Riemann-Hilbert problems and integrable nonlinear partial differential equations, II

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II. Riemann-Hilbert problems as a tool

Given $V(s) : \Sigma \mapsto \mathbb{C}^{n\times n}$, find $\Phi(z) : \mathbb{C} \setminus \Sigma \mapsto \mathbb{C}^{n\times n}$ s.t.

$$\Phi_+(s) = \Phi_-(s)V(s),\ s \in \Sigma; \qquad \Phi(\infty) = I,$$

- In applications, the jump matrix V depends also on certain parameters; then the solution also depends on parameters; e.g., $V(s; x, t) \mapsto \Phi(z; x, t)$.
- Main Idea:
  1. map the data for your problem (e.g., initial or boundary data for a PDE) to a jump matrix for a RHP;
  2. solve the obtained RHP;
  3. "extract" the solution of your problem from solution of RHP
- Main expected benefit:
  - study of properties of the solution of your problem reduces to study of properties of the solution of the associated RHP;
  - having a method for analyzing the solutions of RHPs, one would have a universal tool for analyzing solutions of problems of quite different nature (as far as they are reduced to a RHP!)

Riemann-Hilbert problems as a tool

- For example, in applications to evolution partial differential equations (PDE) in dimension $1+1$:

$$q_t = F(q, q_x, q_{xx}, ...)$$

  with x being the space variable and t the time variable;
- Particular interest: analyze the behavior of the solution of a problem for PDE (e.g., initial or boundary or initial-boundary value problem) as t becomes large.

Example: orthogonal polynomials, I

Let weight $w(s) \geq 0$, $s \in (-\infty, \infty)$ be given. Orthogonal (orthonormal) polynomials $p_n(s)$: $\int_{\mathbb{R}} p_n(s)p_m(s)w(s)ds = \delta_{n,m}$.
Let $\pi_n(s)$ be the associated monic polinomials: $\pi_n(s) = s^n + \ldots$,

$$p_n(s) = a_{n,n}\pi_n(s) = a_{n,n}s^n + a_{n,n-1}s^{n-1} + \ldots$$

Question: $a_{n,n}$, $a_{n,n-1}$ as $n \to \infty$?

RHP for orthogonal polynomials

Consider the RHP for $Y(z,n)$: given $w(s)$, $s \in \mathbb{R}$, find $Y(z,n)$ analytic in $\mathbb{C} \setminus \mathbb{R}$ s.t.

- $Y_+(s,n) = Y_-(s,n) \begin{pmatrix} 1 & w(s) \\ 0 & 1 \end{pmatrix}$, $s \in \mathbb{R}$
- $Y(z,n) = (I + O(z^{-1})) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$, $z \to \infty$.

Example: orthogonal polynomials, II

Solution of the RHP

$$Y(z,n) = \begin{pmatrix} \pi_n(z) & \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\pi_n(s)w(s)}{s-z}ds \\ \gamma_{n-1}\pi_{n-1}(z) & \frac{\gamma_{n-1}}{2\pi i}\int_{\mathbb{R}} \frac{\pi_{n-1}(s)w(s)}{s-z}ds \end{pmatrix}$$

where $\gamma_n = 2\pi i \frac{D_{n-1}}{D_n}$, $D_n = \begin{vmatrix} \int_{\mathbb{R}} s^0 w(s)ds & \dots & \int_{\mathbb{R}} s^n w(s)ds \\ \dots & \dots & \dots \\ \int_{\mathbb{R}} s^n w(s)ds & \dots & \int_{\mathbb{R}} s^{2n} w(s)ds \end{vmatrix}$

- Variation: $w = w(s,n) = e^{-nV(s)}$. Case $V(s) = s^2$: Hermite polynomials.
- Relevant to Random Matrix Theory.

Riemann-Hilbert problem for "linearized nonlinear Schrödinger equation" (LNLS)

Cauchy problem for a linear PDE

Given $q_0(x)$, $-\infty < x < \infty$ s.t. $q_0(x) \to 0$ as $x \to \infty$, find $q(x,t)$:

- $iq_t(x,t) + q_{xx}(x,t) = 0, \quad -\infty < x < \infty,\ t > 0;$
- $q(x,0) = q_0(x), \quad -\infty < x < \infty$ (initial conditions);
- $q(x,t) \to 0$ as $x \to \infty$ for all $t > 0$.

- Problem can be solved using Fourier transform (direct and inverse). But:
- We solve it in another way (having in mind the perspective to develop a similar approach to (some) nonlinear PDE):
  1. Using the fact that PDE can be represented as compatibility condition of 2 (linear) ODE depending on auxiliary (spectral) parameter:
  2. solving Cauchy problem reduces to solving an associated RHP.

Linear PDE and Lax pair

- LNLS is the compatibility condition for Lax pair equations w.r.t $\mu(x,t,k)$:

$$\mu_x(x,t,k) = -ik\mu(x,t,k) + q(x,t)$$
$$\mu_t(x,t,k) = -ik^2\mu(x,t,k) + kq(x,t) + iq_x(x,t)$$

  Indeed, $\mu_{xt} = \mu_{tx}$ reduces to LNLS.
- The Lax pair for a general linear PDE with constant coefficients $iq_t - \omega(i\frac{\partial}{\partial x})q = 0$, where $\omega(k)$ is a polynomial of order n:

$$\mu_x(x,t,k) = -ik\mu(x,t,k) + q(x,t)$$
$$\mu_t(x,t,k) = -i\omega(k)\mu(x,t,k) + \sum_{j=0}^{n-1} c_j(k)\frac{\partial^j q}{\partial x^j}$$

  where $\left(\frac{\omega(k)-\omega(l)}{k-l}\right)\Big|_{l=i\partial_x} = c_j(k)\partial_x^j$.
- Assume that $q(x,t)$ is given. Goal: find representation for $q(x,t)$ in terms of the solution of a RH problem, whose data are uniquely determined by $q_0(x)$.

RHP for LNLS, I

- Introduce $\mu_\pm(x,t,k)$ as solutions of "x-equation" of the Lax pair fixed by conditions at $x=\pm\infty$ ("Jost solutions"):

$$\mu_\pm(x,t,k) = o(1), \qquad x \to \pm\infty.$$

  $\mu_\pm$ are unique and given explicitly:
  $\mu_\pm(x,t,k) = \int_{\pm\infty}^{x} e^{-ik(x-y)} q(y,t)dy$.
  Notice that $\mu_\pm$ needn't solve "t-equation"!

- Analytic properties of $\mu_\pm$ as functions of k:
  1. $\mu_\pm$ is analytic for $k \in \mathbb{C}_\pm = \{k : \pm \operatorname{Im} k > 0\}$;
  2. $\mu_\pm \to 0$ as $k \to \infty$; moreover, $\mu_\pm(x,t,k) = \frac{q(x,t)}{ik} + O(k^{-2})$.

- Define μ by: $\mu = \mu_\pm$ for $k \in \mathbb{C}_\pm$ and calculate the jump of μ across $\mathbb{R}$:

$$\mu_+(x,t,k) - \mu_-(x,t,k) = -e^{-ikx} Q(k,t), \qquad k \in \mathbb{R}$$

  where $Q(k,t) := \int_{-\infty}^{\infty} e^{iky} q(y,t)dy$.

- Let's analyze the t-dependence of $Q(k,t)$:

$$\frac{dQ}{dt} = \int_{-\infty}^{\infty} e^{iky} q_t(y,t)dy = \text{[by PDE]} = i\int_{-\infty}^{\infty} e^{iky} q_{yy}(y,t)dy$$
$$= \text{[int. by parts]} = -ik^2 \int_{-\infty}^{\infty} e^{iky} q(y,t)dy = -ik^2 Q(k,t).$$

Thus $Q(k,t) = e^{-ik^2 t} Q(k,0)$, where
$Q(k,0) = \int_{-\infty}^{\infty} e^{iky} q_0(y)dy$ is determined by initial data.

RHP for LNLS, III

- Thus jump cond. for μ is determined by $q_0(x)$ (via $Q(k,0)$):

$$\mu_+(x,t,k) - \mu_-(x,t,k) = -e^{-ikx-ik^2t}Q(k,0), \qquad k \in \mathbb{R} \tag{1}$$

Complementing the jump condition (1) by the normalization cond.

$$\mu(x,t,k) \to 0, \qquad k \to \infty, \tag{2}$$

we arrive at the (additive) RHP for the Cauchy problem for LNLS equation:

> Given $q_0(x)$, find $\mu(x,t,k)$ satisfying (1)+(2), where $Q(k,0) = \int_{-\infty}^{\infty} e^{iky} q_0(y)dy$.

- Solution of RHP: $\mu(x,t,k) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{-ilx-il^2t}Q(l,0)}{l-k}dl$
- Solution of Cauchy problem for LNLS:
  $q(x,t) = i\lim_{k\to\infty}(k\mu(x,t,k)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ilx-il^2t}Q(l,0)dl$.
- Remark: we have retrieved the solution by Fourier transform!

RHP for LNLS, IV

Now we present an alternative way of derivation of the RH problem, based on **simultaneous solutions of Lax pair equations**.

- Introduce the solution of "background" Lax pair (with $q \equiv 0$): $\mu_0(x,t,k) = e^{-ikx-ik^2t}$.
- Let $\tilde{\mu} := \mu\mu_0^{-1}$, where μ solves the Lax pair equations; then $\tilde{\mu}$ is solution of the following equations (modified Lax pair):

$$\tilde{\mu}_x = qe^{ikx+ik^2t}, \qquad \tilde{\mu}_t = (kq + iq_x)e^{ikx+ik^2t}$$

- Introduce $\tilde{\mu}_\pm(x,t,k) := \int_{\pm\infty}^{x} q(y,t)e^{iky+ik^2t}dy$.
  1. $\tilde{\mu}_\pm$ obviously solve x-equation;
  2. Let's show that $\tilde{\mu}_\pm$ solve t-equation as well:

$$\tilde{\mu}_{\pm,t} = \int_{\pm\infty}^{x} (q_t + ik^2q)e^{iky+ik^2t}dy = [\text{by PDE}] = \int_{\pm\infty}^{x} (iq_{yy} + ik^2q)e^{iky+ik^2t}dy$$

$$= [\text{int. by parts}] = (iq_x + kq)e^{ikx+ik^2t} + \int_{\pm\infty}^{x} (-ik^2 + ik^2)qe^{iky+ik^2t}dy$$

RHP for LNLS, V

- As in the previous approach, $\tilde{\mu}_\pm$ are analytic in $\mathbb{C}_\pm$.
- Now consider the jump on $\mathbb{R}$: since both $\tilde{\mu}_+$ and $\tilde{\mu}_-$ solve x-equation and t-equation, it follows that the difference $\tilde{\mu}_+ - \tilde{\mu}_-$ is independent of x and t!

$$\tilde{\mu}_+(x,t,k) - \tilde{\mu}_-(x,t,k) = C(k)$$

  with some $C(k)$.
- In order to calculate $C(k)$, set $x = t = 0$; then $C(k) = -\int_{-\infty}^{\infty} q(y,0)e^{iky}dy = -Q(k,0)$.
- Finally, in order to have standard normalization ($= 0$ at $k = \infty$, for all x and t), introduce $\mu := \tilde{\mu}e^{-ikx-ik^2t}$; in this way we **we retrieve the formulation of the RHP obtained in the first approach!**

The integral representation for $q(x,t)$ allows studying its large time behavior (via stationary phase/steepest descent method for oscillatory integrals). Let $\xi := \frac{x}{t}$. Then

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Q(l,0)\mathrm{e}^{\mathrm{i}t\Phi(\xi;l)}dl$$

with $\Phi(\xi;l) = -\xi l - l^2$.

- stationary point: $\Phi'(\xi;l_0) = 0 \Longrightarrow l_0 = -\frac{\xi}{2}$
- $\Phi(\xi;l_0) = \frac{\xi^2}{4}$
- main term (as $t \to \infty$) comes from the (vicinity of) stationary point:

$$q(x,t) = \frac{1}{\sqrt{2t}}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}} Q\left(-\frac{\xi}{2},0\right)\mathrm{e}^{\frac{\mathrm{i}t\xi^2}{4}} + O(t^{-1})$$

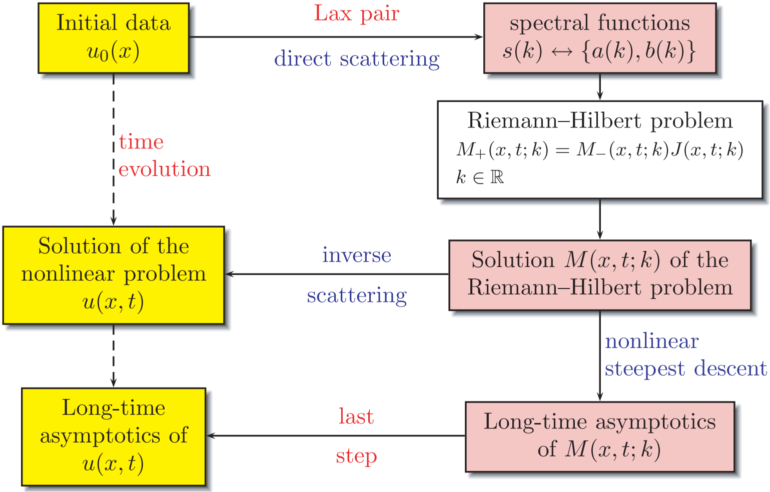
RHP and nonlinear PDE: general remarks, I

- A key point in the scheme for solution Cauchy problem for linear PDE was the Lax pair representation
- Some of nonlinear PDE $q_t = F(q, q_x, q_{xx}, ...)$ also have the Lax pair representation: they are compatibility conditions of pairs of linear equations (depending on a (spectral) parameter).
- Such nonlinear PDE are called integrable: the solution of a nonlinear problem (Cauchy problem for a nonlinear PDE) reduces to certain amount of linear steps (each step is about solving a linear problem).

RHP and nonlinear PDE: general remarks, II

- Linear steps in solving problems for nonlinear PDEs are "similar" to solving problems for linear PDEs by the Fourier transforms (direct and inverse). Because of this, the method of solving the Cauchy problem for an integrable nonlinear PDE based on using the Lax pair representation is sometimes called "Nonlinear Fourier Transform" method (NFT): x-equation in the Lax pair is used to construct the "direct transform" (analogue of direct Fourier transform) whereas the solution of the associated RHP corresponds to the "inverse transform" (analogue of inverse Fourier transform).

RHP and nonlinear PDE: solution scheme



RHP and nonlinear PDE: general remarks, III

- Another name for the "linearization" of the solution of problems for nonlinear PDE is "Inverse Scattering Transform" method (IST):
  - the solutions of x-equation in the Lax pair involved in the construction are fixed by their behavior at $x = \pm\infty$ and thus the relation amongst them can be interpreted as the "scattering" of waves (q being considered as a "scatterer", and the waves are normalized "far away" of the scatterer);
  - t-equation governs a linear evolution of "scattering data";
  - the most involved step is to go back, from scattering data to functions in the physical space. Historically, this step was performed by using Marchenko-Gelfand-Levitan (linear) integral equations of the inverse problem(for the x-equation considered as a spectral (scattering) problem); latter on, the Riemann-Hilbert variant of this step was invented.
- **An advantage of the RHP approach**: efficiency for large time analysis (for other asymptotic regimes: e.g., small dispersion limit).