Syllabus

I. **Course Name**: Sobolev Spaces and Theory of Unbounded Operators

II. **Course description and objective**

The course includes an introduction to the distribution theory and theory of unbounded operators. Then we consider Sobolev spaces in the whole space, half-spaces and bounded domains, as well as Sobolev spaces on the boundary. Finally, we study the elements of interpolation theory for Sobolev spaces.

III. **Elective**

IV. **Bachelor Program, 8th Term, 64 Hours, 4 Credits**

V. **Course content**

<u>Chapter 1. Distribution theory.</u>

Section 1. Space of test functions $\mathfrak{D}$.

Construction lemma. Regularization. Density of $\mathfrak{D}$ *in* L^2.

Section 2. Space of distributions $\mathfrak{D}'$.

Completeness property. Support of a distribution. Regular and singular distributions.
Section 3. Basic operations with distributions.
Linear change of variables, product with a multiplier, derivatives.
Section 4. Direct product and convolution of distributions.
Properties. Existence of a convolution.
Section 5. Tempered distributions.
Space of test functions S. Tempered distributions S'. Fourier transform of tempered distributions.

<u>Chapter 2. Unbounded operators</u>

Section 1. Basic concepts.
Bounded and unbounded operators. Extension of an operator. Inverse operator. Closed operator. Closed extensions.
Section 2. Symmetric and self-adjoint operators
Adjoint operator and its properties. Symmetric operator. Criterion of boundedness. Symmetric extensions. Self-adjoint operators and their properties. Multiplication operator by the independent variable. Differential operator on an interval and its symmetric extensions. Differential operator on a semi-axis.
Section 3. Friedrichs extension.

Section 4. Spectrum of an operator.

Resolvent operator and spectrum. Classification of spectrum. Properties of eigenvalues and classification of the spectrum of self-adjoint operators. Spectrum of multiplication operator by the independent variable and differential operator.

Section 5. Graph of an operator.

Section 6. Compact operators.

Spectrum of compact operators. Spectrum of compact self-adjoint operators.

Chapter 3. Sobolev spaces

Section 1. Sobolev spaces $H^k(\Omega)$.

Sobolev spaces $H^k(\Omega)$ *and their properties. Traces of functions from* $H^k(\Omega)$. *Trace theorem. Sobolev spaces* $H_0^k(\Omega)$ *and their properties. Rellich theorem. Compactness theorem for the set of the traces of functions from* $H^1(\Omega)$. *Discrete spectrum of the Laplace operator. Equivalent norms in* $H^k(\Omega)$, $H_0^k(\Omega)$. *Friedrichs and Poincaré's inequalities.*

Section 2. Sobolev spaces on $\mathbb{R}^m$.

Sobolev spaces $H^s(\mathbb{R}^m)$. *Density of the space* $C_0^\infty(\mathbb{R}^m)$ *in* $H^s(\mathbb{R}^m)$. *Equivalent norms in* $H^s(\mathbb{R}^m)$. *The embedding theorem for* $H^s(\mathbb{R}^m)$. *Trace theorem for functions from* $H^s(\mathbb{R}^m)$ *.Lifting operators from hyperplane to* $\mathbb{R}^m$.

Section 3. Sobolev spaces in half-space

Sobolev spaces $H_0^s(\mathbb{R}_\pm^m)$. *Density of the space* $C_0^\infty(\mathbb{R}_\pm^m)$ *in* $H_0^s(\mathbb{R}_\pm^m)$ *Equivalent norms in* $H_0^s(\mathbb{R}_\pm^m)$. *Dual spaces to* $H^s(\mathbb{R}^m)$.

Section 4. Sobolev spaces on bounded domains and on boundaries.

Spaces $H^s(\Omega)$ *and* $H_0^s(\Omega)$. *Equivalent norms in* $H^s(\Omega)$. *Spaces* $H^s(\Gamma)$. *Trace theorem for functions from* $H^s(\Omega)$. *Lifting theorem about the* extension *of functions from the boundary to the* entire domain. *Theorem on extension from a domain to the whole space* $\mathbb{R}^m$.

Chapter 4. Interpolation spaces

Section 1. Introduction to interpolation theory

Basic notions. Properties of interpolation spaces.

Section 2. Interpolation of Sobolev spaces

Interpolation of spaces $H^S(\Omega)$. *Interpolation of spaces* $H_0^S(\Omega)$ *for non-half-integer indices. Interpolation of spaces* $H_0^S(\Omega)$ *for half-integer indices. Interpolation between* $H_0^{S_1}(\Omega)$ *and* $H^{-S_2}(\Omega)$. *Interpolation between* $H^{S_1}(\Omega)$ *and* $H^{-S_2}(\Omega)$. *Interpolation between* $H^{S_1}(\Omega)$ *and* $(H^{S_2}(\Omega))^*$. *Interpolation between* $H_0^{S_1}(\Omega)$ *and* $(H^{S_2}(\Omega))^*$.

VI. Pre-taken courses

Mathematical Analysis, Measure Theory and Integration, Functional Analysis

VII. Form of the final test: examination (four-level evaluation scale)/test (two-level evaluation scale)

VIII. Teaching materials and reference books

1. Akhiezer N. I., Glazman I. M., Theory of linear operators in Hilbert space, Mineola, NY: Dover, 1993.
2. Lions J. L., Magenes E., Non-Homogeneous Boundary Value Problems and Applications:Vol.1, Berlin: Springer, 1972.
3. Vladimirov V.S., Equations of Mathematical Physics, Marcel Dekker INC., 1971.

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