Riemann-Hilbert problems and integrable nonlinear partial differential equations, II

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II. Riemann-Hilbert problems as a tool

Given $V(s) : \Sigma \mapsto \mathbb{C}^{n\times n}$, find $\Phi(z) : \mathbb{C} \setminus \Sigma \mapsto \mathbb{C}^{n\times n}$ s.t.

$$\Phi_+(s) = \Phi_-(s)V(s),\ \ s \in \Sigma; \qquad \Phi(\infty) = I,$$

- In applications, the jump matrix V depends also on certain parameters; then the solution also depends on parameters; e.g., $V(s; x, t) \mapsto \Phi(z; x, t)$.
- Main Idea:
  1. map the data for your problem (e.g., initial or boundary data for a PDE) to a jump matrix for a RHP;
  2. solve the obtained RHP;
  3. "extract" the solution of your problem from solution of RHP
- Main expected benefit:
  - study of properties of the solution of your problem reduces to study of properties of the solution of the associated RHP;
  - having a method for analyzing the solutions of RHPs, one would have a universal tool for analyzing solutions of problems of quite different nature (as far as they are reduced to a RHP!)

Riemann-Hilbert problems as a tool

- For example, in applications to evolution partial differential equations (PDE) in dimension $1+1$:

$$q_t = F(q, q_x, q_{xx}, ...)$$

  with x being the space variable and t the time variable;
- Particular interest: analyze the behavior of the solution of a problem for PDE (e.g., initial or boundary or initial-boundary value problem) as t becomes large.

Riemann-Hilbert problem for "linearized nonlinear Schrödinger equation" (LNLS)

Cauchy problem for a linear PDE

Given $q_0(x)$, $-\infty < x < \infty$ s.t. $q_0(x) \to 0$ as $x \to \infty$, find $q(x,t)$:

- $iq_t(x,t) + q_{xx}(x,t) = 0, \quad -\infty < x < \infty,\ t > 0;$
- $q(x,0) = q_0(x), \quad -\infty < x < \infty$ (initial conditions);
- $q(x,t) \to 0$ as $x \to \infty$ for all $t > 0$.

- Problem can be solved using Fourier transform (direct and inverse). But:
- We solve it in another way (having in mind the perspective to develop a similar approach to (some) nonlinear PDE):
  1. Using the fact that PDE can be represented as compatibility condition of 2 (linear) ODE depending on auxiliary (spectral) parameter:
  2. solving Cauchy problem reduces to solving an associated RHP.

Linear PDE and Lax pair

- LNLS is the compatibility condition for Lax pair equations w.r.t $\mu(x,t,k)$:

$$\mu_x(x,t,k) = -ik\mu(x,t,k) + q(x,t)$$
$$\mu_t(x,t,k) = -ik^2\mu(x,t,k) + kq(x,t) + iq_x(x,t)$$

  Indeed, $\mu_{xt} = \mu_{tx}$ reduces to LNLS.
- The Lax pair for a general linear PDE with constant coefficients $iq_t - \omega(-i\frac{\partial}{\partial x})q = 0$, where $\omega(k)$ is a polynomial of order n:

$$\mu_x(x,t,k) = -ik\mu(x,t,k) + q(x,t)$$
$$\mu_t(x,t,k) = -i\omega(k)\mu(x,t,k) + \sum_{j=0}^{n-1} c_j(k)\frac{\partial^j q}{\partial x^j}$$

  where $-i\left(\frac{\omega(k)-\omega(l)}{k-l}\right)\big|_{l=-i\partial_x} = c_j(k)\partial_x^j$.
- Assume that $q(x,t)$ is given. Aim: find representation for $q(x,t)$ in terms of the solution of a problem, whose data are uniquely determined by $q_0(x)$.

RHP for LNLS, I

- Introduce $\mu_\pm(x,t,k)$ as solutions of "x-equation" of the Lax pair fixed by conditions at $x = \pm\infty$ ("Jost solutions"):

$$\mu_\pm(x,t,k) = o(1), \qquad x \to \pm\infty.$$

  $\mu_\pm$ can be given explicitly: $\mu_\pm = \int_{\pm\infty}^{x} e^{-ik(x-y)} q(y,t)dy$.
  Notice that $\mu_\pm$ needn't solve "t-equation"!
- Analytic properties of $\mu_\pm$ as functions of k:
  1. $\mu_\pm$ is analytic for $k \in \mathbb{C}_\pm = \{k : \pm \operatorname{Im} k > 0\}$;
  2. $\mu_\pm \to 0$ as $k \to \infty$; moreover, $\mu_\pm(x,t,k) = \frac{q(x,t)}{ik} + O(k^{-2})$.
- Define μ by: $\mu = \mu_\pm$ for $k \in \mathbb{C}_\pm$ and calculate the jump of μ across $\mathbb{R}$:

$$\mu_+(x,t,k) - \mu_-(x,t,k) = -e^{-ikx}\rho(k,t), \qquad k \in \mathbb{R}$$

  where $\rho(k,t) := \int_{-\infty}^{\infty} e^{iky} q(y,t)dy$.

- Let's analyze the t-dependence of $\rho(k,t)$:

$$\frac{d\rho}{dt} = \int_{-\infty}^{\infty} e^{iky} q_t(y,t)dy = [\text{by PDE}] = i\int_{-\infty}^{\infty} e^{iky} q_{yy}(y,t)dy$$
$$= [\text{int. by parts}] = -ik^2 \int_{-\infty}^{\infty} e^{iky} q(y,t)dy = -ik^2\rho(k,t).$$

Thus $\rho(k,t) = e^{-ik^2 t}\rho(k,0)$, where $\rho(k,0) = \int_{-\infty}^{\infty} e^{iky} q_0(y)dy$ is determined by initial data.

RHP for LNLS, III

- Thus jump cond. for μ is determined by $q_0(x)$ (via $\rho(k,0)$):

$$\mu_+(x,t,k) - \mu_-(x,t,k) = -e^{-ikx-ik^2t}\rho(k,0), \qquad k \in \mathbb{R} \tag{1}$$

Complementing the jump condition (1) by the normalization cond.

$$\mu(x,t,k) \to 0, \qquad k \to \infty, \tag{2}$$

we arrive at the (additive) RHP for the Cauchy problem for LNLS equation:

> Given $q_0(x)$, find $\mu(x,t,k)$ satisfying (1)+(2), where
> $\rho(k,0) = \int_{-\infty}^{\infty} e^{iky} q_0(y) dy$.

- Solution of RHP: $\mu(x,t,k) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{-ilx-il^2t}\rho(l,0)}{l-k} dl$
- Solution of Cauchy problem for LNLS:
  $q(x,t) = i \lim_{k\to\infty}(k\mu(x,t,k)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ilx-il^2t}\rho(l,0)dl$.
- Remark: we have retrieved the solution by Fourier transform!

RHP for LNLS, IV

Now we present an alternative way of derivation of the RH problem, based on simultaneous solutions of Lax pair equations.

- Introduce the solution of "background" Lax pair (with $q \equiv 0$): $\mu_0(x,t,k) = e^{-ikx-ik^2t}$.
- Let $\tilde{\mu} := \mu\mu_0^{-1}$, where μ solves the Lax pair equations; then $\tilde{\mu}$ is solution of the following equations (modified Lax pair):

$$\tilde{\mu}_x = qe^{ikx+ik^2t}, \qquad \tilde{\mu}_t = (kq + iq_x)e^{ikx+ik^2t}$$

- Introduce $\tilde{\mu}_\pm(x,t,k) := \int_{\pm\infty}^x q(y,t)e^{iky+ik^2t}$.
  1. $\tilde{\mu}_\pm$ obviously solve x-equation;
  2. Let's show that $\tilde{\mu}_\pm$ solve t-equation as well:

$$\tilde{\mu}_{\pm,t} = \int_{\pm\infty}^{x} (q_t + ik^2q)e^{iky+ik^2t}dy = [\text{by PDE}] = \int_{\pm\infty}^{x} (iq_{yy} + ik^2q)e^{iky+ik^2t}dy$$

$$= [\text{int. by parts}] = (iq_x + kq)e^{ikx+ik^2t} + \int_{\pm\infty}^{x} (-ik^2 + ik^2)qe^{iky+ik^2t}dy$$

- As in the previous approach, $\tilde{\mu}_\pm$ are analytic in $\mathbb{C}_\pm$.
- Now consider the jump on $\mathbb{R}$: since both $\tilde{\mu}_+$ and $\tilde{\mu}_-$ solve x-equation and t-equation, it follows that the difference $\tilde{\mu}_+ - \tilde{\mu}_-$ is independent of x and t!

$$\tilde{\mu}_+(x,t,k) - \tilde{\mu}_-(x,t,k) = C(k)$$

  with some $C(k)$.
- In order to calculate $C(k)$, set $x = t = 0$; then $C(k) = -\int_{-\infty}^{\infty} q(y,0)e^{iky}dy = -\rho(k,0)$.
- Finally, in order to have standard normalization (to 0) at $k = \infty$, the RHP is formulated for $\mu := \tilde{\mu}e^{-ik^2t}$, and **we retrieve the formulation of the RHP obtained in the first approach!**

Large time asymptotics: linear PDE

The integral representation for $q(x,t)$ allows studying its large time behavior (via stationary phase/steepest descent method for oscillatory integrals). Let $\xi := \frac{x}{t}$. Then

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \rho(l,0)\mathrm{e}^{\mathrm{i}t\Phi(\xi;l)}dl$$

with $\Phi(\xi;l) = -\xi l - l^2$.

- stationary point: $\Phi'(\xi;l_0) = 0 \Longrightarrow l_0 = -\frac{\xi}{2}$
- $\Phi(\xi;l_0) = \frac{\xi^2}{4}$
- main term (as $t \to \infty$) comes from the (vicinity of) stationary point:

$$q(x,t) = \frac{1}{\sqrt{2t}}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}\rho\left(-\frac{\xi}{2},0\right)\mathrm{e}^{\frac{\mathrm{i}t\xi^2}{4}} + O(t^{-1})$$

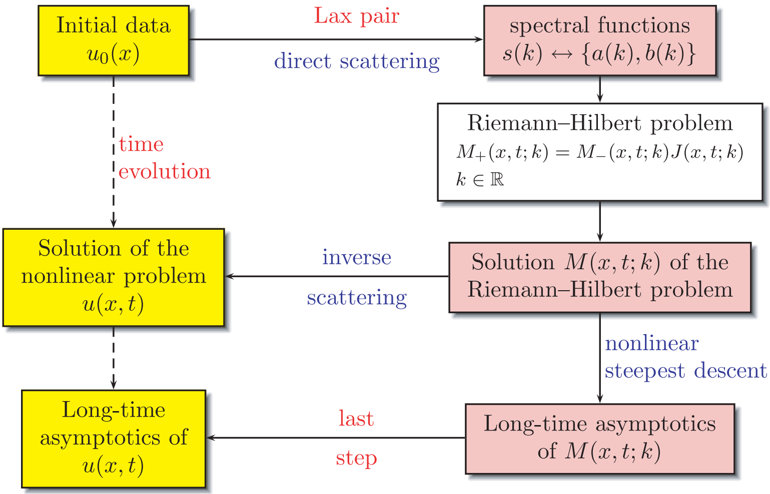
RHP and nonlinear PDE: general remarks, I

- A key point in the scheme for solution Cauchy problem for linear PDE was the Lax pair representation
- Some of nonlinear PDE $q_t = F(q, q_x, q_{xx}, ...)$ also have the Lax pair representation: they are compatibility conditions of pairs of linear equations (depending on a (spectral) parameter).
- Such nonlinear PDE are called intergable: the solution of a nonlinear problem (Cauchy problem for a nonlinear PDE) reduces to certain amount of linear steps (each step is about solving a linear problem).
- Integrable equations are rare; but they are important in physical contexts

RHP and nonlinear PDE: general remarks, II

- Linear steps in solving problems for nonlinear PDEs are similar to solving problems for linear PDEs by the Fourier transforms (direct and inverse). In this way, the method of solving the Cauchy problem for an integrable nonlinear PDE based on using the Lax pair representation is sometimes called "Nonlinear Fourier Transform" method (NFT): x-equation in the Lax pair is used to construct the "direct transform" whereas the solution of the associated RHP corresponds to the "inverse transform".

RHP and nonlinear PDE: solution scheme



RHP and nonlinear PDE: general remarks, III

- Another name for the "linearization" of the solution of problems for nonlinear PDE is "Inverse Scattering Transform" method (IST):
  - the solutions of x-equation in the Lax pair involved in the construction are fixed by their behavior at $x = \pm\infty$ and thus the relation amongst them can be interpreted as the "scattering" of waves (q being considered as a "scatterer", and the waves are normalized "far away" of the scatterer);
  - t-equation governs a linear evolution of "scattering data";
  - the most involved step is to go back, from scattering data to functions in the physical space. Historically, this step was performed by using Marchenko-Gelfand-Levitan (linear) integral equations of the inverse problem; latter on, the Riemann-Hilbert variant of this step was invented.
- An advantage of the RHP approach: efficiency for large time analysis (for other asymptotic regimes: e.g., small dispersion limit).

Integrable nonlinear equations: NLS

From now on, the main object of our study is Nonlinear Schrödinger equation (NLS):

$$\mathrm{i}q_t + q_{xx} - 2\lambda|q|^2 q = 0.$$

Actually, two equations: (i) defocusing NLS, with $\lambda = 1$; and focusing NLS, with $\lambda = -1$.

NLS is compatibility condition for 2 linear equations (Lax pair). They are matrix-valued (2×2) and involve parameter k:

$$\Phi_x(x,t,k) = U(x,t,k)\Phi(x,t,k), \quad \Phi_t(x,t,k) = V(x,t,k)\Phi(x,t,k)$$

where

- $U(x,t,k) = -\mathrm{i}k\sigma_3 + \begin{pmatrix} 0 & q(x,t) \\ \lambda\bar{q}(x,t) & 0 \end{pmatrix}$ with $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$;
- $V(x,t,k) = -2\mathrm{i}k^2\sigma_3 + 2k \begin{pmatrix} 0 & q(x,t) \\ \lambda\bar{q}(x,t) & 0 \end{pmatrix} + \begin{pmatrix} -\mathrm{i}\lambda|q|^2 & \mathrm{i}q_x \\ -\mathrm{i}\lambda\bar{q}_x & \mathrm{i}\lambda|q|^2 \end{pmatrix}$

$$\{\text{NLS for } q\} \Longleftrightarrow \{\Phi_{xt} = \Phi_{tx} \text{ for all } k\} \Longleftrightarrow \boxed{U_t - V_x = [V,U]}$$

where $[V,U] := VU - UV$ (matrix commutator).

Problems for NLS

We are going to study

Cauchy problem for NLS with decaying boundary conds.

Given $q_0(x)$, $-\infty < x < \infty$ s.t. $q_0(x) \to 0$ as $x \to \infty$, find $q(x,t)$:

- $iq_t(x,t) + q_{xx}(x,t) - 2\lambda|q|^2 q = 0, \quad -\infty < x < \infty,\ t > 0;$
- $q(x,0) = q_0(x), \quad -\infty < x < \infty$ (initial conditions);
- $q(x,t) \to 0$ as $x \to \infty$ for all $t > 0$.

- Obtain a representation of solution $q(x,t)$ of Cauchy problem in terms of solution of a RHP;
- Use this representation in order to obtain (the main term of) large time asymptotics of $q(x,t)$

Jost solutions of Lax pair, I

General Idea:

- Assuming that $q(x,t)$ is known, construct appropriate solutions of the Lax pair equations having good control as functions of k (analyticity; asymptotics);
- Use these solutions for constructing multiplicative RH problem s.t. the data for this problem (jump conds.) can be uniquely determined by the data for our Cauchy problem (i.e., initial data $q_0(x)$).

Noticing that Lax pair can be written as

$$\Phi_x = -ik\sigma_3\Phi + \tilde{U}(x,t)\Phi, \quad \Phi_t = -2ik^2\sigma_3\Phi + \tilde{V}(x,t,k)\Phi,$$

where $\tilde{U}, \tilde{V} \to 0$ as $|x| \to \infty$, one can fix the Jost solutions, Φ_- and Φ_+, by their asymptotics as $x \to -\infty$ and as $x \to +\infty$, reducing differential equation to integral equations:

$$\Phi_\pm(x,t,k) = e^{(-ikx-2ik^2t)\sigma_3} + \int_{\pm\infty}^{x} e^{ik(y-x)\sigma_3}\tilde{U}(y,t)\Phi_\pm(y,t,k)dy$$

- It is easily seen that solution of int. equ. satisfy the x-equation of Lax pair;
- The fact that they satisfy also the t-equation of Lax pair: comes from compatibility condition $U_t - V_x + [U,V] = 0$!

Jost solutions of Lax pair, II

- For $k \in \mathbb{R}$, both Φ_+ and Φ_- are well defined. Indeed, for such k, all exponentials in int. equ. are bounded, and, assuming that $q(x,t) \to 0$ as $|x| \to \infty$ sufficiently fast, these int. equ. (which are Volterra integral equations) can be solved by iterations.
- For non-real k, some of exponentials are growing (as $x \to -\infty$ or as $x \to +\infty$, which requires more careful analysis of the int. equations.

It is convenient to introduce $\Psi_\pm := \Phi_\pm e^{(ikx+2ik^2t)\sigma_3}$. Then, the int. equ. for $\Phi_\pm$ reduce to those for $\Psi_\pm$:

$$\Psi_\pm(x,t,k) = I + \int_{\pm\infty}^{x} e^{ik(y-x)\sigma_3}\tilde{U}(y,t)\Psi_\pm(y,t,k)e^{-ik(y-x)\sigma_3}dy$$

(here and below, I is 2×2 identity matrix. An advantage of $\Psi_\pm$ is that it becomes easier to control them (column-wise!) for non-real k.
Introduce notations for columns: $\Psi \equiv (\Psi^{(1)}, \Psi^{(2)})$ and consider, e.g., the int. equ. for $\Psi_-^{(1)} \equiv \begin{pmatrix} \Psi_{-,11} \\ \Psi_{-,21} \end{pmatrix}$:

$$\begin{pmatrix} \Psi_{-,11} \\ \Psi_{-,21} \end{pmatrix}(x,t,k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{-\infty}^{x} \begin{pmatrix} q(y,t)\Psi_{-,21}(y,t,k) \\ \lambda\bar{q}(y,t)\Psi_{-,11}e^{2ik(x-y)} \end{pmatrix} dy$$

Jost solutions of Lax pair, III

Since the only exponentials in the int. equ. for $\Psi_-^{(1)}$ decays to 0 as $y \to -\infty$ for all k with $\operatorname{Im} k > 0$, the follows result holds true.

Theorem 1

$\Psi_-^{(1)}(\cdot, \cdot, k)$ is analytic in $\mathbb{C}_+$ and continuous in $\overline{\mathbb{C}_+}$.

Proof. Rewrite the int. equ. for $h(x) := \Psi_-^{(1)}(x) \equiv \left(\begin{smallmatrix} h_1 \\ h_2 \end{smallmatrix}\right)$ in operator form (omitting t and k):

$$h(x) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) + (Ah)(x) \quad \text{with} \quad A \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}(x) = \int_{-\infty}^{x} \begin{pmatrix} q(y)h_2(y) \\ \lambda \bar{q}(y) h_1(y) e^{2ik(x-y)} \end{pmatrix} dy$$

for $h \in C((-\infty, M], \mathbb{R}^2)$ assuming $q(x) \in L_1(-\infty, M)$.
Introduce $\alpha(x) := \int_{-\infty}^{x} |q(y)| dy$. Then

- $\|(Ah)\|(x) \le \|h\| \alpha(x)$;
- $\|(A^2 h)\|(x) \le \|h\| \int\limits_{-\infty}^{x} |q(y)| \alpha(y) dy = \frac{\|h\|}{2} \int\limits_{-\infty}^{x} \frac{d}{dy}(\alpha^2(y)) dy = \|h\| \frac{\alpha^2(x)}{2}$
- by induction, $\|(A^n h)\|(x) \le \|h\| \frac{\alpha^n(x)}{n!}$

It follow that $\Psi_-^{(1)}(x, t, k) := \sum_{n=0}^{\infty} A^n \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ converges uniformly for $k \in \overline{\mathbb{C}_+}$.
Moreover, integrating by parts, $\Psi_-^{(1)}(x, t, k) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) + O(1/k)$ as $k \to \infty$.

Jost solutions of Lax pair, IV

Similarly, the other columns of Ψ_-, Ψ_+ are analytic (as functions of k) in the respective half-planes, so that they can be combined into two matrices: one is analytic in $\mathbb{C}_+$ and continuous in $\overline{\mathbb{C}_+}$, another in $\mathbb{C}_-$ and continuous in $\overline{\mathbb{C}_-}$; moreover, they approach I at infinity:

$$\left(\Psi_-^{(1)}, \Psi_+^{(2)}\right): \quad \text{analytic in } \mathbb{C}_+; \quad = I + O(\frac{1}{k}) \text{ as } k \to \infty;$$

$$\left(\Psi_+^{(1)}, \Psi_-^{(2)}\right): \quad \text{analytic in } \mathbb{C}_-; \quad = I + O(\frac{1}{k}) \text{ as } k \to \infty;$$

Consequently, these matrices are good candidates for the construction of RH problem, in view of (i) analyticity and (ii) large-k asymptotics. Moreover, more detailed analysis of the large-k behavior of $\Psi_\pm$ reveals that, e. g.,

$$\Psi_{+,12}(x,t,k) = \frac{q(x,t)}{2ik} + O(1/k^2), \qquad k \to \infty.$$

Question: how these matrices are related on $\mathbb{R}$ (which would serve as a contour for the RHP)?

Scattering relation

Let's go back to the Jost solutions $\Phi_{\pm}(x,t,k)$ and write down (formally) the relation

$$\Phi_{+}(x,t,k) = \Phi_{-}(x,t,k)S(x,t,k).$$

Now notice the following:

- Since the both Φ_+ and Φ_- are solutions of an ordinary differential equation w.r.t. x (from Lax pair), it follows that matrix S is independent of x.
- Since the both Φ_+ and Φ_- are solutions of an ordinary differential equation w.r.t. t (again from Lax pair), it follows that matrix S is independent of t.

Therefore we have that S in the relation above depends on k only and thus this relation actually reads

$$\Phi_{+}(x,t,k) = \Phi_{-}(x,t,k)S(k), \qquad k \in \mathbb{R}.$$

This relation is called the scattering relation; matrix $S(k)$ is called the scattering matrix.

Properties of scattering matrix, I

Now let's discuss the properties of $S(k)$.

First, we notice that $\det \Phi_+ = \det \Phi_- \equiv 1$.

Indeed, $\Phi_\pm$ are solutions of the differential equation $\frac{d}{dx}\Phi_\pm = U\Phi_\pm$, where the coefficient matrix U is such that $\operatorname{Tr} U \equiv 0$. It follows that $\frac{d}{dx}(\det \Phi_\pm) \equiv 0$. Similarly, since $\operatorname{Tr} V \equiv 0$ for V in $\frac{d}{dt}\Phi_\pm = V\Phi_\pm$, we have $\frac{d}{dt}(\det \Phi_\pm) \equiv 0$. Thus $\det \Phi_\pm$ is a constant. On the other hand, $\det \Phi_\pm|_{x\to\pm\infty} = 1$. Therefore, $\det(\Phi_\pm) \equiv 1$.

Consequently, $\det S(k) \equiv 1$.

Properties of scattering matrix, II

Further, notice that U satisfies the symmetry conditions:

- If $\lambda = 1$ (the parameter in NLS equation), then $U(x,t,k) = \sigma_1 \overline{U(x,t,\bar{k})} \sigma_1$, where $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.
- If $\lambda = -1$, then $U(x,t,k) = \sigma_2 \overline{U(x,t,\bar{k})} \sigma_2$, where $\sigma_2 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$.

Similarly for V. Since the limiting values of $\Phi_\pm$ as $x \to \pm\infty$ also satisfies these symmetries, it follows (from the ODEs for $\Phi_\pm$) that (for $k \in \mathbb{R}$)

- $\Phi_\pm(x,t,k) = \sigma_1 \overline{\Phi_\pm(x,t,k)} \sigma_1 \quad (\lambda = 1)$;
- $\Phi_\pm(x,t,k) = \sigma_2 \overline{\Phi_\pm(x,t,k)} \sigma_2 \quad (\lambda = -1)$.

It follows that $S(k)$ also has these symmetries: for $k \in \mathbb{R}$,

- $S(k) = \sigma_1 \overline{S(k)} \sigma_1 \quad (\lambda = 1)$;
- $S(k) = \sigma_2 \overline{S(k)} \sigma_2 \quad (\lambda = -1)$.

Properties of scattering matrix, III

The symmetries can be expressed in terms of entries of $S(k)$. Denote $a(k) := S_{22}(k)$ and $b(k) := S_{12}(k)$. Then

- $S(k) = \begin{pmatrix} \bar{a}(k) & b(k) \\ \bar{b}(k) & a(k) \end{pmatrix} \qquad (\lambda = 1)$;
- $S(k) = \begin{pmatrix} \bar{a}(k) & b(k) \\ -\bar{b}(k) & a(k) \end{pmatrix} \qquad (\lambda = -1)$.

$a(k)$ and $b(k)$ can be expressed in terms of determinants of matrices constructed from columns of $\Phi_\pm$ (linear algebra!):

$$a(k) = \det\left(\Phi_-^{(1)}, \Phi_+^{(2)}\right), \quad b(k) = \det\left(\Phi_+^{(2)}, \Phi_-^{(2)}\right).$$

Since $\Phi_-^{(1)}$ and $\Phi_+^{(2)}$ are analytic in $\mathbb{C}_+$, it follows that $a(k)$ is analytic in $\mathbb{C}_+$; moreover,

$$a(k) \to 1 \qquad k \to \infty, \ k \in \overline{\mathbb{C}_+}.$$

As for $b(k)$, it is (in general) defined for $k \in \mathbb{R}$ only, with $b(k) \to 0$ as $k \to \infty$. But in particular situations (e.g. if the support of $q_0(x)$ is finite), $b(k)$ also can be analytically continued from $\mathbb{R}$ into (a part of) $\mathbb{C}$.

Recall the scattering relation

$$\Phi_+(x,t,k) = \Phi_-(x,t,k)S(k), \qquad k \in \mathbb{R}.$$

This relation already has a form of a "multiplicative jump condition". But this relation cannot be interpreted as a jump condition for a RH problem, because neither Φ_+ nor Φ_- are matrices analytic in the respective domains of $\mathbb{C}$! On the other hand, particular columns of Φ_+ and Φ_- do analytic in either $\mathbb{C}_+$ or $\mathbb{C}_-$. This suggests to combine the columns analytic in one or another half-plane into the respective matrices:

$$\left(\Phi_-^{(1)}, \Phi_+^{(2)}\right)(x,t,k) = \left(\Phi_+^{(1)}, \Phi_-^{(2)}\right)(x,t,k)\tilde{S}(k), \qquad k \in \mathbb{R}.$$

Linear algebra problem: express $\tilde{S}(k)$ in terms of $S(k)$ (in terms of $a(k)$ and $b(k)$).

From the scattering matrix to a jump matrix for a RHP, II

Solution of this (algebra) problem:

$$\left(\Phi_-^{(1)}, \Phi_+^{(2)}\right)(x,t,k) = \left(\Phi_+^{(1)}, \Phi_-^{(2)}\right)(x,t,k) \begin{pmatrix} \frac{1}{\bar{a}(k)} & \frac{b(k)}{\bar{a}(k)} \\ -\lambda \frac{\bar{b}(k)}{\bar{a}(k)} & \frac{1}{\bar{a}(k)} \end{pmatrix}, \quad k \in \mathbb{R}.$$

Further problems:

- the determinants of $\left(\Phi_-^{(1)}, \Phi_+^{(2)}\right)$ and $\left(\Phi_+^{(1)}, \Phi_-^{(2)}\right)$ may equal 0 in respectively $\mathbb{C}_+$ and $\mathbb{C}_-$, which is not good for uniqueness (of the solution of RHP).
- large-k asymptotics is not I.

Indeed,

- $\det\left(\Phi_-^{(1)}, \Phi_+^{(2)}\right) = a(k)$ and $\det\left(\Phi_+^{(1)}, \Phi_-^{(2)}\right) = \overline{a(\bar{k})}$;
- recall that $\Psi_\pm := \Phi_\pm e^{(ikx+2ik^2t)\sigma_3} \to I$ as $k \to \infty$.

In order to have all determinants to equal 1: introduce

$$\hat{\Phi} := \begin{cases} \left(\frac{\Phi_-^{(1)}}{a}, \Phi_+^{(2)} \right), & k \in \mathbb{C}_+, \\ \left(\Phi_+^{(1)}, \frac{\Phi_-^{(2)}}{\bar{a}} \right), & k \in \mathbb{C}_-. \end{cases}$$

Then

$$\left(\frac{\Phi_-^{(1)}}{a}, \Phi_+^{(2)} \right) = \left(\Phi_+^{(1)}, \frac{\Phi_-^{(2)}}{\bar{a}} \right) J_0(k),$$

where

$$J_0(k) = \begin{pmatrix} 1 - \lambda |r(k)|^2 & \bar{r}(k) \\ -\lambda r(k) & 1 \end{pmatrix} \qquad \text{with } r(k) := \frac{\bar{b}(k)}{a(k)}.$$

From the scattering matrix to a jump matrix for a RHP, IV

Finally, introduce $M(x,t,k) := \hat{\Phi}(x,t,k)e^{(ikx+2ik^2t)\sigma_3}$. Then

$$M(x,t,k) \to I, \qquad k \to \infty \tag{i}$$

whereas the jump condition for $M(x,t,k)$ takes the form:

$$M_+(x,t,k) = M_-(x,t,k)J(x,t,k), \qquad k \in \mathbb{R}, \tag{ii}$$

where

$$\begin{aligned} J(x,t,k) &= e^{(-ikx-2ik^2t)\sigma_3}J_0(k)e^{(ikx+2ik^2t)\sigma_3} \\ &= \begin{pmatrix} 1-\lambda|r(k)|^2 & \bar{r}(k)e^{-2ikx-4ik^2t} \\ -\lambda r(k)e^{2ikx+4ik^2t} & 1 \end{pmatrix} \end{aligned} \tag{iii}$$

(i)+(ii)+(iii) looks like a RH problem: given $r(k), k \in \mathbb{R}$, find 2×2 matrix function $M(\cdot,\cdot,k)$ analytic in $\mathbb{C} \setminus \mathbb{R}$ and satisfying normalization (i) and jump (ii)+(iii).

Problem

If $a(k) = 0$ for some $k \in \overline{\mathbb{C}_+}$, then M has to have singularities.

The RHP formalism for the NLS equation, I

Assume for a moment that $a(k) \neq 0$ for all $k \in \overline{\mathbb{C}_+}$. Then the analysis above leads to the following algorithm for solving the Cauchy problem for the NLS equation.

1. Given $q_0(x)$, construct $a(k)$ and $b(k)$ from
$$\begin{pmatrix} \bar{a}(k) & b(k) \\ \lambda\bar{b}(k) & a(k) \end{pmatrix} = \Phi_-^{-1}(0,0,k)\Phi_+(0,0,k),$$
where $\Phi_\pm(0,0,k)$ are calculated by solving the linear integral equations
$$\Phi_\pm(x,0,k) = e^{-ikx\sigma_3} + \int_{\pm\infty}^{x} e^{ik(y-x)\sigma_3} \begin{pmatrix} 0 & q_0(y) \\ \lambda\bar{q}_0(y) & 0 \end{pmatrix} \Phi_\pm(y,0,k)dy$$
2. Given $r(k) := \bar{b}(k)/a(k)$, $k \in \mathbb{R}$, construct the jump matrix $J(x,t,k)$ by (iii).
3. Solve the RH problem (i)+(ii) for $M(x,t,k)$.
4. Obtain $q(x,t)$ from the large-k behavior of M:
$$q(x,t) = 2i \lim_{k\to\infty} kM_{12}(x,t,k).$$

The RHP formalism for the NLS equation, II

- Since $\det J(x,t,k) \equiv 1$, it follows that $\det M(x,t,k) \equiv 1$.
- Then, by the Liouville theorem, the solution M of the RHP is unique, if exist.
- Existence of a solution to a matrix RHP: more subtle question. Basically, it can be established in two situations:
  1. "Symmetry case": the jump matrix possesses a particular (Schwartz) symmetry:
     - $J(k) + J^*(k)$ is positive definite for $k \in \mathbb{R}$;
     - $J(k) = J^*(\bar{k})$ for $k \in \Sigma \setminus \mathbb{R}$.

     (* denotes matrix adjoint);
  2. "Small norm case": the jump matrix is "close" to a matrix, for which the RHP has a solution.
     For instance, when $J(k)$ is "close" to I (if $J(k) \equiv I$, then $M(k) \equiv I$).

Zeros of $a(k)$ for $k \in \overline{\mathbb{C}_+}$, I

Cases $\lambda = 1$ and $\lambda = -1$ are different.

Case $\lambda = 1$

If $\lambda = 1$, then $a(k) \neq 0$ for all $k \in \overline{\mathbb{C}_+}$.

Proof. (i) Assume that $a(\hat{k}) = 0$ for some $\hat{k} \in \mathbb{C}_+$. Then, from representation $a(k) = \det(\Phi_-^{(1)}, \Phi_+^{(2)})$ it follows that $\Phi_+^{(2)}(x, 0, \hat{k}) = \Phi_-^{(1)}(x, 0, \hat{k})\hat{b}$ with some $b \in \mathbb{C}$. Now notice that (a) $\Phi_+^{(2)}(x, 0, \hat{k})$ decays exponentially as $x \to +\infty$ and $\Phi_-^{(1)}(x, 0, \hat{k})$ decays exponentially as $x \to -\infty$; (b) both are (vector) solutions of equation $L\Phi = \hat{k}\Phi$, where $L\Phi := i\sigma_3\Phi_x - i\sigma_3 \begin{pmatrix} 0 & q_0(x) \\ \bar{q}_0(x) & 0 \end{pmatrix} \Phi$. But L is self-adjoint operator and, therefore, can't have non-real eigenvalues; this contradicts the existence of $\hat{\Phi}(x) := \Phi_+^{(2)}(x, 0, \hat{k})$.

(ii) For $k \in \mathbb{R}$ we have (determinant relation) $|a(k)|^2 - |b(k)|^2 = 1$. Therefore, $|a(k)|^2 = 1 + |b(k)|^2 > 0$ and thus $a(k) \neq 0$ for $k \in \mathbb{R}$.

Zeros of $a(k)$ for $k \in \overline{\mathbb{C}_+}$, II

Now consider the case $\lambda = -1$.

- The corresponding operator L is not self-adjoint.
- Since $a(k) \to 1$ as $k \to \infty$, it follows that the zeros of $a(k)$ are located in a bounded domain.
- But it is possible that there are infinitely many of them, accumulating to some points on $\mathbb{R}$.
- Determinant relation on $\mathbb{R}$ reads $|a(k)|^2 + |b(k)|^2 = 1$, giving no information about possible real zeros.

From now on, we will assume that in case $\lambda = -1$:

- There are only finite number of zeros of $a(k)$ in $\mathbb{C}_+$ (and thus there are no zeros on $\mathbb{R}$);
- All zeros of $a(k)$ are simple.

Such assumption corresponds to generic situation: the set of initial data for with the assumptions above hold is an open, dense set (in respective topology).

Residue conditions, I

If $a(k)$ has zeros, in $\mathbb{C}_+$, we have to enlarge our notion of Riemann-Hilbert problem, in order to allow singularities of a solution. But then, in order to preserve uniquness of a solution, we need to add more conditions, characterizing possible singularities.

Let $a(k_j)=0$ for some $k_j\in\mathbb{C}_+$, $j=1,\dots,N$. Then $\Phi_+^{(2)}(x,0,k_j)=\Phi_-^{(1)}(x,0,k_j)b_j$ with some $b_j\in\mathbb{C}$. Introduce $c_j:=\frac{1}{\dot{a}(k_j)b_j}$, where dot denotes the derivative w.r.t. k. Then this relation, being considered in terms of M, reads:

$$\operatorname{Res}_{k\to k_j} M^{(1)}(x,t,k)=c_je^{2ik_jx+4ik_j^2t}M^{(2)}(x,t,k_j), j=\overline{1,N}. \tag{iv-a}$$

The corresponding relations in $\mathbb{C}_-$ read

$$\operatorname{Res}_{k\to\bar{k}_j} M^{(2)}(x,t,k)=-\bar{c}_je^{-2i\bar{k}_jx-4i\bar{k}_j^2t}M^{(1)}(x,t,\bar{k}_j),\quad j=\overline{1,N}. \tag{iv-b}$$

Notice that $\{k_j,c_j\}_{j=1}^N$ in these Residue conditions (similarly to $r(k)$, $k\in\mathbb{R}$) are determined by the initial condition $q_0(x)$.

General RH problem for NLS

Thus, in the case $a(k)$ has zeros, the RH problem for the NLS equation is as follows:

RH problem for NLS

Given $r(k)$, $k \in \mathbb{R}$ and $\{k_j, c_j\}_{j=1}^N$, find a 2×2 function M piece-wise meromorphic relative to $k \in \mathbb{R}$ such that it satisfies the normalization (i), the jump conditions (ii)+(iii), and the residue conditions (iv).

Having this RHP solved, the solution of the Cauchy problem for NLS is given by

$$q(x,t) = 2i \lim_{k\to\infty} kM_{12}(x,t,k).$$

Question

Whether conditions (i)-(iv) determine M uniquely?

Residue conditions, II

Proposition

Conditions (i)-(iv) determine M uniquely.

Proof Let $C_j(x,t,k) := c_j e^{2ik_jx+4ik_j^2t}$. (i) First, consider determinant of M. By Residue conds, as $k \to k_j$, $M = \begin{pmatrix} \frac{C_j\alpha}{k-k_j} + O(1) & \alpha + O(k-k_j) \\ \frac{C_j\beta}{k-k_j} + O(1) & \beta + O(k-k_j) \end{pmatrix}$ with some α and β and thus $\det M = O(1)$ as $k \to k_j$ (non-singular!). Similarly as $k \to \bar{k}_j$. Since $\det J \equiv 1$, by the Liouville theorem we have $\det M \equiv 1$.

(ii) Let M and $\tilde{M}$ are two solutions of RHP. Then $\tilde{M}^{-1} = \begin{pmatrix} \tilde{\beta} + O(k-k_j) & -\tilde{\alpha} + O(k-k_j) \\ -\frac{C_j\tilde{\beta}}{k-k_j} + O(1) & \frac{C_j\tilde{\alpha}}{k-k_j} + O(1) \end{pmatrix}$ and thus $M\tilde{M}^{-1} = \frac{C_j}{k-k_j}\begin{pmatrix} \alpha\tilde{\beta} - \alpha\tilde{\beta} & -\alpha\tilde{\alpha} + \alpha\tilde{\alpha} \\ \beta\tilde{\beta} - \beta\tilde{\beta} & \alpha\tilde{\beta} - \alpha\tilde{\beta} \end{pmatrix} + O(1) = O(1)$. Thus $M\tilde{M}^{-1}$ is non-singular; moreover, it has no jump across $\mathbb{R}$ and approaches I as $k \to \infty$; then by the Liouville theorem, $M\tilde{M}^{-1} \equiv I$.

In general, the RH problem can't be solved explicitly: in what follows, we will discuss how to reduce a RHP to a (singular) integral equation.

Cases when the RHP can be solved explicitly: if the jump conditions are trivial ($J(x,t,k) \equiv I$) and thus the only nontrivial information are Residue conditions.

The simplest case: there is only one pair of Res. conds. ($N = 1$), at $k = k_1 \in \mathbb{C}_+$ and $k = \bar{k}_1$:

$$\operatorname{Res}_{k \to k_1} M^{(1)}(x,t,k) = c_1 e^{2ik_1 x + 4ik_1^2 t} M^{(2)}(x,t,k_1),$$
$$\operatorname{Res}_{k \to \bar{k}_1} M^{(2)}(x,t,k) = -\bar{c}_1 e^{-2i\bar{k}_1 x - 4i\bar{k}_1^2 t} M^{(1)}(x,t,\bar{k}_1).$$

Thus in this case, the RHP is as follows: given $\{k_1, c_1\}$, find $M(x,t,k)$ s.t.: (i) it is meromorphic in $\mathbb{C}$; (ii) $M(x,t,k) \to I$ as $k \to \infty$; (iii) it has simple poles at k_1 and $\bar{k}_1$ according to the jump conditions above.

Thus the solution of the RHP is a matrix with rational entries. Moreover, the Res. cond. and the normalization at ∞ dictate the form of M:

$$M(x,t,k) = \begin{pmatrix} \frac{k-B_1(x,t)}{k-k_1} & \frac{D_2(x,t)}{k-\bar{k}_1} \\ \frac{B_2(x,t)}{k-k_1} & \frac{k-D_1(x,t)}{k-\bar{k}_1} \end{pmatrix}$$

with some B_j, D_j. Further, the symmetry $M(\cdot,\cdot,k) = \sigma_2 \overline{M(\cdot,\cdot,\bar{k})}\sigma_2$ implies:

$$D_1(x,t) = \bar{B}_1(x,t), \qquad D_2(x,t) = -\bar{B}_2(x,t).$$

Finally, in order to determine $B_1(x,t)$ and $B_2(x,t)$ we use the Res. conds.: introducing $\hat{C}_1(x,t) := c_1 e^{2ik_1x+4ik_1^2t}$, Res. cond. at $k=k_1$ reduce to the system of 2 linear algebraic equations:

$$k_1 - B_1 = -\frac{\hat{C}_1\bar{B}_2}{k_1-\bar{k}_1}, \qquad B_2 = \frac{\hat{C}_1(k_1-\bar{B}_1)}{k_1-\bar{k}_1}$$

The solution of this algebraic equations:

$$B_1(x,t) = \left(k_1 - \frac{|\hat{C}_1(x,t)|^2}{(k_1-\bar{k}_1)^2}\right)\left(1 - \frac{|\hat{C}_1(x,t)|^2\bar{k}_1}{(k_1-\bar{k}_1)^2}\right)^{-1},$$

$$B_2(x,t) = \frac{\hat{C}_1(x,t)}{1 - \frac{|\hat{C}_1(x,t)|^2}{(k_1-\bar{k}_1)^2}}.$$

Since $q(x,t) = -2i\bar{B}_2(x,t)$, we obtain the corresponding solution q of the NLS:

$$q(x,t) = \frac{-2i\hat{C}_1(x,t)}{1 - \frac{|\hat{C}_1(x,t)|^2}{(k_1-\bar{k}_1)^2}},$$

where $\hat{C}_1(x,t) = c_1 e^{2ik_1x+4ik_1^2t}$.

Denote $k_1 = k_R + ik_I$. One can distinguish two cases:
(i) $k_R = 0$; (ii) $k_R \neq 0$.

Case 1: $k_1 = i\nu$ with $\nu > 0$.
In this case, $\hat{C}_1(x,t) = c_1 e^{-2\nu x - 4i\nu^2 t}$.

$$q(x,t) = -8i\nu^2 \bar{c}_1 \frac{e^{-2\nu x}}{4\nu^2 + |c_1|^2 e^{-4\nu x}} e^{4i\nu^2 t}.$$

Thus $q(x,t) = f(x)g(t)$, where $f(x) > 0$ and $|g(t)| = 1$.
Moreover, $f(x) \sim \cdot e^{-2\nu|x|}$ as $x \to \pm\infty$. Such solution is called stationary soliton.

Soliton – from "solitary wave".

Case 2: $k_1 = k_R + ik_I$ with $k_R \neq 0$.

$$q(x,t) = -8ik_I^2\bar{c}_1 \frac{e^{-2k_I(x+4k_Rt)}}{4k_I^2 + |c_1|^2 e^{-4k_I(x+4k_Rt)}} e^{-2ik_R\left(x+\frac{2(k_R^2-k_I^2)}{k_R}t\right)}.$$

Particularly,

$$|q(x,t)| = const \frac{e^{-2k_I(x+4k_Rt+\phi)}}{1+e^{-4k_I(x+4k_Rt+\phi)}}$$

with some $\phi \in \mathbb{R}$, i.e., the "envelope" of q moves with velocity $v^{(1)} = -4k_R$. At the same time, the "inner oscillations" move with velocity $v^{(2)} = -\frac{2(k_R^2-k_I^2)}{k_R}$:

$$q(x,t) = f(x - v^{(1)}t)g(x - v^{(2)}t)$$

with $f > 0$ and $|g| = 1$. This is a moving soliton.

Solitons, VI

In the general case of N pairs of residue conditions with parameters $\{k_j, c_j\}_{j=1}^N$: obtaining $q(x,t)$ reduces to solving the system of $2N$ linear algebraic equations. If all $\operatorname{Re} k_j$ are different, then as $t \to -\infty$ or $t \to +\infty$, $q(x,t)$ reduces (asymptotically!) to a sum of one-soliton solutions:

$$q(x,t) \sim \sum_{j=1}^{N} f_j(x - v_j^{(1)} t + \phi_j^{(1,-)}) g_j(x - v_j^{(2)} t + \phi_j^{(2,-)})$$

as $t \to -\infty$ and

$$q(x,t) \sim \sum_{j=1}^{N} f_j(x - v_j^{(1)} t + \phi_j^{(1,+)}) g_j(x - v_j^{(2)} t + \phi_j^{(2,+)})$$

as $t \to +\infty$.

In other words, the interaction between solitons can be viewed as elastic collisions, with only effect being the phase change. It is this property that distinguishes "solitons" among "solitary waves".