Syllabus

I. Course Name: Probability Theory

II. Course description and objective

Content: 1) Probability spaces, random variables, and other fundamental concepts,
2) Weak and strong laws of large numbers, convergence of random series,
3) Characteristic functions, central limit theorem,
4) Some random process.

III. Compulsory

IV. Bachelor Program, 3rd Term, 64 Hours, 8 Credits

V. Course content

`Section 1.` Elements of combinatorial analysis.
Ordered samples.
Subsets and partitions.
Bose-Einstein and Fermi-Dirac statistics.
The hypergeometric and geometric distribution.
Binomial coefficients.
Stirling's formula.

`Section 2.` Axioms of probability.
Experiments with chance.
Outcomes and events. Probabilities.
Axioms of probability.
Probability spaces.
Discrete sample space.
Sample spaces having equally likely outcomes.
The continuity of probability measures.

`Section 3.` Conditional probability. Stochastic independence.
Conditional probability.
Bayes' formula.
Basic properties of conditional expectation.
Independent events.
Product spaces. Independent trials.

`Section 4.` The binomial and the Poisson distributions.
Bernoulli trials.
The binomial distribution.
The Bernoulli law of large numbers.
The Poisson approximation.
The Poisson distribution.
Waiting times. The negative binomial distribution.

`Section 5.` The normal approximation to the binomial distribution.

The normal distribution.
The De Moivre-Laplace Limit theorem.
Examples

`Section 6.` Random Variables. Expectation.
Random Variables.
Expectations.
Variance. Variance of a Sum.
Covariance. The correlation coefficient.
Absolutely continuous and singular distributions.
Lebesgue representation theorem.
Independence.
Product Spaces. Sequences of Independent Variables.
Convolutions.
Chebyshev's Inequality.
Kolmogorov's Inequality.

`Section 7.` Some important distributions.
The binomial random variable.
The Poisson random variable.
The geometric random variable.
The uniform random variable.
The normal random variable.
The exponential random variable.
The Cauchy random variable.

`Section 8.` Multivariate distributions and independence.
Bivariate discrete distributions.
Expectation in the multivariate case.
Independence of discrete random variables.
Random vectors and independence.
Joint density functions.
Marginal density functions and independence.
Sums of absolutely continuous random variables.
Conditional density functions.
Expectations of continuous random variables.
Conditional expectation and the bivariate normal distribution.

`Section 9.` Convergence concepts.
Almost sure convergence.
Borel-Cantelli lemma.
Convergence in probability.
Convergence in distribution (weak convergence).

`Section 10.` The weak law of large numbers. Applications in Analysis.
Chebyshev's inequality.
The weak law of large numbers. Theorems of Chebyshev, Khinchin, Bernshtein, Kolmogorov.
Bernstein polynomials.

Law of the iterated logarithm.

`Section 11.` The strong law of large numbers.
Borel-Kantelli lemma.
The strong laws of large numbers of Borel and Kantelli.
Normal numbers in cense of Borel.

`Section 12.` Generating functions.
The generating function metode.
Branching processes.
A model for population growth.

`Section 13.` Characteristic functions.
Definition and basic properties of characteristic functions.
Mixtures and convolutions.
Regularity properties.
Inversion.
Uniqueness.
Convergence theorems.
Simple applications.
Two characterizations of the normal distribution: Cramér theorem, Bernstein-Kac theorem.
Infinite divisibility.
Positive definite functions.

`Section 14.` Central limit theorem.
The normal approximation of the binomial distribution.
Lindeberg-Levy theorem.
Liapunov theorem.
The Lindeberg conditions.
Feller theorem.
Error estimation. Berry-Esséen theorem.

`Section 15.` Branching processes.
Random processes.
A model for population growth.
The generating-function method.
The probability of extinction.

`Section 16.` Random walks.
Zero-or-one law.
The arc-sine law.
One-dimensional random walks.
Transition probabilities.
The Gambler's Ruin problem.
Application of Fourier methods to random walks. Theorem of Chung Kai-Lai and Fuchs.

`Section 17.` Markov chains.
Markov property.
Higher transition probabilities.

Classification of states.
Irreducible chains.
Invariant distributions.
Ergodic theorems for Markov chains.

`Section 18.` Random processes in continuous time.
Poisson processes.
Inter-arrival times and the exponential distribution.
Population growth and the simple birth process.
Birth and death processes.

VI. Pre-taken courses

Mathematical Analysis, Measure Theory and Lebesgue Integral, Complex Analysis, Discrete Mathematics

VII. Form of the final test: test (two-level evaluation scale)

VIII. Teaching materials and reference books

1. W. Feller. *An introduction to probability theory and its applications*, vol. 1 (1968)
2. W. Feller. *An introduction to probability theory and its applications*, vol. 2 (1971)
3. Chung Kai-Lai. *A Course in Probability Theory*, Third Edition-Academic Press (2000)
4. Yu. V. Prokhorov, L. S. Ponomarenko. *Lectures on Probability Theory and Mathematical Statistics*, Moscow University Press, 2012 (in Russian).

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