# Riemann-Hilbert problems and integrable nonlinear partial differential equations, III

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Classical RHP: smooth contour $\Sigma$; smooth jump matrix.
Problem: generalize the RHP formalism to jumps from $L^p(\Sigma)$ $(1 < p < \infty)$; more general contours.
Key tool: Cauchy operators. Principal value integrals

$$S_\Sigma(s^*) = \frac{1}{\pi i} \fint_\Sigma \frac{f(s)}{s - s^*} ds := \frac{1}{\pi i} \lim_{\delta \downarrow 0} \int_{\Sigma \setminus B(s^*, \delta)} \frac{f(s)}{s - s^*} ds,$$

where $B(s^*, \delta) = \{z : |z - s^*| \le \delta\}$.
Singular Cauchy operators

$$\mathcal{C}_\Sigma^\pm f(s^*) := \lim_{\substack{z \to s^* \\ z \in \Omega_\pm}} \mathcal{C}_\Sigma f(z),$$

where the Cauchy integral is defined by

$$\mathcal{C}_\Sigma f(z) := \frac{1}{2\pi i} \int_\Sigma \frac{f(s)}{s - z} ds, \quad z \in \mathbb{C} \setminus \Sigma.$$

Key property to hold: boundedness of $S_\Sigma$ and $\mathcal{C}_\Sigma^\pm$ as operators: $L^p(\Sigma) \mapsto L^p(\Sigma)$.

# Contours for RHP, I

## Definition: Carleson curve

Union of finite number of arcs (homeomorphic to intervals of $\mathbb{R}$; rectifiable); each arc $\Sigma$ satisfies

$$\sup_{z\in\Sigma}\sup_{r>0}\frac{|\Sigma\cap B(z,r))|}{r}<\infty.$$

## Proposition

$S_\Sigma : L^p(\Sigma)\mapsto L^p(\Sigma)$ is a bounded operator $\Leftrightarrow$ $\Sigma$ is a Carleson curve.

# Contours for RHP, II

**Definition: Carleson jump contour $\Sigma$**

Union of finite number of Carleson arcs (with only common points,if intersect, being the end points) s.t.

- $\mathbb{C} \setminus \Sigma = D_+ \cup D_-$; $D_\pm = \cup_j D_{\pm,j}$ finite number of connected components;
- Orientation: $\Sigma = \partial D_+ = -\partial D_-$;
- $\partial D_{\pm,j}$ is a Carleson curve for all $j$.

**Question**

Assume that $v \in L^p(\Sigma)$ is given. How to understand the jump condition $m_+ = m_- v$? How to understand $m_\pm$?

# Classes for solution of RHP

## Definition: Smirnoff classes of analytic functions

(i) Let $D$ be a bounded component of $\mathbb{C} \setminus \Sigma$. We say that $f(z) \in E^p(D)$, if

- $f(z)$ is analytic in $D$;
- there exist $\{C_n\}_1^\infty \subset D$ rectifiable Jordan curves s.t. for any compact $D_c \subset D$ there exists $N$: (a) for all $n > N$, $C_n$ surrounds $D_c$; (b) $\sup_{n \geq 1} \int_{C_n} |f(z)|^p |dz| < \infty$.

(ii) If $D$ is unbounded, then $f \in E^p(D)$, if $f \circ \varphi^{-1} \in E^p(\varphi(D))$, where $\varphi(z) := \frac{1}{z - z_0}$ with some $z_0 \in \mathbb{C} \setminus \bar{D}$.
(iii) $\dot{E}^p(D) := \{f \in E^p(D) : zf(z) \in E^p(D)\}$
(i.e., $f(z) \to 0$ as $z \to \infty$).
(iv) If $D = D_1 \cup \cdots \cup D_n$, then $f \in E^p(D)$ if $f|_{D_j} \in E^p(D_j)$ for all $j$.

**Proposition**

- If $f \in \dot{E}^2(D_\pm)$, then there exist (almost everywhere) non-tangential limits $f_\pm(s) = \lim\limits_{\substack{z \to s \\ z \in D_\pm}} f(z)$, $s \in \Sigma$; moreover, $f_\pm - I \in L^2(\Sigma)$.
- if $h \in L^2(\Sigma)$, then $\mathcal{C}h \in \dot{E}^2(D_+ \cup D_-)$, where $\mathcal{C}h(z) := \frac{1}{2\pi i} \int_\Sigma \frac{h(s)}{s-z} ds$, $z \in \mathbb{C} \setminus \Sigma$ (Cauchy integral).
- $\mathcal{C}_+$ and $\mathcal{C}_-$ are bounded operators $L^2(\Sigma) \mapsto L^2(\Sigma)$; moreover, $\mathcal{C}_+ - \mathcal{C}_- = I$, and Sokhotski-Plemelj formulas hold:

$$\mathcal{C}_+ = \frac{1}{2}(I + S_\Sigma), \qquad \mathcal{C}_- = \frac{1}{2}(-I + S_\Sigma),$$

where $S_\Sigma$ is the principal value Cauchy integral operator.

# Riemann-Hilbert problem in $L^2$ sense

The properties of $E^2$ and $L^2$ justify the correctness of the formulation of the RHP:

**$L^2$ – RHP**

Let $\Sigma$ be a Carleson jump contour, and let a matrix-valued function $v : \Sigma \mapsto GL(n, \mathbb{C})$ is given on $\Sigma$. Then $m(z)$, $z \in \mathbb{C} \setminus \Sigma$ is called the solution of the RHP $\{\Sigma, v\}$, if

- $m \in I + \dot{E}^2(\mathbb{C} \setminus \Sigma)$;
- $m_+(s) = m_-(s)v(s)$ almost everywhere on $\Sigma$, where $m_\pm$ are non-tangential limits of $m$.

**Questions**

What conditions should be imposed on $v$ in order to have solvability/uniqueness of the RHP? How to solve the RHP?

# Reducing RHP to integral equation, I

## Assumption on $v$

We assume that $v(s)$ can be (algebraically!) factorized as $v(s) = (v_-(s))^{-1} v_+(s)$ s.t.

$$v_\pm, (v_\pm)^{-1} \in I + L^2(\Sigma) \cap L^\infty(\Sigma)$$

(it can be many factorizations like this!).
Introduce $w_+ = v_+ - I$, $w_- = I - v_-$ (so that $v_\pm = I \pm w_\pm$ with $w_\pm \in L^2(\Sigma) \cap L^\infty(\Sigma)$).
Introduce the integral operator $\mathcal{C}_w$ by

$$\mathcal{C}_w h(z) = \mathcal{C}_+(h w_-)(z) + \mathcal{C}_-(h w_+)(z), \quad z \in \Sigma.$$

# Reducing RHP to integral equation, II

**Proposition**

$\mathcal{C}_w$ is bounded operator $L^2(\Sigma) \mapsto L^2(\Sigma)$.

*Proof*

$$\begin{aligned} \|\mathcal{C}_w h(z)\|_{L^2(\Sigma)} &\le \|\mathcal{C}_+(hw_-)(z)\|_{L^2(\Sigma)} + \|\mathcal{C}_-(hw_+)(z)\|_{L^2(\Sigma)} \\ &\le C\left(\|hw_-\|_{L^2} + \|hw_+\|_{L^2}\right) \\ &\le C\|h\|_{L^2}\left(\|w_-\|_{L^\infty} + \|w_+\|_{L^\infty}\right), \end{aligned}$$

where $C = \max\{\|\mathcal{C}_+\|_{\mathcal{L}(L^2(\Sigma))}, \|\mathcal{C}_-\|_{\mathcal{L}(L^2(\Sigma))}\}$. Therefore,

$$\|\mathcal{C}_w\|_{\mathcal{L}(L^2(\Sigma))} \le 2C \max\{\|w_-\|_{L^\infty}, \|w_+\|_{L^\infty}\}.$$

Moreover, if $h \in L^\infty(\Sigma)$, then
$\|\mathcal{C}_w h(z)\|_{L^2(\Sigma)} \le C\left(\|w_-\|_{L^2} + \|w_+\|_{L^2}\right)$; therefore,

$$\mathcal{C}_w : L^2(\Sigma) + L^\infty(\Sigma) \mapsto L^2(\Sigma).$$

# Reducing RHP to integral equation, III

Consider the singular integral equation

$$(I - \mathcal{C}_w)\mu = I, \qquad \mu \in I + L^2(\Sigma)$$

(or, equivalently,

$$(I - \mathcal{C}_w)(\mu - I) = \mathcal{C}_w I,$$

where the r.h.s. $\mathcal{C}_w I \in L^2(\Sigma)$, and the solution $\mu - I$ is also sought in $L^2(\Sigma)$).

**Theorem**

If there exists $(I - \mathcal{C}_w)^{-1}$ as a bounded operator $L^2(\Sigma) \mapsto L^2(\Sigma)$ and $\mu \in I + L^2(\Sigma)$ is the unique solution of $(I - \mathcal{C}_w)\mu = I$, then $m(z) := I + (\mathcal{C}(\mu(w_+ + w_-)))(z)$ is a solution of the $L^2$–RHP: $m \in I + \dot{E}^2(\mathbb{C} \setminus \Sigma)$ and $m_+ = m_- v$ a.e. on $\Sigma$.

# Reducing RHP to integral equation, IV

*Proof* (i)

$$\begin{aligned} m_+ &= I + \mathcal{C}_+(\mu(w_+ + w_-)) = I + \mathcal{C}_+(\mu w_-) + \mathcal{C}_+(\mu w_+) \\ &= [\mathcal{C}_+ - \mathcal{C}_- = I] = I + \mathcal{C}_+(\mu w_-) + \mathcal{C}_-(\mu w_+) + \mu w_+ \\ &= I + \mathcal{C}_w \mu + \mu w_+ = [I + \mathcal{C}_w \mu = \mu] = \mu + \mu w_+ \\ &= \mu(I + w_+) = \mu v_+. \end{aligned}$$

Similarly, $m_- = \mu v_- \Rightarrow \mu = m_- v_-^{-1}$ and thus

$$m_+ = \mu v_+ = m_- v_-^{-1} v_+ = m_- v.$$

(ii) Since $\|w_\pm\|_{L^2} < \infty$, we have $\mu(w_+ + w_-) \in L^2(\Sigma)$. Then, by Proposition, $m \in I + \dot{E}^2(\mathbb{C} \setminus \Sigma)$.

Having $\mu(x, t, s)$, $s \in \Sigma$, solution of the NLS equation is given by

$$q(x,t) = 2i \lim_{z \to \infty} z m_{12}(z) = -\frac{1}{\pi} \int_\Sigma \left(\mu(s)(w_+(s) + w_-(s))\right)_{12} ds.$$

# Reducing RHP to integral equation, V

## Theorem

If $I - \mathcal{C}_w$ is a Fredholm operator, then the existence of $(I - \mathcal{C}_w)^{-1}$ is equivalent to the (unique) solvability of the RHP.

Fredholm property: $\exists (I - \mathcal{C}_w)^{-1} \Leftrightarrow \{(I - \mathcal{C}_w)\mu = 0$ has only trivial solution $\mu \equiv 0\}$.

*Proof* Let RHP $\{\Sigma, v\}$ has a unique solution. Let $(I - \mathcal{C}_w)\mu = 0$. Determine $\hat{m} := (\mathcal{C}(\mu w))(z)$ with $w = w_- + w_+$. Then

$$\begin{aligned}
\hat{m}_+ &= \mathcal{C}_+(\mu(w_+ + w_-)) = \mathcal{C}_+(\mu w_-) + \mathcal{C}_+(\mu w_+) \\
&= [\mathcal{C}_+ - \mathcal{C}_- = I] = \mathcal{C}_+(\mu w_-) + \mathcal{C}_-(\mu w_+) + \mu w_+ \\
&= \mathcal{C}_w \mu + \mu w_+ = [\mathcal{C}_w \mu = \mu] = \mu + \mu w_+ = \mu(I + w_+) = \mu v_+.
\end{aligned}$$

Similarly, $\hat{m}_- = \mu v_-$ and thus $\hat{m}_+ = \hat{m}_- v$. Also $\hat{m} \to 0$ as $z \to \infty$. Therefore, $m_\gamma := m + \gamma \hat{m}$ is a solution of the RHP for any $\gamma$. But by assumption, $\exists!$ solution of RHP; hence, $\hat{m} = 0 \Rightarrow \mathcal{C}_\pm(\mu w) = 0$. Then

$$\begin{aligned}
0 &= \mathcal{C}_+(\mu w) = \mathcal{C}_+(\mu w_-) + \mathcal{C}_+(\mu w_+) = \mathcal{C}_+(\mu w_-) + \mathcal{C}_-(\mu w_+) + \mu w_+ \\
&= \mathcal{C}_w \mu + \mu w_+ = \mu + \mu w_+ = \mu(I + w_+) = \mu v_+.
\end{aligned}$$

Since $\exists v_+^{-1}$, it follows that $\mu \equiv 0$.

# Approximations for RHP, I

**Theorem**

Let $v^{(n)} = (v_-^{(n)})^{-1} v_+^{(n)} = (I - w_-^{(n)})^{-1}(I + w_+^{(n)})$, $n = 1, 2, \ldots$
and $v^\infty = (v_-^\infty)^{-1} v_+^\infty = (I - w_-^\infty)^{-1}(I + w_+^\infty)$.
Let $\exists (I - \mathcal{C}_{w^\infty})^{-1}$. Assume that

$$\|w_\pm^{(n)} - w_\pm^\infty\|_{L^\infty(\Sigma)\cap L^2(\Sigma)} \to 0, \qquad n \to \infty.$$

Then:

- $\exists$ solutions of RHP $m^\infty$ and $m^{(n)}$ for all $n > N$.
- $\|m_\pm^{(n)} - m_\pm^\infty\|_{L^2(\Sigma)} \to 0$ as $n \to \infty$.

*Proof* (i) First, discuss (and estimate) $(I - \mathcal{C}_{w^{(n)}})^{-1}$. As $n \to \infty$,

$$\|\mathcal{C}_{w^\infty} f - \mathcal{C}_{w^{(n)}} f\|_{L^2(\Sigma)} = \|\mathcal{C}_+(f(w_-^{(n)} - w_-^\infty)) + \mathcal{C}_-(f(w_-^{(n)} - w_-^\infty))\|$$
$$\leq C\|f\|_{L^2(\Sigma)} \left(\|w_-^{(n)} - w_-^\infty\|_{L^\infty(\Sigma)} + \|w_+^{(n)} - w_+^\infty\|_{L^\infty(\Sigma)}\right) \to 0.$$

Thus $\|\mathcal{C}_{w^\infty} - \mathcal{C}_{w^{(n)}}\|_{L^2(\Sigma) \mapsto L^2(\Sigma)} \to 0$. Now apply the Second Resolvent Identity: if $R(z, A) := (zI - A)^{-1}$, then

$$R(z, A) - R(z, B) = R(z, A)(B - A)R(z, B)$$

and thus $R(z, A) = (I - (A - B)R(z, B))^{-1} R(z, B)$. Hence

1. $\exists (I - \mathcal{C}_{w^{(n)}})^{-1}$, $n > N$;
2. $\|(I - \mathcal{C}_{w^{(n)}})^{-1} - (I - \mathcal{C}_{w^\infty})^{-1}\|_{L^2(\Sigma) \mapsto L^2(\Sigma)} \to 0$

# Approximations for RHP, III

(ii) Next, estimate $\|\mu^{(n)} - \mu^{\infty}\|$.

$$
\begin{aligned}
&\|\mu^{(n)} - \mu^{\infty}\|_{L^2(\Sigma)} = \|(I - \mathcal{C}_{w^{(n)}})^{-1} I - (I - \mathcal{C}_{w^{\infty}})^{-1} I\|_{L^2} \\
&= \|(I - \mathcal{C}_{w^{(n)}})^{-1} (\mathcal{C}_{w^{(n)}} - \mathcal{C}_{w^{\infty}}) (I - \mathcal{C}_{w^{\infty}})^{-1} I\|_{L^2} \\
&= \|(I - \mathcal{C}_{w^{(n)}})^{-1} (\mathcal{C}_{w^{(n)}} - \mathcal{C}_{w^{\infty}}) (I + (I - \mathcal{C}_{w^{\infty}})^{-1} \mathcal{C}_{w^{\infty}} I\|_{L^2} \\
&\leq \|(I - \mathcal{C}_{w^{(n)}})^{-1}\|_{L^2 \mapsto L^2} \|\mathcal{C}_{+}(w_{-}^{(n)} - w_{-}^{\infty}) + \mathcal{C}_{-}(w_{+}^{(n)} - w_{+}^{\infty})\|_{L^2} \\
&+ \|(I - \mathcal{C}_{w^{(n)}})^{-1}\|_{L^2 \mapsto L^2} \|\mathcal{C}_{w^{(n)}} - \mathcal{C}_{w^{\infty}}\|_{L^2 \mapsto L^2} \|(I - \mathcal{C}_{w^{\infty}})^{-1}\| \|\mathcal{C}_{+} w_{-}^{\infty} + \mathcal{C}_{-} w_{+}^{\infty}\|_{L^2} \\
&\leq C \left( \|w_{-}^{(n)} - w_{-}^{\infty}\|_{L^2 \cap L^{\infty}} + \|w_{+}^{(n)} - w_{+}^{\infty}\|_{L^2 \cap L^{\infty}} \right) \to 0.
\end{aligned}
$$

(iii) Finally,

$$
\begin{aligned}
\|m_{\pm}^{(n)} - m_{\pm}^{\infty}\|_{L^2} &= \|\mu^{(n)} v_{\pm}^{(n)} - \mu^{\infty} v_{\pm}^{\infty}\|_{L^2} \\
&\leq \|\mu^{\infty}\|_{L^2} \|v_{\pm}^{(n)} - v_{\pm}^{\infty}\|_{L^{\infty}} + \|v_{\pm}^{(n)}\|_{L^{\infty}} \|\mu^{(n)} - \mu^{\infty}\|_{L^2} \to 0.
\end{aligned}
$$

# Approximations for RHP, IV

## Conclusion

Assume that we deals with a family of regular RHP (no res. conds.) depending on a parameter, and are interested in the situation when the values of this parameter increases (e.g., $t$ as the parameter for the RHP for the NLS). Assume that the jump matrices of these RHPs tend (in the sense $L^2 \cap L^\infty$) to a (limiting) jump matrix, which correspond to a solvable RHP (particularly, having explicit solution). Then

- If the values of the parameter is big enough, then all the RHPs corresponding to these parameters are solvable;
- The solutions of these RHPs tend to the solution of the limiting RHP.

We will apply this, when will study the large-$t$ asymptotics of solutions of the NLS equation.

## Question

What to do in the case when the RHP is non-regular, i.e. it involves the residue conditions?

# Reducing RHP with res. conds. to regular RHP

We consider 2 ways how to reduce a RHP with res. conds. to a regular RHP:

1. Multiplying (basically) from the left, to “kill” the singularities of the solution of the RHP;
2. Multiplying from the right, to augment the contour of the RHP, replacing the residue conditions by the jump conditions on the added parts of the contour.

# From non-regular to regular RHP: the first way, I

**Lemma**

Let $M$ satisfies the RHP with the jump cond.

$$M_+(k) = M_-(k)J(k), \qquad k \in \Sigma.$$

Let $d_j(k)$, $j = 1, 2$ be functions analytic (meromorphic) and non-zero on $\Sigma$. Let $\tilde{M} := M \begin{pmatrix} d_1(k) & 0 \\ 0 & d_2(k) \end{pmatrix}$.
Then $\tilde{M}$ satisfies the jump cond.

$$\tilde{M}_+(k) = \tilde{M}_-(k)\tilde{J}(k),$$

where the entries of $\tilde{J}$ are:

$$\tilde{J}_{11} = J_{11}, \tilde{J}_{22} = J_{22}, \tilde{J}_{12} = J_{12}\frac{d_2}{d_1}, \tilde{J}_{21} = J_{21}\frac{d_1}{d_2}.$$

# From non-regular to regular RHP: the first way, II

## Theorem

Let $M(x,t,k)$ be the solution of the RHP corresponding to the focusing NLS, i.e. it satisfies

- $M_+(x,t,k) = M_-(x,t,k)J(x,t,k)$ with $J = \begin{pmatrix} 1+|r(k)|^2 & \bar{r}(k)e^{-2ikx-4ik^2t} \\ r(k)e^{2ikx+4ik^2t} & 1 \end{pmatrix}$, $k \in \mathbb{R}$
- $M(x,t,k) \to I$ as $k \to \infty$
- $\operatorname{Res}_{k \to k_j} M^{(1)}(x,t,k) = c_j e^{2ik_jx+4ik_j^2t} M^{(2)}(x,t,k_j), j = \overline{1,N}$
  $\operatorname{Res}_{k \to \bar{k}_j} M^{(2)}(x,t,k) = -\bar{c}_j e^{-2i\bar{k}_jx-4i\bar{k}_j^2t} M^{(1)}(x,t,\bar{k}_j), \quad j = \overline{1,N}$.

Let $\tilde{M}$ be the solution of the regular RHP: (a) without res. conds.; (b) with $r(k)$ replaced by $r(k)\prod_{j=1}^{N}(k-k_j)/(k-\bar{k}_j)$. Then $M$ can be expressed in terms of $\tilde{M}$ as follows:

$$M(x,t,k) = (kI+B_N)(kI+B_{N-1})\dots(kI+B_1)\tilde{M}(x,t,k)\begin{pmatrix} \prod_{j=1}^{N}\frac{1}{k-k_j} & 0 \\ 0 & \prod_{j=1}^{N}\frac{1}{k-\bar{k}_j} \end{pmatrix}$$

# From non-regular to regular RHP: the first way, III

Here the $2 \times 2$ matrices $B_j(x,t)$ are independent of $k$; they can be determined recursively:

$$\tilde{M}_0 \mapsto B_1 \mapsto \tilde{M}_1 \mapsto B_2 \mapsto \tilde{M}_2 \mapsto \cdots \mapsto \tilde{M}_{N-1} \mapsto B_N,$$

where $\tilde{M}_0 \equiv \tilde{M}$, $\tilde{M}_j = (kI + B_j)\tilde{M}_{j-1}$, and $B_j$ is the solution of the linear algebraic equations

$$(k_j I + B_j)\tilde{M}_{j-1}(k_j) \begin{pmatrix} 1 \\ -d_j \end{pmatrix} = 0,$$

$$(\bar{k}_j I + B_j)\tilde{M}_{j-1}(\bar{k}_j) \begin{pmatrix} \bar{d}_j \\ 1 \end{pmatrix} = 0,$$

with $d_j(x,t) = c_j \frac{\prod_{l=1, l\neq j}^{N} (k_j - k_l)}{\prod_{l=1}^{N} (\bar{k}_j - k_l)} e^{2ik_j x + 4ik_j^2 t}$.

# From non-regular to regular RHP: the first way, IV

- If $r(k) \equiv 0$ (pure soliton case), then $\tilde{M} \equiv I$, and we arrive at another way to present a pure $N$-soliton solution (but still the calculations reduce to solving linear algebraic equations).

Let's discuss the details in the case $N = 1$. In this case, $d_1(x,t) = c_1 \frac{1}{k_1 - \bar{k}_1} e^{2ik_1 x + 4ik_1^2 t}$ and

$$M(x,t,k) = (kI + B_1)\tilde{M}(x,t,k) \begin{pmatrix} \frac{1}{k-k_1} & 0 \\ 0 & \frac{1}{k-\bar{k}_1} \end{pmatrix}.$$

Thus $\operatorname{Res}_{k\to k_1} M^{(1)} = (k_1 I + B_1)\tilde{M}^{(1)}(k_1)$ and $M^{(2)}(k_1) = (k_1 I + B_1)\tilde{M}^{(2)}(k_1)\frac{1}{k_1 - \bar{k}_1}$. Let's check the residue condition:

$$\begin{aligned} \operatorname{Res}_{k\to k_1} M^{(1)} &= (k_1 I + B_1)\tilde{M}^{(1)}(k_1) = d_1 (k_1 I + B_1)\tilde{M}^{(2)}(k_1) \\ &= d_1 (k_1 - \bar{k}_1) M^{(2)}(k_1) = c_1 e^{2ik_1 x + 4ik_1^2 t} M^{(2)}(k_1). \end{aligned}$$

The equations for determining $B_1$ reduce to $B_1 W = -W \begin{pmatrix} k_1 & 0 \\ 0 & \bar{k}_1 \end{pmatrix}$, where $W$ is the $2\times 2$ matrix: $W = \left( \tilde{M}(k_1)\begin{pmatrix} 1 \\ -d_1 \end{pmatrix}, \tilde{M}(\bar{k}_1)\begin{pmatrix} \bar{d}_1 \\ 1 \end{pmatrix} \right)$. Due to the symmetry, $\det W > 0$ and thus $B_1$ is uniquely determined.

# From non-regular to regular RHP: the second way, I

The second way of reducing non-regular RHP to a regular one is based on adding (small) circles surrounding $k_j$ and $\bar{k}_j$ to the jump contour and respective re-definition of the solution of the RHP in the disks $D_j = \{k : |k - k_j| < \varepsilon\}$ and their complex conjugates $\bar{D}_j$

At the **first step**, define

$$\hat{M} := M \begin{cases} \begin{pmatrix} 1 & 0 \\ -\frac{c_j e^{2ik_j x + 4ik_j^2 t}}{k - k_j} & 1 \end{pmatrix}, & k \in D_j \\ \begin{pmatrix} 1 & \frac{\bar{c}_j e^{-2i\bar{k}_j x - 4i\bar{k}_j^2 t}}{k - \bar{k}_j} \\ 0 & 1 \end{pmatrix}, & k \in \bar{D}_j \\ I, & \text{otherwise} \end{cases}$$

Let's check that $\hat{M}$ is non-singular at $k = k_1$. Indeed, for $k$ near $k_1$,

$$\hat{M}^{(1)}(k) = M^{(1)}(k) - \frac{c_j e^{2ik_j x + 4ik_j^2 t}}{k - k_j} M^{(2)}(k)$$

$$= \frac{c_j e^{2ik_j x + 4ik_j^2 t}}{k - k_j} M^{(2)}(k_1) + O(1) - \frac{c_j e^{2ik_j x + 4ik_j^2 t}}{k - k_j} M^{(2)}(k_1) + O(1) = O(1).$$

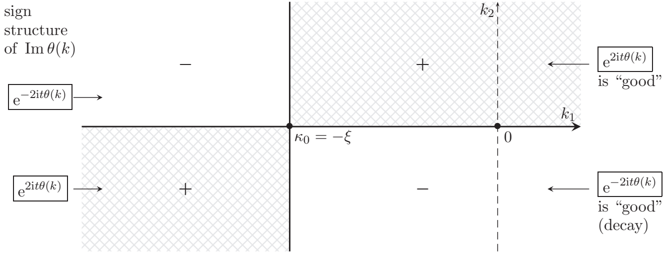
Then $\hat{M}_+ = \hat{M}_- \hat{J}$, $k \in \Sigma = \mathbb{R} \cup_{j=1}^{N} \left(\partial D_j \cap \partial \bar{D}_j\right)$, with $\hat{J} = J$ on $\mathbb{R}$ and the triangular jumps across $\partial D_j$ and $\partial \bar{D}_j$ as above.

# From non-regular to regular RHP: the second way, II

In principal, $\hat{M}$ satisfies a regular RHP, whose solution gives the solution of the NLS. But this RHP is not appropriate for studying properties of this solution, particularly, large-$t$ asymptotics, because the triangular jumps may exponentially grow as $t \to \infty$! Indeed, let's rewrite the exponentials above as

$$e^{2ikx+4ik^2t} = e^{2it\theta(\xi,k)}, \quad \xi := \frac{x}{4t}, \quad \theta(\xi,k) := 2k^2 + 4\xi.$$

Consider the Signature Table, i.e., the distribution of signs of $\operatorname{Im}\theta(\xi,k))$ in the $k$ plane, depending on the value of $\xi$:



Now notice that if, for a given $\xi$, some $k_j$ is located in the upper-right quarter of the signature table, then the corresponding exponentials in the jump matrix on the circle surrounding $k_j$ decays to $0$ as $t \to \infty$ (keeping $\xi$ fixed!); thus one may expect that this part of the jump conditions give exponentially small contribution to $M(x,t,k)$ and thus to $q(x,t)$.

On the other hand, for those $k_j$ located in the upper-left quarter, the exponential will grow. In order to cope with this growth, we need the **second step** in the transformation of the original RHP, based on the matrix identity

$$\begin{pmatrix} 1 & 0 \\ A & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{A} \\ -A & 0 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{A} \\ 0 & 1 \end{pmatrix}.$$

The **second step**: define $\tilde{M} :=$

$$\hat{M}\begin{cases}\begin{pmatrix}1 & -\frac{k-k_j}{c_j}e^{-2it\theta(k_j)}\\ \frac{c_j e^{2it\theta(k_j)}}{k-k_j} & 0\end{pmatrix}\begin{pmatrix}\prod\limits_{j\in K(\xi)}\frac{k-k_j}{k-\bar{k}_j} & 0\\ 0 & \prod\limits_{j\in K(\xi)}\frac{k-\bar{k}_j}{k-k_j}\end{pmatrix}, & k\in D_j\\ \begin{pmatrix}0 & -\frac{\bar{c}_j e^{-2it\theta(\bar{k}_j)}}{k-\bar{k}_j}\\ \frac{k-\bar{k}_j}{\bar{c}_j}e^{2it\theta(\bar{k}_j)} & 1\end{pmatrix}\begin{pmatrix}\prod\limits_{j\in K(\xi)}\frac{k-k_j}{k-\bar{k}_j} & 0\\ 0 & \prod\limits_{j\in K(\xi)}\frac{k-\bar{k}_j}{k-k_j}\end{pmatrix}, & k\in \bar{D}_j\\ \begin{pmatrix}\prod\limits_{j\in K(\xi)}\frac{k-k_j}{k-\bar{k}_j} & 0\\ 0 & \prod\limits_{j\in K(\xi)}\frac{k-\bar{k}_j}{k-k_j}\end{pmatrix}, & \text{otherwise}\end{cases}$$

where $K(\xi)=\{j : \operatorname{Re} k_j < -\xi\}$. Then $\tilde{M}_+ = \tilde{M}_-\tilde{J}$, where the jump matrix $\tilde{J}$ is:

1. $\tilde{J} = \begin{pmatrix}1 & -\frac{k-k_j}{c_j}\prod\limits_{j\in K(\xi)}\left(\frac{k-\bar{k}_j}{k-k_j}\right)^2 e^{-2it\theta(k_j)}\\ 0 & 1\end{pmatrix}$ for $k\in\partial D_j$ (corr. $\partial\bar{D}_j$);
2. $\tilde{J} = \begin{pmatrix}1+|\tilde{r}(k)|^2 & \bar{\tilde{r}}(k)e^{-2it\theta(k)}\\ \tilde{r}(k)e^{2it\theta(k)} & 1\end{pmatrix}$, $k\in\mathbb{R}$, with

$$\tilde{r}(k) := r(k)\prod_{j\in K(\xi)}\left(\frac{k-\bar{k}_j}{k-k_j}\right)^2.$$

Thus, in this way we arrive at the RHP s.t.

- On $\mathbb{R}$, the jump matrix keeps its original structure, with $r$ replaced by $\tilde{r}$ (reflecting the (asymptotic) influence of the discrete spectrum on the continuous one);
- On all $\partial D_j$ and $\partial \bar{D}_j$, the jump matrices decay to $I$ (as $t \to \infty$).

Thus for any direction $x/4t = const$ that does not coincide with $x/4t = -\operatorname{Re} k_j$ for all $j \in \{1, \ldots, N\}$, the large-$t$ asymptotics should have the same nature as in the case without solitons; we will see that this asymptotics is decaying, of order $t^{-\frac{1}{2}}$.

Along $x/4t = -\operatorname{Re} k_j$, the asymptotics is non-decaying; it is dominated by the corresponding soliton.

- The first way (of reducing non-reg. RHP to reg. RHP) is more convenient for calculating the parameters of soliton asymptotics along $x/4t = -\operatorname{Re} k_j$; the second way is more convenient for calculating the parameters of decaying asymptotics.

Previous considerations have demonstrated the flexibility of the RHP: one can change the contour (without changing the quantity that we want to extract from the solution of the RHP), in view of making the RHP with “better” jumps (e.g., approaching $I$ for large values of the parameter).
One of the contour deformations most useful for the asymptotic analysis is called “making lenses”.

Let the original RHP be $M_+ = M_- v$, $k \in \Sigma$. Assume that the jump matrix $v$ admits factorization $v = v_-^{-1} v_1 v_+$ on a part of $\Sigma$ s.t. $v_\pm$ is analytic in $\Omega_\pm$ (analytically continued from $\Sigma$ into the left/right domain). Define $\hat{M}$:

$$\hat{M} = \begin{cases} M v_+, & k \in \Omega_+, \\ M v_-, & k \in \Omega_-, \\ M, & \text{otherwise} \end{cases}$$

Then $\hat{M}$ satisfies the jump conds. $\hat{M}_+ = \hat{M}_- \hat{v}$, $k \in \hat{\Sigma}$, where

$$\hat{v} = \begin{cases} v_+^{-1}, & k \in \Sigma_+, \\ v_-, & k \in \Sigma_-, \\ v_1, & k \in \Sigma \cap \Omega, \\ v, & \text{otherwise} \end{cases}$$

(a) (b)

Particularly, if $v_1 \equiv I$, then a part of $\Sigma$ is erased from the contour.

- Making lenses is useful if $v$ is oscillatory w.r.t. $t$, but the entries of $v_\pm$ are exponentially decaying (analogue of linear steepest descent method).

Our next goal: use the lenses mechanism in order to make the original RHP for the NLS more treatable asymptotically (large $t$). Recall the jump matrix of original RHP: $J(x,t,k) \mapsto J(\xi,t,k)$, where

$$J(\xi,t,k) = \begin{pmatrix} 1-\lambda|r(k)|^2 & \bar{r}(k)e^{-2it\theta(\xi,k)} \\ -\lambda r(k)e^{2it\theta(\xi,k)} & 1 \end{pmatrix}, \quad k\in\mathbb{R} \quad \xi = \frac{x}{4t}.$$

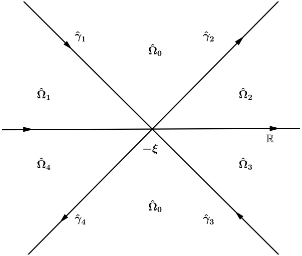
Notice that $J$ admits the (natural) triangular factorization

$$J(\xi,t,k) = \begin{pmatrix} 1 & \bar{r}(k)e^{-2it\theta(\xi,k)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\lambda r(k)e^{2it\theta(\xi,k)} & 1 \end{pmatrix}$$

Recall "signature table": for $\operatorname{Re} k > k_0 = -\xi$, $e^{-2it\theta(\xi,k)}$ decays to $0$ (as $t\to\infty$), if $\operatorname{Im} k < 0$ and $e^{2it\theta(\xi,k)}$ decays to $0$, if $\operatorname{Im} k < 0$.

We assume that $r(k)$ can be analytically continued into $\mathbb{C}_+$.

- Good factorization for $k > -\xi$: introducing




$$\hat{M} = \begin{cases} M \begin{pmatrix} 1 & \bar{r}(k)e^{-2it\theta(\xi,k)} \\ 0 & 1 \end{pmatrix}, & k \in \hat{\Omega}_2 \\ M \begin{pmatrix} 1 & 0 \\ \lambda r(k)e^{2it\theta(\xi,k)} & 1 \end{pmatrix}, & k \in \hat{\Omega}_1 \end{cases}$$

  1. erases $(-\xi, \infty)$ from the contour;
  2. introduces $\hat{\gamma}_2$ and $\hat{\gamma}_3$ as new parts of the contour, where the jump matrices decay to $I$ (as $t \to \infty$) exponentially fast.

- But bad factorization for $k < -\xi$: introducing $\hat{M}$ as above in $\hat{\Omega}_1$ and $\hat{\Omega}_4$ would lead to jumps exponentially increasing on $\hat{\gamma}_1$ and $\hat{\gamma}_4$!

- For $k < -\xi$, it would be nice to have an opposite, "left-triangular" – "right-triangular" factorization; then, due to signature table, the new jump matrices would decay to $I$ as well.
- But we have problem: the "left-triangular" – "right-triangular" factorization involves also a diagonal factor:

$$J = \begin{pmatrix} 1 & 0 \\ -\frac{\lambda r e^{2it\theta(\xi,k)}}{1-\lambda|r|^2} & 1 \end{pmatrix} \begin{pmatrix} 1-\lambda|r|^2 & 0 \\ 0 & \frac{1}{1-\lambda|r|^2} \end{pmatrix} \begin{pmatrix} 1 & \frac{\bar{r}e^{-2it\theta(\xi,k)}}{1-\lambda|r|^2} \\ 0 & 1 \end{pmatrix}$$

- Remedy: remove the diagonal factor, multiplying by an appropriate diagonal matrix having appropriate jump across $k < -\xi$.

In order to construct the appropriate diagonal factor, introduce $\delta(\xi;k)$ ($\xi$ is a parameter) as the solution of the scalar Riemann-Hilbert problem: find a function analytic in $\mathbb{C}\setminus(-\infty,-\xi)$ s.t.

$$\begin{cases}\delta_+(k)=\delta_-(k)(1-\lambda|r(k)|^2), & k\in(-\infty,-\xi)\\ \delta(k)\to 1, & k\to\infty\end{cases}$$

(notice that $1-\lambda|r(k)|^2>0$). The solution of this RHP is given explicitly:

$$\delta(\xi;k)=\exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{-\xi}\frac{\log(1-\lambda|r(s)|^2)}{s-k}ds\right\}.$$

- For $k$ near $-\xi$: $\delta(\xi;k)=(k+\xi)^{i\nu(\xi)}e^{\chi(\xi;k)}$, where $\nu(\xi)=-\frac{1}{2\pi}\log(1-\lambda|r(-\xi)|^2\in\mathbb{R}$ and $\chi(\xi;k)$ is bounded and has a limit as $k\to-\xi$:

$$\chi(\xi;k)=-\frac{1}{2\pi i}\int_{-\infty}^{-\xi}\log(k-s)d_s\log(1-\lambda|r(s)|^2)$$

With $\delta(\xi;k)$ defined as above, the contour/jump deformation process involves 2 steps: (i) $M \mapsto \overset{(1)}{M} := M \begin{pmatrix} \delta^{-1} & 0 \\ 0 & \delta \end{pmatrix}$;
(ii) $\overset{(1)}{M} \mapsto \overset{(2)}{M} := \overset{(1)}{M} \cdot$ (triangular factors).

**First step**: the jump conditions are $\overset{(1)}{M}_+ = \overset{(1)}{M}_- \overset{(1)}{J}$, $k \in \mathbb{R}$, where

$$\overset{(1)}{J} = \begin{cases} \begin{pmatrix} 1 & \bar{r}\delta^2 e^{-2it\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\lambda r \delta^{-2} e^{2it\theta} & 1 \end{pmatrix}, & k \in (-\xi, \infty) \\ \begin{pmatrix} 1 & 0 \\ -\frac{\lambda r \delta_-^{-2} e^{2it\theta}}{1-\lambda|r|^2} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{\bar{r}\delta_+^2 e^{-2it\theta}}{1-\lambda|r|^2} \\ 0 & 1 \end{pmatrix}, & k \in (-\infty, -\xi) \end{cases}$$

**Second step**: use triangular factors from above.

**Second step**: Introduce $\overset{(2)}{M}$ by

$$\overset{(2)}{M}(\xi,t,k) = \overset{(1)}{M}(\xi,t,k) \begin{cases} \begin{pmatrix} 1 & 0 \\ \lambda r(k)\delta^{-2}(k)e^{2it\theta(\xi,k)} & 1 \end{pmatrix}, & k \in \hat{\Omega}_2 \\ \begin{pmatrix} 1 & \bar{r}(k)\delta^{2}(k)e^{-2it\theta(\xi,k)} \\ 0 & 1 \end{pmatrix}, & k \in \hat{\Omega}_3 \\ \begin{pmatrix} 1 & -\frac{\bar{r}(k)\delta_+^{2}(k)e^{-2it\theta(\xi,k)}}{1-\lambda|r|^2} \\ 0 & 1 \end{pmatrix}, & k \in \hat{\Omega}_1 \\ \begin{pmatrix} 1 & 0 \\ -\frac{\lambda r(k)\delta_-^{-2}(k)e^{2it\theta(\xi,k)}}{1-\lambda|r|^2} & 1 \end{pmatrix}, & k \in \hat{\Omega}_4 \end{cases}$$

Then $\overset{(2)}{M}(k)$ satisfies the jump conditions $\overset{(2)}{M}_+(k) = \overset{(2)}{M}_-(k)\overset{(2)}{J}(k)$ across the "cross" $k \in \hat{\Sigma} := \cup_{j=1}^{4}\hat{\gamma}_j$, where $\overset{(2)}{J}(k)$ is as the triangular matrices above respectively on $\hat{\gamma}_2$, $\hat{\gamma}_3$, $\hat{\gamma}_1$, and $\hat{\gamma}_4$.

- $\overset{(2)}{J}(k)$ decays to $I$ as $t \to \infty$ for any $k \in \hat{\Sigma} \setminus \{-\xi\}$. Moreover, it decays uniformly on $\hat{\Sigma}$ outside any small vicinity of $k = -\xi$!

We face a problem: as $t \to \infty$, the jump matrix $\overset{(2)}{J}(\xi, t, k)$ does not approach $I$ in $L^\infty$; thus we can't even state that $M(\xi, t, k)$ approaches $I$ as $t \to \infty$ (which correspond to $q(x,t) \to 0$)!
But we want even more: to obtain the main (decaying!) term of $q(x,t)$ as $t \to \infty$ explicitly. How can this be achieved?

## Idea (of local parametrix)

In a small vicinity $D_\varepsilon(\xi)$ of $k = -\xi$, replace $\overset{(2)}{M}(k)$ by $\tilde{m}_0(k)$ s.t.

- $\tilde{m}_0(k)$ can be constructed explicitly, via the solution of a special RH problem with constant jump matrix;
- $\hat{m} := \begin{cases} \overset{(2)}{M}(k)\tilde{m}_0^{-1}(k), & |k+\xi| < \varepsilon \\ \overset{(2)}{M}(k), & |k+\xi| > \varepsilon \end{cases}$ satisfies a RH problem with jump matrix $\hat{v}(k)$ close to $I$ in $L^2 \cap L^\infty$ (as $t \to \infty$), with an error estimate.

Then:

- estimate the solution $\mu$ of the respective integral equation $\mu - \mathcal{C}_{\hat{w}}\mu = I$ with $\hat{w} = \hat{v} - I$;
- show that in the expression $(\Sigma_\varepsilon(\xi) = \hat{\Sigma} \setminus D_\varepsilon(\xi))$

$$2i \lim_{k\to\infty} k\left(\hat{m}(\xi,t,k) - I\right) = -\frac{1}{\pi}\int_{\partial D_\varepsilon(\xi)} \mu(s)\hat{w}(s)ds - \frac{1}{\pi}\int_{\Sigma_\varepsilon(\xi)} \mu(s)\hat{w}(s)ds$$

  the first term is dominating as $t \to \infty$;
- calculate explicitly this term and thus the main term of the large-$t$ asymptotics of $q(x,t)$.

The (explicit) construction of $\tilde{m}_0$ is based on:

- rescaling the vicinity $D_\varepsilon(\xi)$ of $k = -\xi$ by introducing new spectral variable $z = (k+\xi)\sqrt{8t}$; then $2it\theta(\xi;k) = \frac{iz^2}{2} - 4it\xi^2$.
- approximating $r(k) \approx r(-\xi)$ and $\delta(\xi;k(z)) \approx \left(\frac{z}{\sqrt{8t}}\right)^{i\nu(\xi)} e^{\chi(-\xi)}$

This suggests to introduce the RH problem in the complex $z$-plane
$m_+^p(\xi,z) = m_-^p(\xi,z)J^p(\xi,z),\ z \in C$ (the cross centered at $0$)

$$J^p(\xi,z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ -\lambda r(-\xi) e^{\frac{iz^2}{2}} z^{-2i\nu(\xi)} & 1 \end{pmatrix}, & z \in C_2 \\ \begin{pmatrix} 1 & -\bar{r}(-\xi) e^{\frac{-iz^2}{2}} z^{2i\nu(\xi)} \\ 0 & 1 \end{pmatrix}, & z \in C_3 \\ \begin{pmatrix} 1 & \frac{\bar{r}(-\xi) e^{\frac{-iz^2}{2}} z^{2i\nu(\xi)}}{1-\lambda|r(-\xi)|^2} \\ 0 & 1 \end{pmatrix}, & z \in C_1 \\ \begin{pmatrix} 1 & 0 \\ \frac{\lambda r(-\xi) e^{\frac{iz^2}{2}} z^{-2i\nu(\xi)}}{1-\lambda|r(-\xi)|^2} & 1 \end{pmatrix}, & z \in C_4 \end{cases}$$

with standard normalization: $m^p(\xi,z) \to I$ as $z \to \infty$.

Importance of $m^p(\xi, z)$:

- $\tilde{m}_0(x,t,k)$ can be given explicitly in terms of $m^p(\xi, z)$;
- the RHP for $m^p(\xi, z)$ can be solved explicitly.

## Proposition 1

$$\tilde{m}_0(x,t,k) = \Delta(\xi,t) m^p(\xi, z(k)) \Delta^{-1}(\xi,t),$$

where $\Delta(\xi,t) = e^{(2it\xi^2 + \chi(-\xi))\sigma_3} (8t)^{-\frac{i\nu(\xi)}{2}\sigma_3}$.

## Proposition 2

$$m^p(\xi, z) = m_0(\xi, z) D_j^{-1}(\xi, z), \quad z \in \Omega_j, j = 0, 1, ..., 4$$

where $D_0 = e^{-\frac{iz^2}{4}\sigma_3} z^{i\nu(\xi)\sigma_3}$, $D_1 = D_0 \begin{pmatrix} 1 & \frac{\bar{r}(-\xi)}{1-\lambda|r(-\xi)|^2} \\ 0 & 1 \end{pmatrix}$, $D_2 =$
$D_0 \begin{pmatrix} 1 & 0 \\ -\lambda r(-\xi) & 1 \end{pmatrix}$, $D_3 = D_0 \begin{pmatrix} 1 & -\bar{r}(-\xi) \\ 0 & 1 \end{pmatrix}$, $D_4 = D_0 \begin{pmatrix} 1 & 0 \\ \frac{\lambda r(-\xi)}{1-\lambda|r(-\xi)|^2} & 1 \end{pmatrix}$.

and $m_0(\xi, z)$ is the solution of the RHP with constant jump condition (independent of $z$)

$$m_{0+}(\xi, z) = m_{0-}(\xi, z) j_0(\xi), \quad z \in \mathbb{R},$$

where $j_0(\xi) = \begin{pmatrix} 1 - \lambda |r(-\xi)|^2 & \bar{r}(-\xi) \\ -\lambda r(-\xi) & 1 \end{pmatrix}$, and with normalization condition $m_0(\xi, z) = \left(I + O(\frac{1}{z})\right) e^{-\frac{iz^2}{4}\sigma_3} z^{i\nu(\xi)\sigma_3}$, $z \to \infty$.

The fact that the jump matrix is constant w.r.t. $z$ suggests that this RHP can be solved explicitly. And indeed, this is the case!

**Proposition 3**

$m_0(\xi, z) = \begin{pmatrix} (m_0)_{11} & -\frac{(\frac{d}{dz} - \frac{iz}{2})(m_0)_{22}}{\gamma(\xi)} \\ -\frac{(\frac{d}{dz} + \frac{iz}{2})(m_0)_{11}}{\beta(\xi)} & (m_0)_{22} \end{pmatrix}$ where $(m_0)_{11}$ and $(m_0)_{22}$ are solutions of Bessel-type equations (other names: parabolic cylinder eq.; Weber eq.; Weber-Hermite eq.):

$$\frac{d^2}{dz^2}(m_0)_{11}(\xi, z) + \left(\frac{i}{2} - \nu(\xi) + \frac{z^2}{4}\right)(m_0)_{11}(\xi, z) = 0.$$

Here $\gamma(\xi)$ and $\beta(\xi)$ satisfy $\gamma(\xi)\beta(\xi) = \nu(\xi)$; they can be expressed in terms of $\nu(\xi)$ and $r(-\xi)$:

$$\beta(\xi) = \frac{2\pi e^{-\frac{\pi\nu(\xi)}{2}} e^{\frac{-3\pi i}{4}}}{\lambda r(-\xi)\Gamma(-i\nu(\xi))}.$$

*Hint for proof*: since $j_0$ does not depends on $z$, it follows that $m_0(z)$ and $\frac{d}{dz}m_0(z)$ satisfy the same jump conds.; consequently, $\frac{dm_0}{dz}m_0^{-1}(z)$ is entire function, whose large-$z$ asymptotics is $-\frac{iz}{2}\sigma_3 - \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$ with some $\beta$ and $\gamma$.

Then, by Liouville theorem, $\frac{dm_0}{dz}m_0^{-1}(z)$ equals this expression.

- Thus we have explicit expressions (following from the prescribed large-$z$ asymptotics) for entries of $m_0$: $(m_0)_{11}(\xi, z) = e^{-\frac{3\pi}{4}} D_{i\nu(\xi)}\left(e^{-\frac{3\pi i}{4}} z\right)$ for $z \in \mathbb{C}_+$, etc.
- but what we actually need is the large-$z$ asymptotics of $m_0$ and, subsequently, of $m^p$:

$$m^p(\xi, z) = I + \frac{i}{z}\begin{pmatrix} 0 & \beta(\xi) \\ -\gamma(\xi) & 0 \end{pmatrix} + O(z^{-2}).$$

Indeed, from this asymptotics it follows that

$$\tilde{m}_0^{-1}(x,t,k) = I + \frac{B(\xi,t)}{\sqrt{8t}(k+\xi)} + O(t^{-1}),$$

where $B_{12}(\xi,t) = -i\beta(\xi)e^{4it\xi^2+2\chi(-\xi))}(8t)^{-i\nu(\xi)}$ (notice $|B_{12}| = \sqrt{|\nu(\xi)|}$).

Now recall our consecutive transformations:

$$q(x,t) = 2i \lim_{k\to\infty} k(\overset{(2)}{M} - I)_{12} = 2i \lim_{k\to\infty} k(\hat{m} - I)_{12} =$$

$$= -\frac{1}{\pi} \int\limits_{|k+\xi|=\varepsilon} (\tilde{m}_0^{-1} - I)_{12} dk - \frac{1}{\pi} \int\limits_{|k+\xi|=\varepsilon} (\mu - I)(\tilde{m}_0^{-1} - I)_{12} dk$$

$$- \frac{1}{\pi} \int\limits_{\Sigma_\varepsilon} (\mu \hat{w})_{12} dk - \frac{1}{\pi} \int\limits_{\Sigma_{ext}} (\mu \hat{w})_{12} dk$$

Estimates for $\hat{w}$ and $\mu$ imply that the dominating term is the first one (in red). But it can be computed by Residue Theorem:

$$q(x,t) = -\frac{2iB_{12}(\xi,t)}{\sqrt{8t}} + O(t^{-1}\log t)$$

Recalling the expression for $B_{12}$, we finally arrive at the asymptotic formula for $q(x,t)$: $q(x,t) = q_{as}(\xi,t) + O(t^{-1}\log t)$, where

$$q_{as}(\xi,t) = \frac{\sqrt{|\nu(\xi)|}}{\sqrt{2t}} e^{i(4\xi^2 t - \nu(\xi)\log t + \varphi(\xi))}$$

with

$$\varphi(\xi) = \frac{1}{\pi i}\int_{-\infty}^{-\xi} \log|\xi + s| d_s \log(1 - \lambda |r(s)|^2) - 3\nu(\xi)\log 2 - \frac{3\pi}{4}$$
$$- \arg r(-\xi) + \arg \Gamma(i\nu(\xi))$$

($\Gamma$ is the Gamma function).
Recall that $\xi = \frac{x}{4t}$ and $\nu(\xi) = -\frac{1}{2\pi}\log(1 - \lambda |r(-\xi)|^2)$.