \phi(z) - \tilde{\phi}(z)$. Then $D_+(z) = D_-(z)$ on $\Sigma \implies D(z)$ is entire function. But $D(\infty) = 0 \implies$ by Liouville's theorem, $D(z) \equiv 0$.
- Open contour (arc): same formula; but the solution has logarithmic singularities at the endpoints.
- Unbounded contour (e.g., $\Sigma = \mathbb{R}$): take care of decay of f at ∞ ($|\phi(s)| < \frac{C}{|s|^\rho}$ with $\rho > 0$); same formula.

Multiplicative RH problem, I

Scalar RH problem (closed contour)

Given Σ and $g \in C^{0,\alpha}(\Sigma)$, $g(s) \neq 0$, find ϕ s.t.
$\phi_+(s) = \phi_-(s) g(s)$, $s \in \Sigma$; $\qquad \phi(\infty) = 1$.

Idea: solve via the logarithm, reducing to additive problem.

- If $\log g(s)$ is well-defined, Hölder continuous, then

$$\log \phi_+(s) = \log \phi_-(s) + \log g(s) \Rightarrow \phi(z) = \exp\left\{ \frac{1}{2\pi i} \int_\Sigma \frac{\log g(s)}{s-z} ds \right\}$$

- This works, if $\operatorname{ind}_\Sigma g(s) = 0$, where the index of g w.r.t. Σ is defined by

$$\kappa = \operatorname{ind}_\Sigma g(s) := \frac{1}{2\pi}[\arg g(s)]_\Sigma = \frac{1}{2\pi i}[\log g(s)]_\Sigma = \frac{1}{2\pi i} \int_\Sigma d \log g(s).$$

- Uniqueness of solution with $\phi \neq 0$: again by Liouville's theorem (applied to $\phi(z)/\tilde{\phi}(z)$).

Multiplicative RH problem, II

What to do if $\operatorname{ind}_\Sigma g(s) \neq 0$?
Let $0 \in \Omega_+$ (w.l.g.); then $\operatorname{ind}_\Sigma s^{-\kappa} g(s) = 0$. Thus we can (uniquely) solve the problem

$$\psi_+(s) = \psi_-(s) s^{-\kappa} g(s), s \in \Sigma; \qquad \psi(\infty) = 1$$

as above: $\psi(z) = \exp\left\{\frac{1}{2\pi i}\int_\Sigma \frac{\log s^{-\kappa} g(s)}{s-z} ds\right\}$.

- if $\kappa > 0$, then solution is not unique:

$$\phi(z) = \begin{cases} P(z)\psi(z), & z \in \Omega_+, \\ P(z)\psi(z) z^{-\kappa}, & z \in \Omega_-, \end{cases}$$

  where $P(z)$ is any monic polynomial of degree κ.
  If $P(z) \equiv 1$, then $\phi(z)$ is called the fundamental solution.
- if $\kappa < 0$, then there are no nontrivial bounded solutions.
  Fundamental solution: as above (with $P(z) \equiv 1$);
  $\phi(z) \sim z^{-\kappa}$ as $z \to \infty$; $\phi(z) \neq 0$.

Multiplicative RH problem, III

Inhomogeneous RH problem

Given Σ and $g, f \in C^{0,\alpha}(\Sigma)$, $g(s) \neq 0$, find ϕ s.t.
$\phi_+(s) = \phi_-(s)g(s) + f(s)$, $s \in \Sigma$; $\quad \phi(\infty) = 0$.

- First, find the fundamental solution $\nu(z)$ of the corresponding homogeneous problem $\nu_+(s) = \nu_-(s)g(s)$; can be solved as above.
- The RHP reduces to additive problem:

$$\frac{\phi_+(s)}{\nu_+(s)} = \frac{\phi_-(s)}{\nu_-(s)} + \frac{f(s)}{\nu_+(s)}; \quad \frac{\phi(z)}{\nu(z)} = O(z^{-\kappa-1}), z \to \infty.$$

- if $\kappa \geq 0$, then $\phi(z)/\nu(z)$ is given by the Cauchy integral. It follows that

$$\phi(z) = \nu(z) \left(\frac{1}{2\pi i} \int_\Sigma \frac{f(s)}{\nu_+(s)(s-z)} ds + P(z) \right),$$

where $P(z)$ is any polynomial of degree $< \kappa$.

Multiplicative RH problem, IV

- if $\kappa < 0$, then in order to have

$$\frac{\phi(z)}{\nu(z)} = \frac{1}{2\pi i}\int_{\Sigma}\frac{f(s)}{\nu_+(s)(s-z)}ds = O(z^{\kappa-1}),$$

we need additional moment conditions to be satisfied:

$$\int_{\Sigma} s^n \frac{f(s)}{\nu_+(s)}ds = 0, \qquad n = 0, \ldots, -\kappa - 1.$$

RH problem on open contours

- additive problem: $\phi_+(s) - \phi_-(s) = f(s)$, $s \in (a,b)$. As above, solved by Cauchy integral $\phi(z) = \frac{1}{2\pi i}\int_{(a,b)} \frac{f(s)}{s-z}ds$; unique solution under *local integrability cond.* (with $\log$ singularities at a, b).
- multiplicative problem: $\phi_+(s) = \phi_-(s)g(s)$, $s \in (a,b)$. The Cauchy integral solution $\psi(z) = \exp\left\{\frac{1}{2\pi i}\int_{(a,b)} \frac{\log g(s)}{s-z}ds\right\}$ has power-type singularities (zeros):

$$\psi(z) = (z-b)^{\zeta_b} H_b(z), \qquad \psi(z) = (z-a)^{\zeta_a} H_a(z),$$

where $\zeta_b = \frac{\log g(b)}{2\pi i}$ and $\zeta_a = -\frac{\log g(a)}{2\pi i}$; $H_a(z)$, $H_b(z)$ have definite limits as $z \to a, b$. Let $\zeta_c = \lambda_c + i\mu_c$ and let $n_c \in \mathbb{Z}$ be such that $-1 < n_c + \lambda_c < 1$, $c = a, b$. Then

$$\phi(z) = (z-a)^{n_a}(z-b)^{n_b}\psi(z)$$

is a locally integrable solution.

Example of RH problem

RH problem with constant multiplicative jump

$\phi_+(s) = \alpha\phi_-(s)$, $s \in (a,b)$; ϕ of finite degree at ∞.

- Set $\alpha = |\alpha|e^{i\theta}$ so that $\log\alpha = \log|\alpha| + i\theta$. A solution is given by Cauchy integral

$$\psi(z) = \exp\left\{\frac{\log|\alpha| + i\theta}{2\pi i}\log\left(\frac{z-b}{z-a}\right)\right\} = \left(\frac{z-b}{z-a}\right)^{-i\log|\alpha|/(2\pi)+\theta/(2\pi)}.$$

- The locally integrable solutions are

$$\phi(z) = (z-a)^{n_a}(z-b)^{n_b}\psi(z),$$

where $-1 < n_a - \theta/(2\pi) < 1$, $-1 < n_b + \theta/(2\pi) < 1$.
Thus, 2 solutions, if $\theta/(2\pi)$ is not integer.
For uniqueness: add requirements for $z \sim a$, $z \sim b$.

Matrix Riemann-Hilbert problems

A general form of jump conditions:

$$\Phi_+(s) = \Phi_-(s)G(s) + F(s), \qquad s \in \Sigma,$$

where $\Phi : \mathbb{C} \setminus \Sigma \mapsto \mathbb{C}^{m\times n}$, $G : \Sigma \mapsto \mathbb{C}^{n\times n}$, and $F : \Sigma \mapsto \mathbb{C}^{m\times n}$.
Most often in applications: $m = n = 2$.

- Unlike scalar RHP, matrix RHP cannot, in general, be solved in closed form.
- The general theory involves the analysis of singular integral operators.
- But: three types of RHP can be solved in closed form (by reducing matrix RHP to a sequence of scalar RHP).

Diagonal Riemann-Hilbert problems

$$\Phi_+(s) = \Phi_-(s)D(s),\ s \in \Sigma; \qquad \Phi(\infty) = I,$$

where $\Phi : \mathbb{C} \setminus \Sigma \mapsto \mathbb{C}^{n\times n}$, $D \in C^{0,\alpha}(\Sigma)$,
$D(s) = \mathrm{diag}(d_1(s), \ldots, d_n(s))$ with $\det D(s) \neq 0$;
$\log d_i(s) \in C^{0,\alpha}(\Sigma)$

- The problem decomposes into n scalar RHP:

$$\phi_{i+}(s) = \phi_{i-}(s)d_i(s), s \in \Sigma; \quad \phi_i(\infty) = 1, \quad i = 1, \ldots, n.$$

- Each of these has a solution $\phi_i(z) = \exp\left\{\frac{1}{2\pi i}\int_\Sigma \frac{\log d_i(s)}{s-z}ds\right\}$.
- A solution of RHP: $\Phi(z) = \mathrm{diag}(\phi_1(s), \ldots, \phi_n(s))$.
- If Σ is smooth, bounded, and closed, then the solution is unique. Indeed, if Ψ is another solution, consider $\Psi\Phi^{-1}$:

$$\Psi_+(s)\Phi_+^{-1}(s) = \Psi_-(s)D(s)D^{-1}(s)\Phi_-^{-1}(s) = \Psi_-(s)\Phi_-^{-1}(s);$$

thus $\Psi\Phi^{-1}$ is entire. Also, $\Psi(\infty)\Phi^{-1}(\infty) = I$; then, by Liouville's theorem, $\Psi(z)\Phi^{-1}(z) \equiv I$.

Constant jump matrix Riemann-Hilbert problems, I

$$\Phi_+(s) = \Phi_-(s)A, \ s \in \Sigma; \qquad \Phi \text{ of finite degree at } \infty,$$

where A is invertible, diagonalizable: $A = U\Lambda U^{-1}$ with $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$.

- Reduces to a diagonal RHP for $D(z) := U^{-1}\Phi(z)U$.

Constant jump matrix Riemann-Hilbert problems, II

Example

$$\Phi_+(s) = \Phi_-(s)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; s \in (a,b); \qquad \Phi(\infty) = I$$

- First, diagonalize:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = U \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} U^{-1}, \qquad U = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$$

- Solve auxiliary scalar RHP:

$$h_{1+}(s) = -ih_{1-}(s), \qquad h_{2+}(s) = ih_{2-}(s).$$

Solutions: $h_1(z) = \exp\left\{\frac{1}{2\pi i}\int_a^b \frac{\log(-i)}{s-z}ds\right\} = \left(\frac{z-b}{z-a}\right)^{-1/4}$,
$h_2(z) = \exp\left\{\frac{1}{2\pi i}\int_a^b \frac{\log i}{s-z}ds\right\} = \left(\frac{z-b}{z-a}\right)^{1/4} \equiv h_1^{-1}(z)$.

-

$$\Phi(z) = U\begin{pmatrix} h_1(z) & 0 \\ 0 & h_1^{-1}(z) \end{pmatrix} U^{-1} = \begin{pmatrix} \frac{h_1(z)+h_1^{-1}(z)}{2} & \frac{i(h_1(z)-h_1^{-1}(z))}{2} \\ -\frac{i(h_1(z)-h_1^{-1}(z))}{2} & \frac{h_1(z)+h_1^{-1}(z)}{2} \end{pmatrix}$$

Triangular Riemann-Hilbert problems, I

$$\Phi_+(s) = \Phi_-(s)U(s),\ s \in \Sigma; \qquad \Phi(\infty) = I,$$

where $U(s)$ is upper triangular: $U(s) = (U_{i,j}(s))_{1 \leq i,j \leq n}$, $U_{i,j}(s) = 0$ if $i < j$.

Assume $\operatorname{ind}_\Sigma U_{i,i}(s) = 0$, $i = 1, \dots, n$.

- Reduces to solving successive scalar RHP; each row can be found independently of other rows
- Scalar problems for the first row:

$$\begin{aligned}
&\Phi_{1,1+}(s) = \Phi_{1,1-}(s)U_{1,1}(s), && \Phi_{1,1}(\infty) = 1 \\
&\Phi_{1,2+}(s) = \Phi_{1,2-}(s)U_{2,2}(s) + \Phi_{1,1-}(s)U_{1,2}(s), && \Phi_{1,2}(\infty) = 0 \\
&\dots
\end{aligned}$$

- Here the first equ. is a homogeneous RHP for $\Phi_{1,1}$; having it solved, the second equation is a non-homogeneous RHP for $\Phi_{1,2}$, etc.

Triangular Riemann-Hilbert problems, II

Example

$$\Phi_+(s) = \Phi_-(s)\begin{pmatrix}1 & f(s)\\ 0 & 1\end{pmatrix},\ s\in\Sigma; \qquad \Phi(\infty) = I$$

- Scalar problems for the first row:

$$\begin{aligned}\Phi_{1,1+}(s) &= \Phi_{1,1-}(s), & \Phi_{1,1}(\infty) &= 1\\ \Phi_{1,2+}(s) &= \Phi_{1,2-}(s) + \Phi_{1,1-}(s)f(s), & \Phi_{1,2}(\infty) &= 0\end{aligned}$$

- $\Phi_{1,1}(z)\equiv 1$; then the 2-nd equation above reduces to

$$\Phi_{1,2+}(s) = \Phi_{1,2-}(s) + f(s), \qquad \Phi_{1,2}(\infty) = 0,$$

  whose solution is $\Phi_{1,2}(z) = \frac{1}{2\pi i}\int_\Sigma \frac{f(s)}{s-z}ds$.
- For the second row:

$$\begin{aligned}\Phi_{2,1+}(s) &= \Phi_{2,1-}(s), & \Phi_{2,1}(\infty) &= 0\\ \Phi_{2,2+}(s) &= \Phi_{2,2-}(s) + \Phi_{2,1-}(s)f(s), & \Phi_{2,2}(\infty) &= 1\end{aligned}$$

  Solutions: first, $\Phi_{2,1}(z)\equiv 0$; then, $\Phi_{2,2}(z)\equiv 1$.
- Finally,

$$\Phi(z) = \begin{pmatrix}1 & \frac{1}{2\pi i}\int_\Sigma \frac{f(s)}{s-z}ds\\ 0 & 1\end{pmatrix}$$