

Riemann-Hilbert problems and integrable nonlinear partial differential equations, I

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I. Complex analysis: boundary value problems

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II. Integrable nonlinear PDE; RH problem as a tool

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I. Introduction: Cauchy integrals; RH problems (additive/multiplicative; scalar/matrix)

Riemann–Hilbert problem: boundary value problem in complex analysis

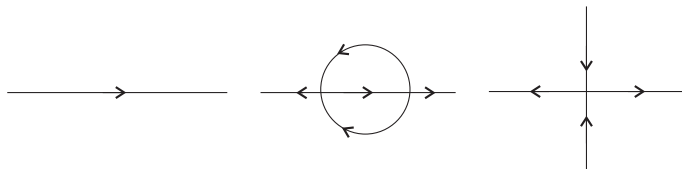
Given (i) contour $\Sigma \in \bar{\mathbb{C}}$, (ii) “jump across Σ ”, (iii) normalization cond., find a function:

- 1 is analytic in $\mathbb{C} \setminus \Sigma$;
- 2 satisfies “jump conditions across Σ ”;
- 3 satisfies normalization cond. at some $z_0 \in \bar{\mathbb{C}}$.

Questions

- what are contours, jump conditions, normalization?
- existence of a solution
- uniqueness

Oriented contours:



- bounded / unbounded
- simple / composed (with self-intersections)
- closed / open (e.g., finite arcs)

Orientation: $\mathbb{C} = \Sigma \cup \Omega_+ \cup \Omega_-$, where Ω_{\pm} are (multicomponent) domains s.t. Σ is the counterclockwise boundary of (components of) Ω_+ and Σ is the clockwise boundary of (components of) Ω_- .

Boundary values: $\Phi_{\pm}(s) := \lim_{\substack{z \rightarrow s \\ z \in \Omega_{\pm}}} \Phi(z)$ (non-tangent limits).

RH problems (scalar)

- **Additive RH problem:** given a smooth, bounded and closed contour Σ and a function $f(s) \in C^{0,\alpha}(\Sigma)$ (α -Hölder continuous), find $\phi(z)$ piecewise analytic in $\mathbb{C} \setminus \Sigma$ s.t.
 - $\phi_+(s) - \phi_-(s) = f(s)$, $s \in \Sigma$ (“jump cond.”);
 - $\phi(\infty) = 0$ (normalization).
- **Multiplicative RH problem (homogeneous):** given contour Σ and a function $g(s) \in C^{0,\alpha}(\Sigma)$ s.t. $g(s) \neq 0$, find $\phi(z)$ piecewise analytic in $\mathbb{C} \setminus \Sigma$ s.t.
 - $\phi_+(s) = \phi_-(s)g(s)$, $s \in \Sigma$ (“jump cond.”);
 - $\phi(\infty) = 1$ (normalization).
- **Multiplicative RH problem (inhomogeneous):** given contour Σ and functions $g(s), f(s) \in C^{0,\alpha}(\Sigma)$ s.t. $g(s) \neq 0$, find $\phi(z)$ piecewise analytic in $\mathbb{C} \setminus \Sigma$ s.t.
 - $\phi_+(s) = \phi_-(s)g(s) + f(s)$, $s \in \Sigma$ (“jump cond.”);
 - $\phi(\infty) = 0$ (normalization).
- **Generalizations:** more complicated contours (unbounded; open arcs; composed); less smooth jump functions.

Cauchy integrals

The Cauchy integral is the **fundamental object (tool) in the theory of RH problems**. Given an oriented contour Σ and a function $f : \Sigma \rightarrow \mathbb{C}$, the **Cauchy integral** is defined by

$$\mathcal{C}_{\Sigma}f(z) := \frac{1}{2\pi i} \int_{\Sigma} \frac{f(s)}{s-z} ds, \quad z \in \mathbb{C} \setminus \Sigma.$$

We are interested in the boundary values of $\mathcal{C}_{\Sigma}f(z)$, when z approach a point on Σ :

$$\mathcal{C}_{\Sigma}^{\pm}f(s^*) := \lim_{\substack{z \rightarrow s^* \\ z \in \Omega_{\pm}}} \mathcal{C}_{\Sigma}f(z).$$

For $s^* \in \Sigma$, define also the **Cauchy principal value integral**:

$$\int_{\Sigma} \frac{f(s)}{s-s^*} ds := \lim_{\delta \downarrow 0} \int_{\Sigma \setminus B(s^*, \delta)} \frac{f(s)}{s-s^*} ds,$$

where $B(s^*, \delta) = \{z : |z - s^*| \leq \delta\}$.

Boundary values of Cauchy integrals, I

Sokhotski-Plemelj formulas

Let Σ be a smooth arc from a to b and let $f \in C^{0,\alpha}(\Sigma)$. Then for $s^* \in \Sigma \setminus \{a, b\}$

$$(\mathcal{C}_{\Sigma}^{+} f)(s^*) = \frac{1}{2} f(s^*) + \frac{1}{2\pi i} \int_{\Sigma} \frac{f(s)}{s - s^*} ds,$$

$$(\mathcal{C}_{\Sigma}^{-} f)(s^*) = -\frac{1}{2} f(s^*) + \frac{1}{2\pi i} \int_{\Sigma} \frac{f(s)}{s - s^*} ds,$$

Notice the local character of these formulas: f has to be “smooth” only in a vicinity of s^* .

On the other hand, the global result holds true:

Cauchy integrals as operators

If Σ is smooth, bounded, and closed, then the Cauchy integral operators $\mathcal{C}_{\Sigma}^{\pm}$ are bounded linear operators on $C^{0,\alpha}(\Sigma)$.

Boundary values of Cauchy integrals, II

Sokhotski-Plemelj formulas: **proof**.

(i) Let $f(s) \equiv 1$. Then $f_{\Sigma} \frac{ds}{s-s^*} = i\pi$.

Indeed, let $l := (s_1, s_2) \subset \Sigma$ s.t. $s^* \in l$ and $|s_1 - s^*| = \delta$, $|s_2 - s^*| = \delta$. Let (a, b) be a part of Σ (arc) s.t. $l \subset (a, b)$. Let $\log(z - s^*)$ be defined with a cut from s^* to ∞ outside Σ . Then

$$\int_{(a,b) \setminus l} \frac{ds}{s-s^*} = \log(s-s^*)|_a^{s_1} + \log(s-s^*)|_{s_2}^b = \log \frac{b-s^*}{a-s^*} + \log \frac{s_1-s^*}{s_2-s^*}.$$

Second term as $\delta \rightarrow 0$:

$$\log \left| \frac{s_1-s^*}{s_2-s^*} \right| + i(\arg(s_1-s^*) - \arg(s_2-s^*)) \rightarrow \log 1 + i\pi = i\pi.$$

Setting $b = a$, the **first term vanishes**.

Boundary values of Cauchy integrals, III

(ii) For arbitrary smooth $f(s)$, introduce $\phi(s^*) := \frac{1}{2\pi i} \int_{\Sigma} \frac{f(s) - f(s^*)}{s - s^*} ds$

(non-singular integral). Then

$$\frac{1}{2\pi i} \int_{\Sigma} \frac{f(s)}{s - s^*} ds = \frac{1}{2\pi i} \int_{\Sigma} \frac{f(s) - f(s^*)}{s - s^*} ds + \frac{f(s^*)}{2\pi i} \int_{\Sigma} \frac{ds}{s - s^*} = \phi(s^*) + \frac{f(s^*)}{2}.$$

(iii) Let $z \in \mathbb{C} \setminus \Sigma$. Then

$$\mathcal{C}_{\Sigma} f(z) = \frac{1}{2\pi i} \int_{\Sigma} \frac{f(s) - f(s^*)}{s - z} ds + \frac{f(s^*)}{2\pi i} \int_{\Sigma} \frac{ds}{s - z}.$$

First term: bounded for z near s^* ; **has limit $\phi(s^*)$ as $z \rightarrow s^*$** along any path!

Second term – by Cauchy Theorem: $\frac{1}{2\pi i} \int_{\Sigma} \frac{ds}{s - z} = 1$ for $z \in \Omega_+$ (inside Σ) and

$\frac{1}{2\pi i} \int_{\Sigma} \frac{ds}{s - z} = 0$ for $z \in \Omega_-$. Then

$$(\mathcal{C}_{\Sigma}^+ f)(s^*) = \phi(s^*) + f(s^*) = \frac{1}{2\pi i} \int_{\Sigma} \frac{f(s)}{s - s^*} ds - \frac{f(s^*)}{2} + f(s^*),$$

$$(\mathcal{C}_{\Sigma}^- f)(s^*) = \phi(s^*) + 0 = \frac{1}{2\pi i} \int_{\Sigma} \frac{f(s)}{s - s^*} ds - \frac{f(s^*)}{2}.$$

Cauchy integrals near endpoints

Let Σ be a smooth arc from a to b and let $f \in C^{0,\alpha}(\Sigma)$. The following holds for any endpoint $c = a$ or b :

- for z near c , $z \in \mathbb{C} \setminus \Sigma$,

$$(\mathcal{C}_{\Sigma}f)(z) = \sigma \frac{f(c)}{2\pi i} \log \frac{1}{z-c} + F_0(z; \sigma),$$

- for s near c , $s \in \Sigma$,

$$(\mathcal{C}_{\Sigma}^{\pm}f)(s) = \sigma \frac{f(c)}{2\pi i} \log \frac{1}{|z-c|} + H_0(s; \sigma),$$

where $\sigma = -1$ if $c = b$ and $\sigma = 1$ if $c = a$; F_0 and H_0 both tend to definite limits as the argument approaches c .

Additive RH problem: solution

If Σ is smooth, bounded, and closed, then the RH problem $\phi_+(s) - \phi_-(s) = f(s)$, $s \in \Sigma$, $f \in C^{0,\alpha}(\Sigma)$; $\phi(\infty) = 0$ is solved by the Cauchy integral:

$$\phi(z) = \frac{1}{2\pi i} \int_{\Sigma} \frac{f(s)}{s-z} ds, \quad z \in \mathbb{C} \setminus \Sigma.$$

- Existence: by construction.
- Uniqueness: Assume $\phi(z)$, $\tilde{\phi}(z)$ are solutions. Define $D(z) := \phi(z) - \tilde{\phi}(z)$. Then $D_+(z) = D_-(z)$ on $\Sigma \implies D(z)$ is entire function. But $D(\infty) = 0 \implies$ by Liouville's theorem, $D(z) \equiv 0$.
- Open contour (arc): same formula; but the solution has logarithmic singularities at the endpoints.
- Unbounded contour (e.g., $\Sigma = \mathbb{R}$): take care of decay of f at ∞ ($|\phi(s)| < \frac{C}{|s|^\rho}$ with $\rho > 0$); same formula.

Multiplicative RH problem, I

Scalar RH problem (closed contour)

Given Σ and $g \in C^{0,\alpha}(\Sigma)$, $g(s) \neq 0$, find ϕ s.t.

$$\phi_+(s) = \phi_-(s)g(s), \quad s \in \Sigma; \quad \phi(\infty) = 1.$$

Idea: solve via the logarithm, reducing to additive problem.

- If $\log g(s)$ is well-defined, Hölder continuous, then

$$\log \phi_+(s) = \log \phi_-(s) + \log g(s) \Rightarrow \phi(z) = \exp \left\{ \frac{1}{2\pi i} \int_{\Sigma} \frac{\log g(s)}{s-z} ds \right\}$$

- This works, if $\text{ind}_{\Sigma} g(s) = 0$, where the **index** of g w.r.t. Σ is defined by

$$\kappa = \text{ind}_{\Sigma} g(s) := \frac{1}{2\pi} [\arg g(s)]_{\Sigma} = \frac{1}{2\pi} [\log g(s)]_{\Sigma} = \frac{1}{2\pi} \int_{\Sigma} d \log g(s).$$

- Uniqueness of solution with $\phi \neq 0$: again by Liouville's theorem (applied to $\phi(z)/\tilde{\phi}(z)$).

Multiplicative RH problem, II

What to do if $\text{ind}_\Sigma g(s) \neq 0$?

Let $0 \in \Omega_+$ (w.l.g.); then $\text{ind}_\Sigma s^{-\kappa} g(s) = 0$. Thus we can (uniquely) solve the problem

$$\psi_+(s) = \psi_-(s) s^{-\kappa} g(s), \quad s \in \Sigma; \quad \psi(\infty) = 1$$

as above: $\psi(z) = \exp \left\{ \frac{1}{2\pi i} \int_\Sigma \frac{\log s^{-\kappa} g(s)}{s-z} ds \right\}$.

- if $\kappa > 0$, then

$$\phi(z) = \begin{cases} P(z)\psi(z), & z \in \Omega_+, \\ P(z)\psi(z)z^{-\kappa}, & z \in \Omega_-, \end{cases}$$

where $P(z)$ is any monic polynomial of degree κ . If $P(z) \equiv 1$, then $\phi(z)$ is called the **fundamental solution**.

- if $\kappa < 0$, then there are no nontrivial **bounded** solutions: the only sectionally analytic function that satisfy the jump condition and is bounded at ∞ is the zero solution.

Multiplicative RH problem, III

Inhomogeneous RH problem

Given Σ and $g, f \in C^{0,\alpha}(\Sigma)$, $g(s) \neq 0$, find ϕ s.t.

$$\phi_+(s) = \phi_-(s)g(s) + f(s), \quad s \in \Sigma; \quad \phi(\infty) = 0.$$

- First, find the fundamental solution $\nu(z)$ of the corresponding **homogeneous problem** $\nu_+(s) = \nu_-(s)g(s)$; can be solved as above.
- The RHP reduces to additive problem:

$$\frac{\phi_+(s)}{\nu_+(s)} = \frac{\phi_-(s)}{\nu_-(s)} + \frac{f(s)}{\nu_+(s)}; \quad \frac{\phi(z)}{\nu(z)} = O(z^{\kappa-1}), \quad z \rightarrow \infty.$$

- if $\kappa \geq 0$, then $\phi(z)/\nu(z)$ is given by the Cauchy integral. It follows that

$$\phi(z) = \nu(z) \left(\frac{1}{2\pi i} \int_{\Sigma} \frac{f(s)}{\nu_+(s)(s-z)} ds + P(z) \right),$$

where $P(z)$ is any polynomial of degree $< \kappa$.

- if $\kappa < 0$, then in order to have

$$\frac{\phi(z)}{\nu(z)} = \frac{1}{2\pi i} \int_{\Sigma} \frac{f(s)}{\nu_+(s)(s-z)} ds = O(z^{\kappa-1}),$$

we need additional **moment conditions** to be satisfied:

$$\int_{\Sigma} s^n \frac{f(s)}{\nu_+(s)} ds = 0, \quad n = 0, \dots, \kappa - 1.$$

RH problem on open contours

- additive problem: $\phi_+(s) - \phi_-(s) = f(s)$, $s \in (a, b)$. As above, solved by Cauchy integral $\phi(z) = \frac{1}{2\pi i} \int_{(a,b)} \frac{f(s)}{s-z} ds$; unique solution under *local integrability cond.* (with log singularities at a, b).
- multiplicative problem: $\phi_+(s) = \phi_-(s)g(s)$, $s \in (a, b)$. The Cauchy integral solution $\psi(z) = \exp \left\{ \frac{1}{2\pi i} \int_{(a,b)} \frac{\log g(s)}{s-z} ds \right\}$ has power-type singularities (zeros):

$$\psi(z) = (z - b)^{\zeta_b} H_b(z), \quad \psi(z) = (z - a)^{\zeta_a} H_a(z),$$

where $\zeta_b = -\frac{\log g(b)}{2\pi i}$ and $\zeta_a = \frac{\log g(a)}{2\pi i}$; $H_a(z)$, $H_b(z)$ have definite limits as $z \rightarrow a, b$. Let $\zeta_c = \lambda_c + i\mu_c$ and let $n_c \in \mathbb{Z}$ be such that $-1 < n_c + \lambda_c < 1$, $c = a, b$. Then

$$\phi(z) = (z - a)^{n_a} (z - b)^{n_b} \psi(z)$$

is a locally integrable solution.

RH problem with constant multiplicative jump

$\phi_+(s) = \alpha\phi_-(s)$, $s \in (a, b)$; ϕ of finite degree at ∞ .

- Set $\alpha = |\alpha|e^{i\theta}$ so that $\log \alpha = \log |\alpha| + i\theta$. A solution is given by Cauchy integral

$$\psi(z) = \exp \left\{ \frac{\log |\alpha| + i\theta}{2\pi i} \log \left(\frac{z-b}{z-a} \right) \right\} = \left(\frac{z-b}{z-a} \right)^{-i \log |\alpha| / (2\pi) + \theta / (2\pi)}.$$

- The locally integrable solution is

$$\phi(z) = (z-a)^{n_a} (z-b)^{n_b} \psi(z),$$

where $-1 < n_a - \theta/(2\pi) < 1$, $-1 < n_b + \theta/(2\pi) < 1$.

A general form of jump conditions:

$$\Phi_+(s) = \Phi_-(s)G(s) + F(s), \quad s \in \Sigma,$$

where $\Phi : \mathbb{C} \setminus \Sigma \mapsto \mathbb{C}^{m \times n}$, $G : \Sigma \mapsto \mathbb{C}^{n \times n}$, and $F : \Sigma \mapsto \mathbb{C}^{m \times n}$.

Most often in applications: $m = n = 2$.

- Unlike scalar RHP, matrix RHP cannot, in general, be solved in closed form.
- The general theory involves the analysis of singular integral operators.
- But: three types of RHP can be solved in closed form (by reducing matrix RHP to a sequence of scalar RHP).

Diagonal Riemann-Hilbert problems

$$\Phi_+(s) = \Phi_-(s)D(s), \quad s \in \Sigma; \quad \Phi(\infty) = I,$$

where $\Phi : \mathbb{C} \setminus \Sigma \mapsto \mathbb{C}^{n \times n}$, $D \in C^{0,\alpha}(\Sigma)$,
 $D(s) = \text{diag}(d_1(s), \dots, d_n(s))$ with $\det D(s) \neq 0$;
 $\log d_i(s) \in C^{0,\alpha}(\Sigma)$

- The problem decomposes into n scalar RHP:

$$\phi_{i+}(s) = \phi_{i-}(s)d_i(s), \quad s \in \Sigma; \quad \phi_i(\infty) = 1, \quad i = 1, \dots, n.$$

- Each of these has a solution $\phi_i(z) = \exp \left\{ \frac{1}{2\pi i} \int_{\Sigma} \frac{\log d_i(s)}{s-z} ds \right\}$.
- A solution of RHP: $\Phi(z) = \text{diag}(\phi_1(z), \dots, \phi_n(z))$.
- If Σ is smooth, bounded, and closed, then the solution is unique. Indeed, if Ψ is another solution, consider $\Psi\Phi^{-1}$:

$$\Psi_+(s)\Phi_+^{-1}(s) = \Psi_-(s)D(s)D^{-1}(s)\Phi_-^{-1}(s) = \Psi_-(s)\Phi_-^{-1}(s);$$

thus $\Psi\Phi^{-1}$ is entire. Also, $\Psi(\infty)\Phi^{-1}(\infty) = I$; then, by Liouville's theorem, $\Psi(z)\Phi^{-1}(z) \equiv I$.

$$\Phi_+(s) = \Phi_-(s)A, \quad s \in \Sigma; \quad \Phi \text{ of finite degree at } \infty,$$

where A is invertible, diagonalizable: $A = U\Lambda U^{-1}$ with $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$.

- Reduces to a diagonal RHP for $D(z) := U^{-1}\Phi(z)U$.

Example

$$\Phi_+(s) = \Phi_-(s) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad s \in (a, b); \quad \Phi(\infty) = I$$

- First, diagonalize:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = U \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} U^{-1}, \quad U = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$$

- Solve auxiliary scalar RHP:

$$h_{1+}(s) = ih_{1-}(s), \quad h_{2+}(s) = -ih_{2-}(s).$$

$$\text{Solutions: } h_1(z) = \exp \left\{ \frac{1}{2\pi i} \int_a^b \frac{\log i}{s-z} ds \right\} = \left(\frac{z-b}{z-a} \right)^{1/4},$$

$$h_2(z) = \exp \left\{ \frac{1}{2\pi i} \int_a^b \frac{\log(-i)}{s-z} ds \right\} = \left(\frac{z-b}{z-a} \right)^{-1/4} \equiv h_1^{-1}(z).$$

-

$$\Phi(z) = U \begin{pmatrix} h_1(z) & 0 \\ 0 & h_1^{-1}(z) \end{pmatrix} U^{-1} = \begin{pmatrix} \frac{h_1(z)+h_1^{-1}(z)}{2} & \frac{i(h_1(z)-h_1^{-1}(z))}{2} \\ -\frac{i(h_1(z)-h_1^{-1}(z))}{2} & \frac{h_1(z)+h_1^{-1}(z)}{2} \end{pmatrix}$$

Triangular Riemann-Hilbert problems, I

$$\Phi_+(s) = \Phi_-(s)U(s), \quad s \in \Sigma; \quad \Phi(\infty) = I,$$

where $U(s)$ is upper triangular: $U(s) = (U_{i,j}(s))_{1 \leq i, j \leq n}$,
 $U_{i,j}(s) = 0$ if $i < j$.

Assume $\text{ind}_\Sigma U_{i,i}(s) = 0$, $i = 1, \dots, n$.

- Reduces to solving successive scalar RHP; each row can be found independently of other rows
- Scalar problems for the first row:

$$\Phi_{1,1+}(s) = \Phi_{1,1-}(s)U_{1,1}(s), \quad \Phi_{1,1}(\infty) = 1$$

$$\Phi_{1,2+}(s) = \Phi_{1,2-}(s)U_{2,2}(s) + \Phi_{1,1-}(s)U_{1,2}(s), \quad \Phi_{1,2}(\infty) = 0$$

...

- Here the first equ. is a homogeneous RHP for $\Phi_{1,1}$; having it solved, the second equation is a non-homogeneous RHP for $\Phi_{1,2}$, etc.

Example

$$\Phi_+(s) = \Phi_-(s) \begin{pmatrix} 1 & f(s) \\ 0 & 1 \end{pmatrix}, \quad s \in \Sigma; \quad \Phi(\infty) = I$$

- Scalar problems for the first row:

$$\Phi_{1,1+}(s) = \Phi_{1,1-}(s), \quad \Phi_{1,1}(\infty) = 1$$

$$\Phi_{1,2+}(s) = \Phi_{1,2-}(s) + \Phi_{1,1-}(s)f(s), \quad \Phi_{1,2}(\infty) = 0$$

- $\Phi_{1,1}(z) \equiv 1$; then the 2-nd equation above reduces to

$$\Phi_{1,2+}(s) = \Phi_{1,2-}(s) + f(s), \quad \Phi_{1,2}(\infty) = 0,$$

whose solution is $\Phi_{1,2}(z) = \frac{1}{2\pi i} \int_{\Sigma} \frac{f(s)}{s-z} ds$.

- For the second row:

$$\Phi_{2,1+}(s) = \Phi_{2,1-}(s), \quad \Phi_{2,1}(\infty) = 0$$

$$\Phi_{2,2+}(s) = \Phi_{2,2-}(s), \quad \Phi_{2,2}(\infty) = 1$$

The solutions are: $\Phi_{2,1}(z) \equiv 0$ and $\Phi_{2,2}(z) \equiv 1$.

- Finally,

$$\Phi(z) = \begin{pmatrix} 1 & \frac{1}{2\pi i} \int_{\Sigma} \frac{f(s)}{s-z} ds \\ 0 & 1 \end{pmatrix}$$