Operator Theory
Advances and Applications
280

Daniel Alpay
Bernd Fritzsche
Bernd Kirstein
Editors

Complex Function Theory, Operator Theory, Schur Analysis and Systems Theory

A Volume in Honor of V.E. Katsnelson

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Contents

Editorial Introduction

Daniel Alpay, Bernd Fritzsche, and Bernd Kirstein

Einmal werd ich die Wahrheit sagen – das meint man, aber die Lüge ist ein Egel, sie hat die Wahrheit ausgesaugt.[1]

Max Frisch, Andorra [17, p. 35]

This volume is dedicated to Victor Emmanuilovich Katsnelson on the occasion of his seventy fifth birthday. The volume contains biographical material written by former students, colleagues and friends, and eleven refereed papers written by experts in their fields. The papers in the biographical part give a picture of the personality and achievements of Victor Emmanuilovich. One can also find relevant information inside the research papers themselves; see for instance the first footnote in the paper of A. Kheifets and P. Yuditskii, and the first section in the paper of B. Fritzsche, B. Kirstein, and C. Mädler.

The life and work of Victor can be divided into three cities, Kharkiv, Leipzig, Rehovot, and his scientific (and personal) impact in each of these cities, and beyond, is very important. We mention in particular his huge influence in Schur analysis. The paper [18] is of special interest. There, together with A. Kheifets and P. Yuditskii, he develops one of the most powerful and versatile method of interpolation for Schur functions, namely the Abstract Interpolation Problem setting. The list of publications in the current MathScinet listing shows how selective Victor is in his

[1] *One day I shall tell the truth-that's what one says; but a lie is a leech, it sucks the truth dry.* Translation by Michael Bullock, see [16, p. 199].

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choice of research topics, not rushing after fashionable topics, and, to quote Harry Dym in his contribution in this volume, *not drilling where the wood is soft.* Too many accomplished mathematicians would stay in their zone of comfort, past the first years after the thesis. This is definitely not the case of Victor.

As is illustrated in the sentence of Max Frisch in exergue of this introduction, it requires character and willpower to go against the crowd and tell what one thinks. Victor never hesitates to tell what he thinks, and his views on mathematical research, and is ready to pay the price for his sincerity and frankness. His criticism can be caustic (as the first named editor has experienced, in particular after a talk in Leipzig in 2000), but they force the opponent to think, to reconsider one's point of view on a domain, or a topic.

Three names should be mentioned in particular, in connection with Victor. Boris Levin, who supervised his PhD thesis, Vladimir Potapov, whose work (and in particular the fundamental inequality method) was made well-known in the west in great part thanks to Victor, and Moshe Livsic, the founder of the characteristic operator function [12, 20–23]. From 1965 to 1990 V.E. witnessed a golden era of mathematics in Kharkiv, marked by the work of N.I. Akhiezer, I.M. Glazman, A.V. Pogorelov, V.A. Marchenko, Yu.I. Lyubich, V.P. Petrenko, L.I. Ronkin, V.S. Azarin, L.A. Pastur, and F.S. Rofe-Beketov, amongst others. The influence of this diverse set of mathematicians is one main cause for V.E.'s impressive universality and his deep understanding for the links between only seemingly distant subdomains of mathematics. In many discussions with the third named editor, V.E. expressed his high esteem for V.I. Matsaev, whom he saw as a special genius.

The first named editor first met Victor at the Weizmann Institute, and recall driving to Tel-Aviv with Victor, where he gave a talk at Gohberg's seminar (left versus right factorizations of Blaschke products). The influence of Victor on the first editor is multifaceted but let us mention the paper [14]; later, using realization methods [2], one could "see" the underlying formulas in [14]. Finding them from scratch is a real *tour de force.*

The first named editor was also very fortunate to have Vladimir Bolotnikov as a doctoral student (Vladimir had begun to work with Victor in the Ukraine prior to his arrival in Israel) and Dan Volok as a postdoctoral fellow after Dan finished his thesis under Victor's supervision. The collaborations with Vladimir and Dan continue to this very day, and comprise, all-together, more than fifty-five papers, of which we would like to mention [3–6] and [7–10] (Fig. 1).

The second and third named editors first met V.E. during a research stay at the Kharkiv University in the spring of 1988. Since then we have established close scientific and personal exchange of ideas with him. The scientific careers of both of us were shaped in an essential way by this exchange. He is a universal erudite mathematician, who not only is an expert on the classical works on Schur analysis published in German (see [19]), but also able to identify striking links between different problems, which significantly guided BF and BK's choice of research problems to work on. At the age of 46, his visit to Leipzig in the fall of 1989 was his first journey abroad. Since this year, V.E. spent a lot of time in Leipzig,

Fig. 1 Picture taken at the Toeplitz Lectures a joint German–Israeli workshop on linear one-dimensional singular integral equations in March 1995 in Tel-Aviv. From left to right: B. Fritzsche, B. Kirstein, V.E., A. Böttcher, I. Gohberg

if added together in total about 3 years. V.E. spent almost the whole year of 1991 in Leipzig, which thus makes up for a third. During that year, V.E. acquired exceptional language skills, which enabled him to start conversations on varying topics with total strangers without much effort when he was on his extended walks through Leipzig's parks. During the second half of 1991, he made the consequential decision to accept a professorship at the Institute for Theoretical Mathematics at the Weizmann Institute associated with the emigration to Israel. It was a tough decision to leave his beloved hometown Kharkiv and its university, which has determined his academic career up to this day. Shortly after his arrival at the Weizmann Institute, he started to establish long-term collaborations between our group on Schur analysis and several Israeli mathematicians working on this field (i.e. H. Dym, I. Gohberg, M.S. Livsic, DA). Two highlights of this collaboration were a conference in August 1994 in Leipzig in honor of the 80th birthday of V.P. Potapov and a seminar in November 1995 in Leipzig on Schur analysis in honor of the 100th birthday of Rolf Nevanlinna.

During V.E.'s first stay in Leipzig in 1989, we organized an excursion to the Wilhelm Ostwald memorial site in Großbothen (a small town to the southeast of Leipzig) for all the participants of the INTSEM Schur analysis. Wilhelm Ostwald (1853–1932) quit university life even before he received the Nobel Prize in chemistry and continued his research until his death as an independent researcher. To this day the museum is worth a visit as his laboratory contains many self-built appliances on display, a comprehensive library and his exchange of letters with eminent figures of his days. Every year the museum attracts many visitors. During V.E.'s visit, he surprised us with his profound knowledge about Ostwald's life and his views on several topics in science. V.E. is a supporter of Ostwald's classification of scientists as classics or romantics, whereby he considered himself as a prototypical romanticist. During his yearlong stay in Leipzig in 1991, he often went to Großbothen and established a close friendship to Gretel Brauer (1918–2008), Wilhelm Ostwald's youngest granddaughter and director of the memorial site. Since 1991 he is an active member of the Wilhelm-Ostwald-Society, which is a charity that advocates for the friends and supporters of Ostwald's scientific heritage.

Many times he co-organized mathematical conferences held on the grounds of the memorial site surrounded by the quiet atmosphere of the Wilhelm-Ostwald-Park in Großbothen. V.E. is also an ardent admirer of the work of the German–American social psychologist and psychoanalyst Erich Fromm (1900–1980). First and foremost his book "Haben oder Sein" (engl. title "*To have or to be*" [15]) made a big impression on him. He considered himself as a typical representative of the "Sein-Mensch" (people of being rather than people of having). To continue this thoughts, V.E. always views mathematics as an art, which should be able to evolve in an environment characterized by the absence of any economic constraints that urge for near-term applicability of theories.

Another topic of V.E.'s interest is sampling theory (see [11]). For this reason he tried to attend important conferences in the field. In 1993, he registered for a conference on sampling theory in Cairo organized by Ahmed I. Zayed. In the meantime V.E. had become a citizen of the state of Israel, which led to the refusal of his entry to Egypt. His counter reaction was to stop shaving his beard and he started to grow a beard. He will only be willing to shave again, when he was able to enter Egypt. To the best of our knowledge, V.E. wears his impressive beard to this very day (Fig. 2).

Fig. 2 December 1994 in Leipzig, from left to right: W. Schempp, V.E., P.R. Masani

A lasting impact on his future scientific career had a conference at the Technion in Haifa in the spring of 1994, where he met Walter Schempp from Siegen, a renowned specialist on harmonic analysis of the Heisenberg group. Schempp's publications on the mathematical foundations of magnetic resonance imaging received special attention. V.E. informed Walter Schempp about a meeting on the occasion of Norbert Wiener's 100th birthday in December 1994 in Großbothen near Leipzig, and asked us to invite him. The special highlight of this meeting was the participation of Wiener's co-author and biograph P.R. Masani (1919–1999). This marked the beginning of a collaboration between Walter Schempp, V.E. and the Schur analysis group in Leipzig around BF and BK which lasts to the present day. Concerning the acquaintance with Walter Schempp later V.E. often recounted the surprising insight of this encounter with a whimsical smile, the optimal route from Siegen to Leipzig can pass Haifa at times.

The conference in Haifa had another surprise for V.E. in store. For the first and only time he met Paul Erdős. The circumstances of this encounter deserve some explanation. Paul Erdős had lost orientation on the Technion campus, when V.E. passed by. In his desperation, Erdős asked V.E. for help—in German. V.E. conciliated him, and explained him the way to the conference rooms—as well in German.

Over the decades, V.E. got to know Leipzig very well. Unfortunately, his health condition repeatedly did not allow him to see the new university building at the Augustusplatz.

We received most valuable advise from V.E. when we were working on [13]. In this paper we study the truncated matricial moment problem on a finite closed interval by using Potapov's FMI method. Therein we are lead to a system of two coupled fundamental matrix inequalities. The effective coupling is brought about by an algebraic identity between two Block Hankel matrices (see Proposition 2.2). Finding this identity was a pivotal step. V.E. had seen similar identities appearing in related problems and thus knew exactly that this fact is meriting particular attention of the reader.

The research papers can be divided, in a somewhat arbitrary manner, in the following overlapping categories:

Function Theory Most, if not all, the papers in the volume involve function theory, but in the following three papers, it is the major topic and tool. In *On a Blaschke-type condition for subharmonic functions with two sets of singularities on the boundary*, **S. Favorov** and **L. Golinskii** prove Blaschke-type conditions for the Riesz measure associated to certain subharmonic functions. A two-dimensional version of the Layer Cake Representation (LCR) theorem from measure theory plays an important role in the arguments. Taylor domination consists in finding bounded for the Taylor coefficients of an analytic function in terms of a number of the first terms. De Branges's theorem for univalent functions (i. e. the solution of the Bieberbach conjecture) is a striking example of this domain. In *Exponential Taylor domination*, **O. Friedland**, **G. Goldman** and **Y. Yomdin** use the Borel transform to study the topic, comparing the valence of a function analytic in the open unit disk, to the valence of its Borel transform. H. Widom gave a characterization of Hardy spaces on a Riemann surface. In the paper *Martin Functions of Fuchsian Groups and Character Automorphic Subspaces of the Hardy Space in the Upper Half Plane*, **A. Kheifets** and **P. Yuditskii** consider the related problem of characterizing subspaces of the Hardy space of the open upper half-plane consisting of functions character automorphic with respect to a discrete subgroup of $SL_2(\mathbb{R})$.

Schur Analysis, Moment Problems and Related Topics Unitary coupling, introduced by V.M. Adamyan, D.Z. Arov in [1] are associated to operator-valued measurable functions contractive in the open unit disk $\mathbb{D}$ (as opposed to analytic contractive functions in $\mathbb{D}$, corresponding to unitary colligations). In *Extensions and defect functions of contractive measurable operator-valued functions*, **S.S. Boiko** and **V.K. Dubovoy** study the regular extensions of unitary couplings. The next two papers associated to Schur analysis deal with matricial moment problems. **Yu.M. Dyukarev** considers the Hamburger moment problem in *On conditions for complete indeterminacy of the matricial Hamburger moment problem*, while **B. Fritzsche**, **B. Kirstein**, and **C. Mädler** present a precise study of the set of solutions of the truncated moment problems for Stieltjes functions in terms of two special solutions, which have underlying measures consisting of finite sets of jumps in *A closer look at the solution of the truncated matricial moment problem*.

Extensions of Linear Operators and Linear Relations Krein's formula describes all compressed resolvents of self-adjoint extensions of a given symmetric relation with equal (and possibly infinite) index in terms of a linear fractional transformation with parameter a $d \times d$ Nevanlinna function. In *Self-adjoint extensions of a symmetric linear relation with finite defect: compressions and Straus subspaces*, **A. Dijksma** and **H. Langer** study relationships between the parameter, the compression of the extension and connections with the Strauss extension of the given symmetric relation. Sectorial relations are the topic of the paper *On a class of sectorial relations and the associated closed forms*, by **S. Hassi** and **H. de Snoo**. In particular the authors give an expression for the extremal maximal sectorial extensions of the sum of sectorial relations, and characterize when the form sum extension is extremal. Finally, in the paper *Spectral decompositions of selfadjoint relations in Pontryagin spaces and factorizations of generalized Nevanlinna functions* by **S. Hassi** and **H.L. Wietsma**, the authors consider the important setting of Pontryagin spaces. One then has to replace Nevanlinna functions by their generalizations, defined in terms of the number of negative squares of an associated kernel.

Non Commutative Analysis In the paper *Interpolation by contractive multipliers between Fock spaces*, **J. Ball** and **Vladimir Bolotnikov** consider interpolation problems for contractive multipliers between noncommutative reproducing kernel Hilbert spaces. They develop in particular a noncommutative multivariable analogue of the Abstract Interpolation Problem [18]. In classical probability theory the Gaussian distribution plays a central role, while in free probability theory the semicircular law is the key actor. In the paper *Free-homomorphic Relations Induced by Certain Free Semicircular Families*, **I. Cho** and **P. Jorgensen** present a new construction of semicircular elements, based on a new analysis on the p-adic number fields $\mathbb{Q}_p$, for primes p, and give a novel approach to calculus of free random variables.

Acknowledgements Daniel Alpay thanks the Foster G. and Mary McGaw Professorship in Mathematical Sciences, which supported this research.

References

1. V.M. Adamyan and D.Z. Arov. On unitary couplings of semiunitary operators. *Mat. Issled*, 1(2):3–64, 1966.
2. D. Alpay, J. Ball, I. Gohberg, and L. Rodman. Interpolation in the Stieltjes class. *Linear Algebra Appl.*, 208-209:485–521, 1994.
3. D. Alpay and V. Bolotnikov. On tangential interpolation in reproducing kernel Hilbert space modules and applications. In H. Dym, B. Fritzsche, V. Katsnelson, and B. Kirstein, editors, *Topics in interpolation theory*, volume 95 of *Operator Theory: Advances and Applications*, pages 37–68. Birkhäuser Verlag, Basel, 1997.
4. D. Alpay, V. Bolotnikov, F. Colombo, and I. Sabadini. Self-mappings of the quaternionic unit ball: multiplier properties, Schwarz-Pick inequality, and Nevanlinna–Pick interpolation problem. *Indiana Univ. Math. J.*, 64:151–180, 2015.
5. D. Alpay, V. Bolotnikov, F. Colombo, and I. Sabadini. Interpolation problems for certain classes of slice hyperholomorphic functions. *Integral Equations Operator Theory*, 86(2):165–183, 2016.
6. D. Alpay, V. Bolotnikov, A. Dijksma, and H.S.V. de Snoo. On some operator colligations and associated reproducing kernel Pontryagin spaces. *J. Funct. Anal.*, 136:39–80, 1996.
7. D. Alpay, A. Dijksma, and D. Volok. Schur multipliers and de Branges-Rovnyak spaces: the multiscale case. *J. Operator Theory*, 61(1):87–118, 2009.
8. D. Alpay, P. Jorgensen, I. Lewkowicz, and D. Volok. A new realization of rational functions, with applications to linear combination interpolation, the Cuntz relations and kernel decompositions. *Complex Var. Elliptic Equ.*, 61(1):42–54, 2016.
9. D. Alpay, M. Shapiro, and D. Volok. Rational hyperholomorphic functions in R^4. *J. Funct. Anal.*, 221(1):122–149, 2005.
10. D. Alpay and D. Volok. Point evaluation and Hardy space on a homogeneous tree. *Integral Equations Operator Theory*, 53:1–22, 2005.
11. L. Bezuglaya and V. E. Katsnelson. The sampling theorem for functions with limited multi-band spectrum. *Z. Anal. Anwendungen*, 12(3):511–534, 1993.
12. M. S. Brodskiĭ and M. S. Livšic. Spectral analysis of non-self-adjoint operators and intermediate systems. *Uspehi Mat. Nauk (N.S.)*, 13(1(79)):3–85, 1958.
13. A.E. Choque Rivero, Y.M. Dyukarev, B. Fritzsche, and B. Kirstein. A truncated matricial moment problem on a finite interval. In *Interpolation, Schur functions and moment problems*, volume 165 of *Oper. Theory Adv. Appl.*, pages 121–173. Birkhäuser, Basel, 2006.
14. Yu. Dyukarev and V.E. Katsnelson. Multiplicative and additive classes of Stieltjes analytic matrix–valued functions and interpolation problems associated with them. I. *American Mathematical Society Translations*, 131:55–70, 1986.
15. E. Fromm *Haben oder Sein. Die seelischen Grundlagen einer neuen Gesellschaft*. Deutsche Verlags Anstalt, 1976.
16. Max Frisch. *Three plays*. Methuen & co Ldt 36 Essex Street London WC2, 1962.
17. Max Frisch. *Andorra*. Suhrkamp, 1975 (first published 1961).

18. V.E. Katsnelson, A. Kheifets, and P. Yuditskii. An abstract interpolation problem and the extension theory of isometric operators. In H. Dym, B. Fritzsche, V. Katsnelson, and B. Kirstein, editors, *Topics in interpolation theory*, volume 95 of *Operator Theory: Advances and Applications*, pages 283–297. Birkhäuser Verlag, Basel, 1997. Translated from: Operators in function spaces and problems in function theory, p. 83–96 (Naukova–Dumka, Kiev, 1987. Edited by V.A. Marchenko).
19. V.E. Katsnelson and B. Kirstein. 25 years of Schur analysis in Leipzig. *Complex Anal. Oper. Theory*, 5(2):325–330, 2011.
20. M. S. Livšic. On the reduction of linear non-Hermitian operator to "triangular" form. *Doklady Akad. Nauk SSSR (N.S.)*, 84:873–876, 1952.
21. M. S. Livšic. On spectral decomposition of linear nonself-adjoint operators. *Mat. Sbornik N.S.*, 34(76):145–199, 1954.
22. M.S. Livšic. Isometric operators with equal deficiency indices, quasi-unitary operators. *Mat. Sbornik N.S.*, 26(68):247–264, 1950.
23. M.S. Livšic. On a class of linear operators in Hilbert spaces. *Math. USSR-Sb.*, 61:239–262, 1946. English translation in: American mathematical society translations (2), vol. 13, p. 61–84 (1960).

Part I
Personal Recollections

Victor Comes to Rehovot

Harry Dym

To Victor on the occasion of his seventy fifth birthday with best wishes

Some recollections of Victor Katsnelson and how he came to join the Department of Mathematics of the Weizmann Institute. I knew his name because Professor Tsuyoshi Ando of Hokkaido University had very generously translated and distributed a number of papers on themes connected with the work of the Potapov school from Russian to English, and one of these was a 150 page monograph by Victor: *Methods of J-theory in continuous interpolation problems of analysis, Part I*. Consequently I was very much in favor of this visit, and managed to arrange support for a 1 month visit. The logistics turned out to be somewhat complicated. This was still in the early days of Perestroika and there were no direct flights between Ukraine and Israel. Victor had to make his way to Budapest, where a prepaid ticket was arranged for him on an El Al flight to Tel Aviv. Somehow we managed to coordinate this (though his part was much more difficult than mine) and I went to meet him at Ben Gurion airport. He arrived in the early hours of the morning carrying two linen bags. I believe one was for spare clothing and the other was for books and papers.

At that time Victor's knowledge of English was somewhat limited. Nevertheless he managed to deliver a couple of impressive lectures, at least one of which was on his work with his former students Alexander Kheifets and Peter Yuditskii on the Abstract Interpolation Problem and some of its applications. Also, in private discussions, he exhibited familiarity with a wide array of subjects, including an understanding of Louis de Branges' fundamental work on the inverse problem for 2×2 canonical systems. During that visit Victor walked around with a Russian-English dictionary attached to his belt, which he seemed to be able to thumb though at lightning speed.

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Victor returned to Kharkov and sometime thereafter applied for a permanent position at the Weizmann Institute. The application was successful, and in the latter half of 1991 he was offered a Full Professorship in the Department of Mathematics.

Victor arrived in Rehovot with his wife and (I think) one of his three daughters towards the end of February in 1992.

Victor adapted rapidly to his new environment. He thoroughly mastered latex and became a fountain of knowledge in that area. He convinced me to switch from Unix to WinEdt, for which I shall be forever grateful. He also learned to drive, with perhaps a little less proficiency than his skill in latex. But he survived Israeli traffic, which is no mean feat.

Although Victor can be very critical, he has a generous nature. He helped to support a postdoc with a sizeable chunk of the funds that had been made available to him upon joining the Institute, even though there was absolutely no benefit to him. This generosity manifested itself in other ways that led him on occasion to agree to supervise students at the PhD level who were not fully qualified and hence were unable to finish.

The decade starting in the early nineties was one of the most active in the general area of operator theory and analysis in the department. Michael Solomyak had just recently joined the department and there was an influx of talented young doctoral students (Michael Shmoish, Boris Freydin, Iosif Polterovich, Katherine Naimark, Victor Olevskii) and postdoctoral fellows (Alexander Kheifets, Michael Gekhtman, Alexander Stolin, Ilia Videnskii, Michael Shapiro, Nikolai Bykov, Michael Simbirskii) from the former Soviet Union. Peter Yuditskii visited and a number of established professors from the Odessa circle also visited. (Lev Sakhnovich, Vadim Adamjan, Israel Kac for short periods and Damir Arov, who became a regular visitor.)[1]

Victor has an impressively wide range of interests. Although many of his publications focus on the interplay between complex analysis and operator theory, he has also made serious contributions to other areas, including Fuchsian systems of linear differential equations (with his doctoral student Dan Volok), many aspects of time frequency analysis (partially with Ronny Haim Machluf), the BMV conjecture and a probabilistic analysis of Schur parameters, to mention just a few.

Victor does not *drill where the wood is soft*. Here are two excerpts from Math Reviews:

The review of his 2002 paper *A generic Schur function is an inner one (2002)*, in which the Schur parameters of nonrational Schur functions are identified as independent identically distributed random variables, notes that

> ... the author's proofs involve skillful exploitation of the intricate connection between [certain classes] of Schur functions and its sequence of Schur parameters. On the probabilistic side the main tools are theorems from multiplicative ergodic theory of H. Furstenberg and H. Kesten, H. Furstenberg, and V. I. Oseledets. The author presents in admirably clear detail the preliminary ideas and results that form the basis of his proofs.

[1] These three lists of names are based on incomplete records. My apologies for oversights.

The reviewer of his 2016 paper *On the roots of a hyperbolic polynomial pencil*, writes that it

> includes an independent proof of a simplified version of the BMV theorem, ... which is ... practically a model proof ... In this way Katsnelson obtains the connection of the Laplace transformation with the trace function, in a better and deeper way than was done previously in the literature. .... the paper is full of technical masterpieces

Over the years, Victor also offered a wide range of courses in the Feinberg Graduate School (which is the graduate school of the Weizmann Institute). In addition to courses on Complex Analysis, Functional Analysis and Harmonic Analysis, which one might expect, the list of his offerings includes courses on Differential Geometry, Riemannian Geometry, Topology, Inverse Problems of Spectral Analysis and Random Matrices.

To my great regret we only collaborated together on one paper, a tutorial on I. Schur's contributions to analysis, that appeared in a volume *Studies in Memory of Issai Schur*; most of the rest of this volume has a distinctly algebraic flavour. We did, however, serve together as editors of a volume OT 95 (jointly with Bernd Kirstein and Bernd Fritsche from Leipzig) *Topics in Interpolation Theory*.

In recent years Victor has suffered from ill health which has seriously affected his vision, but he still continues to work when he can. In addition to the great personal difficulties that this entails, it is also a great loss for the mathematical community that he cannot work at his normal full pace.

My Teacher Viktor Emmanuilovich Katsnelson

Yu. M. Dyukarev

Dedicated to Viktor Emmanuilovich Katsnelson, with gratitude, on occasion of his 75th birthday

It was early September 1971 when I sighted Viktor Emmanuilovich for the first time. I had just began my studies at the Faculty of Mechanics and Mathematics at Kharkov National University. An utterly extraordinary man entered the lecture hall to teach the course of Mathematical Analysis. He was very young, rather short, had long wavy hair and expressive, intelligent eyes. Viktor Emmanuilovich's appearance alone was an indication of the strong and exceptional personality before us. But most fascinating was his lecturing style. Viktor Emmanuilovich constantly moved around the lecture hall and accentuated essential moments via intonation and gestures. His enthusiasm used to captivate everyone to such an extent that not until the end of the lecture did we notice his suit, his hair even, all whitened with chalk. Viktor Emmanuilovich quickly became my favorite teacher, my idol. Viktor Emmanuilovich's presentation style, manners and even his very appearance are forever ingrained in my memory as a multitude of bright events and images, making up for the deepest impressions of my university years.

When I was close to finishing my studies, Viktor Emmanuilovich proposed that I write my dissertation under his supervision. I agreed, of course. It was a very pleasant surprise for me, somehow even an honor. But that was only the first surprise of many. After the completion of my dissertation, Viktor Emmanuilovich offered me to become his PhD student. Thereafter began our collaboration and friendship of many years. Upon recommendation of Viktor Emmanuilovich, in 1977 I took up a role of researcher at the Department of Mechanics and Mathematics at Kharkov National University. Then in 1978 I became a PhD student. For bureaucratic reasons, Viktor Emmanuilovich could not act as my academic

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supervisor. Therefore, Vladimir Petrovich Potapov became the official supervisor of my PhD dissertation by request of Viktor Emmanuilovich, who remained my de-facto supervisor. However, Vladimir Petrovich did have an interest in my research as well. I recall a conversation between Vladimir Petrovich and Viktor Emmanuilovich about the topic of my future dissertation. This took place in Vladimir Petrovich's apartment in the summer of 1978. Viktor Emmanuilovich suggested the study of interpolation problems in the Stieltjes class as my dissertation topic. Vladimir Petrovich approved but then suddenly said: "How about he tries to take the bull by the horns and tackle the problem on deficiency numbers of differential operators?" I did not understand what they were talking about, I was prepared to accept any topic. But Viktor Emmanuilovich replied: "No, he should better work of Stieltjes interpolation." Thus, with the easy hand of Viktor Emmanuilovich, the topic of my scientific research was set. To my knowledge, the problem of deficiency numbers of differential operators has not been fully solved to this day.

At that time, and we are talking about the second half of the 1970s and the beginning of the 1980s, mathematical life in Kharkov was in full swing. In 1976, Vladimir Petrovich Potapov and Irina Vasilevna Kovalishina relocated from Odessa to Kharkov. Vladimir Petrovich, Irina Vasilevna and Viktor Emmanuilovich were intensely engaged in the study of analytic interpolation problems; an important role in their research played the fundamental results of V. P. Potapov in the theory of J-contractive analytic matrix functions. With those results and ideas they became a center of attraction of many mathematicians in Kharkov. Amongst the beginner and young mathematicians who under supervision of Vladimir Petrovich and Viktor Emmanuilovich actively worked on interpolation problems, were L. B. Golinskii, I. V. Mikhailova, P. M. Yuditskii, A. Ya. Kheifets, Yu. M. Dyukarev, M. F. Bessmertnyi and others. More experienced mathematicians V. K. Dubovoy, V. A. Zolotarev, A. G. Rutkas, A. A. Yantsevich, I. E. Ovcharenko and others also developed interest in this subject matter.

In the Soviet Union many people would often meet in the kitchens of their small apartments in the evenings. The conversation topics were most diverse: science, literature, history, politics, etc. It was possible to bring up any topics that led to an interesting discussion. I remember well conversations in the kitchens of apartments of Viktor Emmanuilovich and Vladimir Petrovich, where I got to know so much, learnt a lot, and used to get a huge pleasure from the connection with such different, but extraordinarily bright and talented people. Typically, these kitchen conversations were accompanied with tea, alcoholic beverages, and cigarettes. Viktor Emmanuilovich did not smoke and did not like alcohol. In his kitchen, the discussions were usually over tea. The still very young Viktor Emmanuilovich used to sit at the table, his intelligent eyes gleaming. He surprised me with his ability to comment on any topic unpredictably, brightly and often in a somehow paradoxical manner. He kept everyone on their toes and often joked sharp-wittedly when someone expressed their thoughts inaccurately. I recollect working on our only joint mathematical article, which became my first scientific publication. Upon resolving all issues with regards to the core content, the phrasing of the text was next. I was not yet skilled at it at all, but Viktor Emmanuilovich began teaching me the knacks. In the evenings, we made tea and settled in the kitchen. I had to write the text. Virtually after each sentence Viktor Emmanuilovich would interrupt me and say: “But what exactly are you trying to express with this phrase?” I explained thoroughly. “Correct”, Viktor Emmanuilovich replied. “But what have you written?!...” In this way we worked for many days until finally Viktor Emmanuilovich suggested to finish the draft of the article saying “The best is the worst enemy of the good.”

In spring 1980 Viktor Emmanuilovich suggested that I take a position teaching higher mathematics at the Department of Physics at Kharkov National University. I replied that I did not want to leave the Department of Mechanics and Mathematics. But Viktor Emmanuilovich noted that at that moment there was no vacant teaching position at the Department of Mechanics and Mathematics, and added: “Maybe you should switch to the Department of Physics for awhile?” In this suggestion I liked the sound of the word “awhile” and I agreed. But, as Viktor Emmanuilovich often said, temporary circumstances are the most constant circumstances in life. I have already been working at the Department of Physics for nearly 40 years.

In December 1980 Vladimir Petrovich Potapov passed away. The death of this indisputable leader and bright exceptional personality was a big loss to us. We, the mathematicians who stood at the beginning of their mathematical careers, now glanced with hope at Viktor Emmanuilovich, and he more than lived up to our hopes. In spring 1981, a big conference was held in Odessa in honor of Vladimir Petrovich Potapov. The legendary mathematician Mark Grigorievich Krein took part in that conference. During the plenary lectures he sat in proud solitude in the middle of the hall. He made various remarks during the speeches. His comments were frequently along the lines of: “This follows from my more general results” or “That is not interesting”. Viktor Emmanuilovich also presented. He was visibly nervous and all through the presentation, he kept clutching a folder with his left hand whilst writing essential notes on the blackboard with his right. Mark Grigorievich commented on Viktor Emmanuilovich‘s presentation with two remarks. Evidently jokingly he asked: “Did the folder that you kept holding in your hand the whole time help you with your presentation?” But then he added earnestly that new ways for mathematical research have been shown in the presentation. Such an evaluation delighted us greatly. We were going back to Kharkov full of new hopes and expectations. I received my PhD title in the fall of 1982. Viktor Emmanuilovich providing me with all-round help and support. That way, my years as a student ended. During that time, the major foundations for my later life were established. And in many of the important events of those years, my teacher Viktor Emmanuilovich Katsnelson played a big role.

Some Impressions of Viktor Emmanuilovich Katsnelson

G. M. Feldman

Viktor Emmanuilovich in the 1960s

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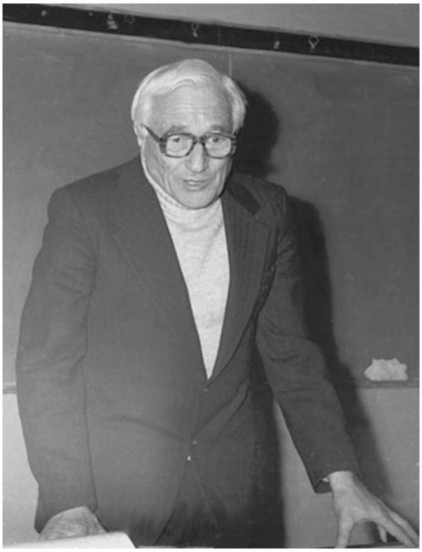
After graduating high school in 1965, I began my studies at the Faculty of Mechanics and Mathematics of the Kharkiv University. Around that time, Viktor Emmanuilovich Katsnelson finished his degree at the Faculty of Mechanics and Mathematics of the Kharkiv University and got employed as assistant in the Department of Mathematical Analysis. I heard that Boris Yakovlevich Levin, who was the Head of the Department of Mathematical Analysis back then, had wanted to have V. E. as a PhD student, but due to his last name, which sounded very Jewish, he was unable to. This is why it took great effort to find a workplace for him at the university. Thus, it happened that, chronologically speaking, I became one of V. E.'s first students. Since the beginning of our first year of study V. E. has taught the exercise class for mathematical analysis. Then I wrote my first scientific work under his supervision and later we published a joint work. He was also the supervisor of my diploma thesis. What follows, are my fragmentary memories of V. E.



Boris Yakovlevich Levin (22.12.1906–24.08.1993)

The 1960s were the golden age of Kharkiv's mathematics. Back then, many extraordinary mathematicians worked in Kharkiv. Among the young mathematicians Viktor Emmanuilovich was not only one of the leading ones but he was the best. No-one has ever expressed doubts about that. Of course, there was Vladimir Igorevich Matsaev, who was considered as a mathematical genius. However, Matsaev was 6 years older than Viktor Emmanuilovich and had already moved from Kharkiv to Chernogolovka at that time.

In B. Ya. Levin's seminar, V. E. has always been very active. I myself did not visit this seminar very often. Each time I did attend it though, V. E. was the one person


Vladimir Igorevich Matsaev (30.04.1937–25.02.2013)

in the audience, who stood out the most. He often asked very profound questions and it turned out that he understood the nature of the subject better than the lecturer himself.

V. E. has never particularly cared about his attire nor his appearance and oftentimes he did not look very well groomed. He was not at all bothered by that.

He has once said to me, and ever since I have remembered it, that role models in mathematics are such mathematicians whose example he could follow and mentioned V. I. Arnold, since Arnold is not only an exceptional mathematician, but also has excellent knowledge of physics and mechanics.

I attended numerous courses of specialization lectures of V. E. He talked about difficult topics with great enthusiasm. He loved Privalov's book "Boundary properties of analytic functions" and a few times he held a special lecture based on this book. He also held a very profound lecture series on operator theory, which contained sufficiently advanced topics as e. g. Hellinger's theory of spectral types.

V. E. was very proud that his famous work "Conditions under which systems of eigenvectors of some classes of operators form a basis" was published in one of the first issues of the journal "Functional Analysis and Its Applications". Back then, the journal had only just been established. He told me that his work was originally supposed to appear in the very first issue of the journal, but for technical reasons it was only included in the second one. At the same issue of the journal there were published articles by F. A. Berezin, D. P. Zhelobenko, V. D. Lidskii, R. A. Minlos,

G. E. Shilov, I. Ts. Gokhberg, and among these famous mathematicians was very young V. E. At that time he did not defend his PhD thesis yet.

During my fourth year of studies, I prepared my first work for publication under guidance of V. E. There have not been computers yet, the manuscripts were drafted on type writers, and in the end the formulas were written by hand. Adding bulky calculations by hand accurately was not easy. V. E. demonstrated me how to do this and, eventually, it resulted in him adding formulas on a few pages of my article himself in order to show me how it has to be done.

V. E. was particularly proud that he was able to do various manual works himself, e. g., regarding electronics or metalworks. He was convinced that he was pretty well versed in techniques.

V. E. loved giving characterizations of mathematicians. In my opinion they have not always been quite objective. Oftentimes, on varying occasions, he expressed paradox evaluations, but arguing with him was useless.

V. E. with Vladimir Solomonovich Azarin

Finally, I want to add some words about the Thursday Seminar at Kharkiv University in analysis which was started by B. Ya. Levin in the year 1956. For about 40 years it has been a school for Kharkiv mathematics working in analysis and has been a center of active mathematical research. The major parts of seminar talks concerned complex analysis and its applications. Nevertheless, there was no restriction on the subject: There were talks on Banach spaces, spectral theory of operators, and differential and integral equations. A meeting of the seminar usually lasted more than 2 h, with a short break. In most cases detailed proofs were presented. Its active participants in different years included P. Z. Agranovich, V. S. Azarin, G. R. Belitskii, G. P. Chistyakov, A. E. Eremenko, S. Yu. Favorov, A. E. Fryntov, L. B. Golinskii, A. F. Grishin, V. P. Gurarii, M. I. Kadets, V. E. Katsnelson, A. Ya. Kheifets, S. A. Kupin, V. N. Logvinenko, Yu. I. Lyubarskii, Yu. I. Lyubich,

V. I. Matsaev, V. D. Milman, M. V. Novitskii, I. V. Ostrovskii, I. E. Ovcharenko, V. P. Petrenko, A. Yu. Rashkovskii, L. I. Ronkin, A. M. Rusakovskii, M. L. Sodin, V. A. Tkachenko, A. M. Ulanovskii, P. M. Yuditskii, and many others. B. Ya. Levin has always been proud and delighted with achievements of the participants of his seminar.

The Good Fortune of Maintaining a Long-Lasting Close Friendship and Scientific Collaboration with V. E. Katsnelson

Bernd Kirstein

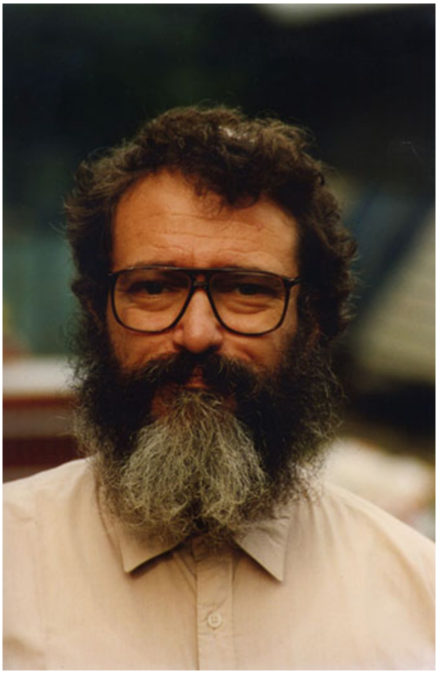


Fig. 1 V. E. in Großbothen 1994

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1 First Steps in Schur Analysis

After the defense of our joint dissertation on problems of the filter theory of multidimensional stationary sequences in December 1983, Bernd Fritzsche and I decided to aim our future research at the analytic foundation of prediction theory of multivariate stationary sequences. Against this background, we took up intense studies of the trend-setting works of the Soviet school (Kolmogorov, Rozanov, Matveev) as well as of American scholars (Wiener, Masani, Helson, Lowdenslager). During this process, we became aware of V. P. Potapov's fundamental work [45] about the multiplicative structure of J-contractive matrix functions for the first time and we began to study the basics of J-theory systematically. Our choice of this research field was considerably encouraged by P. R. Masani. During Masanis's visit of Leipzig University in May 1986 we had profound discussions about the state of prediction theory at that time and its prospects. P. R. Masani revealed to us that in collaboration with Norbert Wiener, following the works [43, 44, 50, 51], further research on an application of the results of V. P. Potapov in prediction theory was planned. However, the realization of this intention became unattainable due to Norbert Wiener's death on March 18, 1964. Without Norbert Wiener P. R. Masani was reluctant to tackle this project and he turned towards a systematic elaboration of the theory of measures with orthogonal values in a Hilbert space or rather of the theory of orthoprojector-valued measures. P. R. Masani encouraged us to get in direct touch with the students of V. P. Potapov, who had passed away in the year 1980. V. P. Potapov had been employed at the FTINT (Russian abbreviation for *B. Verkin Institute for Low Temperature Physics and Engineering of the National Academy of Sciences of Ukraine in Kharkov*) during the last period of his life (1976–1980) and he was able to contribute significantly to the popularization of J-Theory. In particular, he managed to assemble a group of exceedingly committed mathematicians in Kharkov who devoted themselves with great enthusiasm to the application of Potapov's method of fundamental matrix inequalities to matricial versions of classical interpolation and moment problems. Among others, I. V. Kovalishina, V. E. Katsnelson, V. K. Dubovoy, L. B. Golinskii, I. V. Mikhailova, and Yu. M. Dyukarev belonged to this circle of mathematicians. In the summer of 1986, Bernd Fritzsche and I decided to invite one representative of the circle of the above mentioned mathematicians to a month-long work visit at Leipzig University in the year 1987.

Luckily, in the 1980s, numerous alumni of the University of Kharkov were employed at the mathematical department of Leipzig University, whom we were now able to ask for their advice. In particular, we contacted Rainer Kaiser and Frank Löffler from the department of "Functional Analysis and Mathematical Physics". Both recommended to extend an invitation to V. K. Dubovoy first, as he had already held office as dean of the Mechanical Mathematical Faculty at the University of Kharkov for a certain period and because he was the one most likely to be able to avail himself of the invitation. This hope was fulfilled.

In May 1987, V. K. Dubovoy stayed about a month in Leipzig. This was the onset of a scientific partnership and friendship of our group and several mathematicians from Kharkov, which should last right up to the present day. During V. K. Dubovoy's visit in Leipzig, essential steps for the future collaboration were established. In this regard, two points are worth to be emphasized. First, our joint monograph (see [16]) on the matricial Schur problem and second, Bernd Fritzsche and I spent one semester for postgraduate studies at the University of Kharkov in 1988. The exact German notion for these studies was "Zusatzstudium", which carried the meaning of additional studies after graduation. These postgraduate studies were effectively realized from mid-March to mid-May of 1988 due to the invitation of V. K. Dubovoy. We were accommodated in a guest apartment of the University of Kharkov in the Street Otokara Yarosha 16A, where we found good facilities for undisturbed research. For this reason, Bernd Fritzsche and I typically only went to the university, which was accessible from our housing in about 25 min by bus, for lectures or agreed meetings. Our scientific work was divided into two lines. On the one hand, we worked on elaborating the manuscript for our joint Teubner-Text with V. K. Dubovoy, whereas, the second line fully measured up to the term "additional studies". That is because we studied several manuscripts on J-theory in thorough detail, which were deposited in special Soviet or Ukrainian institutions and which were, as a result, accessible to foreigners only with extreme difficulties. In a first step, we translated these texts from Russian to German. In the second step, we tried to comprehend the respective proofs in detail.

Our first encounter with V. E. Katsnelson (Fig. 1) happened, to our surprise, on April 2, 1988, precisely 2 weeks after arriving in Kharkov. Viktor Emmanuilovich showed up in our accommodation and brought us a journal containing Yu. L. Shmulyan's classical work [48] on operator balls. Just at that time, Bernd and I were engaged in drafting the section on matrix balls for our Teubner-Text and, accordingly, we had asked V. K. Dubovoy whether he could provide the work of Shmulyan. Since the journal in question was not available in the library of the University of Kharkov, V. K. Dubovoy asked V. E. for help. This launched a scientific collaboration which lasts for more than 30 years, and even more importantly, a close friendship. During that first encounter, he informed us that the inventory of mathematical literature in the university library of Kharkov was limited and, moreover, he let us know that, concerning current literature, his private library was many times richer. In the continuation of this thought, he offered to extensively use his private library during our stay in Kharkov. In order to convince us entirely, he spontaneously invited us to come along to his apartment in the Prospect Pravdy 5A to have a look at everything ourselves. Once we had arrived, we were deeply astonished when we found ourselves in a huge room which reminded us of the stack-rooms of our institute in Leipzig. It was densely filled with book shelves tightly packed, which V. E. had built on his own, as we found out later. The books in the shelves were carefully and systematically sorted by topic. The following day, Bernd and I were invited to lunch at his home. It was Sunday April 3, 1988, accordingly, it was the 81st birthday of Mark Grigorevich Krein. Over the course of the afternoon, V. E. noticed this fact and sent a congratulatory telegram to Mark

Grigorevich adding both congratulators. This was the only personal contact Bernd and I had ever had with Mark Grigorevich, who was one of those mathematicians that had the strongest influence on the direction and aims of our research, in retrospect. At the end of our postgraduate studies in Kharkov, Bernd and I outlined further steps of our collaboration with V. K. Dubovoy as well as the establishment of a long-term research contact to V. E. in regard to the topics of Schur analysis.

2 Viktor Emmanuilovich's First Visit to Leipzig

Over the course of planning the research activities for the year 1989 in fall 1988, Bernd Fritzsche and I intended to hold a week-long international seminar on Schur analysis. This request was complied with by the administration of the Center for Theoretical Sciences (NTZ). The week from October 16–20, 1989 was specified as the date of the event. In connection with this seminar, we requested a 3-week stay for V. E. at Leipzig University. This request was granted as well. On September 23, 1989, he arrived in Leipzig. Aged 46 years it was his very first travel abroad. At the beginning of his stay it was unforeseeable that during his visit certain incidents would happen in Leipzig, which should stir up the political situation in the GDR significantly. V. E. became an eye-witness of the massive Monday protests in the city center of Leipzig on both October 9th and October 16th that set the decay of the GDR in motion. Within a personal evaluation of these events, he reasoned that the reunification of Germany was the only consequence that seemed logical. We ourselves considered his prognosis very utopian at the time. History proved, though, that he had predicted everything completely correctly. Less than 1 year later, on October 3, 1990, the reunification of Germany was in fact enforced. During the first 3 weeks of his stay in Leipzig, Bernd and I had profound mathematical discussions with him, where he drew our attention to central problems in Schur analysis and, moreover, imparted to us fundamental aspects of the research of the Kharkov school. At that time, his lectures were held in Russian and I acted as interpreter into German for the audience (Fig. 2). At the end of his stay in Leipzig, the INTSEM (International Seminar) on Schur Analysis took place. It was P. R. Masani who had suggested the event during his first visit to Leipzig in 1986. The aim of this seminar was to gather leading specialists from the East and the West working on Schur analysis. This goal was successfully pursued. Among the Western participants were P. R. Masani, A. Dijksma, H. S. V. de Snoo, S. Hassi, and others. The list of Soviet participants included I. V. Kovalishina, V. E. Katsnelson, V. K. Dubovoy, Yu. L. Shmulyan, and I. M. Spitkovskii (Fig. 3). On October 17, 1989, the second day of the seminar, Mark Grigorevich Krein, one of the greatest mathematicians of the twentieth century, who had made fundamental contributions to Schur analysis and numerous other fields passed away. For this reason, D. Z. Arov has not been

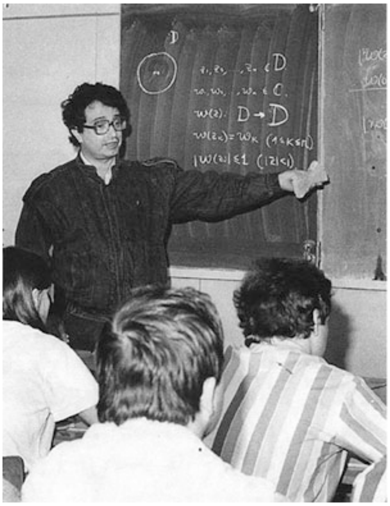

Fig. 2 V. E. in Leipzig, autumn 1989

Fig. 3 INTSEM Schur Analysis, Leipzig October 16–20, 1989. Front row from left: B. Kirstein, B. Fritzsche, Yu. L. Shmulyan, P. R. Masani, H. J. Girlich, G. Heinig, V. E. Katsnelson. Second row from left: A. Böttcher, far right I. V. Kovalishina

able to come to Leipzig for the seminar in time. He arrived on October 21st, that is, 1 day after the end of the seminar. Following the seminar, it was intended that he would stay in Leipzig for another 3 weeks. During this time, the foundation for a long-term scientific collaboration with D. Z. Arov was set.

3 Viktor Emmanuilovich's Year as a Visiting Professor at Leipzig University

In order to deepen the scientific collaboration with Viktor Emmanuilovich, Bernd and I intensively thought about possibilities to invite him to Leipzig for a longer period. This aim could indeed be pursued in 1991 on a large scale. At this point a few further words are advisable. One of the most famous scientists in history of Leipzig University is undoubtedly Wilhelm Ostwald (1853–1932), one of the founding fathers of physical chemistry, who was honored with the Nobel prize in chemistry in 1909. In honor of Wilhelm Ostwald, a chair named after him was established at Leipzig University, which was assigned to exceptionally renowned foreign guest researchers by the Faculty of Sciences of Leipzig University. Due to V. E.'s extensive publications in function theory and functional analysis, he had already gained a high reputation in the 1980s. Among other things, this was particularly shown by an invitation to a week-long guest stay at the famous Weizmann Institute of Science Rehovot in summer 1990. This sparked the idea to put forward the proposal to assign the Ostwald Chair to V. E. in the first half of 1991. To our great delight, the proposal was accepted by the Faculty of Sciences of Leipzig University. From today's perspective, Viktor Emmanuilovich turned out to be the last holder of the Ostwald Chair. The profound changes at Leipzig University after the political turnaround resulted in the abolishment of the Ostwald Chair for guest researchers. During the time of his visit to Leipzig, a big part of V. E.'s family emigrated to Israel. For this reason, he was unsure how and where he could continue his academic career. In order to support him, Bernd and I considered it advisable to extend his stay in Leipzig beyond the duration of the Ostwald Chair. The realization of this idea was complemented by a fortunate circumstance. The DFG offered multiple funding opportunities in order to support universities in the former GDR. Taking advantage of one of these programs, Bernd and I managed to arrange a DFG visiting professorship at Leipzig University for V. E. for the second half of 1991. Eventually, he stayed in Leipzig for an entire year. On January 22, 1992, he then left Leipzig for the Weizmann Institute, where he was offered a professorship for Theoretical Mathematics (see Fig. 4).

I now want to address the mathematical achievements of V. E. in Leipzig in 1991. Due to his presence in Leipzig, Bernd and I decided to organize an international seminar on Schur analysis on short notice . It took place from March 12–13, 1991. Among others, H. Langer, A. Dijksma, H. S. V. de Snoo, S. A. M. Marcantognini, and P. Bruinsma appeared on the list of participants. As holder of the Ostwald Chair, V. E. naturally offered a series of special lectures. Roughly spoken, he talked about two topics, the first was Schur analysis, of course. In particular, he touched upon the method of fundamental matrix inequalities created by V. P. Potapov and its applications to continuous problems in analysis. The second topic contained special problems of complex analysis. Thanks to the work of B. Ya. Levin and I. V. Ostrovskii, this field had a long tradition in Kharkov. As B. Ya. Levin's Ph. D. student and longtime employee in the department of complex analysis

Fig. 4 V. E. in Leipzig, January 1992

at the University of Kharkov, Viktor Emmanuilovich had an exceptional level of knowledge on complex analysis and its application to certain problems of probability theory. The intersection of these topics was extensively studied by the group led by I. V. Ostrovskii in Kharkov and by the group led by Hans-Joachim Roßberg in Leipzig.

When talking about the scientific activities of V. E. in Leipzig in 1991, it is inevitable to mention the monograph [31], which is available as an unpublished manuscript in Russian. This book treats various extremal problems for holomorphic functions, which are closely related to topics in mathematical analysis such as the asymptotic behavior of orthogonal polynomials, Beurling's theorem about shift-invariant subspaces, various problems on positive definiteness (in particular, the trigonometric moment problem as well as power moment problems), problems of weighted approximation, Wiener-Hopf factorization of positive functions, which are defined on the unit circle or the real axis, as well as problems about the analytical background in prediction theory of weakly stationary stochastic processes. V. E. put special emphasis on identifying the connections between complex analytical interpolation problems (coefficient problems by Schur and Carathéodory, Nevanlinna-Pick problem, etc.). It is also worth mentioning that he focused strongly on the historical retrospective and emphasized the largely unnoticed but significant contribution of V. I. Smirnov. While diligently working out the historical contexts, V. E. drew upon the rich inventory of the library of the Mathematical Institute at Leipzig University and received a wide range of assistance from the librarian Ina Letzel. We had the opportunity to observe the development of that manuscript right from the beginning and to discuss many of the arising questions with V. E. This way, we witnessed his working methods over several months. It left a lasting impression on Bernd and me for our future career in mathematics. He finished the script for the monograph [31] in August 1991. He felt relieved that the political coup in Moscow during those days could be subdued and dedicated the monograph to that incident.

One experience that remained unforgettable for V. E. was the visit of Ilya Prigogine (1917–2003) to Leipzig at the beginning of September 1991. In 1977, Prigogine received the Nobel Prize in chemistry for his work on non-equilibrium thermodynamics. On September 6, 1991, V. E. had already attended a lecture of Prigogine at Leipzig University. Impressed by this lecture, he convinced me to attend a second lecture of Prigogine on September 7, 1991 that took place in the course of the discussion series at the Wilhelm-Ostwald memorial in Großbothen, which lies approximately 30 km south-east of Leipzig. Wilhelm Ostwald, who received the Nobel Prize in chemistry in 1909 after retiring from Leipzig University as an independent researcher, worked there until his death. The title of Prigogine's lecture in Großbothen was "The time paradox and its resolution". This topic determined Prigogine's research significantly. In Kharkov, V. E. had already studied several of Prigogine's works thoroughly and, in particular, he had analyzed the role of the arrow of time in chaos theory. Therein lied one of the main emphases of Prigogine's lecture in Großbothen. Finally, it should be mentioned that, in the mid-1990s, V. E. gave a talk at the Institute for Physics and Chemistry in Brussels, which was headed by Prigogine, and this way he was able to get in personal contact with Prigogine.

From September 16–20, 1991, another INTSEM on Schur Analysis took place in Leipzig. Among others, L. A. Sakhnovich, V. Pták, M. Dritschel, A. Dijksma, H. S. V. de Snoo, S. A. M. Marcantognini, H. Waadeland, D. Alpay, H.-J. Runckel participated. In the second half of 1991, V. E. worked on profound problems of the theory of pseudocontinuable meromorphic matrix functions in special consideration of the phenomenon of Arov-singularity.

Already in his native language Russian V. E. is fond of wordplays. During his year long stay in Leipzig he made great efforts to learn German. This efforts went to such a level of mastery, that he was quickly able to carry over his love for wordplays into German. Whilst extended walks through Leipzig's parks he not only acquired a good sense of orientation in the city, but furthermore started conversations with strangers, which helped him to improve even further. This enabled V. E. to cultivate his wit and led to a myriad of anecdotes. Let me share some of them.

For instance, one day we worked at my place, when my younger son approached me with a question. Overhearing our conversation, V. E. commented: "Remarkable boy, this young Niels, only 4 years of age and already so versed in German."

Even years later, V. E. was still capable of his typical jests. One of V. E.'s favorite jokes involved an imaginary mathematician Hermann Amandus Lunke. Note the first names are taken from the famous German mathematician Hermann Amandus Schwarz (1843–1921). He used to register himself and Mr. Lunke at conferences and was happy to see when a certain H. A. Lunke appeared in the list of participants. In the English-speaking world, this imaginary mathematician would correspond for example to Thomas Rickster or T. Rickster. This proves yet another time his extraordinary level of understanding for the subtleties of the German language.

Another joke reads as follows: In the fall of 1996, we attended a conference on complex function theory in Trondheim. On arrival at the hotel, we were asked to complete a form and had to state the reason for our stay. V. E. quickly realized, that likely nobody will ever have a look on the forms and took the bet that his statement will not cause any trouble. He had written: “um eine Bank auszurauben”, which is German for “to rob a bank”. Completely relaxed, he told me: “It is in German, nobody will notice.” Of course, nobody read it and we could also leave Norway again.

Shortly before the end of V. E.’s year-long stay in Leipzig, Bernd and I managed to fulfill a particular scientific wish of his. In fact, we succeeded in inviting Herbert Stahl to Leipzig for the colloquium lecture on January 8, 1992. In 1974, Herbert Stahl had done his doctorate on Padé approximations at the TU Berlin with Christian Pommerenke. Thereafter, he wrote highly regarded papers at the intersection of complex analysis, potential theory, and approximation theory. Soon after their publication, V. E. had already realized the importance of Herbert Stahl’s work and was hence eager to get to know him personally. The colloquium lecture in Leipzig and the get-together thereafter with our research group on Schur analysis in Leipzig played an essential role in the realization of this plan. It quickly became obvious that the chemistry between V. E. and Herbert Stahl was right. This close connection with Herbert Stahl is particularly substantially demonstrated by V. E.’s reaction to Herbert Stahl’s last great mathematical thunderbolt. Herbert Stahl died on April 22nd, 2013. Shortly before his death, he had received confirmation from the journal *Acta Mathematica* that his work [49] about the proof of the BMV conjecture would be published. However, he was not able to lay eyes upon the published version. Inspired by a problem in quantum physics, the BMV conjecture was stated in 1975 by D. Bessis, P. Moussa, and M. Villani in their work [9] and withstood all attempts to prove it until Stahl’s breakthrough. Initially, the BMV conjecture could solely be proved for a few special cases. The proof by Herbert Stahl is based on very profound considerations of Riemann surfaces of algebraic functions. In various of his more recent examinations (see [34–38]), V. E. addressed different aspects of the nature of the BMV conjecture. For example, he once again investigated various special cases, which reveal new aspects of the underlying phenomenon. In this regard, I refer, in particular, to the work [37] in which V. E. pointed out a connection of the Laplace transform and the trace function of a matrix, which had not been observed deeply in the mathematical literature before. Regarding further contributions to the proof of the BMV conjecture by Herbert Stahl, I refer to Eremenko [20] and Clivaz [13].

4 Viktor Emmanuilovich’s Views on Mathematics

In 1991 I had the chance to spent a lot of time with V. E. when we shared an office at the Institute of Mathematics. We spent many days discussing mostly mathematical content until the early evening. He had arrived in Leipzig with seven suitcases, which carried predominantly books. These books again represented only a tiny fraction of his enormous private library. The suitcase selection conveyed an objective idea of what was dearest to him. As a first approximation one could come to the conclusion that the area of spectral function theory—a mathematical field at the intersection of complex function theory and functional analysis—was of special importance. The function classes named after Hardy, Nevanlinna, Smirnov and others were of special interest to him. Thereby he was interested in the unit circle as well as in the half-planes. V. E. was also an expert on the following monographs by [17, 22, 24, 27, 40, 46, 47].

A special status had the just published masterpiece on *The Logarithmic Integral I* (see [41]) by Paul Koosis, who personally send V. E. a copy of this book (later he received also the second part [42] once it got published). Koosis' books treated in particular the interplay between harmonic analysis and potential theory, which originated from the work of Arne Beurling (see esp. Beurling/Malliavin [12]). V. E. is a profound adept of Beurling's work. He praised the extraordinary depth of Beurling's thoughts and as well as the fact that the library of our research group on Schur analysis contained the two volumes of the Collected Works of Arne Beurling [10, 11].

During our conversations I quickly realized his exceptional knowledge of the classical texts on the origin of Schur analysis and his interest in their matricial generalizations. This foundation was the essential motivation for our scientific collaboration over the following decades. In particular he is a passionate admirer and excellent connoisseur of Issai Schur's work (see the survey article Dym/Katsnelson [19]). Another joint research topic is the theory of outer matrix functions. This function class is of central importance in prediction theory of stationary sequences and thus moved quickly into the focus of attention of Bernd Fritzsche and me. Viktor Emmanuilovich was led to this function class while studying certain extremal problems in the context of factorization problems, which also led him to the work of Norbert Wiener and Masani (see [43, 44, 50, 51]) and Helson/Lowdenslager [25, 26].

Another of V. E.'s favorite topics are de Branges-Hilbert spaces of entire functions (see de Branges [15]). For V. P. Potapov this theory was "one of the main achievements in mathematical analysis of the 20th century", which is why he strived for a clearer exposition of this theory compared to the presentation in de Branges' monograph [15]. A result of this effort is the paper by L. B. Golinskii and I. V. Mikhailova [23], initially published in Russian as preprint No. 28-80 of the Kharkov Institute for Low Temperature Physics and Engineering, 1980. Throughout the genesis of this preprint, V. E. was in close contact to the project's mentor V. P. Potapov and the authors L. B. Golinskii and I. V. Mikhailova. Later V. E. translated this preprint into English and catered its publication in [18] in order to popularize V. P.'s view that this topic belongs by its nature to J-theory. Apparently, the essential theorems of de Branges' theory are statements about analytic 2×2 matrix functions, which often allow for generalizations to $q \times q$ matrix functions. Regrettably, on December 21, 1980 this line of work was put to an abrupt end with the untimely death of V. P. Potapov.

Viktor Emmanuilovich's interest in the theory of de Branges-Hilbert spaces of entire functions remained vivid, especially after Louis de Branges' announced a

Fig. 5 V. E. with Louis de Branges in fall 2005

proof of the famous Riemann hypothesis—which is about nontrivial roots of the Riemann zeta function—-with the help of this theory. In the fall of 2005, V. E. followed an invitation of A. E. Eremenko and spent several months at Purdue University in Lafayette (Indiana), where Louis de Branges worked. He interacted closely with de Branges, to the extent that he was his only listener in a series of lectures on Hilbert spaces of entire functions (Fig. 5).

He mentioned later the very positive impression de Branges' seminars left on him. Replying to my specific enquiry, he revealed that he didn't believe, that de Branges' ansatz is capable to generate a proof of the Riemann hypothesis. His prophecy is correct to this day.

At this point I want to emphasize, that V. E. due to his many years of attendance of B. Ya. Levin's legendary Thursday seminars in Kharkov, acquired a superb knowledge about many attempts to prove the Riemann hypothesis. In particular, he is an expert on the classical studies by J. L. W. V. Jensen, G. Pólya, I. Schur and of more recent work by T. Craven, G. Csordas, T. S. Norfolk, R. S. Varga and others. For comments on the work of G. Pólya and I. Schur we refer to [19, Section 8].

An important aspect V. E. taught us while referring to V. P. Potapov is that often the nature of a scalar holomorphic function is better understood if it can be determined whether it is possible to embed it as an element of a holomorphic matrix function of a distinguished class. For instance, D. Z. Arov was led to such a situation of the problem in [3], when he studied the problem of Darlington realization. This problem is equivalent to the problem of embedding a function as an element of an inner matrix function. Hence, the original function needs to be pseudocontinuable, which is even not only necessary but also sufficient. In the proof of this statement the maximum principle of V. I. Smirnov is instrumental. V. E. as well as D. Z. often reminded me about its relevance.

If I am asked to talk about V. E.'s views on mathematics, I feel urged to mention a particular aspect he repeatedly pointed out in numerous personal conversations—his extraordinary fascination for the number π and its mysticism. As a profound expert of the history of mathematics and the work of its leading figures, he realized the tremendous influence this number had in the works of many geniuses like Euler, Gauß and Ramanujan; and how this generated a feedback effect on the direction of mathematics. Therefore, V. E. likes to think of the works of mathematicians in terms of the relation of their oeuvre and the number π.

I felt particularly honored by a very special vote of confidence, when V. E. entrusted me with exceptionally valuable literature for safekeeping, which he considered of eminent significance. These are a copy of V. P. Potapov's habilitation in which V. P. himself added the formulae by hand, one copy of his own PhD thesis which was supervised by B. Ya. Levin and which he defended in 1967, and finally the Russian edition of the collected works in four volumes of S. N. Bernstein (1880–1968).

At the end of this section I want to add some remarks about V. E.'s affection for books of all kind. As the owner of a private library of gigantic size, he found in me a congenial collocutor. His gift of Hermann Hesse's "Das Glasperlenspiel" ("The Glas Bead Game") marked my 38th birthday on July 9, 1991 as a very special occasion. This novel played an important role in the last years of V. P. Potapov's life. Especially the poem "Die Stufen" ("The Steps") contains many messages which not only moved Vladimir Petrovich but also V. E. Furthermore, V. E. always supported my endeavor to acquaint my sons Mark and Niels with literature. On the occasion of one of his frequent visits to our place, he gave them the German edition of Winnie-the-Pooh by A. A. Milne as a present with the Russian epigraph "To Mark and Niels from a small furry teddy bear". The deeper meaning behind this inscription was that V. E. carried "Pu" as a nickname himself during his youth. Among others B. Ya. Levin called him "Pu". Another remarkable book gift has been the famous Latin-German glossary "Stowasser". His message was to reiterate the importance of mastering foreign languages, which he not only led by example but underpinned it with a funny story from the animal kingdom pasted into the front binding.

5 My First Visit to Israel in March 1992

Upon invitation of Daniel Alpay, from March 17 to 24, 1992, I visited Israel for the first time. This invitation had already been expressed at the time of V. E.'s visit to Leipzig and had initially been referring to the Ben Gurion University of the

Negev in Beer Sheva. However, at the actual time of the travel, V. E. had already been nominated as professor for Theoretical Mathematics at the Weizmann Institute in Rehovot Fig. 6. This is why I spent about half of this week-long stay in Israel in Beer Sheva and the other half in Rehovot Fig. 7, where I was lodged in the accommodation of V. E. This trip should become unforgettable for me. A main reason therefore were all the personal conversations with renowned mathematicians like H. Dym, V. I. Matsaev, and N. S. Landkof. One afternoon, which V. E. and I spent in M. S. Livsic's office at the University of Beer Sheva, however, left the most long-lasting impression Fig. 8. That afternoon, Mikhail Samuilovich talked a lot about his student years in Odessa and his many years of friendship and scientific collaboration with V. P. Potapov. V. E. mentioned that the work of V. P. Potapov was met with great interest from Leipzig University and that, as a result, the idea of arranging an international conference on Schur analysis in Leipzig in 1994 on the occasion of the 80th birthday of Vladimir Petrovich had emerged. It was an

Fig. 6 V. E. at the campus of the Weizmann Institute in Summer 1992

Fig. 7 At the campus of the Weizmann Institute, from left to right: D. Alpay, V. E., B. Kirstein, H.-J. Runckel, H. Dym

Fig. 8 From left to right: V. E., N. S. Landkof, B. Kirstein, M. S. Livsic

unforgettable moment when M. S. Livsic supported this thought and spontaneously declared his willingness to participate in this event. Thereby, the decisive impulse for the realization of this conference was given. Together with V. E., Bernd Fritzsche and I began to work out the scientific conception in detail. In preparation of this conference, V. E. visited Leipzig several times.

6 Mathematician from Israel in Leipzig – Prof. Viktor Katsnelson: Expanding Joint Work

This section contains the English translation of an article in "Journal Universität Leipzig, May 1994, 3/94". This article was based on an interview that the editor Bärbel Adams conducted in Leipzig in March 1994 with V. E. and me (Fig. 9).

> In February/March, Jewish mathematician Prof. Dr. Viktor E. Katsnelson stayed in Leipzig once again. Native of Kharkov, he has lived in Israel for two years and has had the fortune to be employed at the renowned Weizmann Institute in Rehovot on a permanent basis. However, he continues to maintain close scientific contact to his Ukrainian home and Leipzig University at which he has already worked as a guest professor several times, including in fall 1989, for the entire year of 1991, and during the summer semester of 1993.
>
> Thus, Prof. Katsnelson has become a witness of the profound changes of the past five years. He sees the increased opportunities for scientific work and the exchange between scientists, but at the same time he notices the dangers for science occurring from a monetarily oriented society. "Pure science" will quickly lose reputation; its instrumentalization for career advancement will become the sole purpose for a merely shallow activity in it. In the old states of Germany he heard many colleagues complain about this. Not only does he plead for a strict scientificity in research and studies, but at the same time for a literally universal education at universities. A too narrow specialization will be just as

Fig. 9 V. E. with B. Kirstein in March 1994

dangerous as the commercialization of science. However, he has not yet noticed that the problems of commercial science play a role in Leipzig. The young people deciding to study mathematics or wanting to work in this field scientifically are highly motivated, so that he has great interest in intensifying and expanding the working relationship. Currently, two young scientists of the mathematical institute are using a two-month research stay at the Weizmann Institute in Rehovot for a self-optimization in Schur analysis, Viktor Katsnelson's field of work. Schur analysis, named after the Jewish mathematician Issai Schur who, native to Belarus, spent most of his life in Germany before having to emigrate in 1939, requires a universal mathematical knowledge. Being located at the interface between different mathematical disciplines at once, Schur analysis helps to demonstrate the interrelationship of the subfields as well as the oneness of mathematics as a whole. Nowadays, Prof. Katsnelson is considered one of the protagonists in this area. In 1992, he received the Barecha award for his merits concerning Schur analysis that is awarded every two to three years to outstanding Jewish scientists, who have immigrated to Israel – an honor that the "thoroughbred scientists" as his Leipzig partner Prof. Dr. Bernd Kirstein calls him can be particularly proud of. Back in the days, Viktor Katsnelson emigrated via Germany to Israel with seven suitcases full of books. The joint work of the two mathematicians reaches back to the year 1988 when the then junior scientists from Leipzig Dr. Bernd Kirstein and Dr. Bernd Fritzsche (current dean of the Faculty of Mathematics and Computer Science) stayed in Kharkov for an exchange regarding the topics of Schur analysis. The work group emerging thereof has continued to exist until today and includes others as well. The exchange of scientists is almost a part of the routine: with the aid of Minerva scholarships and the DAAD a few students were able to go to Israel; conversely, the Science and Technology Centre (STC) of Leipzig University funded guest stays here. Joint research projects in Schur analysis and related problems are already an integral part of the scientific program of the STC. There is an international conference planned for August that is dedicated to the 80th birthday of the Odessa mathematician Vladimir Petrovich Potapov, one of the fathers of Schur analysis. Around 40 guests from all over the world are expected, among which are many students and contemporaries of Potapov, inter alia, Michail Samuilovich Livsic, who is considered the last living legend among mathematicians. Viktor Katsnelson is one of the initiators of the event that cannot take place in the Ukraine for financial reasons.

7 The Years 1994 and 1995: Three Remarkable Conferences in Leipzig Under the Influence of V. E. Katsnelson

The year 1994 held two highlights for Leipzig's Schur analysis group, that were significantly influenced by V. E. First, the conference "Recent Developments in Schur Analysis – A Seminar in Honor of the 80th Birthday of V. P. Potapov" from August 22 to 26, 1994 needs to be mentioned. Numerous mathematicians connected with V. P. Potapov attended the conference. First and foremost, I mention M. S. Livsic, who, together with V. P. Potapov and M. S. Brodskii, had been working on fundamental problems in the interface of operator theory and complex analysis after World War II. Further participants of the conference in Leipzig, that were considerably inspired by their personal contact to V. P. Potapov, were L. A. Sakhnovich, D. Z. Arov, A. A. Nudelman, I. V. Kovalishina, V. E. Katsnelson, and V. K. Dubovoy. In the context of a special memorial day for V. P. Potapov, that took place at the Wilhelm Ostwald memorial site in Großbothen (see Fig. 10), the

Fig. 10 August 24, 1994, Potapov memorial session in Großbothen, from left to right: V. E., Gretel Brauer (youngest granddaughter of Wilhelm Ostwald), M. S. Livsic

mentioned scientists remembered the scientist and person V. P. Potapov by sharing personal anecdotes. V. E. was the last of these people to talk about V. P. Potapov's years in Kharkov and he particularly touched upon the thoughts that had moved Vladimir Petrovich during the last months of his life. In this context, the novel "The Glas Bead Game" by Nobel Laureate in literature Hermann Hesse played a special role. In particular, it was the poem "The Stairs" therein, which left an indelible impression on V. P. Potapov. At the end of the memorial service in Großbothen, V. E. recited both the Russian and the English translation of this poem and asked me afterwards to read the original German version of it. This marked the end of an unforgettable afternoon in remembrance of V. P. Potapov.

The conference itself reflected the current state of Schur analysis in the mid 1990s. Fundamental contributions are contained in the book "Topics in Interpolation Theory", which was published as volume 95 in the OT-series (see [18]).

After the conference V. E. and also D. Z. Arov stayed for some more weeks in Leipzig. In this context, it is particularly notable that he celebrated his 51st birthday on September 3, 1994 in Leipzig. We gathered for a small celebration in the apartment of Bernd Fritzsche and were joined, besides by V. E., also by Damir Zjamovich Arov and his wife Natalya Grigorevna (Fig. 11).

The second highlight in 1994 at the mathematical institute of Leipzig University that V. E. played an influential part in was a 2-day workshop in December 1994 on the occasion of the 100th birthday of Norbert Wiener. The focus of this event lay on the interdisciplinary and universal nature of the work of Norbert

Fig. 11 September 3, 1994: V. E. with D. Z. Arov

Wiener, which reached far beyond mathematics and included, among other things, cybernetics, control theory, physiology, and philosophy. The talk of V. E. touched upon more recent developments in harmonic analysis, that were originally initiated by Norbert Wiener. The workshop turned out to be particularly attractive thanks to the attendance of P. R. Masani, one of the main students of Norbert Wiener, who shared many personal impressions of the views and visions of his teacher. Incidentally, that was the third and last visit of P. R. Masani to Leipzig. After a relaxed get-together in a restaurant with P. R. Masani on the last evening after the conference and after then saying good-bye to him, V. E. commented that this might have been our last encounter with P. R. Masani. This was in fact the case, as P. R. Masani passed away on October 18, 1999 and none of us had met him again.

In the early stages of the development of Schur analysis, Rolf Nevanlinna (1895–1980) made fundamental contributions by successfully applying Schur's idea to an interpolation problem that is now called the Nevanlinna-Pick problem as well as to the Hamburger moment problem. In honor of the 100th birthday of Rolf Nevanlinna, an international seminar on Schur analysis took place at Leipzig University from November 6 to 10, 1995. This event was jointly co-organized by Leipzig University and the Weizmann Institute. On part of the Israeli side, V. E., D. Alpay, and Y. Yomdin were the ones in charge. Due to the assassination of Israel's prime minister Yitzhak Rabin on November 4, 1995, Yosef Yomdin was not able to come to Leipzig himself. Striking achievements of his research group were presented by Dmitri Novikov and Nina Roytvarf. V. E. himself gave a two-part lecture on continuous analogues of the theorems of Hamburger and Nevanlinna in which he explained the main contents of his cycle of substantial work on the topic (Fig. 12).

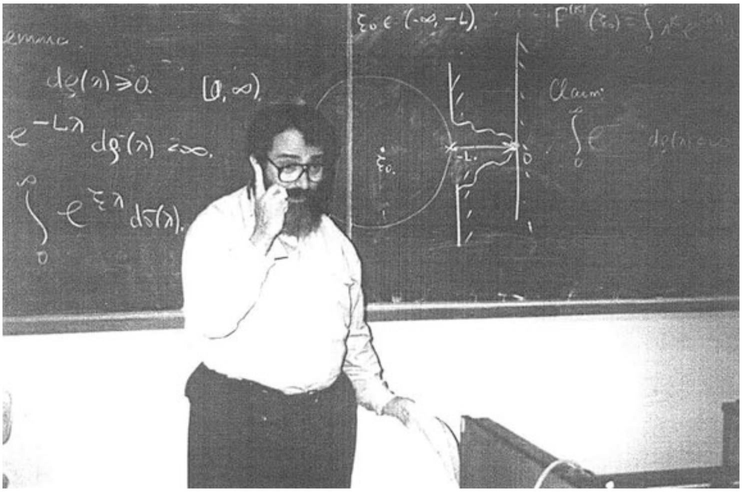

Fig. 12 V. E. lecturing at the Nevanlinna seminar

8 The 60th Birthday of Viktor Emmanuilovich

In honor of V. E.'s 60th birthday on September 3, 2003, we organized a workshop at Leipzig University which was among others attended by V. K. Dubovoy, W. Schempp, B. Silbermann, G. Heinig, A. Lasarow. Of course, a scientific contribution of the jubilarian was a must as well. He gave a two-part lecture about rational solutions of Schlesinger's equation and their tau functions. At the end of the workshop on September 10, 2003, V. K. Dubovoy gave a very appropriate description of V. E. in form of an entry in the guest book of the Mathematical Institute of Leipzig University:

> First and foremost, I would like to cordially thank Professor Bernd Kirstein and Professor Bernd Fritzsche for the invitation to Leipzig and the opportunity to speak at the conference in honor of the 60th birthday of Viktor Emmanuilovich Katsnelson.
>
> I first encountered Viktor Emmanuilovich in spring 1963. Forty years have already passed since then. The predominant part of these years, I stood in close contact with Viktor Emmanuilovich. How many different topics were elucidated throughout!!! Regardless of a certain severity in his judgments, Viktor Emmanuilovich is very democratic company and also willing to expose himself to sharp criticism, which he did sufficiently often compared to others, too. To me, the contact with the mathematician Viktor Emmanuilovich was extremely valuable. I learnt a lot from him; talking to him enriched me extraordinarily and allowed me to give up a whole series of illusions. I know Viktor Emmanuilovich as a person, who feels mathematics deeply and subtly and who strives to convey this feeling to others. He has written scientific papers that identify him as a great master.
>
> You can compare Viktor Emmanuilovich to a singular point in our lives from which a mighty stream of energies emerges. It has not always been easy (just how it is not always easy for him), but without him the world would be poorer.
>
> I express my wishes for Vitja through a passage from a poem by Boris Pasternak, which he loves a lot:
>
> "... but be alive – this only matters – alive and burning to the end."

9 Schur Analysis Workshop at Leipzig University in October 2006 in Honor of the 60th Birthday of V. K. Dubovoy

On July 19, 2006 it was V. K. Dubovoy's 60th birthday. In recognition of his merits in developing Schur analysis and his long standing collaboration with the Leipzig group, we decided to organize a workshop on Schur analysis in Leipzig shortly before the winter semester started (see Fig. 13). This workshop took place from October 4 to 6, 2006. V. E., D. Alpay, A. Böttcher, A. Dijksma, A. Lasarow, A. L. Sakhnovich, W. Schempp, E. Wegert and many others followed our invitation (see Figs. 14 and 15).

In many respects, V. E.'s participation has been a highlight. First, his mathematical contribution, of course. But even apart from that, V. E. knows V. K. very well and treasures his friendship since the mid 1960s. He enriched the whole event beyond its mere scientific programme to an extent we as organizers couldn't foresee. V. E. shared many anecdotes about his friendship and shared experiences with V. K. Thus many participants got a first-hand impression of V. K. the person behind his

Fig. 13 V. K. Dubovoy lecturing on October 6, 2006

Fig. 14 V. K. Dubovoy with D. Alpay in the office of B. Kirstein

Fig. 15 From left to right: A. Dijksma, L. A. Ostromukhov, V. E., D. Alpay

mathematics. During their many previous research visits in Leipzig V. E. as well as V. K. developed an affection for the valley of the river Mulde, a lovely scenery located in the southeast of Leipzig. For this reason we organized an excursion with the participants of the workshop into this area (see Fig. 16). The social highlight was a guided tour through Colditz castle, which was used as a detention center for prisoners of war of the rank of officers of the allied forces during WWII. There were

Fig. 16 V. E. with D. Alpay during lunch in Lastau guesthouse

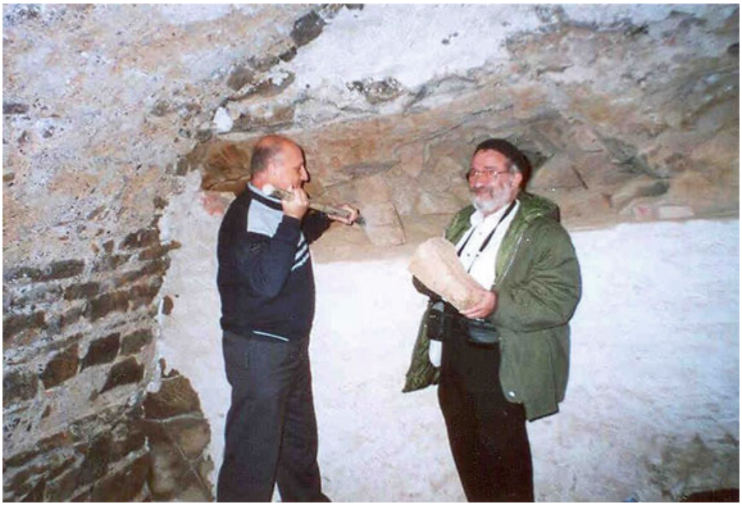

Fig. 17 V. E. with V. K. Dubovoy reenacting an attempted escape during the excursion to Colditz castle

many attempts to escape from this detention center. A TV series that run for many years in Great Britain may have contributed to the castle's particular fame there. With this in mind, please enjoy V. E. and V. K.'s reenacting of an attempted escape in Fig. 17. This excursion with a savory lunch break in the Lastau guesthouse left the participants and the organizers with a pleasant impression of these memorable workshop.

10 The Workshop "25 Years of Schur Analysis in Leipzig" in Fall 2009

The year 2009 has a special place in the history of Leipzig University. On December 2, 2009, the 600th anniversary of its foundation was celebrated. Tied in to this anniversary, numerous scientific conferences were held at Leipzig University in 2009. One of these events was significantly shaped by V. E. More precisely, it was the workshop "25 years of Schur analysis in Leipzig", which took place from September 29 to October 1, 2009 in Leipzig. In the opening speech for this workshop, I outlined major milestones of the development of Schur analysis in Leipzig between 1984 and 2009 (see also [39]). The close collaboration with representatives of the schools of V. P. Potapov and M. G. Krein played a special role in that process. Besides V. E., one should in particular mention D. Z. Arov, L. A. Sakhnovich, and V. K. Dubovoy, whom we owe important impulses. It was particularly delightful that as renowned representatives of the Ukrainian school A. L. Sakhnovich and A. Ya. Kheifets also participated in the conference, who both belonged to the following generation. The main contributions of the workshop are contained in Complex Analysis and Operator Theory 5 (2011), Issue 2. The paper by V. E. treated problems concerning the stability of certain classes of entire functions, which are of interest in connection with the classical Stieltjes moment problem (see [33]).

Figures 18, 19, 20, 21, 22, 23, and 24 convey some impressions of the Workshop "25 years of Schur analysis in Leipzig":

Fig. 18 V. E. giving comments to the lecture of M. Yu. Tyaglov

Fig. 19 October 2009 in Leipzig, from left to right: A. Ya. Kheifets, V. E., V. K. Dubovoy

Fig. 20 S. V. Khrushchev in discussion with V. E.

11 Viktor Emmanuilovich's Last Visit to Leipzig in July 2010

In July 2010, the 21st IWOTA took place in Berlin. The list of participants included V. E., who had been planning this trip to Germany for a while. Due to the geographical proximity of Berlin and Leipzig, it was not surprising that he spent some days in Leipzig before the IWOTA. In particular, he was present at the celebration of my 57th birthday at my home on July 9th, 2010. So far, this

Fig. 21 V. E. with B. Fritzsche

has been my last personal encounter with V. E. On July 11th, 2010, he took the train from Leipzig to Berlin to attend the IWOTA. Apart from me, the rest of the Schur analysis group of Leipzig was there as well some of the days for selected lectures of the IWOTA. V. E. met numerous colleagues from the former Soviet Union again. Being pretty familiar with the city of Berlin since 1991, he served as a tour guide for two of his former students from Kharkov, O. M. Katkova and A. M. Vishnyakova. On that occasion, he also got to know Tanja Eisner (née Lobova), who wrote her diploma thesis at the University of Kharkov in 2002 under supervision of A. M. Vishnyakova before moving to Germany together with her family after completing her studies. Tanja then continued her mathematical career in Tübingen under the supervision of Rainer Nagel and eventually acquired her Ph. D. in 2007 with the work “Stability of operators and $\mathcal{C}_0$-semigroups”. Since September 1, 2013, Tanja Eisner has been holding the chair for functional analysis at Leipzig

Fig. 22 D. Z. Arov with V. K. Dubovoy

Fig. 23 From left to right: A. L. Sakhnovich, D. Alpay, B. Kirstein, V. E., B. Fritzsche

University. This marks an extraordinary milestone in the history of the connection of the mathematical faculties of the two universities in Kharkov and Leipzig. For the first time a female Ukrainian mathematician, who had graduated in Kharkov, was appointed to a chair at Leipzig University. V. E. was particularly delighted by this circumstance (Fig. 25).

Much to our regret, V. E. was not able to accept various invitations for research visits to Leipzig due to health reasons. In honor of V. E.'s 70th birthday on

Fig. 24 From left to right: V. E., V. K. Dubovoy, A. L. Sakhnovich, L. A. Ostromukhov

Fig. 25 IWOTA 2010 in Berlin, from left to right: O. M. Katkova, V. E., A. M. Vishnyakova, T. Eisner

September 3rd, 2013, his former Ph. D. student P. M. Yuditskii organized a mini-workshop on complex analysis and spectral theory at the Johannes Kepler University Linz from May 13 to 14, 2014. V. E. intended to attend this event and to give a lecture on the topic of his work [33]. Unfortunately, he had to cancel his participation due to medical reasons.

12 The Influence of V. E. Katsnelson and D. Z. Arov on the Direction of Our Research Group

While working on generalized matricial Nehari problems (see [21]), Bernd and I made first contact with the works of V. E. Katsnelson. The problem is stated as follows:

GENERALIZED MATRICIAL NEHARI PROBLEM: Let $p, q \in \mathbb{N}$. Further, let F_{11} and F_{22} be a non-negative Hermitian $p \times p$ and a $q \times q$ measure, respectively, on the Borelian σ-Algebra $\mathfrak{B}_{\mathbb{T}}$ on $\mathbb{T} := \{z \in \mathbb{C} \colon |z| = 1\}$ and let $(\beta_k)_{k=0}^{\infty}$ be a sequence of complex $p \times q$ matrices. Describe the set $\mathcal{F}(F_{11}, F_{22}, (\beta_k)_{k=0}^{\infty})$ of all σ-additive mappings F_{12} from $\mathfrak{B}_{\mathbb{T}}$ into the set of all complex $p \times q$ matrices fulfilling the conditions

$$\int_{\mathbb{T}} z^{-k} F_{12}(\mathrm{d}z) = \beta_k, \qquad k = 0, 1, 2, \ldots$$

and for which

$$\begin{pmatrix} F_{11} & F_{12} \\ F_{12}^* & F_{22} \end{pmatrix}$$

is a non-negative Hermitian $(p+q) \times (p+q)$ measure on $\mathfrak{B}_{\mathbb{T}}$. In particular, state necessary and sufficient conditions such that the set $\mathcal{F}(F_{11}, F_{22}, (\beta_k)_{k=0}^{\infty})$ is non-empty.

The problem stated above leads one to studying kernels on $\mathbb{N}_0 \times \mathbb{N}_0$ of so-called mixed Toeplitz-Hankel type. To see this, for all $k \in \mathbb{Z}$, set

$$\alpha_k := \int_{\mathbb{T}} z^{-k} F_{11}(\mathrm{d}z) \qquad \text{and} \qquad \delta_k := \int_{\mathbb{T}} z^{-k} F_{22}(\mathrm{d}z)$$

and, for all $(m, n) \in \mathbb{N}_0 \times \mathbb{N}_0$, define

$$K(m, n) := \begin{pmatrix} \alpha_{m-n} & \beta_{m+n} \\ \beta_{m+n}^* & \delta_{n-m} \end{pmatrix}.$$

The kernel K being non-negative definite turns out to be necessary and sufficient for the set $\mathcal{F}(F_{11}, F_{22}, (\beta_k)_{k=0}^{\infty})$ to be non-empty.

The just defined kernel K is also important because of the following observation.

GENERALIZED HERGLOTZ-BOCHNER THEOREM: Let $p, q \in \mathbb{N}$ and let $(\alpha_k)_{k=0}^{\infty}$, $(\beta_k)_{k=0}^{\infty}$, and $(\delta_k)_{k=0}^{\infty}$ be sequences belonging to $\mathbb{C}^{p\times p}$, $\mathbb{C}^{p\times q}$, and $\mathbb{C}^{q\times q}$, respectively. Then there exists a non-negative Hermitian $(p+q) \times (p+q)$ Borelian measure on $\mathbb{T}$ such that for all $m, n \in \{0, 1, 2, \ldots\}$ the equation

$$K(m, n) = \int_{\mathbb{T}} \left[\operatorname{diag}(z^{-m} I_p, z^m I_q)\right] F(\mathrm{d}z) \left[\operatorname{diag}(z^{-n} I_p, z^n I_q)\right]^*$$

is satisfied if and only if K is non-negative definite.

In opposition to the classical matricial Herglotz-Bochner theorem the non-negative Hermitian $(p+q) \times (p+q)$ measure is not uniquely determined by the above integral formulas.

Fig. 26 From the left: V. E., C. Sadosky, V. Vinnikov, B. Fritzsche

The above introduced kernel K appears for the first time in works of R. Arocena, M. Cotlar and C. Sadosky [1, 2, 14], where it is referred to as the generalized Toeplitz kernel. This kernel was studied and renamed by V. E. in [29] who called it kernel of mixed Toeplitz-Hankel type which brings the nature of this object better to the point. The generalized Herglotz-Bochner theorem appears for the first time in [14]. In autumn 1990, I spent a month at the Mittag-Leffler Institute in Djursholm and encountered Mischa Cotlar. In our conversations I realized immediately, how highly he spoke of V. E. When I later told V. E., he asserted me how highly he thought of him. I am not aware of any personal encounters of V. E. with Mischa Cotlar (01.08.1913–16.01.2007). In any case, I became an eye-witness of many encounters of V. E. and Cora Sadosky (23.05.1940–03.10.2010), who in the summer of 1993 visited Leipzig University which is where she also met V. E. A second place of many encounters of V. E. with Cora was the IWOTA 1993 in Vienna (Fig. 26).

Cora Sadosky and V. E.'s relation has always been a very warm one, characterized by their great respect for each other's great mathematical achievements.

By using the matricial version of the F. Riesz-Herglotz theorem, V. E. observed in [34] that the above formulated generalized Nehari problem is closely related to the following problem of block completion for holomorphic matrix functions:

PROBLEM OF THE TROIKA: Let $p, q \in \mathbb{N}$ and let $\alpha\colon \mathbb{D} \to \mathbb{C}^{p\times p}$, $\beta\colon \mathbb{D} \to \mathbb{C}^{p\times q}$, and $\delta\colon \mathbb{D} \to \mathbb{C}^{q\times q}$ matrix functions, which are holomorphic in $\mathbb{D} := \{z \in \mathbb{C}\colon |z| < 1\}$. Describe the set $\mathcal{NC}(\alpha, \beta, \delta)$ of all $q \times p$ matrix functions γ, which are holomorphic in $\mathbb{D}$ and for which

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

is a $(p + q) \times (p + q)$ matrix function that belongs to the Carathéodory class. In particular, characterize the case that the set $\mathcal{NC}(\alpha, \beta, \delta)$ is non-empty.

In [21] we obtained a description of the solution set of the "Problem of the Troika" in terms of their Taylor coefficients. Our approach is based on a combination of Schur analysis methods with an application of the theory of matrix balls developed by Shmulyan in [48].

In this case, drawing on the theory of matrix balls developed by Shmulyan [48], a description of the set $\mathcal{NC}(\alpha, \beta, \delta)$ was given by means of the sequences of their Taylor coefficients.

In continuation of the subject matter of [21], we then intensely studied the works of V. E. on integral representations of non-negative definite kernels in the light of Potapov's method of fundamental matrix inequalities.

During the study visit in Kharkov in spring 1988, Bernd and I received access to numerous deposited manuscripts of V. E., which were in general very difficult to access for foreigners. Among others, this included the monograph [28], which features a comprehensible introduction to the techniques developed by V. E. regarding the usage of methods of J-theory in continuous interpolation problems in analysis. In particular, problems fall within this class of problems which are connected with integral representations for various classes of holomorphic matrix functions. In this context, the moment problems named after Hamburger and Stieltjes, respectively, are particularly remarkable special cases. At that time, Bernd and I had not yet sensed that the matricial versions of these moment problems would dominate our joint research with Conrad Mädler from 2005 until the immediate present. In the second half of the 1980s, we were mainly interested in matricial versions of the interpolation problems named after Schur and Carathéodory, respectively, and the studies of the Weyl matrix balls associated with those. Against this background, the corresponding explanations in [28] concerning the theory of Weyl balls were hence particularly useful to us.

In the year 1991, which V. E. spent entirely in Leipzig apart from one 2-week work visit of A. Dijksma and H. S. V. de Snoo in Groningen, Bernd and I experienced the best opportunity to be a part of his mathematical thought processes. One problem area that V. E. dealt with distinctively at the time was the factorization theory of J-inner functions. Regarding this subject matter, in the second half of the 1980s essential new findings were achieved by D. Z. Arov and proved to be an enrichment of the factorization theory of V. P. Potapov [45]. Following his studies of the matricial version of the classical Nehari interpolation problem, D. Z. Arov encountered fundamental classes of J-inner functions, which had not been observed before, namely, the so-called Arov-regular as well as Arov-singular J-inner functions, respectively. D. Z. Arov showed that each J-inner function can essentially be represented as a product of an Arov-regular and an Arov-singular J-inner function. (The terminologies "Arov-regular" and "Arov-singular" were introduced upon recommendation of V. E. Earlier D. Z. Arov simply called these types of functions regular and singular.) More precisely, one actually has to distinguish between left-hand Arov-regular and right-hand Arov-regular J-inner functions. An example for a right-hand (resp. left-hand) Arov-regular J-inner matrix function are left (resp. right) Blaschke-Potapov products. In [28], V. E. found remarkable connections between left and right Blaschke-Potapov products. E. g.

he showed the existence of a left Blaschke-Potapov product $B^{(l)}$ that is no right Blaschke-Potapov product, i. e. $B^{(l)}$ has a non-constant Arov-singular right-hand divisor. Thus, he obtained a factorization $B^{(l)} = E \cdot B^{(r)}$ with a non-constant Arov-singular J-inner function E and a right Blaschke-Potapov product $B^{(r)}$ (see [30, Theorem I]). Even more surprising seems the following result: If E is a given Arov-singular J-inner matrix function, then there always exists a right Blaschke-Potapov product $B^{(r)}$ such that the function $W := E \cdot B^{(r)}$ is a left Blaschke-Potapov product (see [30, Theorem II]). Unfortunately, the work [30] does not contain any proofs, because it was written during a short stay of V. E. at the Weizmann Institute by invitation of Harry Dym, which was his second trip abroad ever after his visit of Leipzig in fall 1989. In the year 1992, V. E. wrote an elaborate Russian manuscript in which he picked up the topic of the work [30] once again and which provided proofs and numerous illustrating examples. We were able to watch the arrangement of this manuscript step by step. This way, we got an immediate idea of his impressive creativity combined with the ability to efficiently apply various mathematical methods. We will never forget these months of 1990 and are extremely thankful to V. E. for sharing his mathematical thoughts. At this point, I would like to point out that, as far as I know, V. E. has unfortunately not yet published the Russian manuscript mentioned above with its proofs for the work [30].

In his work [32], V. E. took up the subject of the main results in [30] once again. There, he explained the strategy to justify the proof of Theorem II in [30]. In order to reduce technical difficulties, he focused on the special case of the 2×2 signature matrix $j_{1,1} := \operatorname{diag}(1, -1)$ and elucidated an important intermediate step that had to be done in the proof of [30, Theorem II]. The statement of Theorem II in Proposition 19 is formulated as follows:

Theorem *Let E be an Arov-singular $j_{1,1}$-inner matrix function. Then there exists an infinite right Blaschke-Potapov product $B^{(r)}$ with poles in the open unit disc $\mathbb{D} := \{z \in \mathbb{C}\colon |z| < 1\}$ such that the function $W := E \cdot B^{(r)}$ is an infinite left Blaschke-Potapov product.*

We will now outline some of the core concepts of the proof strategy of V. E. The first consideration is based on the observation that an infinite left Blaschke-Potapov product is always the resolvent matrix of some infinite Carathéodory problem with infinitely many solutions. It hence matters to determine the interpolation data $(z_k, p_k)_{k\in\mathbb{N}}$ on the basis of the given Arov-singular $j_{1,1}$-inner function E. Consequently, the problem is to find a description of the set of all functions $f \in \mathcal{C}(\mathbb{D})$ satisfying the condition $f(z_k) = p_k$ for all $k \in \mathbb{N}$, where $\mathcal{C}(\mathbb{D})$ denotes the set of all functions with non-negative real part, which are holomorphic in $\mathbb{D}$. The determination of the data sequence $(z_k, p_k)_{k\in\mathbb{N}}$ can be broken down into two subtasks. Determining the sequence $(p_k)_{k\in\mathbb{N}}$ is decidedly easier. To do this, let

$$E = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be the element representation of the matrix function E. Due to the choice of E, the function

$$P_{\mathrm{pr}} := (a \cdot 1 + b)(c \cdot 1 + d)^{-1}$$

belongs to the class $\mathcal{C}(\mathbb{D})$. Let $\mathbb{T} := \{z \in \mathbb{C}\colon |z| = 1\}$ be the unit circle and let m denote the normalized Lebesgue measure on the Borelian σ-algebra of $\mathbb{T}$. Since P_{pr} belongs to $\mathcal{C}(\mathbb{D})$, there then exist a measurable function $\underline{P_{\mathrm{pr}}}\colon \mathbb{T} \to \mathbb{C}$ and a m-zero set B_0 such that the condition

$$\lim_{r \to 1-0} P_{\mathrm{pr}}(rz) = \underline{P_{\mathrm{pr}}}(z), \qquad z \in \mathbb{T} \setminus B_0$$

is fulfilled. Setting

$$w_{\mathrm{pr}} := \mathrm{Re}\, \underline{P_{\mathrm{pr}}},$$

it follows

$$\int_{\mathbb{T}} \ln w_{\mathrm{pr}} \mathrm{d}m > -\infty.$$

Considering a Blaschke sequence $(z_k)_{k \in \mathbb{N}}$, i. e., a sequence $(z_k)_{k \in \mathbb{N}}$ of distinct points in $\mathbb{D}$ which fulfill the Blaschke condition

$$\sum_{k \in \mathbb{N}} (1 - |z_k|) < \infty,$$

then the Carathéodory problem with interpolation data $(z_k, P_{\mathrm{pr}}(z_k))_{k \in \mathbb{N}}$ has infinitely many solutions. Now the Blaschke sequence $(z_k)_{k \in \mathbb{N}}$ needs to be chosen in such a manner that the desired connection with the function E is reached. This can be achieved via an appropriately constructed problem of approximating pseudo-continuable holomorphic functions by rational functions with prescribed poles. The study of this approximation problem is the actual central topic of [32]. Once the construction of the sequence $(z_k)_{k \in \mathbb{N}}$ is done, it follows the consideration of the sequence of resolvent matrices $(B_n)_{n \in \mathbb{N}}$ belonging to the interpolation problem $((z_k, P_{\mathrm{pr}}(z_k))_{k \geq n})_{n \in \mathbb{N}}$ for the class $\mathcal{C}(\mathbb{D})$. It turns out that this sequence $(B_n)_{n \in \mathbb{N}}$ converges and satisfies the condition $\lim_{n \to \infty} B_n(z) = E(z)$ for all $z \in \mathbb{D}$. In a natural way, this consideration of the sequence $(B_n)_{n \in \mathbb{N}}$ can be viewed as an inverse Schur-type algorithm.

It shall be mentioned that Viktor Emmanuilovich submitted the work to the journal “Zeitschrift für Analysis und ihre Anwendungen” on March 16, 1992. Recalling that he left Leipzig for Israel on January 22, 1992, this shows that he has already written the work [32] in pretty much its entirety in Leipzig and that, in fact, he let us partake in every phase of its creation. This way, we got to convince ourselves first-handedly of the exceptional creativity, the broad mathematical knowledge, and the virtuous mastery of even highly demanding mathematical techniques. The time in 1991, which Bernd and I spent with him in Leipzig, shaped our way of approaching mathematical problems significantly. It is our most inner need to express our deepest gratitude to V. E.

Another fortunate circumstance in our mathematical career was working together with Damir Zjamovich Arov. Our first direct contact to Damir Zjamovich occurred in early summer of 1989 during the annually organized summer school by Vlastimir Pták on functional analysis, which took place in Liptovský Ján in the Tatra Mountains that year. There, I met D. Z. and invited him to an international seminar on Schur analysis in Leipzig planned for October 1989. I also had first conversations with D. Z., which have already indicated that there are numerous overlaps regarding joint research interests. A deepening of this discussion was arranged for D. Z.'s first visit in Leipzig in October 1989. Due to the decease of Mark Grigorevich Krein on October 17, 1989, this visit was delayed by 1 week. D. Z. only arrived in Leipzig 1 day after the end of the seminar on Schur analysis. Following the seminar, a 3-week work stay in Leipzig was planned for D. Z. This plan could then be realized. In the first of those 3 weeks in Leipzig, D. Z. had the chance to talk to P. R. Masani several times. Masani's results about analytic foundations of prediction theory of multivariate stationary sequences and related problems had provided important sources for the works of V. M. Adamyan and D. Z. Arov. During the International Congress of Mathematicians in Moscow in 1966, both V. M. Adamyan and D. Z. Arov had already had profound professional conversations with P. R. Masani. Between the years 1989 and 1997, Bernd and I were fortunate to work on various problems regarding J-inner functions with D. Z. Arov. This included, e. g. the analysis of the block structure of J-inner functions as well as the construction of J-inner functions from given blocks (see [4–8]).

In the joint work [7] with D. Z., we addressed some of the problems raised by V. E. in [30]. In particular, we obtained an alternative proof of a theorem by V. E. Katsnelson about the Potapov factorization. Moreover, an inverse problem for Arov-singular J-inner functions in the case of the special signature matrix $J := \operatorname{diag}(1, -1)$ was solved via crucial usage of [30, Theorem II] (see [7, Theorem 12]).

Looking back, Bernd and I are very thankful that we had the opportunity to gather plenty of helpful information about the essence of J-inner functions and their role in Schur analytical contexts during our conversations with V. E. Katsnelson and D. Z. Arov. It should be mentioned that D. Z. Arov pointed out an essential difference regarding their respective conceptions of this object to us. While he himself focused more on Arov-regular J-inner functions, V. E. was particularly interested in the Arov-singular world.

V. E.'s view of mathematics can be accurately described by him mainly considering distinctive special cases, which were not overpowered by technical difficulties and which thus conveyed the nature of the underlying phenomenon more clearly. His principle has always been:

> The best is the worst enemy of the good.

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References

1. R. Arocena and M. Cotlar. Generalized Toeplitz kernels and Adamjan-Arov-Kreĭn moment problems. In *Toeplitz centennial (Tel Aviv, 1981)*, volume 4 of *Operator Theory: Adv. Appl.*, pages 37–55. Birkhäuser, Basel-Boston, Mass., 1982.
2. R. Arocena, M. Cotlar, and C. Sadosky. Weighted inequalities in L^2 and lifting properties. In *Mathematical analysis and applications, Part A*, volume 7 of *Adv. in Math. Suppl. Stud.*, pages 95–128. Academic Press, New York-London, 1981.
3. D. Z. Arov. Realization of matrix-valued functions according to Darlington. *Izv. Akad. Nauk SSSR Ser. Mat.*, 37:1299–1331, 1973.
4. D. Z. Arov, B. Fritzsche, and B. Kirstein. Completion problems for j_{pq}-inner functions. I. *Integral Equations Operator Theory*, 16(2):155–185, 1993.
5. D. Z. Arov, B. Fritzsche, and B. Kirstein. Completion problems for j_{pq}-inner functions. II. *Integral Equations Operator Theory*, 16(4):453–495, 1993.
6. D. Z. Arov, B. Fritzsche, and B. Kirstein. On block completion problems for j_{qq}-J_q-inner functions. I. The case of a given block column. *Integral Equations Operator Theory*, 18(1):1–29, 1994.
7. D. Z. Arov, B. Fritzsche, and B. Kirstein. On block completion problems for j_{qq}-J_q-inner functions. II. The case of a given $q \times q$ block. *Integral Equations Operator Theory*, 18(3):245–260, 1994.
8. D. Z. Arov, B. Fritzsche, and B. Kirstein. On some aspects of V. E. Katsnelson's investigations on interrelations between left and right Blaschke-Potapov products. In *Operator theory and boundary eigenvalue problems (Vienna, 1993)*, volume 80 of *Oper. Theory Adv. Appl.*, pages 21–41. Birkhäuser, Basel, 1995.
9. D. Bessis, P. Moussa, and M. Villani. Monotonic converging variational approximations to the functional integrals in quantum statistical mechanics. *J. Mathematical Phys.*, 16(11):2318–2325, 1975.
10. A. Beurling. *The collected works of Arne Beurling. Vol. 1. Complex analysis, Edited by L. Carleson, P. Malliavin, J. Neuberger and J. Wermer*. Contemporary Mathematicians. Birkhäuser Boston, Inc., Boston, MA, 1989.
11. A. Beurling. *The collected works of Arne Beurling. Vol. 2. Harmonic analysis, Edited by L. Carleson, P. Malliavin, J. Neuberger and J. Wermer*. Contemporary Mathematicians. Birkhäuser Boston, Inc., Boston, MA, 1989.
12. A. Beurling and P. Malliavin. On Fourier transforms of measures with compact support. *Acta Math.*, 107:291–309, 1962.
13. F. Clivaz. Stahl's theorem (aka BMV conjecture): insights and intuition on its proof. In *Spectral theory and mathematical physics*, volume 254 of *Oper. Theory Adv. Appl.*, pages 107–117. Birkhäuser/Springer, [Cham], 2016.
14. M. Cotlar and C. Sadosky. On the Helson-Szegő theorem and a related class of modified Toeplitz kernels. In *Harmonic analysis in Euclidean spaces (Proc. Sympos. Pure Math., Williams Coll., Williamstown, Mass., 1978), Part 1*, Proc. Sympos. Pure Math., XXXV, Part, pages 383–407. Amer. Math. Soc., Providence, R.I., 1979.
15. L. de Branges. *Hilbert spaces of entire functions*. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1968.
16. V. K. Dubovoy, B. Fritzsche, and B. Kirstein. *Matricial version of the classical Schur problem*, volume 129 of *Teubner-Texte zur Mathematik [Teubner Texts in Mathematics]*. B. G. Teubner Verlagsgesellschaft mbH, Stuttgart, 1992. With German, French and Russian summaries.

17. P. L. Duren. *Theory of H^p spaces*. Pure and Applied Mathematics, Vol. 38. Academic Press, New York-London, 1970.
18. H. Dym, B. Fritzsche, V. E. Katsnelson, and B. Kirstein, editors. *Topics in interpolation theory*, volume 95 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 1997. Including papers from the Workshop on Recent Developments in Schur Analysis held in honor of the 80th anniversary of the birth of V. P. Potapov at Leipzig University, Leipzig, August 1994.
19. H. Dym and V. Katsnelson. Contributions of Issai Schur to analysis. In *Studies in memory of Issai Schur (Chevaleret/Rehovot, 2000)*, volume 210 of *Progr. Math.*, pages xci–clxxxviii. Birkhäuser Boston, Boston, MA, 2003.
20. A. E. Eremenko. Herbert Stahl's proof of the BMV conjecture. *Mat. Sb.*, 206(1):97–102, 2015.
21. B. Fritzsche and B. Kirstein. On generalized Nehari problems. *Math. Nachr.*, 138:217–237, 1988.
22. J. B. Garnett. *Bounded analytic functions*, volume 96 of *Pure and Applied Mathematics*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1981.
23. L. Golinskii and I. Mikhailova. Hilbert spaces of entire functions as a J theory subject. Edited by V. P. Potapov [Preprint No. 28-80, Inst. Low Temp. Phys. Engrg., Khar′kov, 1980]. In *Topics in interpolation theory (Leipzig, 1994)*, volume 95 of *Oper. Theory Adv. Appl.*, pages 205–251. Birkhäuser, Basel, 1997. Translated from the Russian.
24. H. Helson. *Lectures on invariant subspaces*. Academic Press, New York-London, 1964.
25. H. Helson and D. Lowdenslager. Prediction theory and Fourier series in several variables. *Acta Math.*, 99:165–202, 1958.
26. H. Helson and D. Lowdenslager. Prediction theory and Fourier series in several variables. II. *Acta Math.*, 106:175–213, 1961.
27. K. Hoffman. *Banach spaces of analytic functions*. Prentice-Hall Series in Modern Analysis. Prentice-Hall, Inc., Englewood Cliffs, N. J., 1962.
28. V. E. Katsnelson. Methods of J-theory in continuous interpolation problems of analysis. Part I (Russian). Deposited in VINITI, 1983.
29. V. E. Katsnelson. Integral representation of Hermitian positive kernels of mixed type and the generalized Nehari problem. I. *Teor. Funktsiĭ Funktsional. Anal. i Prilozhen.*, (43):54–70, 1985.
30. V. E. Katsnelson. Left and right Blaschke-Potapov products and Arov-singular matrix-valued functions. *Integral Equations Operator Theory*, 13(6):836–848, 1990.
31. V. E. Katsnelson. Extremal problems and other related problems of analysis. Part I. The scalar case (Russian). Leipzig, 1991.
32. V. E. Katsnelson. Weighted spaces of pseudocontinuable functions and approximations by rational functions with prescribed poles. *Z. Anal. Anwendungen*, 12(1):27–67, 1993.
33. V. E. Katsnelson. Stieltjes functions and Hurwitz stable entire functions. *Complex Anal. Oper. Theory*, 5(2):611–630, 2011.
34. V. E. Katsnelson. The function $\cosh(\sqrt{a\,t^2+b})$ is exponentially convex. *Integral Equations Operator Theory*, 85(3):381–398, 2016.
35. V. E. Katsnelson. The matrix function e^{tA+B} is representable as the Laplace transform of a matrix measure. *Integral Equations Operator Theory*, 86(3):439–452, 2016.
36. V. E. Katsnelson. On a special case of the Herbert Stahl theorem. *Integral Equations Operator Theory*, 86(1):113–119, 2016.
37. V. E. Katsnelson. On the roots of a hyperbolic polynomial pencil. *Arnold Math. J.*, 2(4):439–448, 2016.
38. V. E. Katsnelson. On the BMV conjecture for 2×2 matrices and the exponential convexity of the function $\cosh(\sqrt{at^2+b})$. *Complex Anal. Oper. Theory*, 11(4):843–855, 2017.
39. V. E. Katsnelson and B. Kirstein. 25 years of Schur analysis in Leipzig. *Complex Anal. Oper. Theory*, 5(2):325–330, 2011.
40. P. Koosis. *Introduction to H_p spaces*, volume 40 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge-New York, 1980. With an appendix on Wolff's proof of the corona theorem.

41. P. Koosis. *The logarithmic integral. I*, volume 12 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1988.
42. P. Koosis. *The logarithmic integral. II*, volume 21 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1992.
43. P. R. Masani. The prediction theory of multivariate stochastic processes. III. Unbounded spectral densities. *Acta Math.*, 104:141–162, 1960.
44. P. R. Masani. Shift invariant spaces and prediction theory. *Acta Math.*, 107:275–290, 1962.
45. V. P. Potapov. The multiplicative structure of J-contractive matrix functions. *Trudy Moskov. Mat. Obšč.*, 4:125–236, 1955.
46. M. Rosenblum and J. Rovnyak. *Hardy classes and operator theory*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1985. Oxford Science Publications.
47. M. Rosenblum and J. Rovnyak. *Topics in Hardy classes and univalent functions*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Verlag, Basel, 1994.
48. Yu. L. Shmulyan. Operator balls. *Integral Equations Operator Theory*, 13(6):864–882, 1990.
49. H. R. Stahl. Proof of the BMV conjecture. *Acta Math.*, 211(2):255–290, 2013.
50. N. Wiener and P. R. Masani. The prediction theory of multivariate stochastic processes. I. The regularity condition. *Acta Math.*, 98:111–150, 1957.
51. N. Wiener and P. R. Masani. The prediction theory of multivariate stochastic processes. II. The linear predictor. *Acta Math.*, 99:93–137, 1958.

A Piece of Victor Katsnelson's Mathematical Biography

Mikhail Sodin

Abstract We give an overview of several works of Victor Katsnelson published in 1965–1970, and pertaining to the complex and harmonic analysis and the spectral theory.

1 A Preamble

As a mathematician, Victor Katsnelson was raised within a fine school of function theory and functional analysis, which was blossoming in Kharkov starting the second half of 1930s. He studied in the Kharkov State University in 1960–1965. Among his teachers were Naum Akhiezer, Boris Levin, Vladimir Marchenko. That time he became acquainted with Vladimir Matsaev whom Victor often mentions as one of his teachers. In 1965 Katsnelson graduated with the master degree, Boris Levin supervised his master thesis. Since then and till 1990, he teaches at the Department of Mathematics and Mechanics of the Kharkov State University. In 1967 he defends the PhD Thesis "Convergence and Summability of Series in Root Vectors of Some Classes of Non-Selfadjoint Operators" also written under Boris Levin guidance. Until he left Kharkov in the early 1990s, Katsnelson remained an active participant of the Kharkov function theory seminar run on Thursdays by Boris Levin and Iossif Ostrovskii. His talks, remarks and questions were always interesting and witty.

Already in the 1960s Victor established himself among the colleagues as one of the finest Kharkov mathematicians of his generation, if not the finest one. Nevertheless, he was not appointed as a professor and was never allowed to travel abroad.

Most of Katsnelson's work pertain to the spectral theory of functions and operators. I will touch only a handful of his results, mostly published in 1965–1970,

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that is, at the very beginning of his mathematical career. A big portion of his works written in Kharkov appeared in the local journal "Function Theory, Functional Analysis and Their Applications" and were never translated in English. Today, this journal is available at http://dspace.univer.kharkov.ua/handle/123456789/43.

In this occasion, let me mention two wonderful books carefully written by Katsnelson [18, 19]. They exist only as manuscripts, and curiously, both have "Part I" in their titles, though, as far as I know, no continuations appeared. In both books mathematics interlaces with interesting historical comments. Last but not least, let me also mention an extensive survey of Issai Schur's works in analysis written jointly by Dym and Katsnelson [7].

2 A Paley-Wiener-Type Theorem

The paper [14] was, probably, the first published work of Katsnelson. Therein, he studied the following question raised by Boris Levin. Given a convex compact set $K \subset \mathbb{C}$ with the boundary $\Gamma = \partial K$, let $\mathsf{L}^2(\Gamma)$ be the L^2-space of function on Γ with respect to the Lebesgue length measure. *How to characterize entire functions F represented by the Laplace integral*

$$F(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma f(w) e^{wz}\, \mathrm{d}w, \tag{2.1}$$

with $f \in \mathsf{L}^2(\Gamma)$?

In the case when $K = \Gamma$ is an interval, the answer is provided by the classical Paley-Wiener theorem. In this case, it is convenient to assume that $\Gamma \subset \mathrm{i}\mathbb{R}$. Then we can rewrite (2.1) as follows

$$F(z) = \frac{1}{2\pi \mathrm{i}} \int_{\mathrm{i}a}^{\mathrm{i}b} f(w) e^{wz}\, \mathrm{d}w = \frac{1}{2\pi} \int_a^b \varphi(t) e^{\mathrm{i}tz}\, \mathrm{d}t\,, \qquad \varphi \in \mathsf{L}^2(a,b)\,,$$

and, by the Paley-Wiener theorem, a necessary and sufficient condition for this representation with some $a < b$ is that F is an entire function of exponential type (EFET, for short) and $F \in \mathsf{L}^2(\mathbb{R})$.

Now, assume that the convex compact K is not an interval, that is, is a closure of its interior, and put $\Omega_K = \overline{\mathbb{C}} \setminus K$. Note that the Laplace transform of F coincides with the Cauchy integral of f:

$$\int_0^\infty F(z) e^{-\lambda z}\, \mathrm{d}z = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{f(w)}{\lambda - w}\, \mathrm{d}w\,.$$

The RHS is analytic in Ω_K, vanishes at infinity, and belongs to the Smirnov space $\mathsf{E}^2(\Omega_K)$, which can be defined, for instance, as the closure in $\mathsf{L}^2(\Gamma)$ of analytic functions in Ω_K, continuous up to the boundary, and vanishing at infinity. Thus, Levin's question can be reformulated as follows: *Given a convex compact set K with non-empty interior, find a complete normed space B_K of EFET such that the Laplace integral $\mathcal{L}$ defined in* (2.1) *gives a bounded bijection* $\mathsf{E}^2(\Omega_K) \overset{\mathcal{L}}{\to} \mathsf{B}_K$. Note that the representation (2.1) yields that all functions $F \in \mathsf{B}_K$ have the growth bound

$$|F(re^{\mathrm{i}\theta})| \leqslant C(\Gamma)\, \|f\|_{\mathsf{L}^2(\Gamma)}\, \exp\big[\max_{w\in K} \mathrm{Re}(we^{\mathrm{i}\theta})r\big] = C(\Gamma)\, \|f\|_{\mathsf{L}^2(\Gamma)}\, e^{h_K(-\theta)r},$$

where $h_K(\theta)$ is the supporting function of K.

The first result in that direction is due to Levin himself who considered in [24, Appendix I, Section 3] the case when K is a convex polygon and noticed that in this case the answer is a straightforward consequence of the classical version of the Paley-Wiener theorem. Then, M. K. Liht [26] considered the case when K is a disk centered at the origin and of radius h. He showed that in this case one can take B_K being a Bargmann-Fock-type space, which consists of entire functions F satisfying

$$\int_0^\infty \int_{-\pi}^{\pi} |F(re^{\mathrm{i}\theta})|^2\, e^{-2hr}\, \sqrt{r}\, \mathrm{d}r\, \mathrm{d}\theta < \infty\,.$$

The starting point of Katsnelson's work [14] was a remark that a more accurate version of the Liht argument yields an isometry

$$\int_\Gamma |f|^2\, |\mathrm{d}w| = \int_0^\infty \int_{-\pi}^{\pi} |F(re^{\mathrm{i}\theta})|^2\, e^{-2hr}\, \rho(hr)\, \mathrm{d}r\, \mathrm{d}\theta\,,$$

where

$$\rho(r) = 2r \int_0^\infty \frac{e^{-2tr}}{\sqrt{2t+t^2}}\, \mathrm{d}t\,.$$

Then, Katsnelson proves that representation (2.1) yields a uniform bound

$$\sup_{|\theta|\leqslant\pi} \int_0^\infty |F(re^{\mathrm{i}\theta})|^2 e^{-2h_K(-\theta)r}\, \mathrm{d}r \leqslant C(\Gamma)\, \|f\|^2_{\mathsf{L}^2(\Gamma)}\,.$$

The proof is based on the following lemma close in the spirit to known estimates due to Gabriel and Carlson.

Lemma 2.1 *Suppose that K is a convex compact set, $\Gamma = \partial K$, $\Omega_K = \overline{\mathbb{C}} \setminus K$, and $f \in \mathsf{E}^2(\Omega_K)$. Then, for any supporting line ℓ to Γ,*

$$\int_\ell |f|^2\, |\mathrm{d}w| \leqslant C(\Gamma) \int_\Gamma |f|^2\, |\mathrm{d}w|\,.$$

The constant on the RHS does not depend on ℓ and f.

One can modify Levin's question replacing the space $\mathsf{E}^2(\Omega_K)$ by another space of functions analytic in Ω_K. If functions in that space do not have boundary values on Γ, then one needs to replace the integral over Γ on the RHS of (2.1) by the contour integral

$$\frac{1}{2\pi \mathrm{i}} \int_\gamma f(w) e^{wz}\, \mathrm{d}w,$$

where γ is a simple closed contour in Ω_K, which contains K in its interior. This integral is called the Borel transform of f. It acts on the Taylor coefficients as follows:

$$f(w) = \sum_{n\geqslant 0} \frac{a_n}{w^{n+1}} \mapsto F(z) = \sum_{n\geqslant 0} \frac{a_n}{n!}\, z^n.$$

One of possible modifications of Levin's question was considered in [14]. Katsnelson introduces the weight

$$\rho_\Gamma(w) = \frac{1}{|w - a_1(w)| + |w - a_2(w)|}\,,$$

where $a_j(w)$, $j = 1, 2$, are supporting points for the line supporting to Γ that passes through $w \in \Omega_K$ (the weight $\rho_\Gamma(w)$ is not defined when w belongs to the supporting line to Γ that has a common segment with Γ). The last result proven in [14] is a curious isometry

$$\int_0^\infty \int_{-\pi}^{\pi} |F(re^{\mathrm{i}\theta})|^2\, e^{-2h_K(-\theta)r}\, \mathrm{d}r\, \mathrm{d}\theta = \frac{1}{2\pi} \iint_{\Omega_K} |f(w)|^2 \rho_\Gamma(w)\, \mathrm{d}\sigma(w)\,,$$

where σ is the Lebesgue area measure.

Works of Liht and Katsnelson had follow-ups. In [30], Lyubarskii extended Liht's theorem to convex compact sets K with smooth boundary. The decisive word was said by Lutsenko and Yulmukhametov. In [29] they proved that the Laplace integral $\mathcal{L}$ defines an isomorphism[1] between $\mathsf{E}^2(\Omega_K)$ and a space of EFET such that

$$\int_0^\infty \int_{-\pi}^{\pi} |F(re^{\mathrm{i}\theta})|^2\, \frac{\mathrm{d}r\, \mathrm{d}\Delta(\theta)}{K(re^{\mathrm{i}\theta})}\,,$$

where

$$K(z) = \|e^{wz}\|^2_{\mathsf{E}^2(\Omega_K)} = \int_\Gamma e^{2\,\mathrm{Re}(wz)}\, |\mathrm{d}w|\,,$$

[1]Understood as an isomorphism between Banach spaces.

and $d\Delta(\theta) = (h''(-\theta) + h(-\theta))d\theta$ (understood as a distribution). One of the novelties in their work is the fact that the identity map provides an isomorphism between the Smirnov space $\mathsf{E}^2(\Omega_K)$ and the space of analytic functions in Ω_K vanishing at infinity, with finite Dirichlet-type integral

$$\iint_{\Omega_K} |f'(w)|^2 \,\mathrm{dist}(w, \Gamma)\, d\sigma(w) < \infty.$$

The proof of that fact relies on Lemma 2.1. We mention that Yulmukhametov together with his pupils and collaborators proved several other non-trivial results related to Levin's question (see, for instance, [13, 28]) and that Lindholm [27] extended the Lutsenko-Yulmukhametov theorem to analytic functions of several complex variables.

3 Riesz Bases of Eigenvectors of Non-selfadjoint Operators

One of the central questions in the spectral theory is the expansion in eigenfunctions (more generally, in root vectors) of non-selfadjoint operators. It originates in the theories of ordinary and partial differential equations and of integral equations. In the middle of the 1960s the corresponding completeness problem was already understood relatively well, first of all, due to the pioneering works by Keldysh and Matsaev. A portion of their works can be found in the classical Gohberg-Krein book [10], another portion became available later in [22] and in [31, 32]. The situation with convergence of the series of eigenfunctions was understood much less clearly. Though a few results, due to Glazman, Mukminov, and Markus, were known (all of them were summarized in [10, Chapter VI]), no general methods existed until in [15] Katsnelson discovered a novel approach to the Riesz basis property of eigenfunctions of arbitrary contractions and dissipative operators. His approach is based on a deep result of Carleson pertaining to the interpolation by bounded analytic functions in the unit disk.

We start with some definitions. First, we remind the notion of Riesz basis of a system of subspaces (X_k) of a Hilbert space H. The details can be found in [10, Chapter VI]. In the case when all subspaces (X_k) are one-dimensional, this notion reduces to the usual notion of the Riesz basis of vectors in H.

Let (X_k) be a collection of linear subspaces of H, and X be the closure of their linear span. *The subspaces* (X_k) *form a basis in* X if any vector $x \in \mathsf{X}$ has a unique decomposition into a convergent series $x = \sum_k x_k,\ x_k \in \mathsf{X}_k$. To simplify notation, we assume that the linear span of the subspaces (X_k) is dense in H, i.e., that $\mathsf{X} = \mathsf{H}$. Let P_k be projectors on X_k. Then the system (X_k) forms a basis if and only if $\mathsf{P}_k\mathsf{P}_j = \delta_{kj}\mathsf{P}_k$, and $\sup_n \left\| \sum_{k=1}^n \mathsf{P}_k \right\| < \infty$. The subspaces (X_k) form *an orthogonal basis* if all P_ks are orthogonal projectors, that is, for any x, $\|x\|^2 = \|\mathsf{P}_k x\|^2 + \|(\mathsf{I} - \mathsf{P}_k)x\|^2$. The subspaces (X_k) form *a Riesz basis* if there exists an invertible operator A from H onto H such that subspaces $(\mathsf{A}\mathsf{X}_k)$ form an orthogonal basis. Gelfand's theorem [10, Chapter VI, $ 5] says that *a basis of subspaces* (X_k) *is a Riesz basis if and only if it remains a basis after any permutation of its elements*.

In [15] Katsnelson studies the question when the collection of root subspaces of a non-selfadjoint operator is a Riesz basis in the closure of its linear span. He considers two general classes of non-selfadjoint operators, contractions and dissipative operators. A linear operator T on a Hilbert space H is called *a contraction* if $\|\mathsf{T}\| \leqslant 1$. Given an eigenvalue $\lambda \in \overline{\mathbb{D}}$, the linear space

$$\mathsf{X}(\lambda) = \bigcup_{n \geqslant 1} \mathrm{Ker}\big[(\mathsf{T} - \lambda \mathsf{I})^n\big]$$

is called *the root subspace* corresponding to λ. The eigenvalue λ *has finite order* if there exists a positive integer m such that

$$\mathsf{X}(\lambda) = \bigcup_{1 \leqslant n \leqslant m} \mathrm{Ker}\big[(\mathsf{T} - \lambda \mathsf{I})^n\big] = \mathrm{Ker}\big[(\mathsf{T} - \lambda \mathsf{I})^m\big].$$

The least value m is called *the order* $m(\lambda)$ *of the eigenvalue* λ. The following theorem is the main result of [15].

Theorem 3.1 *Let* (λ_k) *be some eigenvalues of a contraction* T*, let* (X_k) *be the corresponding root subspaces, and let* (m_k) *be the orders of* (λ_k)*. Suppose that*

$$\inf_j \prod_{k \neq j} \Big| \frac{\lambda_k - \lambda_j}{1 - \lambda_j \bar{\lambda}_k} \Big|^{m_j m_k} \geqslant \delta > 0\,, \tag{3.1}$$

Then the system of root subspaces (X_k) *forms a Riesz basis in the closure of its linear span.*

In [15] Katsnelson only sketches the proof of this result, some details can be found in Nikolskii's survey paper [36, § 3]. Here are the main steps of the proof.

First, Katsnelson observes that in the assumptions of Theorem 3.1 only those λ_k that lie in the open unit disk matter, while the unitary part of the operator T can be discarded. He also assumes that the linear span of the root subspaces $\mathsf{X}(\lambda_k)$ is dense in H (otherwise, he considers the restriction of T on the closure of this linear span). Keeping in mind Gelfand's theorem, it suffices to find projectors $\mathsf{P}_j : \mathsf{H} \to \mathsf{X}(\lambda_j)$ such that $\mathsf{P}_j \mathsf{X}(\lambda_k) = \{0\}$ for $j \neq k$, and

$$\sup_J \Big\| \sum_{j \in J} \mathsf{P}_j \Big\| < \infty\,,$$

where the supremum is taken over all finite subsets J of the set of all indices j.

Fix a finite set J. Suppose that we succeeded to find an analytic in the unit disk function f_J such that $f_J(\lambda_j) = 1$ for $j \in J$, $f_J^{(\nu)}(\lambda_j) = 0$ for $j \notin J$ and $0 \leqslant \nu \leqslant m(\lambda_j) - 1$, and $\sup_{\mathbb{D}} |f_J| \leqslant M$, with a constant M independent of J. Suppose momentarily that the function f_J is analytic on a neighbourhood of the closed unit disk (that is, that the set (λ_j) is finite). Then, by the F. Riesz operator calculus [38, Chapter IX], $f_J(\mathsf{T})$ is well-defined and, by von Neumann's theorem [38, Section 153, Theorem A], $\|f_J(\mathsf{T})\| \leqslant M$.

At the next step, Katsnelson again uses a piece of the F. Riesz operator calculus. The projectors P_j can be defined by the contour integrals

$$\mathsf{P}_j = -\frac{1}{2\pi \mathrm{i}} \int_{C_j} (\mathsf{T} - \zeta \mathsf{I})^{-1}\, \mathrm{d}\zeta\,,$$

where C_j is a circumference of a small radius which separates the point λ_j from the rest of the spectrum and traversed counterclockwise. Whence,

$$f_J(\mathsf{T}) = -\frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}} f_J(\zeta)(\mathsf{T} - \zeta \mathsf{I})^{-1}\, \mathrm{d}\zeta = \sum_{j\in J} \mathsf{P}_j\,,$$

and therefore,

$$\Big\| \sum_{j\in J} \mathsf{P}_j \Big\| = \|f_J(\mathsf{T})\| = \sup_{\mathbb{D}} |f_J| \leqslant M\,.$$

To get rid of the assumption that the function f_J is analytic on the neighbourhood of the closed unit disk, Katsnelson applies a classical result due to Pick and Schur, which says that *given a bounded analytic function f in the unit disk and given a finite set of points $\Lambda \subset \mathbb{D}$, there exists a rational function R which interpolates f at Λ, that is, $R(\lambda) = f(\lambda)$, $\lambda \in \Lambda$, and* $\max_{\mathbb{D}} |R| = \sup_{\mathbb{D}} |f|$, see, for instance, [8, Corollary IV.1.8].

At the final step, Katsnelson deduces the existence of the analytic function f_J with the properties as above from Carleson's "$0 - 1$-interpolation theorem", which, in turn, was the main step in his solution to the corona problem [5, Theorem 2]. □

This chain of arguments discovered in [15] had a significant impact on works of many mathematicians, notably from the Saint Petersburg school, cf. Nikolskii-Pavlov [34, 35] (apparently, Nikolskii and Pavlov rediscovered some of Katsnelson's results), Treil [40, 41], Vasyunin [45], see also [37, Lectures IX and X].

Katsnelson also notes that condition (3.1) in Theorem 3.1 cannot be weakened. Given a sequence $(\lambda_k) \subset \mathbb{D}$ satisfying the Blaschke condition $\sum_k (1 - |\lambda_k|) < \infty$ and such that

$$\inf_j \prod_{k\ne j} \Big| \frac{\lambda_k - \lambda_j}{1 - \lambda_j \bar{\lambda}_k} \Big|^{m_j m_k} = 0,$$

he brings a simple construction (the idea of which, according to [15], is due to Matsaev) of a contraction T such that

(i) (λ_k) are simple eigenvalues of T and the whole spectrum of T coincides with (λ_k), and
(ii) the eigenvalues of T are complete in H but are not uniformly minimal.[2]

Furthermore in this construction, the operator $\mathsf{I} - \mathsf{T}^*\mathsf{T}$ is one-dimensional.

Among other results brought in [15], there is a version of Theorem 3.1 for dissipative operators, i.e., the operators A such that $\mathrm{Im}\langle \mathsf{A}x, x\rangle \geqslant 0$, for any x in the domain of A. This version is reduced to Theorem 3.1 by an application of the Caley transform $\mathsf{A} \mapsto (\mathsf{A} - i\mathsf{I})(\mathsf{A} + i\mathsf{I})^{-1}$.

4 Series of Simple Fractions

Let $\mathsf{C}_0(\mathbb{R})$ be the Banach space of complex-valued continuous functions on $\mathbb{R}$, tending to zero at infinity, equipped with the uniform norm $\|f\| = \sup_{\mathbb{R}} |f|$. Fix finite subsets in the upper and lower half-planes $\{z_k\}_{1\leqslant k\leqslant n} \subset \mathbb{C}_+$ and $\{w_k\}_{1\leqslant k\leqslant m} \subset \mathbb{C}_-$ and denote by $\mathsf{E}_+ = \mathsf{E}_+(\mathsf{w}_1, \dots, \mathsf{w}_\mathsf{m})$, $\mathsf{E}_- = \mathsf{E}_-(\mathsf{z}_1, \dots, \mathsf{z}_\mathsf{n})$ the subspaces in $\mathsf{C}_0(\mathbb{R})$ generated by the simple fractions $\{1/(t-w_k)\colon 1 \leqslant k \leqslant m\}$ and $\{1/(t-z_k)\colon 1 \leqslant k \leqslant n\}$. The functions in E_+ are analytic on $\mathbb{C}_+$, the functions in E_- are analytic on $\mathbb{C}_-$. Furthermore, $\mathsf{E}_+ \cap \mathsf{E}_- = \varnothing$ and the sum $\mathsf{E} = \mathsf{E}_+ + \mathsf{E}_-$ is a direct one, i.e., for any function $f \in \mathsf{E}$, there exists a unique decomposition $f = f_+ + f_-$ with $f_\pm \in \mathsf{E}_\pm$. In [16] Katsnelson estimates the norms of the projectors $\mathsf{P}_\pm = \mathsf{P}_\pm(\mathsf{z}_1, \dots, \mathsf{z}_\mathsf{n}; \mathsf{w}_1, \dots, \mathsf{w}_\mathsf{m})$ from E onto the corresponding subspace $\mathsf{E}_\pm$. The main result of that work is the following theorem:

Theorem 4.1 *There exists a positive numerical constant C such that*

$$\|\mathsf{P}_\pm\| \leqslant C \min(m,n)\,(m+n)\,.$$

The main point in this theorem is that the upper bound it gives does not depend on the positions of z_ks and w_ks.

Note that in the space $\mathsf{L}^2(\mathbb{R})$, by one of the versions of the Paley-Wiener theorem, the functions analytic in the upper and lower half-planes are orthogonal to each other, which makes the corresponding projectors orthogonal. In view of this remark, it is quite natural that the proof of Theorem [16] uses the Fourier transform. The proof is nice and not too long and the reader can find its details in [16]. Bochtejn and Katsnelson bring in [2] a counterpart of Theorem 4.1 for the unit circle $\mathbb{T}$ instead of the real line $\mathbb{R}$.

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[2] A system of vectors $\{x_n\}$ in the Hilbert space H is called *uniformly minimal* if there exists $\delta > 0$ such that for all n the distance between x_n and the linear span of $\{x_k\colon k \neq n\}$ is at least δ.

Likely, the upper bound given in Theorem 4.1 is not sharp. In [16], Katsnelson conjectures that, for $m = n$, the sharp upper bound should be $C\ \log(n+1)$, and, as far as we know, this conjecture remains open till today. As a supporting evidence towards this conjecture, he brings the following result.

Theorem 4.2 *Given two finite sets of points* $\{z_k\}_{1\leqslant k\leqslant n} \subset \mathbb{C}_+$ *and* $\{w_k\}_{1\leqslant k\leqslant n} \subset \mathbb{C}_-$, *consider the functions*

$$g_+(t) = \sum_{k=1}^{n} \frac{1}{t-w_k}\,, \quad g_-(t) = \sum_{k=1}^{n} \frac{1}{t-z_k}\,,$$

and let $g(z) = g_+(z) + g_-(z)$. *Then*

$$\max_{\mathbb{R}} |g_\pm| \leqslant C \log(n+1) \cdot \max_{\mathbb{R}} |g|\,.$$

with a positive numerical constant C.

A simple example shows that the order of growth of the RHS cannot be improved. Put

$$g_+(t) = \sum_{k=1}^{n} \frac{1}{t+\mathrm{i}k}\,, \quad g_-(t) = \sum_{k=1}^{n} \frac{1}{t-\mathrm{i}k}\,.$$

Then

$$g(t) = g_+(t) + g_-(t) = \sum_{k=1}^{n} \frac{2t}{t^2+k^2}\,,$$

and

$$\max_{\mathbb{R}} |g| \leqslant \int_0^\infty \frac{2t}{t^2+x^2}\,\mathrm{d}x = \pi,$$

while

$$\max_{\mathbb{R}} |g_-| = |g_-(0)| = \sum_{k=1}^{n} \frac{1}{k} = \log n + O(1)\,.$$

The proof of Theorem 4.2 is short and elegant (and accessible to undergraduate students). As a byproduct of that proof he obtains

Theorem 4.3 *Let* P *be a polynomial of degree* $n \geqslant 2$ *such that*

$$\sup_{\mathbb{R}} \left| \frac{P'}{P} \right| \leqslant M\,.$$

Then P has no zeroes in the strip

$$|\operatorname{Im} z| \leqslant \frac{c}{M \log n}\,,$$

where c is a positive numerical constant.

Slightly earlier a similar estimate was obtained by Gelfond [9]. Gelfond's proof was rather different (and more involved). The question about the size of the strip around the real axis free of zeroes of P was raised by Gorin in [11], first results in that direction were obtained by him and then by Nikolaev in [33]. The final word in this question was said by Danchenko, who proved in [6] that under assumption of Theorem 4.3, the polynomial P has no zeroes in the strip

$$|\operatorname{Im} z| \leqslant \frac{c}{M} \cdot \frac{\log\log n}{\log n}\,,$$

and that the order of decay of the RHS cannot be improved.

Twenty five years later, Katsnelson returned in [20, 21] to the linear spans of simple fraction but from a different point of view. That time his work was motivated by Potapov's results on factorization of J-contractive matrix functions.

Let m be the Lebesgue measure on the unit circle $\mathbb{T}$, and $w\colon \mathbb{T} \to [0, \infty]$ be an m-integrable weight, satisfying the Szegő condition

$$\int_{\mathbb{T}} \log w \,\mathrm{d}m > -\infty. \tag{4.1}$$

By $\mathsf{PCH}^2(\mathsf{w})$ Katsnelson denotes the Hilbert space of functions f analytic on $\mathbb{C} \setminus \mathbb{T}$ and satisfying the following conditions

(a) The restriction of f onto $\mathbb{D}_+$ and $\mathbb{D}_-$ belongs to the Smirnov class, i.e., $\log_+ |f|$ has positive harmonic majorants both in $\mathbb{D}_+$ and $\mathbb{D}_-$.
(b) The boundary values of $f\big|_{\mathbb{D}_+}$ and $f\big|_{\mathbb{D}_-}$ coincide m-a.e. on $\mathbb{T}$, that is,

$$\lim_{r\uparrow 1} f(rt) = \lim_{r\downarrow 1} f(rt) \;(=: f(t)) \quad m - \text{a.e. on } \mathbb{T}.$$

(Conditions (a) and (b) together provide the so called pseudocontinuation property of the function f.)
(c)

$$\|f\|_w^2 = \int_{\mathbb{T}} |f|^2 w \,\mathrm{d}m < \infty\,.$$

Note that whenever $w^{-1} \in \mathsf{L}^1(m)$ the space $\mathsf{PCH}^2(\mathsf{w})$ is trivial, i.e., contains only constant functions f. Indeed, convergence of the integrals

$$\int_{\mathbb{T}} |f|^2 w \, \mathrm{d}m < \infty, \quad \int_{\mathbb{T}} \frac{\mathrm{d}m}{w} < \infty$$

yields that $f \in \mathsf{L}^1(m)$, and then, by a version of the removable singularity theorem that goes back to Carleman, the function f is entire, and since it is bounded, by Lioville's theorem it is a constant function.

Given a set of points $S \subset \mathbb{D}_+ \cup \mathbb{D}_-$ satisfying the Blaschke condition

$$\sum_{\lambda \in S \cap \mathbb{D}_+} (1 - |\lambda|) < \infty \,, \qquad \sum_{\lambda \in S \cap \mathbb{D}_-} (1 - |\lambda|^{-1}) < \infty \,, \tag{4.2}$$

denote by $\mathsf{R}(\mathsf{S}; \mathsf{w})$ the closure of the linear span of the simple fractions $\{(t - \lambda)^{-1}\}_{\lambda \in S}$ together with the constant functions in the space $\mathsf{L}^2(\mathsf{w})$. Let $S = S_1 \supset S_2 \supset \ldots$ be a chain of sets such that $\bigcap_n S_n = \{\varnothing\}$.

The starting point of Katsnelson's work [20] is the inclusion $\bigcap_n \mathsf{R}(\mathsf{S_n}; \mathsf{w}) \subset \mathsf{PCH}^2(\mathsf{w})$, which follows from classical results of Tumarkin [42], see also [43, 44] and [46]. Katsnelson observes that this inclusion might be a strict one, that is, generally speaking, not every function in the space $\mathsf{PCH}^2(\mathsf{w})$ can be approximated in $\mathsf{L}^2(\mathsf{w})$ by a sequence of functions $r_n \in \mathsf{R}(\mathsf{S_n}; \mathsf{w})$. The main result of [20] is the following approximation theorem.

Theorem 4.4 *For any non-negative m-integrable function w on $\mathbb{T}$ satisfying the Szegő condition* (4.1)*, there exists a set $S \subset \mathbb{D}_+ \cup \mathbb{D}_-$ satisfying the Blaschke condition* (4.2) *such that $\bigcap_n \mathsf{R}(\mathsf{S_n}; \mathsf{w}) = \mathsf{PCH}^2(\mathsf{w})$.*

In [21] Katsnelson extends this result to a more general approximation scheme by simple fractions with poles at a given table of points in $\mathbb{C} \setminus \mathbb{T}$. In [23] Kheifets used Katsnelson's construction to answer a question raised by Sarason.

5 Spectral Radius of Hermitian Elements in Banach Algebras and the Bernstein Inequality

One of the most important properties of EFET is the classical Bernstein inequality, which states that *if F is an entire function of exponential type σ, then*

$$\sup_{\mathbb{R}} |F'| \leqslant \sigma \sup_{\mathbb{R}} |F| \,,$$

and the equality sign attains if and only if $F(z) = c_1 \cos \sigma z + c_2 \sin \sigma z$. Different proofs, deep extensions, and various applications of the Bernstein inequality can be found in the books [1, 24, 25] and in the survey paper [12]. Interestingly, Bernstein's inequality is also closely related to the theory of Banach algebras.

An element a of a Banach algebra A is called *hermitian* if $\|e^{\mathrm{i}at}\| = 1$ for every $t \in \mathbb{R}$. For instance, hermitian elements of the algebra of all bounded operators in a Hilbert space are self-adjoint operators. Another, more special, example is the differentiation operator $\mathsf{D} = \frac{1}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}x}$ considered in various Banach spaces of EFET equipped with some translation-invariant norm, in which case the exponent $e^{\mathrm{i}\mathsf{D}t}$ is realized by the translation. It is well-known that the operator norm of a self-adjoint operator in a Hilbert space coincides with its spectral radius. Making use of the Bernstein inequality, Katsnelson proved in [17] the following result, which, independently (and more or less simultaneously), was also found by Browder [4] and Sinclair [39].

Theorem 5.1 *For every hermitian element in a Banach algebra, the norm coincides with the spectral radius.*

Moreover, as both Katsnelson and Browder observed, this result is *equivalent* to the Bernstein inequality, that is, *the latter follows from the former, applied to the differentiation operator* D *in the Bernstein space* B_σ *of EFET at most* σ *bounded on the real axis and equipped with the uniform norm.*

The proof of Theorem 5.1 is short and elegant: Let a be a hermitian element in a Banach algebra A. Take an arbitrary linear functional $\varphi \in \mathsf{A}^*$ with the unit norm, and consider the EFET

$$F(z) \stackrel{\text{def}}{=} \varphi\big(e^{\mathrm{i}az}\big) = \sum_{n\geqslant 0} \varphi(a^n)\,\frac{z^n}{n!}\,.$$

Applying, first, the formula, which expresses the exponential type of an entire function via its Taylor coefficients, then a crude estimate of $n!$, and then Gelfand's formula for the spectral radius, we estimate the exponential type of F:

$$\begin{aligned}\sigma_F &= \frac{1}{e}\,\limsup_{n\to\infty} n\Big(\frac{|\varphi(a^n)|}{n!}\Big)^{1/n} \\ &\qquad\qquad \leqslant \frac{1}{e}\,\limsup_{n\to\infty} n\Big(\frac{\|a^n\|}{n!}\Big)^{1/n} = \limsup_{n\to\infty}\|a^n\|^{1/n} = \rho(a)\,,\end{aligned}$$

where $\rho(a)$ denotes the spectral radius of a. Since the element a is hermitian, we have $|F(x)| = |\varphi\big(e^{\mathrm{i}ax}\big)| \leqslant \|e^{\mathrm{i}ax}\| = 1$, whence, by the Bernstein inequality,

$$|\varphi(a)| = |F'(0)| \leqslant \sigma_F \sup_{\mathbb{R}} |F| \leqslant \rho(a)\,,$$

and then, by the Hahn-Banach theorem, $\|a\| \leqslant \rho(a)$. This completes the proof of Theorem 5.1 since the converse inequality $\rho(a) \leqslant \|a\|$ is obvious. $\square$

The proof of the result which goes in the opposite direction is also quite simple. Let D be the differentiation operator in the Bernstein space B_σ of EFET at most σ bounded on $\mathbb{R}$. As we have already mentioned, the exponential function $e^{\mathrm{i}\mathsf{D}t}$ acts on B_σ as the translation by t, so D is a hermitian operator in B_σ. To evaluate the spectral radius of D, we need to estimate from above the norms $\|\mathsf{D}^n\|$, that is, $\sup_{\mathbb{R}} |F^{(n)}|$, $F \in \mathsf{B}_\sigma$. By Cauchy's estimate for the derivatives of analytic functions, combined with the bound $|F(x+w)| \leqslant e^{\sigma|w|} \sup_{\mathbb{R}} |F|$ valid for any $F \in \mathsf{B}_\sigma$, we obtain $|F^{(n)}(x)| \leqslant n!\,r^{-n} e^{\sigma r} \|F\|$ for any $r > 0$ and any $x \in \mathbb{R}$. Optimising the RHS, we get $|F^{(n)}(x)| \leqslant n! \exp[n - n\log n + n\log\sigma]\,\|F\|$, that is, $\|\mathsf{D}^n\| \leqslant n! \exp[n - n\log n + n\log\sigma]$, and finally, $\rho(\mathsf{D}) = \lim \|\mathsf{D}^n\|^{1/n} = \sigma$. Thus, for any function $F \in \mathsf{B}_\sigma$ and any $x \in \mathbb{R}$, we have

$$|F'(x)| = |(\mathsf{D}F)(x)| \leqslant \|\mathsf{D}\| \cdot \|F\| = \rho(\mathsf{D})\|F\| \leqslant \sigma\|F\|,$$

proving the Bernstein inequality. □

In this context, it is also worth mentioning that a bit later Bonsall and Crabb [3] found a simple direct proof of Theorem 5.1, which yields another proof of the Bernstein inequality. Their proof is based on the following lemma, which is a simple exercise on the functional calculus in Banach algebras:

Lemma 5.2 *Let a be a hermitian element in a Banach algebra with $\rho(a) < \pi/2$. Then, $a = \arcsin(\sin a)$.*

Now, Theorem 5.1 follows almost immediately. Proving Theorem 5.1, it suffices, assuming that a is an arbitrary hermitian element with $\rho(a) < \pi/2$, to show that $\|a\| \leqslant \pi/2$. Let c_n be the n-th Taylor coefficient of the function $z \mapsto \arcsin z$, $|z| \leqslant 1$. The values c_n are positive and their sum equals $\arcsin(1) = \pi/2$. By Lemma 5.2, $\|a\| \leqslant \sum_{n\geqslant 1} c_n \|\sin a\|^n$. Since the element a is hermitian, $\|\sin a\| \leqslant 1$, and therefore, $\|a\| \leqslant \sum_{n\geqslant 1} c_n = \pi/2$. □

* * *

In the reference list, referring to the papers in Russian published in journals translated from cover to cover, we mention only the translations. Today, the original Russian versions of these papers can be found at the Math-Net.Ru site (http://www.mathnet.ru).

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References

1. N. I. Akhiezer, Lectures in the theory of approximation. Second, revised and enlarged edition. (Russian). Nauka, Moscow, 1965.
2. A. M. Bochtejn, V. Katsnelson, Estimation of the norm of a projection in a certain space of analytic functions. (Russian) Teor. Funkciĭ Funkcional. Anal. i Priložen. **12** (1970), 81–85.

3. F. F. BONSALL, M. J. CRABB, The spectral radius of a Hermitian element of a Banach algebra. Bull. London Math. Soc. **2** (1970), 178–180.
4. A. BROWDER, On Bernstein's inequality and the norm of Hermitian operators. Amer. Math. Monthly **78** (1971), 871–873.
5. L. CARLESON, Interpolations by Bounded Analytic Functions and the Corona Problem, Annals of Math. **76** (1962), 547–560.
6. V. I. DANCHENKO, Estimates for the distances from the poles of the logarithmic derivatives of polynomials to straight lines and circles. Russian Acad. Sci. Sb. Math. **82** (1995), 425–440.
7. H. DYM, V. KATSNELSON, Contributions of Issai Schur to analysis. In: Studies in memory of Issai Schur (Chevaleret/Rehovot, 2000), Progr. Math. **210**, Birkhäuser Boston, Boston, MA, 2003.
8. J. B. GARNETT, Bounded analytic functions, Academic Press, 1981.
9. A. O. GELFOND, Estimation of the imagninary parts of roots of polynomials with derivatives of their logarithms bounded on the real axis. (Russian) Mat. Sb. (N.S.) **71** (1966), 289–296.
10. I. TS. GOHBERG, M. G. KREIN, Introduction to the theory of linear nonselfadjoint operators. Translations of Mathematical Monographs, **18**. American Mathematical Society, Providence, R.I. 1969
11. E. A. GORIN, Partially hypoelliptic partial differential equations with constant coefficients. (Russian) Sibirsk. Mat. Zh. **3** (1962), 500–526.
12. E. A. GORIN, Bernstein's inequality from the point of view of operator theory. (Russian) Vestnik Khar'kov. Gos. Univ. **205** (1980), 77–105. English translation in Selecta Math. Soviet. **7** (1988), 191–219.
13. K. P. ISAEV, R. S. YULMUKHAMETOV, The Laplace transform of functionals on Bergman spaces. Izv. Math. **68** (2004), 3–41.
14. V. KATSNELSON, Generalization of the Wiener-Paley theorem on the representation of entire functions of finite degree. (Russian) Teor. Funkciĭ Funkcional. Anal. i Priložen. **1** (1965), 99–110.
15. V. KATSNELSON, Conditions for a system of root vectors of certain classes of operators to be a basis. Functional Anal. Appl. **1** (1967), 122–132.
16. V. KATSNELSON, Certain operators acting on spaces generated by the functions $1/(t - z_k)$. (Russian) Teor. Funkciĭ Funkcional. Anal. i Priložen. **4** 1967 58–66.
17. V. KATSNELSON, A conservative operator has norm equal to its spectral radius. (Russian) Mat. Issled. **5:3** (1970), 186–189.
18. V. KATSNELSON, Methods of J-theory in continuous interpolation problems of analysis. Part I. Translated from the Russian and with a foreword by T. Ando. T. Ando, Hokkaido University, Sapporo, 1985.
19. V. KATSNELSON, Extremal problems of G. Szegő and M. Riesz, factorization problems, and other related problems of analysis. Part I. Scalar case. (Russian) Leipzig, 1991.
20. V. KATSNELSON, Weighted spaces of pseudocontinuable functions and approximations by rational functions with prescribed poles. Z. Anal. Anwendungen **12** (1993), 27–67.
21. V. KATSNELSON, Description of a class of functions which admit an approximation by rational functions with preassigned poles. I. In: Matrix and operator valued functions, 87–132, Oper. Theory Adv. Appl., **72**, Birkhäuser, Basel, 1994.
22. M. V. KELDYSH, The completeness of eigenfunctions of certain classes of nonselfadjoint linear operators. Russian Math. Surveys **26** (1971), no 4, 15–44.
23. A. YA. KHEIFETS, On Regularization of γ-Generating Pairs. J. Funct. Anal. **130** (1995), 310–333.
24. B. YA. LEVIN, Distribution of zeros of entire functions. Revised edition. Translations of Mathematical Monographs, **5**. American Mathematical Society, Providence, R.I., 1980.
25. B. YA. LEVIN, Lectures on entire functions. In collaboration with and with a preface by Yu. Lyubarskii, M. Sodin and V. Tkachenko. Translations of Mathematical Monographs, **150**. American Mathematical Society, Providence, RI, 1996.
26. M. K. LIHT, A remark on the theorem of Paley and Wiener on entire functions of finite type. (Russian) Uspehi Mat. Nauk **19** (1964), 169–171.

27. N. LINDHOLM, A Paley-Wiener theorem for convex sets in $\mathbb{C}^n$. Bull. Sci. Math. **126** (2002), 289–314.
28. V. I. LUTSENKO, R. S. YULMUHAMETOV, A generalization of the Paley-Wiener theorem on weighted spaces, Math. Notes **48** (1990), 1131–1136.
29. V. I. LUTSENKO, R. S. YULMUHAMETOV, A generalization of the Paley-Wiener theorem to functionals on Smirnov spaces, Proc. Steklov Inst. Math. **200** (1993), 271–280.
30. YU. LYUBARSKII, The Paley-Wiener theorem for convex sets. Izv. Akad. Nauk Armyan. SSR Ser. Mat. **23** (1988), 163–172; English translation in Soviet J. Contemporary Math. Anal. **23** (1988), no. 2, 64–74.
31. V. I. MATSAEV, E. Z. MOGULSKII, Certain criteria for the multiple completeness of the system of eigen- and associated vectors of polynomial operator pencils. (Russian) Teor. Funkciĭ Funkcional. Anal. i Priložen. **13** (1971), 3–45.
32. V. I. MATSAEV, E. Z. MOGULSKII, Completeness of weak perturbations of self-adjoint operators. (Russian) Zapiski nauchnyh seminarov LOMI **56** (1976), 90–103; English translation in: J. Soviet Math. **14** (1980), 1091–1103.
33. E. G. NIKOLAEV A geometrical property of the roots of polynomials. (Russian). Vestnik Moskov. Univ. Ser. I Mat. Meh. (1965), no. 5, 23–26.
34. N. K. NIKOLSKII, B. S. PAVLOV, Eigenvector expansions of nonunitary operators, and the characteristic function. (Russian) Zap. Nauchn. Sem. LOMI **11** (1968), 150–203.
35. N. K. NIKOLSKII, B. S. PAVLOV, Bases of eigenvectors of completely nonunitary contractions, and the characteristic function. Mathematics of the USSR-Izvestiya, **4** (1970), no 1, 91–134.
36. N. K. NIKOLSKII, Bases of invariant subspaces and operator interpolation. Trudy Mat. Inst. Steklov. **130** (1978), 50–123.
37. N. K. NIKOLSKII, Treatise on the shift operator. Spectral function theory. With an appendix by S. V. Khrushchev and V. V. Peller. Springer, Berlin, 1986.
38. F. RIESZ, B. SZ.-NAGY, Functional analysis. Dover, New York, 1990.
39. A. M. SINCLAIR, The norm of a hermitian element in a Banach algebra. Proc. Amer. Math. Soc. **28** (1971), 446–450.
40. S. TREIL, Hankel operators, embedding theorems and bases of co-invariant subspaces of the multiple shift operator. Leningrad Math. J. **1** (1990), 1515–1548.
41. S. TREIL, Unconditional bases of invariant subspaces of a contraction with finite defects. Indiana Univ. Math. J. **46** (1997), 1021–1054.
42. G. TS. TUMARKIN, The description of the class of functions which can be approximated by fractions with preassigned poles (Russian), Izv. Akad. Nauk Armjan. SSR Ser. Mat. **1** (1966), 89–105.
43. G. TS. TUMARKIN, Approximation with respect to various metrics of functions defined on the circumference by sequences of rational fractions with fixed poles (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **30** (1966), 721–766; English translation in Amer. Math. Soc. Transl. Ser. 2 **77** (1968), 183–233.
44. G. TS. TUMARKIN, Necessary and sufficient conditions for the possibility of approximating a function on a circumference by rational fractions, expressed in terms directly connected with the distribution of poles of the approximating fractions (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **30** (1966), 969–980; English translation in Amer. Math. Soc. Transl. Ser. 2 **77** (1968), 235–248.
45. V. I. VASYUNIN, Unconditionally convergent spectral decompositions and interpolation problems. (Russian) Trudy Mat. Inst. Steklov. **130** (1978), 5–49.
46. A. VOLBERG, Thin and thick families of rational fractions. In: Complex analysis and spectral theory (Leningrad, 1979/1980), pp. 440–480, Lecture Notes in Math., **864**, Springer, Berlin-New York, 1981.

Part II
Scientific Papers

Interpolation by Contractive Multipliers Between Fock Spaces

Joseph A. Ball and Vladimir Bolotnikov

Dedicated to Professor Victor Katsnelson on the occasion of his 75th birthday

Abstract We survey various Nevanlinna-Pick type interpolation problems for contractive multipliers between two vector Fock spaces of noncommutative formal power series. An adaptation of Potapov's method leads to a chain-matrix linear-fractional parametrization for the set of all solutions for the case where the Pick operator is invertible. The most general problem considered here is a noncommutative multivariable analogue of the Abstract Interpolation Problem formulated by Katsnelson, Kheifets and Yuditskii for the single-variable case; we obtain a Redheffer-type linear-fractional parametrization for the set of all solutions (including in degenerate cases) via an adaptation of ideas of Arov and Grossman.

Keywords Abstract Interpolation Problem · Linear-fractional map · Stein equation · Left tangential interpolation with operator argument

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1 Introduction

To define the vector-valued-coefficient Fock space $H^2_{\mathcal{Y}}(\mathbb{F}^+_d)$ as in (1.2) below (our free noncommutative analogue of the vector-valued Hardy space $H^2_{\mathcal{Y}}$ over the unit disk), we first introduce the unital free semigroup (i.e., monoid) generated by the set of d letters $\{1, \dots, d\}$ which we denote as $\mathbb{F}^+_d$. Elements of $\mathbb{F}^+_d$ are words of the form $i_N \cdots i_1$ where $i_\ell \in \{1, \dots, d\}$ for each $\ell \in \{1, \dots, N\}$ with multiplication given by concatenation. The unit element of $\mathbb{F}^+_d$ is the empty word denoted by $\emptyset$. For $\alpha = i_N i_{N-1} \cdots i_1 \in \mathbb{F}^+_d$, we let $|\alpha|$ denote the number N of letters in α and we let $\alpha^\top := i_1 \cdots i_{N-1} i_N$ denote the *transpose* of α. We let $z = (z_1, \dots, z_d)$ to be a collection of d formal noncommuting variables and given a Hilbert space $\mathcal{Y}$, let $\mathcal{Y}\langle\langle z \rangle\rangle$ denote the set of noncommutative formal power series $\sum_{\alpha \in \mathbb{F}^+_d} f_\alpha z^\alpha$ where $f_\alpha \in \mathcal{Y}$ and where

$$z^\alpha = z_{i_N} z_{i_{N-1}} \cdots z_{i_1} \quad \text{if} \quad \alpha = i_N i_{N-1} \cdots i_1. \tag{1.1}$$

The Fock space $H^2_{\mathcal{Y}}(\mathbb{F}^+_d)$ is defined as

$$H^2_{\mathcal{Y}}(\mathbb{F}^+_d) = \left\{ \sum_{\alpha \in \mathbb{F}^+_d} f_\alpha z^\alpha \in \mathcal{Y}\langle\langle z \rangle\rangle \colon \ \sum_{\alpha \in \mathbb{F}^+_d} \| f_\alpha \|^2_{\mathcal{Y}} < \infty \right\}. \tag{1.2}$$

We let $\mathbf{R}_z$ denote the tuple of right coordinate-variable multipliers

$$\mathbf{R}_z = (R_{z_1}, \dots, R_{z_d}), \quad R_{z_j} : f(z) \mapsto f(z) z_j \tag{1.3}$$

(called the *shift operator-tuple* of $H^2(\mathbb{F}^+_d)$), whereas we refer to the tuple

$$\mathbf{R}^*_z = (R^*_{z_1}, \dots, R^*_{z_d}), \quad R^*_{z_j} : \sum_{\alpha \in \mathbb{F}^+_d} f_\alpha z^\alpha \mapsto \sum_{\alpha \in \mathbb{F}^+_d} f_{\alpha j} z^\alpha \tag{1.4}$$

consisting of the adjoints (in the metric of $H^2_{\mathcal{Y}}(\mathbb{F}^+_d)$) as the *backward shift operator-tuple*.

Given two coefficient Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$, we let $\mathcal{M}_{\mathrm{nc},d}(\mathcal{U}, \mathcal{Y})$ denote the space of multipliers from $H^2_{\mathcal{U}}(\mathbb{F}^+_d)$ to $H^2_{\mathcal{Y}}(\mathbb{F}^+_d)$, that is, formal power series

$$S(z) = \sum_{\alpha \in \mathbb{F}^+_d} S_\alpha z^\alpha$$

with coefficients $S_\alpha \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ such that the associated multiplication operator

$$M_S : u(z) = \sum_{\alpha \in \mathbb{F}^+_d} u_\alpha z^\alpha \mapsto S(z) u(z) := \sum_{\alpha \in \mathbb{F}^+_d} \left(\sum_{\beta\gamma = \alpha} S_\beta u_\gamma \right) z^\alpha \tag{1.5}$$

is a bounded operator from $H^2_{\mathcal{U}}(\mathbb{F}_d^+)$ to $H^2_{\mathcal{Y}}(\mathbb{F}_d^+)$. It is not hard to show that $\mathcal{M}_{\mathrm{nc},d}(\mathcal{U},\mathcal{Y})$ is the intertwining space for the shift tuples $\mathbf{R}_z \otimes I_{\mathcal{U}}$ and $\mathbf{R}_z \otimes I_{\mathcal{Y}}$. More precisely, an operator $\Psi \in \mathcal{L}(H^2_{\mathcal{U}}(\mathbb{F}_d^+), H^2_{\mathcal{Y}}(\mathbb{F}_d^+))$ equals $\Psi = M_S$ for some $S \in \mathcal{M}_{\mathrm{nc},d}(\mathcal{U},\mathcal{Y})$ if and only if

$$(R_{z_j} \otimes I_{\mathcal{Y}})\Psi = \Psi(R_{z_j} \otimes I_{\mathcal{U}}) \quad \text{for} \quad j = 1, \dots, d.$$

We define the *noncommutative Schur class* $\mathcal{S}_{\mathrm{nc},d}(\mathcal{U},\mathcal{Y})$ to be the closed unit ball of $\mathcal{M}_{\mathrm{nc},d}(\mathcal{U},\mathcal{Y})$:

$$\mathcal{S}_{\mathrm{nc},d}(\mathcal{U},\mathcal{Y}) = \{S \in \mathcal{M}_{\mathrm{nc},d}(\mathcal{U},\mathcal{Y}) \colon \ \|M_S\|_{\mathrm{op}} \le 1\},$$

that is, the set of all *contractive multipliers* from $H^2_{\mathcal{U}}(\mathbb{F}_d^+)$ to $H^2_{\mathcal{Y}}(\mathbb{F}_d^+)$. If $\mathcal{M}$ is a closed subspace of $H^2_{\mathcal{U}}(\mathbb{F}_d^+)$ which is invariant under the backward shift $\mathbf{R}_z^*$ (i.e., $R_{z_j}^*\mathcal{M} \subset \mathcal{M}$ for $j = 1, \dots, d$) and $S \in \mathcal{M}_{\mathrm{nc},d}(\mathcal{U},\mathcal{Y})$, then the operator $\Phi = \mathcal{P}_{\mathcal{M}} M_S \in \mathcal{L}(H^2_{\mathcal{U}}(\mathbb{F}_d^+), \mathcal{M})$ (where $\mathcal{P}_{\mathcal{M}}$ denotes the orthogonal projection of $H^2_{\mathcal{U}}(\mathbb{F}_d^+)$ onto $\mathcal{M}$) satisfies conditions $\|\Phi\| \le \|M_S\|$ and (the details will be given below)

$$\mathcal{P}_{\mathcal{M}} R_{z_j}\Phi = \Phi R_{z_j} \quad (j = 1, \dots, d). \tag{1.6}$$

Recovering $S \in \mathcal{M}_{\mathrm{nc},d}(\mathcal{U},\mathcal{Y})$ from its projection onto a backward-shift invariant subspace is a problem of interpolation nature. This problem is *norm-constrained* if in addition we impose a bound for $\|M_S\|$. The normalized version of this problem is: *given* $\mathbf{R}_z^*$*-invariant subspace* $\mathcal{M} \subset H^2_{\mathcal{Y}}(\mathbb{F}_d^+)$ *and given* $\Phi \in \mathcal{L}(H^2_{\mathcal{U}}(\mathbb{F}_d^+), \mathcal{M})$ *subject to relations* (1.6)*, find all contractive multipliers* $S \in \mathcal{S}_{\mathrm{nc},d}(\mathcal{U},\mathcal{Y})$ *such that* $\mathcal{P}_{\mathcal{M}} M_S = \Phi$. A clear necessary condition $\|\Phi\| \le 1$ for the existence of such S turns out to be also sufficient, by the Commutant Lifting theorem for Fock spaces [28]. Motivation for this noncommutative Commutant Lifting theorem arose from noncommutative interpolations problems in the Sarason operator formulation from [35] (see [2]); the notion of evaluation at a noncommutative point and associated Nevanlinna-Pick interpolation problems came later (see [1, 8, 13]).

In the single-variable case, the Commutant Lifting interpolation problem appeared as a natural and brilliant generalization and uniformization of then already existing classical interpolation problems of Nevanlinna-Pick type, and we have seen that a similar motivation was behind Popescu's noncommutative Commutant Lifting theorem for the Fock-space setting. Already in the single-variable setting it was known that the Commutant Lifting theorem did not apply to all interpolation problems, in particular boundary interpolation and moment problems: this was one of the motivations for the development of the Abstract Interpolation Problem of Katsnelson et al. [20].

In the present noncommutative setting, we take the Commutant Lifting problem as our starting point. After some preliminaries concerning formal noncommutative reproducing kernel Hilbert spaces and Schur multipliers in Sect. 2, in Sect. 3 we develop Kreĭn-space analogues of the ideas introduced in Sect. 2, and introduce and use the Potapov-Ginzburg transform (at various levels) as a tool for reducing the indefinite setting to the definite setting (when possible), as well as material on linear-fractional transformations (in chain-scattering and Redheffer form) which will be used later. In Sect. 4 we formulate the Operator Argument interpolation Problem **OAP** with the interpolation condition given in terms of certain noncommutative tangential evaluation and show how Commutant Lifting and the Sarason Interpolation Problem can be seen as particular cases of **OAP**. In Sect. 5 we use a suitable adaptation of the Potapov method of Fundamental Matrix Inequalities to obtain a characterization of when solutions exist in terms of positivity of an associated Pick matrix; when this Pick matrix is strictly positive definite, we obtain a linear-fractional-transformation parametrization for the set of all solutions. In Sect. 6 we refine the formulation of the **OAP** by expressing the interpolation condition in more implicit form leading to a still more general problem called the **analytic Abstract Interpolation Problem** (**aOAP**) and show how a more careful analysis of the proofs in the previous section leads to a solution of this problem. In Sect. 7 we introduce our most general interpolation problem, called the **Abstract Interpolation Problem** (**AIP**) in analogy with that studied for the single-variable case in [20, 22–24], identify solutions as corresponding to unitary-colligation extensions of a partially defined isometry determined by the problem data, and use results of Arov-Grossman [4, 5] to obtain linear-fractional-transformation parametrizations for the set of all solutions. Sections 4–7 have much in parallel with some of the exposition in the paper [6] where the same hierarchy of interpolation problems and linear-fractional-transformation parametrizations for their solution sets was discussed in the commutative-variable setting of contractive multipliers on the Drury-Arveson space.

2 Formal Noncommutative Reproducing Kernel Hilbert Spaces and Schur Multipliers

In this section we collect miscellaneous preliminary results needed for the work in the sequel. We start with formal positive kernels and associated reproducing kernel Hilbert spaces. For more complete details we refer to [15]; see also Section 2.1 of the forthcoming [9]. Such spaces are also closely related to the more general notion of noncommutative reproducing kernel Hilbert space introduced in [12]; for the precise connection between the two notions, we refer to Section 3.5 in [12].

2.1 Formal Reproducing Kernel Hilbert Spaces

In general, for any coefficient linear space $\mathfrak{X}$, we denote by $\mathfrak{X}\langle\langle z\rangle\rangle$ the space of all formal power series $f(z) = \sum_{\alpha\in\mathcal{X}} f_\alpha z^\alpha$ with coefficients f_α in the space $\mathfrak{X}$. We shall say for simplicity that an element $f(z) = \sum_{\alpha\in\mathbb{F}_d^+} f_\alpha z^\alpha$ in $\mathfrak{X}\langle\langle z\rangle\rangle$ is a $\mathfrak{X}$-valued formal power series even though more precisely $f(z)$ is a formal power series with coefficients in the space $\mathfrak{X}$. In particular, given a coefficient Hilbert space $\mathcal{Y}$, we define the space $\mathcal{L}(\mathcal{Y})\langle\langle z, \overline{\zeta}\rangle\rangle$ consisting of formal power series

$$K(z,\zeta) = \sum_{\alpha,\beta\in\mathbb{F}_d^+} K_{\alpha,\beta} z^\alpha \overline{\zeta}^{\beta^\top}, \quad K_{\alpha,\beta} \in \mathcal{L}(\mathcal{Y})\rangle, \tag{2.1}$$

in the freely noncommuting indeterminates $z = (z_1, \dots, z_d)$ and $\overline{\zeta} = (\overline{\zeta}_1, \dots, \overline{\zeta}_d)$ such that each z_k commutes with each $\overline{\zeta}_j$. Any such power series will be referred to as a *formal kernel*. The kernel (2.1) is called *positive* if for any finitely supported $\mathcal{Y}$-valued function $\alpha \mapsto y_\alpha$ on $\mathbb{F}_d^+$,

$$\sum_{\alpha,\beta\in\mathbb{F}_d^+} \langle K_{\alpha,\beta}\, y_\alpha, y_\beta\rangle_{\mathcal{Y}} \geq 0.$$

For example, if $\mathcal{X}$ is another coefficient Hilbert space and $H(z)$ is an element in $\mathcal{L}(\mathcal{X},\mathcal{Y})\langle\langle z\rangle\rangle$, then the formula

$$K(z,\zeta) = H(z)H(\zeta)^* \tag{2.2}$$

defines a positive formal kernel $K(z,\zeta)$. Here we use the conventions

$$(\zeta^\beta)^* = \overline{\zeta}^{\beta^\top}, \quad H(\zeta)^* = \left(\sum H_\beta \zeta^\beta\right)^* = \sum H_\beta^* \overline{\zeta}^{\beta^\top}.$$

Conversely, any $\mathcal{L}(\mathcal{Y})$-valued formal positive kernel K admits a factorization (2.2) (Kolmogorov decomposition of K) for some Hilbert space $\mathcal{X}$ and an $\mathcal{L}(\mathcal{X},\mathcal{Y})$-valued formal power series $H(z)$.

Suppose that $\mathcal{H}$ is a Hilbert space consisting of formal power series

$$f(z) = \sum_{\alpha\in\mathbb{F}_d^+} f_\alpha z^\alpha \in \mathcal{Y}\langle\langle z\rangle\rangle$$

with coefficients in the coefficient Hilbert space $\mathcal{Y}$. We say that $\mathcal{H}$ is a *noncommutative formal reproducing kernel Hilbert space* (NFRKHS) if, for each $\beta \in \mathbb{F}_d^+$ it is the case that the linear operator $\Phi_\beta : \mathcal{H} \to \mathcal{Y}$ defined by

$$\Phi_\beta : f(z) = \sum_{\alpha \in \mathbb{F}_d^+} f_\alpha z^\alpha \mapsto f_\beta$$

is continuous. As any such power series is completely determined by the list of its coefficients $\alpha \mapsto f_\alpha$ for $\alpha \in \mathbb{F}_d^+$, we may view an element $f(z) \in \mathcal{H}$ as the corresponding $\mathcal{Y}$-valued function $\alpha \mapsto f_\alpha$ on $\mathbb{F}_d^+$. By the standard Aronszajn theory of reproducing kernel Hilbert spaces [3], there is a positive kernel $K : \mathbb{F}_d^+ \times \mathbb{F}_d^+ \to \mathcal{L}(\mathcal{Y})$ so that $\mathcal{H}$ is the reproducing kernel Hilbert space associated with K.

For the present context it is preferable to rephrase all this as follows. Let us write $K_{\alpha,\beta}$ rather than $K(\alpha, \beta)$ for the Aronszajn kernel derived as above. We view the element $\Phi_\beta^* y \in \mathcal{H}$ as a formal power series rather than as a function on $\mathbb{F}_d^+$, namely:

$$\Phi_\beta^* y = \sum_{\alpha \in \mathbb{F}_d^+} K_{\alpha,\beta}\, y\, z^\alpha.$$

Then the reproducing property becomes

$$\langle f, K_\beta(\cdot) y \rangle_{\mathcal{H}} = \langle f, \Phi_\beta^* y \rangle_{\mathcal{H}} = \langle \Phi_\beta f, y \rangle_{\mathcal{Y}} = \langle f_\beta, y \rangle_{\mathcal{Y}}. \tag{2.3}$$

We can make the notation more suggestive of the classical case as follows. Let $\overline{\zeta} = (\overline{\zeta}_1, \dots, \overline{\zeta}_d)$ be a second d-tuple of noncommuting indeterminates. Given a coefficient Hilbert space $\mathcal{C}$, we can use the $\mathcal{C}$-inner product to define pairings

$$\langle \cdot, \cdot \rangle_{\mathcal{C} \times \mathcal{C}\langle\langle \overline{\zeta} \rangle\rangle} \mapsto \mathbb{C}\langle\langle \zeta \rangle\rangle \quad \text{and} \quad \langle \cdot, \cdot \rangle_{\mathcal{C}\langle\langle \zeta \rangle\rangle \times \mathcal{C}} \mapsto \mathbb{C}\langle\langle \zeta \rangle\rangle$$

(where $\mathbb{C}\langle\langle \zeta \rangle\rangle$ is the space of formal power series in the set of formal conjugate indeterminates $\zeta = (\zeta_1, \dots, \zeta_d)$ with coefficients in $\mathbb{C}$) by

$$\Big\langle c, \sum f_\alpha \overline{\zeta}^\alpha \Big\rangle_{\mathcal{C} \times \mathcal{C}\langle\langle \overline{\zeta} \rangle\rangle} = \sum \langle c, f_\alpha \rangle_{\mathcal{C}}\, \zeta^{\alpha^\top},$$
$$\Big\langle \sum f_\alpha \zeta^\alpha, c \Big\rangle_{\mathcal{C}\langle\langle \zeta \rangle\rangle \times \mathcal{C}} = \sum \langle f_\alpha, c \rangle_{\mathcal{C}}\, \zeta^\alpha.$$

These pairings can be seen as special cases of the more general pairing

$$\left\langle \sum_{\alpha \in \mathbb{F}_d^+} f_\alpha \zeta^\alpha, \sum_{\beta \in \mathbb{F}_d^+} g_\beta \overline{\zeta}^\beta \right\rangle_{\mathcal{C}\langle\langle \zeta \rangle\rangle \times \mathcal{C}\langle\langle \overline{\zeta} \rangle\rangle} = \sum_{\alpha \in \mathbb{F}_d^+} \left[\sum_{\beta, \gamma :\, \alpha = \gamma^\top \beta} \langle f_\beta, g_\gamma \rangle_{\mathcal{C}} \right] \zeta^\alpha,$$

which can be written more suggestively as

$$\begin{aligned}\langle f(\zeta), g(\overline{\zeta})\rangle_{\mathcal{C}\langle\langle\zeta\rangle\rangle\times\mathcal{C}\langle\langle\overline{\zeta}\rangle\rangle} &= \Big\langle \sum f_\alpha \zeta^\alpha, \sum g_\beta \overline{\zeta}^\beta \Big\rangle_{\mathcal{C}\langle\langle\zeta\rangle\rangle\times\mathcal{C}\langle\langle\overline{\zeta}\rangle\rangle} \\ &= g(\overline{\zeta})^* f(\zeta),\end{aligned}$$

if we set

$$g(\overline{\zeta})^* = \left(\sum g_\beta \overline{\zeta}^\beta\right)^* = \sum g_\beta^* \zeta^{\beta^\top},$$

where we view $g_\beta^* \in \mathcal{L}(\mathcal{C}, \mathbb{C})$ as a linear functional on $\mathcal{C}$ so that

$$g_\beta^* f_\alpha = \langle f_\alpha, g_\beta\rangle_{\mathcal{C}} \quad \text{for any} \quad f_\alpha \in \mathcal{C}.$$

Then, if $S(\zeta) \in \mathcal{L}(\mathcal{U}, \mathcal{Y})\langle\langle\zeta\rangle\rangle$, $f(\zeta) \in \mathcal{U}\langle\langle\zeta\rangle\rangle$ and $g(\overline{\zeta}) \in \mathcal{Y}\langle\langle\overline{\zeta}\rangle\rangle$, we see that

$$\begin{aligned}\langle S(\zeta) f(\zeta), g(\overline{\zeta})\rangle_{\mathcal{Y}\langle\langle\zeta\rangle\rangle\times\mathcal{Y}\langle\langle\overline{\zeta}\rangle\rangle} &= g(\overline{\zeta})^* \left(S(\zeta) f(\zeta)\right) \\ &= \left(g(\overline{\zeta})^* S(\zeta)\right) f(\zeta) \\ &= \langle f(\zeta), S(\zeta)^* g(\overline{\zeta})\rangle_{\mathcal{U}\langle\langle\zeta\rangle\rangle\times\mathcal{U}\langle\langle\overline{\zeta}\rangle\rangle}.\end{aligned} \tag{2.4}$$

The reproducing kernel property (2.3) can be written more suggestively as

$$\langle f, K(\cdot, \zeta) y\rangle_{\mathcal{H}\times\mathcal{H}\langle\langle\overline{\zeta}\rangle\rangle} = \langle f(\zeta), y\rangle_{\mathcal{Y}\langle\langle\zeta\rangle\rangle\times\mathcal{Y}}, \tag{2.5}$$

where we set

$$K(z, \zeta) = \sum_{\alpha,\beta\in\mathbb{F}_d^+} K_{\alpha,\beta} z^\alpha \overline{\zeta}^{\beta^\top} \in \mathcal{L}(\mathcal{Y})\langle\langle z, \overline{\zeta}\rangle\rangle. \tag{2.6}$$

We note that, for each $y \in \mathcal{Y}$, the formal power series

$$K(z, \zeta) y = \sum_{\alpha,\beta\in\mathbb{F}_d^+} K_{\alpha,\beta}\, y\, z^\alpha \overline{\zeta}^{\beta^\top} = \sum_{\beta\in\mathbb{F}_d^+} \left[\sum_{\alpha\in\mathbb{F}_d^+} K_{\alpha,\beta^\top}\, y\, z^\alpha\right] \overline{\zeta}^\beta$$

is an element of $\mathcal{H}\langle\langle\overline{\zeta}\rangle\rangle$. Furthermore, K so constructed has a factorization $K(z, \zeta) = H(z) H(\zeta)^*$ where $H(\zeta)^* \in \mathcal{L}(\mathcal{Y}, \mathcal{H})\langle\langle\overline{\zeta}\rangle\rangle$ is given by

$$H(\zeta)^* = \sum_\beta \Phi_\beta^* \overline{\zeta}^{\beta^\top},$$

thereby verifying that K is a positive formal kernel in the sense given above. When K and $\mathcal{H}$ are related in this way, we say that K is the reproducing kernel for the FNRKHS $\mathcal{H}$ and we write $\mathcal{H} = \mathcal{H}(K)$.

Conversely, any positive formal kernel arises as the reproducing kernel for a FNRKHS. To construct the FNRKHS associated with a given kernel K, one can use the noncommutative Aronszajn construction [15], or instead directly in terms of the power series $H(z)$ appearing in the Kolmogorov decomposition (2.2):

$$\mathcal{H}(K) = \{H(z)h_0 \colon h_0 \in \mathcal{X}\}$$

with norm taken to be the "lifted norm"

$$\|H(\cdot)h_0\|_{\mathcal{H}(K)} = \|Qh_0\|_{\mathcal{H}_0}$$

where Q is the orthogonal projection of $\mathcal{H}_0$ onto the orthogonal complement of the kernel of the map $M_H : \mathcal{X} \to \mathcal{Y}\langle\langle z\rangle\rangle$ given by $M_H : h_0 \mapsto H(z) \cdot h_0$.

Remark 2.1 The Fock space $H^2_{\mathcal{Y}}(\mathbb{F}^+_d)$ introduced in (1.2) is the NFRKHS with reproducing kernel $k_{\rm Sz} \otimes I_{\mathcal{Y}}$ where $k_{\rm Sz}$ is the *noncommutative Szegő kernel*

$$k_{\rm Sz}(z, \zeta) = \sum_{\alpha \in \mathbb{F}^+_d} z^{\alpha} \overline{\zeta}^{\alpha^{\top}}.$$

Indeed, for $f(z) = \sum_{\alpha \in \mathbb{F}^+_d} f_{\alpha} z^{\alpha} \in H^2_{\mathcal{Y}}(\mathbb{F}^+_d)$ and $y \in \mathcal{Y}$, we have

$$\begin{aligned} \langle f, k_{\rm Sz}(\cdot, \zeta)y\rangle_{H^2_{\mathcal{Y}}(\mathbb{F}^+_d) \times H^2_{\mathcal{Y}}(\mathbb{F}^+_d)\langle\langle\overline{\zeta}\rangle\rangle} &= \sum_{\alpha \in \mathbb{F}^+_d} \langle f(z),\ y\, z^{\alpha}\rangle_{H^2_{\mathcal{Y}}(\mathbb{F}^+_d)} \zeta^{\alpha} \\ &= \sum_{\alpha \in \mathbb{F}^+_d} \langle f_{\alpha}, y\rangle_{\mathcal{Y}} \zeta^{\alpha} = \langle f(\zeta), y\rangle_{\mathcal{Y}\langle\langle\zeta\rangle\rangle \times \mathcal{Y}}. \end{aligned}$$

Given two positive kernels K and K' with values in $\mathcal{L}(\mathcal{Y})$ and $\mathcal{L}(\mathcal{U})$ respectively, and the associated NFRKHSs $\mathcal{H}(K)$ and $\mathcal{H}(K')$, a formal power series $F \in \mathcal{L}(\mathcal{U}, \mathcal{Y})\langle\langle z\rangle\rangle$ is called a *multiplier from* $\mathcal{H}(K')$ *to* $\mathcal{H}(K)$ if the multiplication operator M_F defined as in (1.5) is bounded from $\mathcal{H}(K')$ to $\mathcal{H}(K)$. The action of the adjoint operator $M_F^* : \mathcal{H}(K) \to \mathcal{H}(K')$ extended to the kernel elements $K(\cdot, \zeta)y \in \mathcal{H}(K)\langle\langle\overline{\zeta}\rangle\rangle$ is given by the formula

$$M_F^* : K(\cdot, \zeta)y \mapsto K'(\cdot, \zeta)F(\zeta)^*y \in \mathcal{H}(K')\langle\langle\overline{\zeta}\rangle\rangle \text{ for all } y \in \mathcal{Y}, \tag{2.7}$$

which is verified by the formal version of a standard inner-product computation making use of the identity (2.4). The multiplier F is called *contractive*, *inner* or *strictly inner* if the operator $M_F : \mathcal{H}(K') \to \mathcal{H}(K)$ is a contraction, a partial isometry or an isometry, respectively.[1] For details of the proof of the next result, we refer to Proposition 3.2 in [9].

Proposition 2.2 *Let* $K \in \mathcal{L}(\mathcal{Y})\langle\langle z, \overline{\zeta}\rangle\rangle$ *and* $K' \in \mathcal{L}(\mathcal{U})\langle\langle z, \overline{\zeta}\rangle\rangle$ *be two positive formal kernels. A formal power series* $F \in \mathcal{L}(\mathcal{U}, \mathcal{Y})\langle\langle z\rangle\rangle$ *is a contractive multiplier from* $\mathcal{H}(K')$ *to* $\mathcal{H}(K)$ *if and only if*

$$K_F(z,\zeta) = K(z,\zeta) - F(z)K'(z,\zeta)F(\zeta)^* \in \mathcal{L}(\mathcal{Y})\langle\langle z, \overline{\zeta}\rangle\rangle$$

is a positive formal kernel.

2.2 Noncommutative Schur Class

The noncommutative functional calculus (1.1) extends to a d-tuple of operators $\mathbf{A} = (A_1, \dots, A_d)$ by letting

$$\mathbf{A}^\alpha := A_{i_N} A_{i_{N-1}} \cdots A_{i_1} \quad \text{if} \quad \alpha = i_N i_{N-1} \cdots i_1 \in \mathbb{F}_d^+,$$

where the multiplication is now operator composition. Letting

$$Z(z) = \begin{bmatrix} z_1 & \cdots & z_d \end{bmatrix} \otimes I_{\mathcal{X}}, \quad A = \begin{bmatrix} A_1 \\ \vdots \\ A_d \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ \vdots \\ B_d \end{bmatrix}, \tag{2.8}$$

we next observe that

$$(Z(z)A)^j = \Big(\sum_{i=1}^d z_i A_i\Big)^j = \sum_{\alpha \in \mathbb{F}_d^+ :\, |\alpha| = j} \mathbf{A}^\alpha z^\alpha \quad \text{for all} \quad j \geq 0$$

and therefore,

$$(I - Z(z)A)^{-1} = \sum_{j=0}^\infty (Z(z)A)^j = \sum_{j=0}^\infty \sum_{\alpha \in \mathbb{F}_d^+ :\, |\alpha| = j} \mathbf{A}^\alpha z^\alpha = \sum_{\alpha \in \mathbb{F}_d^+} \mathbf{A}^\alpha z^\alpha.$$

[1] We note that the term *McCullough-Trent (McCT) inner* rather than *inner* is used in [9] for additional emphasis of the distinction between these different notions of *inner*, but here for simplicity we contract *McCT-inner* to *inner* (see [26] where this notion of inner appears in the commutative context of multipliers on the Drury-Arveson space).

The tuple $\mathbf{A} = (A_1, \ldots, A_d) \in \mathcal{L}(\mathcal{X})^d$ is called *strongly stable* if

$$\lim_{N\to\infty} \sum_{\alpha\in\mathbb{F}_d^+ :\, |\alpha|=N} \|\mathbf{A}^\alpha x\|^2 = 0 \quad \text{for all} \quad x \in \mathcal{X}. \tag{2.9}$$

Given a d-tuple $\mathbf{A} \in \mathcal{L}(\mathcal{X})^d$ as above and given an output operator $C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, the output pair $(C, \mathbf{A})$ is said to be *output-stable* if the associated observability operator

$$\mathcal{O}_{C,\mathbf{A}} \colon\, x \mapsto \sum_{\alpha\in\mathbb{F}_d^+} (C\mathbf{A}^\alpha x) z^\alpha = C(I_{\mathcal{X}} - Z(z)A)^{-1} x \tag{2.10}$$

maps $\mathcal{X}$ into $H^2_{\mathcal{Y}}(\mathbb{F}_d^+)$ and is bounded. In this case, it makes sense to introduce the *observability gramian*

$$\mathcal{G}_{C,\mathbf{A}} := \mathcal{O}^*_{C,\mathbf{A}} \mathcal{O}_{C,\mathbf{A}} = \sum_{\alpha\in\mathbb{F}_d^+} \mathbf{A}^{*\alpha^\top} C^* C \mathbf{A}^\alpha. \tag{2.11}$$

The strong convergence of the power series in (2.11) follows from the power-series expansion (2.10) for the observability operator together with the characterization (1.2) of the $H^2_{\mathcal{Y}}(\mathbb{F}_d^+)$-norm. An important property of $\mathcal{G}_{C,\mathbf{A}}$ is that it satisfies the Stein equation

$$H - \sum_{j=1}^{d} A_j^* H A_j = C^* C \tag{2.12}$$

as can be seen by plugging in the series expansion (2.11). The following result appears as Theorem 3.1 in [11].

Theorem 2.3 *Let $S(z) \in \mathcal{L}(\mathcal{U}, \mathcal{Y})\langle\langle z\rangle\rangle$. The following are equivalent:*

1. *$S \in \mathcal{S}_{nc,d}(\mathcal{U}, \mathcal{Y})$, i.e., $M_S \colon H^2_{\mathcal{U}}(\mathbb{F}_d^+) \to H^2_{\mathcal{Y}}(\mathbb{F}_d^+)$ is a contraction.*
2. *There exist an auxiliary Hilbert space $\mathcal{X}$ and a power series $H \in \mathcal{L}(\mathcal{X}, \mathcal{Y})\langle\langle z\rangle\rangle$ such that*

$$\begin{aligned} K_S(z,\zeta) &:= k_{\rm Sz}(z,\zeta) \otimes I_{\mathcal{Y}} - S(z)(k_{\rm Sz}(z,\zeta) \otimes I_{\mathcal{U}}) S(\zeta)^* \\ &= H(z) H(\zeta)^*, \end{aligned} \tag{2.13}$$

i.e., K_S associated with S via formula (2.13) is a positive formal kernel.

3. *There exist a Hilbert space $\mathcal{X}$ and a unitary connection operator* $\mathbf{U}$ *of the form*

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ \vdots & \vdots \\ A_d & B_d \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \vdots \\ \mathcal{X} \\ \mathcal{Y} \end{bmatrix} \tag{2.14}$$

so that $S(z)$ can be realized as a formal power series in the form

$$\begin{aligned} S(z) &= D + \sum_{j=1}^{d} \sum_{\alpha \in \mathbb{F}_d^+} C\mathbf{A}^{\alpha} B_j z^{\alpha} \cdot z_j \\ &= D + C(I - Z(z)A)^{-1} Z(z) B, \end{aligned} \tag{2.15}$$

where $Z(z)$, A and B are defined as in (2.8). When the formal power series $S(z)$ arises from the connection matrix (also called colligation) $\mathbf{U}$ *as in (2.15), we also say that $S(z)$ is the* characteristic formal power series *of the colligation* $\mathbf{U}$ *and we write $S = T_{\mathbf{U}}$.*

4. *There exist a Hilbert space $\mathcal{X}$ and a contractive block operator matrix* $\mathbf{U}$ *as in (2.14) such that $S(z)$ is given as in (2.15).*

Note that for S of the form (2.15) with unitary connection matrix $\mathbf{U}$, the Kolmogorov decomposition in (2.13) holds with

$$H(z) = C\,(I_{\mathcal{X}} - Z(z)A)^{-1}. \tag{2.16}$$

Note also that formulas (2.15) and (2.16) can be written directly in terms of the unitary operator $\mathbf{U}$ as follows:

$$\begin{aligned} S(z) &= \mathcal{P}_{\mathcal{Y}} \mathbf{U} \left(I_{\mathcal{X} \oplus \mathcal{U}} - \mathcal{P}_{\mathcal{X}}^* Z(z) \mathcal{P}_{\mathcal{X}^d} \mathbf{U} \right)^{-1} |_{\mathcal{U}}, \\ H(z) &= \mathcal{P}_{\mathcal{Y}} \mathbf{U} \left(I_{\mathcal{X} \oplus \mathcal{U}} - \mathcal{P}_{\mathcal{X}}^* Z(z) \mathcal{P}_{\mathcal{X}^d} \mathbf{U} \right)^{-1} |_{\mathcal{X}}, \end{aligned} \tag{2.17}$$

where $\mathcal{P}_{\mathcal{Y}}$ and $\mathcal{P}_{\mathcal{X}^d}$ are the orthogonal projections of the space $\mathcal{X}^d \oplus \mathcal{Y}$ onto $\mathcal{Y}$ and $\mathcal{X}^d$, respectively, and $\mathcal{P}_{\mathcal{X}}^*$ is the inclusion map of $\mathcal{X}$ into $\mathcal{X} \oplus \mathcal{U}$.

We now recall a useful procedure for constructing an element in $\mathcal{S}_{\mathrm{nc},d}(\mathcal{U}, \mathcal{Y})$ with the prescribed output pair $(C, \mathbf{A})$ in its contractive realization, and furthermore identifying when this S is inner (i.e., $M_S : H^2_{\mathcal{U}}(\mathbb{F}_d^+) \to H^2_{\mathcal{Y}}(\mathbb{F}_d^+)$ is a partial isometry) or even strictly inner (i.e., M_S is an isometry). This result appears as Theorem 3.8 in [9].

Theorem 2.4 *Given a tuple $\mathbf{A} = (A_1, \ldots, A_d) \in \mathcal{L}(\mathcal{X})^d$ and $C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, let $H \in \mathcal{L}(\mathcal{X})$ be a strictly positive definite solution to the Stein equation (2.12). Let*

A and $Z(z)$ be defined as in (2.8) and let $\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right] : \mathcal{U} \to \left[\begin{smallmatrix} \mathcal{X}^d \\ \mathcal{Y} \end{smallmatrix}\right]$ be a solution of the Cholesky factorization problem

$$\begin{bmatrix} B \\ D \end{bmatrix} \begin{bmatrix} B^* & D^* \end{bmatrix} = \begin{bmatrix} H^{-1} \otimes I_d & 0 \\ 0 & I_{\mathcal{Y}} \end{bmatrix} - \begin{bmatrix} A \\ C \end{bmatrix} H^{-1} \begin{bmatrix} A^* & C^* \end{bmatrix}. \tag{2.18}$$

1. *Then the pair $(C, \mathbf{A})$ is output-stable and the power series*

$$S(z) = D + C(I - Z(z)A)^{-1} Z(z) B$$

belongs to $\mathcal{S}_{\mathrm{nc},d}(\mathcal{U}, \mathcal{Y})$. Moreover,

$$\begin{aligned} & k_{\mathrm{Sz}}(z, \zeta) \otimes I_{\mathcal{Y}} - S(z)(k_{\mathrm{Sz}}(z, \zeta) \otimes I_{\mathcal{U}}) S(\zeta)^* \\ & = C(I - Z(z)A)^{-1} H^{-1} (I - A^* Z(\zeta)^*)^{-1} C^*. \end{aligned}$$

2. *If $\mathbf{A}$ is strongly stable, then S is inner. Conversely, any inner multiplier arises in this way.*
3. *If $\mathbf{A}$ is strongly stable and the solution $\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]$ of (2.18) is normalized to be injective, then S is strictly inner. Conversely, any strictly inner multiplier arises in this way.*

The next factorization result is the noncommutative version the Leech theorem (see [25]); the necessity is an immediate consequence of Theorem 2.3, for the proof of sufficiency, see [9].

Theorem 2.5 *Given formal power series $G \in \mathcal{L}(\mathcal{Y}, \mathcal{X})\langle\langle z \rangle\rangle$, $F \in \mathcal{L}(\mathcal{U}, \mathcal{X})\langle\langle z \rangle\rangle$, the formal kernel*

$$K_{G,F}(z, \zeta) := G(z)(k_{\mathrm{Sz}}(z, \zeta) \otimes I_{\mathcal{Y}}) G(\zeta)^* - F(z)(k_{\mathrm{Sz}}(z, \zeta) \otimes I_{\mathcal{U}}) F(\zeta)^*$$

is positive if and only if there exists an $S \in \mathcal{S}_{\mathrm{nc},d}(\mathcal{U}, \mathcal{Y})$ such that $F(z) = G(z) S(z)$.

2.3 de Branges-Rovnyak Spaces

Associated with any $S \in \mathcal{S}_{\mathrm{nc},d}(\mathcal{U}, \mathcal{Y})$ is the *de Branges-Rovnyak space* $\mathcal{H}(K_S)$, the NFRKHS with reproducing kernel K_S (which is positive by Theorem 2.3). Just as in the classical case, the de Branges-Rovnyak space $\mathcal{H}(K_S)$ has several equivalent characterizations. The original characterization of $\mathcal{H}(K_S)$, as the space of all formal power series $f(z) \in \mathcal{Y}\langle\langle z\rangle\rangle$ with finite $\mathcal{H}$-norm

$$\|f\|_{\mathcal{H}}^2 = \sup_{g \in H^2_{\mathcal{U}}(\mathbb{F}^+_d)} \left\{ \|f + Sg\|^2_{H^2_{\mathcal{Y}}(\mathbb{F}^+_d)} - \|g\|^2_{H^2_{\mathcal{U}}(\mathbb{F}^+_d)} \right\} \tag{2.19}$$

is due to de Branges and Rovnyak [16] (for the case $d = 1$); see [11] for the general case. In particular, it follows from (2.19) that $\|f\|_{\mathcal{H}(K_S)} \geq \|f\|_{H^2_{\mathcal{Y}}(\mathbb{F}^+_d)}$ for every $f \in \mathcal{H}(K_S)$, i.e., that $\mathcal{H}(K_S)$ is contained in $\mathcal{H}_{\mathcal{Y}}(\mathbb{F}^+_d)$ contractively. On the other hand, the general complementation theory applied to the contractive operator M_S provides the characterization of $\mathcal{H}(K_S)$ as the operator range

$$\mathcal{H}(K_S) = \operatorname{Ran}(I - M_S M_S^*)^{\frac{1}{2}} \tag{2.20}$$

with the lifted norm

$$\|(I - M_S M_S^*)^{\frac{1}{2}} f\|_{\mathcal{H}(K_S)} = \|(I - Q)f\|_{\mathcal{H}_{\mathcal{Y}}(\mathbb{F}^+_d)} \tag{2.21}$$

for all $f \in H^2_{\mathcal{Y}}(\mathbb{F}^+_d)$, where Q is the orthogonal projection onto $\operatorname{Ker}(I - M_S M_S^*)^{\frac{1}{2}}$. Upon setting $f = (I - M_S M_S^*)^{\frac{1}{2}} h$ in (2.21) we get

$$\|(I - M_S M_S^*)h\|^2_{\mathcal{H}(K_S)} = \langle (I - M_S M_S^*)h,\, h\rangle_{H^2_{\mathcal{Y}}(\mathbb{F}^+_d)}.$$

Finally, if S is realized as in (2.15) with a unitary colligation matrix $\mathbf{U}$, then the kernel K_S admits a Kolmogorov decomposition (2.13) with $H(z)$ defined as in (2.16) and hence, $\mathcal{H}(K_S)$ can be characterized as the range space of the observability operator $\mathcal{O}_{C,\mathbf{A}}$ with lifted norm

$$\mathcal{H}(K_S) = \left\{\mathcal{O}_{C,\mathbf{A}} x \,:\, x \in \mathcal{X}\right\} \quad \text{and} \quad \|\mathcal{O}_{C,\mathbf{A}} x\|_{\mathcal{H}(K_S)} = \|Qx\|_{\mathcal{X}}, \tag{2.22}$$

where Q is the orthogonal projection of $\mathcal{X}$ onto $(\operatorname{Ker} \mathcal{O}_{C,\mathbf{A}})^{\perp}$. More complete details concerning the spaces $\mathcal{H}(K_S)$ and related matters of realization and the model theory for commutative row contractions can be found in [11].

If $\Theta \in \mathcal{S}_{\mathrm{nc},d}(\mathcal{U}, \mathcal{Y})$ is inner, then the associated de Branges-Rovnyak space $\mathcal{H}(K_\Theta)$ is isometrically included in $H^2_{\mathcal{Y}}(\mathbb{F}^+_d)$ and

$$\mathcal{H}(K_\Theta) = H^2_{\mathcal{Y}}(\mathbb{F}^+_d) \ominus \Theta H^2_{\mathcal{U}}(\mathbb{F}^+_d). \tag{2.23}$$

Moreover, the orthogonal projection $\mathcal{P}_{\mathcal{H}(K_\Theta)}$ of $H^2_{\mathcal{Y}}(\mathbb{F}^+_d)$ onto $\mathcal{H}(K_\Theta)$ is given by

$$\mathcal{P}_{\mathcal{H}(K_\Theta)} = I_{H^2_{\mathcal{Y}}(\mathbb{F}^+_d)} - M_\Theta M_\Theta^*.$$

Since the space $\Theta H^2_{\mathcal{F}}(\mathbb{F}^+_d)$ is shift invariant (i.e., R_{z_j}-invariant for $j = 1, \dots, d$), it follows from (2.23) that the space $\mathcal{H}(K_\Theta)$ is backward shift invariant (i.e., $R^*_{z_j}$-invariant for $j = 1, \dots, d$). The Beurling-Lax theorem for $H^2_{\mathcal{Y}}(\mathbb{F}^+_d)$ (see [10, 17, 27]) asserts that *any shift invariant closed subspace* $\mathcal{M}$ *of* $H^2_{\mathcal{Y}}(\mathbb{F}^+_d)$ *necessarily has the form* $\Theta H^2_{\mathcal{U}}(\mathbb{F}^+_d)$ *for some strictly inner multiplier* $\Theta \in \mathcal{S}_{\mathrm{nc},d}(\mathcal{F}, \mathcal{Y})$; in this situation we say that Θ is a *Beurling-Lax representer* for the shift-invariant subspace $\mathcal{M}$. Therefore any backward-shift-invariant subspace $\mathcal{M}$ of $H^2_{\mathcal{Y}}(\mathbb{F}^+_d)$ has the form $\mathcal{M} = \mathcal{H}(K_\Theta)$. Let us finally note that there is a simple procedure for cutting down an inner Θ such that $\mathcal{M} = \Theta H^2_{\mathcal{U}}$ to a Beurling-Lax representer for $\mathcal{M}$, i.e., to a strictly inner Θ_0 so that $\mathcal{M} = \Theta_0 H^2_{\mathcal{U}_0}$: see Remark 3.11 in [9].

3 Indefinite Noncommutative Schur Class and Linear-Fractional Transformations

If $\mathcal{X}$ is a Hilbert space and G is a selfadjoint operator on $\mathcal{X}$, we use the notation $(\mathcal{X}, G)$ to denote the space $\mathcal{X}_G$ with the indefinite inner product induced by G:

$$\langle x, y\rangle_{\mathcal{X}_G} := \langle Gx, y\rangle_{\mathcal{X}}.$$

As a further abuse of notation we shall on occasion write $\|x\|^2_{\mathcal{X}_G}$ for $\langle x, x\rangle_G$ even though the result $\|\cdot\|_{\mathcal{X}_G}$ so defined is not a norm if G is indefinite. Usually it is assumed that G is invertible, so $(\mathcal{X}, G)$ is a Hilbert space if G is positive definite and a Kreĭn space in general. In what follows, the indefinite metric will be primarily determined by signature operators

$$J_{\mathcal{Y},\mathcal{U}} = \begin{bmatrix} I_{\mathcal{Y}} & 0 \\ 0 & -I_{\mathcal{U}} \end{bmatrix} \quad \text{and} \quad J_{\mathcal{F},\mathcal{U}} = \begin{bmatrix} I_{\mathcal{F}} & 0 \\ 0 & -I_{\mathcal{U}} \end{bmatrix}. \tag{3.1}$$

A bounded operator

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} : \begin{bmatrix} \mathcal{F} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix} \tag{3.2}$$

is called a $(J_{\mathcal{F},\mathcal{U}}, J_{\mathcal{Y},\mathcal{U}})$-*bicontraction* if

$$W^* J_{\mathcal{Y},\mathcal{U}} W \preceq J_{\mathcal{F},\mathcal{U}} \quad \text{and} \quad W J_{\mathcal{F},\mathcal{U}} W^* \preceq J_{\mathcal{Y},\mathcal{U}}. \tag{3.3}$$

If the first (second) relation in (3.3) holds with equality, the operator W is called $(J_{\mathcal{F},\mathcal{U}}, J_{\mathcal{Y},\mathcal{U}})$-*isometry* (*coisometry*). Two equalities in (3.3) define a $(J_{\mathcal{F},\mathcal{U}}, J_{\mathcal{Y},\mathcal{U}})$-*unitary* operator.

Relations (3.3) imply that the operator $W_{22} \in \mathcal{L}(\mathcal{U})$ is an invertible bijection; see [18, Theorem 1.3.4]. Note that the first relation in (3.6) alone guarantees only that W_{22} is injective and has closed range. Furthermore, if the first relation in (3.6) holds (i.e., W is a $(J_{\mathcal{F},\mathcal{U}}, J_{\mathcal{Y},\mathcal{U}})$-contraction) then the second one holds (i.e., W is a $(J_{\mathcal{F},\mathcal{U}}, J_{\mathcal{Y},\mathcal{U}})$-bicontraction) *if and only if* the operator $W_{22} \in \mathcal{L}(\mathcal{U})$ is invertible; see again [18].

3.1 Potapov-Ginzburg Transform

A convenient tool to study Krein space bi-contractions is the *Potapov-Ginzburg transform* introduced in [19] and defined for any W of the form (3.2) with boundedly invertible block $W_{22} = \mathcal{P}_{\mathcal{U}} W|_{\mathcal{U}}$ as follows:

$$T_{\mathcal{PG}}(W) := \begin{bmatrix} W_{11} & W_{12} \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ W_{21} & W_{22} \end{bmatrix}^{-1} \tag{3.4}$$
$$= \begin{bmatrix} W_{11} - W_{12}W_{22}^{-1}W_{21} & W_{12}W_{22}^{-1} \\ -W_{22}^{-1}W_{21} & W_{22}^{-1} \end{bmatrix}$$
$$= \begin{bmatrix} I & -W_{12} \\ 0 & -W_{22} \end{bmatrix}^{-1} \begin{bmatrix} W_{11} & 0 \\ W_{21} & -I \end{bmatrix},$$

where the second and the third equalities are easily verified. One can see from (3.4) that $T_{\mathcal{PG}}(W)$ is an involution. Straightforward computations show that

$$I_{\mathcal{F}\oplus\mathcal{U}} - T_{\mathcal{PG}}(W)^* T_{\mathcal{PG}}(W) = \begin{bmatrix} I & W_{21}^* \\ 0 & W_{22}^* \end{bmatrix}^{-1} (J_{\mathcal{F},\mathcal{U}} - W^* J_{\mathcal{Y},\mathcal{U}} W) \begin{bmatrix} I & 0 \\ W_{21} & W_{22} \end{bmatrix}^{-1},$$
$$I_{\mathcal{Y}\oplus\mathcal{U}} - T_{\mathcal{PG}}(W) T_{\mathcal{PG}}(W)^* = \begin{bmatrix} I & -W_{12} \\ 0 & -W_{22} \end{bmatrix}^{-1} (J_{\mathcal{Y},\mathcal{U}} - W J_{\mathcal{F},\mathcal{U}} W^*) \begin{bmatrix} I & 0 \\ -W_{12}^* & -W_{22}^* \end{bmatrix}^{-1}$$

and therefore $T_{\mathcal{PG}}$ establishes a bijection between $(J_{\mathcal{F},\mathcal{U}}, J_{\mathcal{Y},\mathcal{U}})$-contractions and contractions $S \in \mathcal{L}(\mathcal{F} \oplus \mathcal{U}, \mathcal{Y} \oplus \mathcal{U})$ with boundedly invertible compression $S_{22} = \mathcal{P}_{\mathcal{U}} S|_{\mathcal{U}}$. Furthermore, W is $(J_{\mathcal{F},\mathcal{U}}, J_{\mathcal{Y},\mathcal{U}})$-contractive (isometric, coisometric, unitary) if and only if $S = T_{\mathcal{PG}}(W)$ is contractive (respectively, isometric, coisometric, unitary). The Potapov-Ginzburg transform amounts to a partial inversion (with respect to the $(2, 2)$-block entry) of W in the following sense: assuming that W_{22} is invertible, then *the collection of signals (vectors)* u, y, z, w *satisfies the system of equations*

$$\begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix} = \begin{bmatrix} z \\ y \end{bmatrix}$$

if and only if the same u, y, z, w *also satisfy the system of equations*

$$\begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} z \\ w \end{bmatrix}$$

where $\Theta = \left[\begin{smallmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{smallmatrix}\right]$ *is equal to the Potapov-Ginzburg transform* $T_{\mathcal{PG}}(W)$ *of* W.
In circuit-theoretic terms, this formalism amounts to the transformation from the chain formalism to the scattering formalism; see [18, Section 1.3] and the references cited there in the Notes section at the end of the chapter.

In [33] and [18], the transform (3.4) is formulated for operator-valued functions. We now discuss its extension to formal power series. In what follows, we shall often have use for the operator $J_{\mathcal{Y},\mathcal{U}} \otimes I_{H^2_{\mathcal{Y}\oplus\mathcal{U}}(\mathbb{F}^+_d)}$ acting on $H^2_{\mathcal{Y}\oplus\mathcal{U}}(\mathbb{F}^+_d)$; we shall abuse notation and still write this operator as $J_{\mathcal{Y},\mathcal{U}}$.

Definition 3.1 Given coefficient Hilbert spaces $\mathcal{U}$, $\mathcal{Y}$, $\mathcal{F}$, the *noncommutative indefinite Schur class* $\mathcal{S}_{\mathrm{nc},d}(J_{\mathcal{Y},\mathcal{U}}, J_{\mathcal{F},\mathcal{U}})$ consists of formal power series

$$\mathfrak{A}(z) = \begin{bmatrix} \mathfrak{A}_{11}(z) & \mathfrak{A}_{12}(z) \\ \mathfrak{A}_{21}(z) & \mathfrak{A}_{22}(z) \end{bmatrix} \in \mathcal{L}\left(\begin{bmatrix} \mathcal{F} \\ \mathcal{U} \end{bmatrix}, \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix}\right)\langle\langle z \rangle\rangle \tag{3.5}$$

such that the multiplication operator

$$M_{\mathfrak{A}} : (H^2_{\mathcal{F}\oplus\mathcal{U}}(\mathbb{F}^+_d), J_{\mathcal{F},\mathcal{U}}) \to (H^2_{\mathcal{Y}\oplus\mathcal{U}}(\mathbb{F}^+_d), J_{\mathcal{Y},\mathcal{U}})$$

is a $(J_{\mathcal{F},\mathcal{U}}, J_{\mathcal{Y},\mathcal{U}})$-bicontraction:

$$M^*_{\mathfrak{A}} J_{\mathcal{Y},\mathcal{U}} M_{\mathfrak{A}} \preceq J_{\mathcal{F},\mathcal{U}} \quad \text{and} \quad M_{\mathfrak{A}} J_{\mathcal{F},\mathcal{U}} M^*_{\mathfrak{A}} \preceq J_{\mathcal{Y},\mathcal{U}}. \tag{3.6}$$

By [18], the first relation in (3.6) alone guarantees that $M_{\mathfrak{A}_{22}}$ is injective and has closed range, and if this is the case, then the second relation holds if and only if $M_{\mathfrak{A}_{22}} \in \mathcal{L}(H^2_{\mathcal{U}}(\mathbb{F}^+_d))$ is an invertible bijection.

The first relation in (3.6) means that for any element $g = \left[\begin{smallmatrix} g_1 \\ g_2 \end{smallmatrix}\right] \in H^2_{\mathcal{F}\oplus\mathcal{U}}(\mathbb{F}^+_d)$,

$$\|\mathfrak{A}g\|^2_{(H^2_{\mathcal{Y}\oplus\mathcal{U}}(\mathbb{F}^+_d), J_{\mathcal{Y},\mathcal{U}})} \le \|g\|^2_{(H^2_{\mathcal{F}\oplus\mathcal{U}}(\mathbb{F}^+_d), J_{\mathcal{F},\mathcal{U}})}$$

which, on account of (3.1) and (3.5), can be written in more detail as

$$\|\mathfrak{A}_{11}g_1 + \mathfrak{A}_{12}g_2\|^2_{H^2_{\mathcal{Y}}(\mathbb{F}^+_d)} - \|\mathfrak{A}_{21}g_1 + \mathfrak{A}_{22}g_2\|^2_{H^2_{\mathcal{U}}(\mathbb{F}^+_d)} \le \|g_1\|^2_{H^2_{\mathcal{F}}(\mathbb{F}^+_d)} - \|g_2\|^2_{H^2_{\mathcal{U}}(\mathbb{F}^+_d)}.$$

Write the latter inequality equivalently as

$$\|\mathfrak{A}_{11}g_1 + \mathfrak{A}_{12}g_2\|^2_{H^2_{\mathcal{Y}}(\mathbb{F}^+_d)} + \|g_2\|^2_{H^2_{\mathcal{U}}(\mathbb{F}^+_d)} \le \|\mathfrak{A}_{21}g_1 + \mathfrak{A}_{22}g_2\|^2_{H^2_{\mathcal{U}}(\mathbb{F}^+_d)} + \|g_1\|^2_{H^2_{\mathcal{F}}(\mathbb{F}^+_d)},$$

or as

$$\left\| \begin{bmatrix} \mathfrak{A}_{11} & \mathfrak{A}_{12} \\ 0 & I \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \right\|_{H^2_{\mathcal{Y}\oplus\mathcal{U}}(\mathbb{F}^+_d)} \leq \left\| \begin{bmatrix} I & 0 \\ \mathfrak{A}_{21} & \mathfrak{A}_{22} \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \right\|_{H^2_{\mathcal{F}\oplus\mathcal{U}}(\mathbb{F}^+_d)}. \tag{3.7}$$

Since the range of $\left[\begin{smallmatrix} I & 0 \\ M_{\mathfrak{A}_{21}} & M_{\mathfrak{A}_{22}} \end{smallmatrix} \right]$ is the whole $H^2_{\mathcal{F}\oplus\mathcal{U}}(\mathbb{F}^+_d)$, it follows from (3.7) that

$$\Sigma(z) = T_{\mathcal{PG}}(\mathfrak{A})(z) := \begin{bmatrix} \mathfrak{A}_{11}(z) & \mathfrak{A}_{12}(z) \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ \mathfrak{A}_{21}(z) & \mathfrak{A}_{22}(z) \end{bmatrix}^{-1} \tag{3.8}$$

$$= \begin{bmatrix} I & -\mathfrak{A}_{12}(z) \\ 0 & -\mathfrak{A}_{22}(z) \end{bmatrix}^{-1} \begin{bmatrix} \mathfrak{A}_{11}(z) & 0 \\ \mathfrak{A}_{21}(z) & -I \end{bmatrix}, \tag{3.9}$$

the *Potapov-Ginzburg transform* of $\mathfrak{A}$, belongs to the noncommutative Schur class $S_{\mathrm{nc},d}(\mathcal{F} \oplus \mathcal{U}, \mathcal{Y} \oplus \mathcal{U})$. Similarly to the case of constant operators (3.4), the transform (3.8) is an involution that establishes a bijection between the elements of the noncommutative indefinite Schur class $\mathcal{S}_{\mathrm{nc},d}(J_{\mathcal{Y},\mathcal{U}}, J_{\mathcal{F},\mathcal{U}})$ and Schur-class power series Σ with the block Σ_{22} boundedly invertible in $\mathcal{M}_{\mathrm{nc},d}(\mathcal{U})$.

3.2 Indefinite de Branges-Rovnyak Spaces and Potapov-Ginzburg Transform of Kernels and Realizations

The second relation in (3.6) can be reformulated in terms of positive kernels as follows. By (2.7), the action of the operator $M^*_{\mathfrak{A}} : H^2_{\mathcal{Y}\oplus\mathcal{U}}(\mathbb{F}^+_d) \to H^2_{\mathcal{F}\oplus\mathcal{U}}(\mathbb{F}^+_d)$ on the kernel elements $k_{\mathrm{Sz}}(\cdot, \zeta) \left[\begin{smallmatrix} y \\ u \end{smallmatrix} \right]$ ($y \in \mathcal{Y}$, $u \in \mathcal{U}$) is given by the formula

$$M^*_{\mathfrak{A}} k_{\mathrm{Sz}}(\cdot, \zeta) \left[\begin{smallmatrix} y \\ u \end{smallmatrix} \right] = k_{\mathrm{Sz}}(\cdot, \zeta) \mathfrak{A}(\zeta)^* \left[\begin{smallmatrix} y \\ u \end{smallmatrix} \right].$$

Therefore,

$$J_{\mathcal{Y},\mathcal{U}} - M_{\mathfrak{A}} J_{\mathcal{F},\mathcal{U}} M^*_{\mathfrak{A}} : k_{\mathrm{Sz}}(\cdot, \zeta) \left[\begin{smallmatrix} y \\ u \end{smallmatrix} \right] \mapsto k_{\mathrm{Sz}}(\cdot, \zeta) J_{\mathcal{Y},\mathcal{U}} \left[\begin{smallmatrix} y \\ u \end{smallmatrix} \right] - \mathfrak{A}(\cdot) k_{\mathrm{Sz}}(\cdot, \zeta) J_{\mathcal{F},\mathcal{U}} \mathfrak{A}(\zeta)^* \left[\begin{smallmatrix} y \\ u \end{smallmatrix} \right].$$

By linearity, the latter formula extends to linear combinations of kernel elements, and it is not hard to show that the second condition in (3.6) (i.e., $J_{\mathcal{Y},\mathcal{U}} - M_{\mathfrak{A}} J_{\mathcal{F},\mathcal{U}} M^*_{\mathfrak{A}} \succeq 0$) is equivalent to the formal kernel

$$K^{J_{\mathcal{F},\mathcal{U}}, J_{\mathcal{Y},\mathcal{U}}}_{\mathfrak{A}}(z, \zeta) := k_{\mathrm{Sz}}(z, \zeta) J_{\mathcal{Y},\mathcal{U}} - \mathfrak{A}(z)(k_{\mathrm{Sz}}(z, \zeta) J_{\mathcal{F},\mathcal{U}}) \mathfrak{A}(\zeta)^* \tag{3.10}$$

be positive. Therefore, it makes sense to associate the de Branges-Rovnyak space $\mathcal{H}(K_{\mathfrak{A}}^{J_{\mathcal{F},\mathcal{U}},J_{\mathcal{Y},\mathcal{U}}})$ with an indefinite Schur-class power series $\mathfrak{A}$.

Furthermore the formal kernel $K_{\mathfrak{A}}^{J_{\mathcal{F},\mathcal{U}},J_{\mathcal{Y},\mathcal{U}}}(z,\zeta)$ for the indefinite de Branges-Rovnyak space associated with the indefinite Schur-class power series $\mathfrak{A}$ can be recovered from the formal kernel $K_\Sigma(z,\zeta)$ for the associated definite de Branges-Rovnyak space associated with the Schur-class power series $\Sigma = T_{\mathcal{PG}}(\mathfrak{A})$ according to the following formula.

Proposition 3.2 *If the power series Σ is the Potapov-Ginzburg transform of $\mathfrak{A} \in \mathcal{S}_{\mathrm{nc},d}(J_{\mathcal{Y},\mathcal{U}}, J_{\mathcal{F},\mathcal{U}})$, then the associated kernel K_Σ can be recovered as a kernel-level Potapov-Ginzberg transform of the kernel $K_{\mathfrak{A}}^{J_{\mathcal{F},\mathcal{U}},J_{\mathcal{Y},\mathcal{U}}}$ according to the following formula:*

$$
\begin{aligned}
K_{\mathfrak{A}}^{J_{\mathcal{F},\mathcal{U}},J_{\mathcal{Y},\mathcal{U}}}(z,\zeta) &= T_{\mathcal{PG}}^{\mathfrak{A}}(K_\Sigma)(z,\zeta) \\
&:= \begin{bmatrix} I & -\mathfrak{A}_{12}(z) \\ 0 & -\mathfrak{A}_{22}(z) \end{bmatrix} K_\Sigma(z,\zeta) \begin{bmatrix} I & 0 \\ -\mathfrak{A}_{12}(\zeta)^* & -\mathfrak{A}_{22}(\zeta)^* \end{bmatrix}. \qquad (3.11)
\end{aligned}
$$

Proof The equality (3.11) can be seen to follow from the definitions (2.13), (3.10) of the respective kernels and the formula (3.9). □

We now indicate how to extend the Potapov-Ginzburg transform to realization matrices $\mathbf{U}$ so that we can recover the Potapov-Ginzburg transform $\mathfrak{A} := T_{\mathcal{PG}}(\Sigma) = T_{\mathcal{PG}}(T_{\mathbf{U}})$ of the characteristic formal power series of $T_{\mathbf{U}}$ in realization form as

$$
T_{\mathcal{PG}}(T_{\mathbf{U}}) = T_{T_{\mathcal{PG}}(\mathbf{U})} \qquad (3.12)
$$

where $\widetilde{\mathbf{U}} = T_{\mathcal{PG}}(\mathbf{U})$ is the appropriate version of the Potapov-Ginzburg transform of the colligation matrix $\mathbf{U}$.

Theorem 3.3 *Suppose that*

$$
\mathbf{U} = \mathbf{U} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] = \left[\begin{array}{c|cc} A_1 & B_{11} & B_{21} \\ \vdots & \vdots & \vdots \\ A_d & B_{d1} & B_{d2} \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] : \begin{bmatrix} \mathcal{X} \\ \hline \mathcal{F} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X}^d \\ \hline \mathcal{Y} \\ \mathcal{U} \end{bmatrix} \qquad (3.13)
$$

is a colligation matrix with characteristic formal power series

$$
\begin{aligned}
T_{\mathbf{U}}(z) = \Sigma(z) &= \begin{bmatrix} \Sigma_{11}(z) & \Sigma_{12}(z) \\ \Sigma_{21}(z) & \Sigma_{22}(z) \end{bmatrix} \\
&= \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} + \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (I - Z(z)A)^{-1} Z(z) \begin{bmatrix} B_1 & B_2 \end{bmatrix}.
\end{aligned}
$$

Assume that D_{22} is invertible, so Σ_{22} is invertible in the algebra of formal power series $\mathcal{L}(\mathcal{U})\langle\langle z\rangle\rangle$. Let $\mathfrak{A} = T_{\mathcal{P}G}(\Sigma) \in \mathcal{L}(\mathcal{F} \oplus \mathcal{U}, \mathcal{Y} \oplus \mathcal{U})\langle\langle z\rangle\rangle$ be equal to the Potapov-Ginzburg transform of Σ as in (3.4). Let us set

$$\widetilde{\mathbf{U}}' = T_{\mathcal{P}G}(\mathbf{U}')$$

where $\mathbf{U}'$ is the same as $\mathbf{U}$ but organized differently as a block 2×2 matrix:

$$\mathbf{U}' = \left[\begin{array}{cc|c} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ \hline C_2 & D_{21} & D_{22} \end{array}\right].$$

Application of the Potapov-Ginzburg transform to $\mathbf{U}'$ gives us a block 3×3 matrix $\widetilde{\mathbf{U}}'$ organized as a block 2×2 matrix of the form

$$\widetilde{\mathbf{U}}' = \left[\begin{array}{cc|c} \widetilde{A} & \widetilde{B}_1 & \widetilde{B}_2 \\ \widetilde{C}_1 & \widetilde{D}_{11} & \widetilde{D}_{12} \\ \hline \widetilde{C}_2 & \widetilde{D}_{21} & \widetilde{D}_{22} \end{array}\right]. \tag{3.14}$$

Define a new colligation matrix $\widetilde{\mathbf{U}}$ by reorganizing $\widetilde{\mathbf{U}}'$ as a block-2×2 matrix according to

$$\widetilde{\mathbf{U}} = \left[\begin{array}{c|cc} \widetilde{A} & \widetilde{B}_1 & \widetilde{B}_2 \\ \hline \widetilde{C}_1 & \widetilde{D}_{11} & \widetilde{D}_{12} \\ \widetilde{C}_2 & \widetilde{D}_{21} & \widetilde{D}_{22} \end{array}\right]. \tag{3.15}$$

Then we recover $\mathfrak{A}(z)$ in realization form as the characteristic formal power series $T_{\widetilde{\mathbf{U}}}$ of $\widetilde{\mathbf{U}}$:

$$\mathfrak{A}(z) = T_{\widetilde{\mathbf{U}}}(z) \tag{3.16}$$

$$= \begin{bmatrix} \widetilde{D}_{11} & \widetilde{D}_{12} \\ \widetilde{D}_{21} & \widetilde{D}_{22} \end{bmatrix} + \begin{bmatrix} \widetilde{C}_1 \\ \widetilde{C}_2 \end{bmatrix} (I - Z(z)\widetilde{A})^{-1} \begin{bmatrix} \widetilde{B}_1 & \widetilde{B}_2 \end{bmatrix}. \tag{3.17}$$

Thus, if we are not fussy about specifying how to view a block 3×3 matrix as a block 2×2 matrix, the formula (3.12) holds.

Proof As we are assuming that the operator $D_{22} \in \mathcal{L}(\mathcal{U})$ is boundedly invertible, we may form the Potapov-Ginzburg transform $\widetilde{\mathbf{U}}' := T_{\mathcal{P}\mathcal{G}}(\mathbf{U}')$ via the rule (3.4):

$$\widetilde{\mathbf{U}}' = \begin{bmatrix} \widetilde{A} & \widetilde{B}_1 & \widetilde{B}_2 \\ \widetilde{C}_1 & \widetilde{D}_{11} & \widetilde{D}_{12} \\ \widetilde{C}_2 & \widetilde{D}_{21} & \widetilde{D}_{22} \end{bmatrix} = T_{\mathcal{P}\mathcal{G}}(\mathbf{U}') = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ 0 & 0 & I_{\mathcal{U}} \end{bmatrix} \begin{bmatrix} I_{\mathcal{X}} & 0 & 0 \\ 0 & I_{\mathcal{F}} & 0 \\ C_2 & D_{21} & D_{22} \end{bmatrix}^{-1},$$

or in more detail,

$$\widetilde{A} = A - B_2 D_{22}^{-1} C_2, \tag{3.18}$$

$$\widetilde{B} = \begin{bmatrix} \widetilde{B}_1 & \widetilde{B}_2 \end{bmatrix} = \begin{bmatrix} B_1 & B_2 \end{bmatrix} \begin{bmatrix} I_{\mathcal{F}} & 0 \\ D_{21} & D_{22} \end{bmatrix}^{-1}, \tag{3.19}$$

$$\widetilde{C} = \begin{bmatrix} \widetilde{C}_1 \\ \widetilde{C}_2 \end{bmatrix} = \begin{bmatrix} I_{\mathcal{Y}} & -D_{12} \\ 0 & -D_{22} \end{bmatrix}^{-1} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \tag{3.20}$$

$$\widetilde{D} = \begin{bmatrix} \widetilde{D}_{11} & \widetilde{D}_{12} \\ \widetilde{D}_{21} & \widetilde{D}_{22} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} \\ 0 & I_{\mathcal{U}} \end{bmatrix} \begin{bmatrix} I_{\mathcal{F}} & 0 \\ D_{21} & D_{22} \end{bmatrix}^{-1}. \tag{3.21}$$

As a Potapov-Ginzburg transform, $\widetilde{\mathbf{U}}'$ is a block 2×2 matrix of the form (3.14). To reinterpret it as a colligation matrix, we view it instead as the same matrix but organized as in (3.15). We then let $\widetilde{\mathfrak{A}}(z)$ be the characteristic formal power series of $\widetilde{\mathbf{U}}$ as given by the right-hand side of (3.17). It remains to show that $\widetilde{\mathfrak{A}}$ equals the original power series $\mathfrak{A} = T_{\mathcal{PG}}(\Sigma)$. To this end, we read off from (3.26)

$$\begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ 0 & I \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} \\ 0 & I \end{bmatrix} + \begin{bmatrix} C_1 \\ 0 \end{bmatrix} (I - Z(z)A)^{-1} Z(z) \begin{bmatrix} B_1 & B_2 \end{bmatrix}, \tag{3.22}$$
$$\begin{bmatrix} I & 0 \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ D_{21} & D_{22} \end{bmatrix} + \begin{bmatrix} 0 \\ C_2 \end{bmatrix} (I - Z(z)A)^{-1} Z(z) \begin{bmatrix} B_1 & B_2 \end{bmatrix},$$

and then, upon making use of (3.18)–(3.21), we compute

$$\begin{aligned}
&\widetilde{\mathfrak{A}}(z) \begin{bmatrix} I & 0 \\ \Sigma_{21}(z) & \Sigma_{22}(z) \end{bmatrix} \\
&= \left(\widetilde{D} + \widetilde{C}(I - Z(z)\widetilde{A})^{-1} Z(z) \widetilde{B}\right) \\
&\qquad \cdot \left(\begin{bmatrix} I & 0 \\ D_{21} & D_{22} \end{bmatrix} + \begin{bmatrix} 0 \\ C_2 \end{bmatrix} (I - Z(z)A)^{-1} Z(z) \begin{bmatrix} B_1 & B_2 \end{bmatrix}\right) \\
&= \begin{bmatrix} D_{11} & D_{12} \\ 0 & I \end{bmatrix} + \widetilde{D} \begin{bmatrix} 0 \\ C_2 \end{bmatrix} (I - Z(z)A)^{-1} Z(z) \begin{bmatrix} B_1 & B_2 \end{bmatrix} \\
&\qquad + \widetilde{C}(I - Z(z)\widetilde{A})^{-1} Z(z) \begin{bmatrix} B_1 & B_2 \end{bmatrix} \\
&\qquad + \widetilde{C}(I - Z(z)\widetilde{A})^{-1} Z(z) \widetilde{B}_2 C_2 (I - Z(z)A)^{-1} Z(z) \begin{bmatrix} B_1 & B_2 \end{bmatrix}.
\end{aligned} \tag{3.23}$$

Since $\widetilde{B}_2 C_2 = B_2 D_{22}^{-1} C_2 = A - \widetilde{A}$ (by (3.18) and (3.19)), the sum of the two last terms on the right side of (3.23) equals

$$\widetilde{C}(I - Z(z)A)^{-1} Z(z) \begin{bmatrix} B_1 & B_2 \end{bmatrix}.$$

Besides, it follows from (3.20) and (3.21) that

$$\widetilde{C} + \widetilde{D}\begin{bmatrix} 0 \\ C_2 \end{bmatrix} = \begin{bmatrix} I & -D_{12}D_{22}^{-1} \\ 0 & -D_{22}^{-1} \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} + \begin{bmatrix} D_{12} \\ I \end{bmatrix} D_{22}^{-1}C_2 = \begin{bmatrix} C_1 \\ 0 \end{bmatrix}.$$

Taking the latter into account, we simplify the right side of (3.23) to

$$\begin{bmatrix} D_{11} & D_{12} \\ 0 & I \end{bmatrix} + \begin{bmatrix} C_1 \\ 0 \end{bmatrix} (I - Z(z)A)^{-1}Z(z)\begin{bmatrix} B_1 & B_2 \end{bmatrix}.$$

In other words and in view of (3.22), we just verified the identity

$$\widetilde{\mathfrak{A}}(z)\begin{bmatrix} I & 0 \\ \Sigma_{21}(z) & \Sigma_{22}(z) \end{bmatrix} = \begin{bmatrix} \Sigma_{11}(z) & \Sigma_{12}(z) \\ 0 & I \end{bmatrix}.$$

Therefore, $\widetilde{\mathfrak{A}} = T_{\mathcal{PG}}(\Sigma) = \mathfrak{A}$ and hence, the original power series $\mathfrak{A}$ admits the realization (3.17). □

We are now ready to present the indefinite analogue of Theorem 2.3 as an application of Theorem 3.3.

Theorem 3.4 *Let $\mathfrak{A} \in \mathcal{L}(\mathcal{F} \oplus \mathcal{U}, \mathcal{F} \oplus \mathcal{Y})\langle\langle z\rangle\rangle$ be decomposed as in (3.5). The following are equivalent:*

1. $\mathfrak{A} \in \mathcal{S}_{\mathrm{nc},d}(J_{\mathcal{Y},\mathcal{U}}, J_{\mathcal{F},\mathcal{U}})$.
2. $M_{\mathfrak{A}_{22}} \in \mathcal{L}(H^2_{\mathcal{U}}(\mathbb{F}^+_d))$ *is a bijection and the formal noncommutative kernel* $K_{\mathfrak{A}}^{J_{\mathcal{F},\mathcal{U}}, J_{\mathcal{Y},\mathcal{U}}}(z,\zeta)$ *defined in (3.10) is positive.*
3. *There exists a Hilbert space $\mathcal{X}$ and a $(J_{\mathcal{X}\oplus\mathcal{F},\mathcal{U}}, J_{\mathcal{X}^d\oplus\mathcal{Y},\mathcal{U}})$-unitary connection operator $\widetilde{\mathbf{U}}$ of the form*

$$\widetilde{\mathbf{U}} = \begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{F}\oplus\mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X}^d \\ \mathcal{Y}\oplus\mathcal{U} \end{bmatrix} \tag{3.24}$$

so that $\mathfrak{A}(z)$ can be realized as a formal power series in the form

$$\mathfrak{A}(z) = \widetilde{D} + \widetilde{C}(I - Z(z)\widetilde{A})^{-1}Z(z)\widetilde{B} \tag{3.25}$$

where $Z(z)$, $\widetilde{A}$ and $\widetilde{B}$ are defined as in (2.8).
4. *There exists a Hilbert space $\mathcal{X}$ and a $(J_{\mathcal{X}\oplus\mathcal{F},\mathcal{U}}, J_{\mathcal{X}^d\oplus\mathcal{Y},\mathcal{U}})$-bicontractive operator $\widetilde{\mathbf{U}}$ as in (3.24) such that $\mathfrak{A}(z)$ is given as in (3.17).*

Proof The equivalence (1)⇔(2) has been discussed in the previous section.

To verify the implication (1)⇒(3), we write a realization

$$\begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} + \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (I - Z(z)A)^{-1} Z(z) \begin{bmatrix} B_1 & B_2 \end{bmatrix} \tag{3.26}$$

with the unitary connection matrix

$$\mathbf{U} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] = \left[\begin{array}{c|cc} A_1 & B_{11} & B_{21} \\ \vdots & \vdots & \vdots \\ A_d & B_{d1} & B_{d2} \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] : \begin{bmatrix} \mathcal{X} \\ \hline \mathcal{F} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X}^d \\ \hline \mathcal{Y} \\ \mathcal{U} \end{bmatrix} \tag{3.27}$$

for the Schur-class multiplier $\Sigma = T_{\mathcal{PG}}(\mathfrak{A})$. By general principles of the Potapov-Ginzburg transform, we can recover $\mathfrak{A}$ from Σ as the Potapov-Ginzburg transform of Σ:

$$\mathfrak{A} = T_{\mathcal{PG}}(\Sigma).$$

By Theorem 3.3 we know that $\mathfrak{A} = T_{\widetilde{\mathbf{U}}}$ where $\widetilde{\mathbf{U}}' = T_{\mathcal{PG}}(\mathbf{U}')$ (where we use the notation in the proof of Theorem 3.3). As $\mathbf{U}$ is unitary, so also is $\mathbf{U}'$. By general properties of the Potapov-Ginzburg transform, it then follows that $\widetilde{\mathbf{U}}'$ (and also $\widetilde{\mathbf{U}}$) is $(J_{\mathcal{X}\oplus\mathcal{F},\mathcal{U}}, U_{\mathcal{X}^d\oplus\mathcal{U},\mathcal{U}})$-unitary, i.e., we have a connection matrix $\widetilde{\mathbf{U}}$ so that condition (3) in Theorem 3.4 holds. This completes the proof of (1)⇒(3).

The implication (3)⇒(4) is trivial. To verify (4)⇒(1), let us assume that $\mathfrak{A}$ admits a realization (3.17) with the $(J_{\mathcal{X}\oplus\mathcal{F},\mathcal{U}}, J_{\mathcal{X}^d\oplus\mathcal{Y},\mathcal{U}})$-bicontractive connecting matrix $\widetilde{\mathbf{U}}$. Then the matrix $\mathbf{U} = T_{\mathcal{PG}}(\widetilde{\mathbf{U}})$ of the form (3.13) is a contraction and therefore the power series $\Sigma(z)$ defined as in (3.26) is in the Schur class, by Theorem 2.3. Since $\Sigma = T_{\mathcal{PG}}(\mathfrak{A})$ and hence $\mathfrak{A} = T_{\mathcal{PG}}(\Sigma)$ (again by Theorem 3.3), it follows that $\mathfrak{A} \in \mathcal{S}_{\text{nc},d}(J_{\mathcal{Y},\mathcal{U}}, J_{\mathcal{F},\mathcal{U}})$. □

We next discuss the construction of $\mathfrak{A} \in \mathcal{S}_{\text{nc},d}(J_{\mathcal{Y},\mathcal{U}}, J_{\mathcal{F},\mathcal{U}})$ with the prescribed output pair $(\widetilde{C}, \widetilde{\mathbf{A}})$. It is tempting to use the Potapov-Ginzburg transform to derive such a construction from Theorem 2.4. However, the formulas (3.18)–(3.21) show that this approach requires $\widetilde{D}$ and $\widetilde{B}_2$ which are not known explicitly. We thus need a parallel construction; the ideas follow the proof of the corresponding result in the context of the Drury-Arveson space rather than the Fock space as is done in [6].

It is convenient to change notation. Recall the definition of *output-stable pair* as in the discussion surrounding the introduction of the observability operator (2.10). Let us consider an output-stable pair $(C, \mathbf{T})$ where

$$C = \begin{bmatrix} E \\ N \end{bmatrix} : \mathcal{X} \to \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix} \quad \text{and} \quad \mathbf{T} = (T_1, \ldots, T_d) \in \mathcal{L}(\mathcal{X})^d. \tag{3.28}$$

We define the $J_{\mathcal{Y},\mathcal{U}}$-gramian $\mathcal{G}_{C,\mathbf{T}}^{J_{\mathcal{Y},\mathcal{U}}}$ of the pair $(C, \mathbf{T})$ by

$$\begin{aligned}\mathcal{G}_{C,\mathbf{T}}^{J_{\mathcal{Y},\mathcal{U}}} &:= \mathcal{O}^*_{\left[\begin{smallmatrix} E \\ N \end{smallmatrix}\right],\mathbf{T}} J_{\mathcal{Y},\mathcal{U}} \mathcal{O}_{\left[\begin{smallmatrix} E \\ N \end{smallmatrix}\right],\mathbf{T}} \\ &= \mathcal{O}^*_{E,\mathbf{T}}\mathcal{O}_{E,\mathbf{T}} - \mathcal{O}^*_{N,\mathbf{T}}\mathcal{O}_{N,\mathbf{T}} \\ &= \mathcal{G}_{E,\mathbf{T}} - \mathcal{G}_{N,\mathbf{T}}. \end{aligned} \tag{3.29}$$

An important property of $\mathcal{G}_{C,\mathbf{T}}^{J_{\mathcal{Y},\mathcal{U}}}$ is that it solves the Stein equation

$$P - \sum_{j=1}^{d} T_j^* P T_j = C^* J_{\mathcal{Y},\mathcal{U}} C, \tag{3.30}$$

as follows easily from the fact that $\mathcal{G}_{E,\mathbf{T}}$ and $\mathcal{G}_{N,\mathbf{T}}$ satisfy Stein equations of the type (2.12), or by plugging in the infinite series representations

$$\begin{aligned}\mathcal{G}_{C,\mathbf{T}}^{J_{\mathcal{Y},\mathcal{U}}} &= \mathcal{G}_{E,\mathbf{T}} - \mathcal{G}_{N,\mathbf{T}} \\ &= \sum_{\alpha\in\mathbb{F}_d^+} \mathbf{T}^{*\alpha^\top} \left(E^*E - N^*N\right) \mathbf{T}^{\alpha} \\ &= \sum_{\alpha\in\mathbb{F}_d^+} \mathbf{T}^{*\alpha^\top} C^* J_{\mathcal{Y},\mathcal{U}} C \mathbf{T}^{\alpha}.\end{aligned}$$

Proposition 3.5 *Let us assume that the pair $(C, \mathbf{T})$ as in (3.28) is output-stable and that the gramian $P = \mathcal{G}_{C,\mathbf{T}}^{J_{\mathcal{Y},\mathcal{U}}}$ given by (3.29) is strictly positive definite. Then the operator $\mathcal{O}_{C,\mathbf{T}} : (\mathcal{X}, P) \to (H^2_{\mathcal{Y}\oplus\mathcal{U}}(\mathbb{F}_d^+), J)$ is a contraction. This operator is isometric if and only if $\mathbf{T}$ is strongly stable.*

Proof By the definition of $H^2(\mathbb{F}_d^+)$-inner product, we have

$$\begin{aligned}&\left\langle J\mathcal{O}_{C,\mathbf{T}}x,\ \mathcal{O}_{C,\mathbf{T}}x\right\rangle_{H^2_{\mathcal{Y}\oplus\mathcal{U}}(\mathbb{F}_d^+)} = \sum_{\alpha\in\mathbb{F}_d^+} \left\langle \mathbf{T}^{*\alpha^\top} C^* J C \mathbf{T}^{\alpha} x, x\right\rangle_{\mathcal{X}} \\ &= \lim_{N\to\infty} \sum_{\alpha\in\mathbb{F}_d^+ :\, |\alpha|\le N} \left\langle \mathbf{T}^{*\alpha^\top}(P - \sum_{j=1}^{d} T_j^* P T_j)\mathbf{T}^{\alpha} x, x\right\rangle_{\mathcal{X}} \\ &= \lim_{N\to\infty} \Big(\langle Px, x\rangle_{\mathcal{X}} - \sum_{\alpha\in\mathbb{F}_d^+ :\, |\alpha| = N+1} \langle \mathbf{T}^{*\alpha^\top} P \mathbf{T}^{\alpha} x, x\rangle_{\mathcal{X}}\Big) \le \langle Px, x\rangle\end{aligned}$$

with equality in the last step for all $x \in \mathcal{X}$ if and only if $\mathbf{T}$ is strongly stable. □

The proof of the next proposition can be found in [6, p. 312]. Note that in its formulation, P is *any* strictly positive solution to the Stein equation (3.30).

Proposition 3.6 *Let us assume that* (3.30) *holds for* $C, T_1, \ldots, T_d$ *as in* (3.28) *and a strictly positive definite* $P \in \mathcal{L}(\mathcal{X})$. *Then there exist an auxiliary Hilbert space* $\mathcal{F}$ *and an injective* $\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right] : \mathcal{F} \oplus \mathcal{U} \to \left[\begin{smallmatrix} \mathcal{X}^d \\ \mathcal{Y} \end{smallmatrix}\right]$ *solving the* J*-Cholesky factorization problem*

$$\begin{bmatrix} B \\ D \end{bmatrix} J_{\mathcal{F},\mathcal{U}} \begin{bmatrix} B^* & D^* \end{bmatrix} = \begin{bmatrix} P^{-1} \otimes I_d & 0 \\ 0 & J_{\mathcal{Y},\mathcal{U}} \end{bmatrix} - \begin{bmatrix} T \\ C \end{bmatrix} P^{-1} \begin{bmatrix} T^* & C^* \end{bmatrix}, \tag{3.31}$$

where we have set $T = \begin{bmatrix} T_1 \\ \vdots \\ T_d \end{bmatrix}$.

We are now ready to present the promised result on the construction of $\mathfrak{A} \in \mathcal{S}_{\mathrm{nc},d}(J_{\mathcal{Y},\mathcal{U}}, J_{\mathcal{F},\mathcal{U}})$ with the prescribed output pair $(\widetilde{C}, \widetilde{\mathbf{A}})$.

Theorem 3.7 *Let us assume that* (3.30) *holds for* $C, T_1, \ldots, T_d$ *as in* (3.28) *and a strictly positive definite* $P \in \mathcal{L}(\mathcal{X})$. *Let* $\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right] : \mathcal{F} \oplus \mathcal{U} \to \left[\begin{smallmatrix} \mathcal{X}^d \\ \mathcal{Y} \end{smallmatrix}\right]$ *be a solution to the Cholesky factorization problem* (3.31). *Then*

1. *The space* $\mathcal{M} := \operatorname{Ran} \mathcal{O}_{C,\mathbf{T}}$ *endowed with the lifted norm*

$$\left\| \mathcal{O}_{C,\mathbf{T}} x \right\|^2 = \langle Px, x \rangle_{\mathcal{X}},$$

is isometrically equal to the NFRKHS with reproducing kernel

$$K^P_{C,\mathbf{T}}(z,\zeta) = C(I - Z(z)T)^{-1} P^{-1} (I - T^* Z(\zeta)^*)^{-1} C^*. \tag{3.32}$$

2. *The operator*

$$\mathbf{U} = \begin{bmatrix} T & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{F} \oplus \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X}^d \\ \mathcal{Y} \oplus \mathcal{U} \end{bmatrix} \tag{3.33}$$

satisfies

$$\mathbf{U} \begin{bmatrix} P^{-1} & 0 \\ 0 & J_{\mathcal{F},\mathcal{U}} \end{bmatrix} \mathbf{U}^* = \begin{bmatrix} P^{-1} \otimes I_d & 0 \\ 0 & J_{\mathcal{Y},\mathcal{U}} \end{bmatrix}, \tag{3.34}$$

and the kernel $K^P_{C,\mathbf{T}}(z,\zeta)$ *appearing in* (3.32) *can be expressed as*

$$\begin{aligned} K^P_{C,\mathbf{T}}(z,\zeta) &= K_{\mathfrak{A}}^{J_{\mathcal{F},\mathcal{U}}, J_{\mathcal{Y},\mathcal{U}}}(z,\zeta) \\ &:= k_{\mathrm{Sz}}(z,\zeta) \otimes J_{\mathcal{Y},\mathcal{U}} - \mathfrak{A}(z)(k_{\mathrm{Sz}}(z,\zeta) \otimes J_{\mathcal{F},\mathcal{U}})\mathfrak{A}(\zeta)^* \end{aligned} \tag{3.35}$$

where $\mathfrak{A}(z)$ *is the characteristic formal power series of the colligation* (3.33)*:*

$$\mathfrak{A}(z) = D + C(I - Z(z)T)^{-1}Z(z)B. \tag{3.36}$$

3. *If the operator* $\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]$ *is injective, then* $\mathbf{U}$ *in* (3.33) *in addition satisfies*

$$\mathbf{U}^* \begin{bmatrix} P \otimes I_d & 0 \\ 0 & J_{\mathcal{Y},\mathcal{U}} \end{bmatrix} \mathbf{U} = \begin{bmatrix} P & 0 \\ 0 & J_{\mathcal{F},\mathcal{U}} \end{bmatrix},$$

and $\mathfrak{A}(z)$ *belongs to* $\mathcal{S}_{\mathrm{nc},d}(J_{\mathcal{Y},\mathcal{U}}, J_{\mathcal{F},\mathcal{U}})$.

Proof Statement (1) follows by standard reproducing kernel Hilbert space considerations; for this we refer the reader to [10].

Let us now assume that $\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]$ satisfies (3.31) which in a more compact form can be written as (3.34). Then for $\mathfrak{A}(z)$ defined as in (3.36), we compute

$$\begin{aligned}
&\mathfrak{A}(z)J_{\mathcal{F},\mathcal{U}}\mathfrak{A}(\zeta)^* \\
&= \begin{bmatrix} C(I - Z(z)T)^{-1}Z(z) & I \end{bmatrix} \begin{bmatrix} B \\ D \end{bmatrix} J_{\mathcal{F},\mathcal{U}} \begin{bmatrix} B^* & D^* \end{bmatrix} \begin{bmatrix} Z(\zeta)^*(I - T^*Z(\zeta)^*)^{-1}C^* \\ I \end{bmatrix} \\
&= \begin{bmatrix} C(I - Z(z)T)^{-1}Z(z) & I \end{bmatrix} \begin{bmatrix} P^{-1} \otimes I_d & 0 \\ 0 & J_{\mathcal{Y},\mathcal{U}} \end{bmatrix} \begin{bmatrix} Z(\zeta)^*(I - T^*Z(\zeta)^*)^{-1}C^* \\ I \end{bmatrix} \\
&\quad - \begin{bmatrix} C(I - Z(z)T)^{-1}Z(z) & I \end{bmatrix} \begin{bmatrix} T \\ C \end{bmatrix} P^{-1} \begin{bmatrix} T^* & C^* \end{bmatrix} \begin{bmatrix} Z(\zeta)^*(I - T^*Z(\zeta)^*)^{-1}C^* \\ I \end{bmatrix} \\
&= J_{\mathcal{Y},\mathcal{U}} + C(I - Z(z)T)^{-1}Z(z)(P^{-1} \otimes I_d)Z(\zeta)^*(I - T^*Z(\zeta)^*)^{-1}C^* \\
&\quad - (C(I - Z(z)T)^{-1}Z(z)T + I)P^{-1}(T^*Z(\zeta)^*(I - T^*Z(\zeta)^*)^{-1} + I)C^* \\
&= J_{\mathcal{Y},\mathcal{U}} + C(I - Z(z)T)^{-1}Z(z)(P^{-1} \otimes I_d)Z(\zeta)^*(I - T^*Z(\zeta)^*)^{-1}C^* \\
&\quad - C(I - Z(z)T)^{-1}P^{-1}(I - T^*Z(\zeta)^*)^{-1}C^*,
\end{aligned}$$

which can be equivalently written as

$$\begin{aligned}
&J_{\mathcal{Y},\mathcal{U}} - \mathfrak{A}(z)J_{\mathcal{F},\mathcal{U}}\mathfrak{A}(\zeta)^* \\
&= C(I - Z(z)T)^{-1}\big((1 - \sum_{j=1}^{d} z_j\overline{\zeta}_j)P^{-1}\big)(I - T^*Z(\zeta)^*)^{-1}C^*.
\end{aligned}$$

Multiplying the latter equality by z^α on the right and by $\overline{\zeta}^{\alpha^\top}$ on the left, and then summing up these equalities over all $\alpha \in \mathbb{F}_d^+$ leads to

$$\begin{aligned}&k_{\rm Sz}(z,\zeta)\otimes J_{\mathcal{Y},\mathcal{U}} - \mathfrak{A}(z)(k_{\rm Sz}(z,\zeta)\otimes J_{\mathcal{F},\mathcal{U}})\mathfrak{A}(\zeta)^* \\ &= C(I - Z(z)T)^{-1}P^{-1}(I - T^*Z(\zeta)^*)^{-1}C^*,\end{aligned}$$

which proves (3.35). If the operator $\left[\begin{smallmatrix} B \\ D\end{smallmatrix}\right]$ is injective, then the operator

$$\widetilde{\mathbf{U}} = \begin{bmatrix} P^{\frac{1}{2}}\otimes I_d & 0 \\ 0 & I\end{bmatrix}\mathbf{U}\begin{bmatrix} P^{-\frac{1}{2}}\otimes I_d & 0 \\ 0 & I\end{bmatrix}$$

is $(J_{\mathcal{X}\oplus\mathcal{F},\mathcal{U}}, J_{\mathcal{X}^d\oplus\mathcal{Y},\mathcal{U}})$-unitary, and its characteristic formal power series is the same $\mathfrak{A}$ as in (3.36). But then $\mathfrak{A}$ belongs to $\mathcal{S}_{{\rm nc},d}(J_{\mathcal{Y},\mathcal{U}}, J_{\mathcal{F},\mathcal{U}})$, by Theorem 3.4. □

3.3 Linear Fractional Transformations

Another consequence of the second relation in (3.6) (obtained upon compressing the latter relation to $\mathcal{U}$) is

$$M_{\mathfrak{A}_{21}}M_{\mathfrak{A}_{21}}^* + M_{\mathfrak{A}_{22}}M_{\mathfrak{A}_{22}}^* \preceq I_{H^2_{\mathcal{U}}(\mathbb{F}_d^+)}.$$

Since $M_{\mathfrak{A}_{22}}$ is invertible and $M_{\mathfrak{A}_{22}}^{-1} = M_{\mathfrak{A}_{22}^{-1}}$, we can rewrite this last inequality as

$$M_{\mathfrak{A}_{22}^{-1}\mathfrak{A}_{21}}M^*_{\mathfrak{A}_{22}^{-1}\mathfrak{A}_{21}} \preceq I_{H^2_{\mathcal{U}}(\mathbb{F}_d^+)} - M_{\mathfrak{A}_{22}^{-1}}M^*_{\mathfrak{A}_{22}^{-1}} \prec I_{H^2_{\mathcal{U}}(\mathbb{F}_d^+)}.$$

Therefore, $\|M_{\mathfrak{A}_{22}^{-1}\mathfrak{A}_{21}}\| < 1$ and hence,

$$\|M_{\mathfrak{A}_{22}^{-1}\mathfrak{A}_{21}\mathcal{E}}\| < 1 \quad \text{for any} \quad \mathcal{E}\in\mathcal{S}_{{\rm nc},d}(\mathcal{U},\mathcal{F}).$$

Therefore, the operator $I_{H^2_{\mathcal{U}}(\mathbb{F}_d^+)} - M_{\mathfrak{A}_{22}^{-1}\mathfrak{A}_{21}\mathcal{E}}$ is invertible on $H^2_{\mathcal{U}}(\mathbb{F}_d^+)$ with inverse

$$(I_{H^2_{\mathcal{U}}(\mathbb{F}_d^+)} - M_{\mathfrak{A}_{22}^{-1}\mathfrak{A}_{21}\mathcal{E}})^{-1} = M_{(I_{\mathcal{U}} - \mathfrak{A}_{22}^{-1}\mathfrak{A}_{21}\mathcal{E})^{-1}}.$$

Thus, the formal series $I_{\mathcal{U}} - \mathfrak{A}_{22}^{-1}\mathfrak{A}_{21}\mathcal{E}$ is invertible in $\mathcal{M}_{{\rm nc},d}(\mathcal{U})$ for any $\mathcal{E}\in\mathcal{S}_{{\rm nc},d}(\mathcal{U},\mathcal{F})$ as well as the series

$$\mathfrak{A}_{21}(z)\mathcal{E}(z) + \mathfrak{A}_{22}(z) = \mathfrak{A}_{22}(z)(\mathfrak{A}_{22}(z)^{-1}\mathfrak{A}_{21}(z)\mathcal{E}(z) + I_{\mathcal{U}}).$$

We conclude: *if* $\mathfrak{A} \in \mathcal{S}_{\mathrm{nc},d}(J_{\mathcal{Y},\mathcal{U}}, J_{\mathcal{F},\mathcal{U}})$ *and* $\mathcal{E} \in \mathcal{S}_{\mathrm{nc},d}(\mathcal{U},\mathcal{F})$, *then the linear fractional transform of* $\mathcal{E}$

$$T_{\mathfrak{A}}[\mathcal{E}](z) = (\mathfrak{A}_{11}(z)\mathcal{E}(z) + \mathfrak{A}_{12}(z))(\mathfrak{A}_{21}(z)\mathcal{E}(z) + \mathfrak{A}_{22}(z))^{-1} \tag{3.37}$$

is a well-defined element of $\mathcal{L}(\mathcal{U},\mathcal{Y})\langle\langle z\rangle\rangle$. In other words, for any $\mathfrak{A} \in \mathcal{S}_{\mathrm{nc},d}(J_{\mathcal{Y},\mathcal{U}}, J_{\mathcal{F},\mathcal{U}})$, the linear fractional map $T_{\mathfrak{A}}$ given by (3.37) is well-defined on $\mathcal{S}_{\mathrm{nc},d}(\mathcal{U},\mathcal{F})$. The next theorem gives a useful characterization of its range.

Theorem 3.8 *Let* $\mathfrak{A} \in \mathcal{S}_{\mathrm{nc},d}(J_{\mathcal{Y},\mathcal{U}}, J_{\mathcal{F},\mathcal{U}})$. *Then an* $S \in \mathcal{L}(\mathcal{U},\mathcal{Y})\langle\langle z\rangle\rangle$ *has the form*

$$S = T_{\mathfrak{A}}[\mathcal{E}]$$

for some $\mathcal{E} \in \mathcal{S}_{\mathrm{nc},d}(\mathcal{U},\mathcal{F})$ *if and only if* $S \in \mathcal{S}_{\mathrm{nc},d}(\mathcal{U},\mathcal{Y})$ *and the operator*

$$\begin{bmatrix} I & -M_S \end{bmatrix} : \begin{bmatrix} y(z) \\ u(z) \end{bmatrix} \mapsto y(z) - S(z)u(z)$$

maps $\mathcal{H}(K_{\mathfrak{A}}^{J_{\mathcal{F},\mathcal{U}}, J_{\mathcal{Y},\mathcal{U}}})$ *contractively into the de Branges-Rovnyak space* $\mathcal{H}(K_S)$.

Proof Suppose that $S \in \mathcal{S}_{\mathrm{nc},d}(\mathcal{U},\mathcal{Y})$ and that $\begin{bmatrix} I & -M_S \end{bmatrix}$ maps $\mathcal{H}(K_{\mathfrak{A}}^{J_{\mathcal{F},\mathcal{U}}, J_{\mathcal{Y},\mathcal{U}}})$ contractively into $\mathcal{H}(K_S)$. By Proposition 2.2 (applied to $F = \begin{bmatrix} I & -S \end{bmatrix}$, $K = K_S$ and $K' = K_{\mathfrak{A}}^{J_{\mathcal{F},\mathcal{U}}, J_{\mathcal{Y},\mathcal{U}}}$), we conclude that the kernel given by

$$K_S(z,\zeta) - \begin{bmatrix} I & -S(z) \end{bmatrix} K_{\mathfrak{A}}^{J_{\mathcal{F},\mathcal{U}}, J_{\mathcal{Y},\mathcal{U}}}(z,\zeta) \begin{bmatrix} I \\ -S(\zeta)^* \end{bmatrix} \succeq 0 \tag{3.38}$$

is positive, where K_S is given by (2.13). Combining (2.13) with the definition (3.1) of $J_{\mathcal{Y},\mathcal{U}}$ gives us

$$K_S(z,\zeta) = \begin{bmatrix} I & -S(z) \end{bmatrix} k_{\mathrm{Sz}}(z,\zeta) \otimes J_{\mathcal{Y},\mathcal{U}} \begin{bmatrix} I \\ -S(\zeta)^* \end{bmatrix}. \tag{3.39}$$

Substituting the latter formula in (3.38), we then factor out $\begin{bmatrix} I & -S(z) \end{bmatrix}$ on the left and its adjoint on the right to get

$$\begin{bmatrix} I & -S(z) \end{bmatrix} \left(k_{\mathrm{Sz}}(z,\zeta) \otimes J_{\mathcal{Y},\mathcal{U}} - K_{\mathfrak{A}}^{J_{\mathcal{F},\mathcal{U}}, J_{\mathcal{Y},\mathcal{U}}}(z,\zeta) \right) \begin{bmatrix} I \\ -S(\zeta)^* \end{bmatrix} \succeq 0, \tag{3.40}$$

which on account of the definition (3.10), can be written as

$$\begin{bmatrix} I & -S(z) \end{bmatrix} \mathfrak{A}(z)(k_{\mathrm{Sz}}(z,\zeta) \otimes J_{\mathcal{F},\mathcal{U}})\mathfrak{A}(\zeta)^* \begin{bmatrix} I \\ -S(\zeta)^* \end{bmatrix} \succeq 0. \tag{3.41}$$

If we set

$$\begin{bmatrix} u(z) & -v(z) \end{bmatrix} := \begin{bmatrix} I & -S(z) \end{bmatrix} \mathfrak{A}(z) \in \mathcal{L}(\mathcal{F} \oplus \mathcal{U}, \mathcal{Y})\langle\langle z \rangle\rangle, \tag{3.42}$$

then we get, on account of (3.1),

$$u(z)(k_{\rm Sz}(z,\zeta) \otimes I_{\mathcal{F}})u(\zeta)^* - v(z)(k_{\rm Sz}(z,\zeta) \otimes I_{\mathcal{U}})v(\zeta)^* \succeq 0, \tag{3.43}$$

where, from (3.42), we have

$$u(z) = \mathfrak{A}_{11}(z) - S(z)\mathfrak{A}_{21}(z), \qquad -v(z) = \mathfrak{A}_{12}(z) - S(z)\mathfrak{A}_{22}(z).$$

By Theorem 2.5, there is a $\mathcal{E} \in \mathcal{S}_{{\rm nc},d}(\mathcal{U}, \mathcal{F})$ so that $v(z) = u(z)\mathcal{E}(z)$, i.e.,

$$-(\mathfrak{A}_{12}(z) - S(z)\mathfrak{A}_{22}(z)) = (\mathfrak{A}_{11}(z) - S(z)\mathfrak{A}_{21}(z))\mathcal{E}(z)$$

which can be rearranged as

$$S(z)(\mathfrak{A}_{21}(z)\mathcal{E}(z) + \mathfrak{A}_{22}(z)) = \mathfrak{A}_{11}(z)\mathcal{E}(z) + \mathfrak{A}_{12}(z).$$

It now follows that we recover S as $S = T_{\mathfrak{A}}[\mathcal{E}]$.

Conversely, suppose that $\mathcal{E} \in \mathcal{S}_{{\rm nc},d}(\mathcal{U}, \mathcal{F})$ and $S = T_{\mathfrak{A}}[\mathcal{E}]$. By reversing the steps in the argument above and using that condition (3.43) is necessary as well as sufficient in Theorem 2.5, we arrive at (3.41) which, due to the definition (3.10) can be rewritten in the form (3.40). From the identity (3.39) we are then led to (3.38). As $K_{\mathfrak{A}}^{J_{\mathcal{F},\mathcal{U}}, J_{\mathcal{Y},\mathcal{U}}}$ is a positive kernel by assumption, we conclude that K_S is a positive kernel, i.e., that $S \in \mathcal{S}_{{\rm nc},d}(\mathcal{U}, \mathcal{Y})$. Then the inequality (3.38) means, by Proposition 2.2, that $\begin{bmatrix} I & -M_S \end{bmatrix}$ maps $\mathcal{H}(K_{\mathfrak{A}}^{J_{\mathcal{F},\mathcal{U}}, J_{\mathcal{Y},\mathcal{U}}})$ contractively into $\mathcal{H}(K_S)$. □

3.4 Redheffer Transformation

In addition to the linear-fractional transformations of chain-matrix form (3.37) as discussed above we shall also have use of linear-fractional transformations of Redheffer form. To define these, we suppose that we are given the formal power series

$$\Sigma(z) = \begin{bmatrix} \Sigma_{11}(z) & \Sigma_{12}(z) \\ \Sigma_{21}(z) & \Sigma_{22}(z) \end{bmatrix} \in \mathcal{L}(\mathcal{U} \oplus \widetilde{\Delta}_*, \mathcal{Y} \oplus \widetilde{\Delta})\langle\langle z \rangle\rangle \tag{3.44}$$

for some Hilbert spaces $\mathcal{U}$, $\widetilde{\Delta}_*$, $\mathcal{Y}$, $\widetilde{\Delta}$. We assume that $\Sigma_{22} \in \mathcal{S}_{{\rm nc},d}(\widetilde{\Delta}_*, \widetilde{\Delta})$ and that moreover, $\|M_{\Sigma_{22}}\| < 1$. Then, as was explained above, for any Schur-class power series $\mathcal{W} \in \mathcal{S}_{{\rm nc},d}(\widetilde{\Delta}, \widetilde{\Delta}_*)$, the formal inverse of $(I - \mathcal{W}\Sigma_{22})$ exists in $\mathcal{M}_{{\rm nc},d}(\Delta_*)$, and we define the associated *Redheffer linear-fractional map* $\mathfrak{R}_\Sigma$ acting from $\mathcal{S}_{\mathrm{nc},d}(\widetilde{\Delta}, \widetilde{\Delta}_*)$ to $\mathcal{L}(\mathcal{X}, \mathcal{X}')\langle\langle z\rangle\rangle$ by

$$\mathfrak{R}_\Sigma[\mathcal{W}] := \Sigma_{11}(z) + \Sigma_{12}(z)(I - \mathcal{W}(z)\Sigma_{22}(z))^{-1}\mathcal{W}(z)\Sigma_{21}(z). \tag{3.45}$$

The following criterion for a given power series S to be in the range of $\mathfrak{R}_\Sigma$, while less explicit than the criterion in Theorem 3.8, nevertheless is useful in some applications (see Theorem 7.3 below). For this purpose we say that a pair of formal power series

$$\mathbf{a} \in \mathcal{L}(\widetilde{\Delta}_*, \mathcal{Y})\langle\langle z\rangle\rangle, \qquad \mathbf{c} \in \mathcal{L}(\widetilde{\Delta}, \mathcal{Y})\langle\langle z\rangle\rangle$$

is a *Schur-pair* if the associated $\mathcal{L}(\mathcal{X})$-valued formal kernel below is positive:

$$\mathbf{a}(z)(k_{\mathrm{Sz}}(z,\zeta) \otimes I_{\widetilde{\Delta}_*})\mathbf{a}(\zeta)^* - \mathbf{c}(z)(k_{\mathrm{Sz}}(z,\zeta) \otimes I_{\widetilde{\Delta}})\mathbf{c}(\zeta)^* \succeq 0. \tag{3.46}$$

Theorem 3.9 *Given $\Sigma(z)$ as in (3.44) with $\|M_{\Sigma_{22}}\| < 1$, a formal power series $S \in \mathcal{L}(\mathcal{U}, \mathcal{Y})\langle\langle z\rangle\rangle$ is of the from $S = \mathfrak{R}_\Sigma[\mathcal{W}]$ for some $\mathcal{W} \in \mathcal{S}_{\mathrm{nc},d}(\widetilde{\Delta}, \widetilde{\Delta}_*)$ if and only if there exists a Schur-pair $(\mathbf{a}(z), \mathbf{c}(z))$ so that*

$$\begin{bmatrix} I_{\mathcal{Y}} & \mathbf{c}(z) \end{bmatrix} \Sigma(z) = \begin{bmatrix} S(z) & \mathbf{a}(z) \end{bmatrix}. \tag{3.47}$$

Proof Suppose that $(\mathbf{a}(z), \mathbf{c}(z))$ is a Schur-pair satisfying (3.47). Due to condition (3.46) and Theorem 2.5, there is a $\mathcal{W} \in \mathcal{S}_{\mathrm{nc},d}(\widetilde{\Delta}, \widetilde{\Delta}_*)$ so that

$$\mathbf{c}(z) = \mathbf{a}(z)\mathcal{W}(z). \tag{3.48}$$

Then (3.47) can be written as

$$\begin{aligned} \Sigma_{11}(z) + \mathbf{a}(z)\mathcal{W}(z)\Sigma_{21}(z) &= S(z), \\ \Sigma_{12}(z) + \mathbf{a}(z)\mathcal{W}(z)\Sigma_{22}(z) &= \mathbf{a}(z). \end{aligned} \tag{3.49}$$

From the second of equations (3.49) we can solve for $\mathbf{a}(z)$:

$$\mathbf{a}(z) = \Sigma_{12}(z)(I - \mathcal{W}(z)\Sigma_{22}(z))^{-1}. \tag{3.50}$$

If we plug this expression into the first of equations (3.49), we get

$$S(z) = \Sigma_{11}(z) + \Sigma_{12}(z)(I - \mathcal{W}(z)\Sigma_{22}(z))^{-1}\mathcal{W}(z)\Sigma_{21}(z) = \mathfrak{R}_\Sigma[\mathcal{W}](z)$$

as wanted. For the converse direction, given that $S = \mathfrak{R}_\Sigma[\mathcal{W}]$, if we define $(\mathbf{a}, \mathbf{c})$ by (3.48) and (3.50), then $(\mathbf{a}, \mathbf{c})$ is a Schur-pair meeting the criterion (3.47). □

Remark 3.10 Under the assumption $\|M_{\Sigma_{22}}\| < 1$, a straightforward computation shows that for $S = \mathfrak{R}_\Sigma[\mathcal{W}]$, the kernels K_S, $K_{\mathcal{W}}$ and K_Σ defined via the formula (2.13) are related as follows:

$$K_S(z,\zeta) = \Psi(z)K_{\mathcal{W}}(z,\zeta)\Psi(\zeta)^* + \begin{bmatrix} I & \Psi(z) \end{bmatrix} K_\Sigma(z,\zeta) \begin{bmatrix} I \\ \Psi(\zeta)^* \end{bmatrix}$$

where we have set

$$\Psi(z) = \Sigma_{12}(z)(I - \mathcal{W}(z)\Sigma_{22}(z))^{-1}.$$

In particular, if Σ and $\mathcal{W}$ are in respective noncommutative Schur classes, then the kernel K_S is positive and then $S \in \mathcal{S}_{\text{nc},d}(\mathcal{U},\mathcal{Y})$, by Theorem 2.3.

4 Operator-Argument Interpolation Problem: Problem Statement and Connections with Other Problems

For an output-stable pair $(E,\mathbf{T})$ with $E \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ and $\mathbf{T} = (T_1,\ldots,T_d) \in \mathcal{L}(\mathcal{X})^d$, we define a *left-tangential functional calculus* $f \to (E^*f)^{\wedge L}(\mathbf{T}^*)$ on $H^2_{\mathcal{Y}}(\mathbb{F}^+_d)$ by

$$(E^*f)^{\wedge L}(\mathbf{T}^*) = \sum_{\alpha\in\mathbb{F}^+_d} \mathbf{T}^{*\alpha^\top}E^*f_\alpha \quad \text{if} \quad f = \sum_{\alpha\in\mathbb{F}^+_d} f_\alpha z^\alpha \in H^2_{\mathcal{Y}}(\mathbb{F}^+_d). \tag{4.1}$$

The computation

$$\Big\langle \sum_{\alpha\in\mathbb{F}^+_d} \mathbf{T}^{*\alpha^\top}E^*f_\alpha,\ x\Big\rangle_{\mathcal{X}} = \sum_{\alpha\in\mathbb{F}^+_d} \big\langle f_\alpha,\ E\mathbf{T}^\alpha x\big\rangle_{\mathcal{Y}} = \langle f,\ \mathcal{O}_{E,\mathbf{T}}x\rangle_{H^2_{\mathcal{Y}}(\mathbb{F}^+_d)}$$

shows that the output-stability of the pair $(E,\mathbf{T})$ (recall (2.10)) is exactly what is needed to verify that the infinite series in the definition (4.1) of $(E^*f)^{\wedge L}(\mathbf{T}^*)$ converges in the weak operator topology on $\mathcal{X}$. In fact the left-tangential evaluation with operator argument $f \to (E^*f)^{\wedge L}(\mathbf{T}^*)$ amounts to the adjoint of the observability operator:

$$(E^*f)^{\wedge L}(\mathbf{T}^*) = \mathcal{O}^*_{E,\mathbf{T}}f \quad \text{for} \quad f \in H^2_{\mathcal{Y}}(\mathbb{F}^+_d). \tag{4.2}$$

The evaluation map (4.2) extends to multipliers $S \in \mathcal{M}_{\text{nc},d}(\mathcal{U},\mathcal{Y})$ by

$$(E^*S)^{\wedge L}(\mathbf{T}^*) = \mathcal{O}^*_{E,\mathbf{T}}M_S : \mathcal{U} \to \mathcal{X}$$

and suggests the interpolation problem with operator argument $\mathbf{OAP}(\mathbf{T}, E, N)$ whose data set consists of a d-tuple $\mathbf{T} = (T_1, \dots, T_d) \in \mathcal{L}(\mathcal{X})^d$ and operators $E \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $N \in \mathcal{L}(\mathcal{X}, \mathcal{U})$ such that the pair $(E, \mathbf{T})$ is output stable.

Operator Argument Interpolation Problem ($\mathbf{OAP}(\mathbf{T}, E, N)$) *Given the data set* $\{\mathbf{T}, E, N\}$ *as above, find all* $S \in \mathcal{S}_{\mathrm{nc},d}(\mathcal{U}, \mathcal{Y})$ *such that*

$$(E^*S)^{\wedge L}(\mathbf{T}^*) := \mathcal{O}^*_{E,\mathbf{T}} M_S|_{\mathcal{U}} = N^*. \tag{4.3}$$

Such problems have been considered in [7, 8, 29, 30] for the commutative and noncommutative setting. First we observe simple necessary conditions for a problem $\mathbf{OAP}(\mathbf{T}, E, N)$ to have a solution.

Proposition 4.1 *Let* $(E, \mathbf{T})$ *be an output-stable pair with* $E \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$*, let* $S \in \mathcal{M}_{\mathrm{nc},d}(\mathcal{U}, \mathcal{Y})$ *and let* N *be defined as in* (4.3)*. Then:*

1. *The pair* $(N, \mathbf{T})$ *is output stable and*

$$\mathcal{O}^*_{E,\mathbf{T}} M_S = \mathcal{O}^*_{N,\mathbf{T}} : H^2_{\mathcal{U}}(\mathbb{F}^+_d) \to \mathcal{X}. \tag{4.4}$$

2. *If* $S \in \mathcal{S}_{\mathrm{nc},d}(\mathcal{U}, \mathcal{Y})$*, then* $\mathcal{O}^*_{N,\mathbf{T}} \mathcal{O}_{N,\mathbf{T}} \preceq \mathcal{O}^*_{E,\mathbf{T}} \mathcal{O}_{E,\mathbf{T}}$.

Hence, if the problem $\mathbf{OAP}(\mathbf{T}, E, N)$ *has a solution* $S \in \mathcal{S}_{\mathrm{nc},d}(\mathcal{U}, \mathcal{Y})$*, then* $(N, \mathbf{T})$ *is also output-stable and*

$$P := \mathcal{O}^*_{E,\mathbf{T}} \mathcal{O}_{E,\mathbf{T}} - \mathcal{O}^*_{N,\mathbf{T}} \mathcal{O}_{N,\mathbf{T}} \succeq 0. \tag{4.5}$$

Proof If the pair $(E, \mathbf{T})$ is output-stable, it follows by a simple inner-product computation (see e.g., [10, Theorem 2.8]) that

$$R^*_{z_j} \mathcal{O}_{E,T} x = \mathcal{O}_{E,T} T_j x \quad \text{for all} \quad x \in \mathcal{X} \text{ and } j = 1, \dots, d,$$

where $R^*_{z_j}$ is a backward shift (1.4) on $H^2_{\mathcal{Y}}(\mathbb{F}^+_d)$. Iterating the latter relations we conclude that for any $\alpha \in \mathbb{F}^+_d$ and any $x \in \mathcal{X}$,

$$\mathbf{R}^{*\alpha}_z \mathcal{O}_{E,T} x = \mathcal{O}_{E,T} \mathbf{T}^{\alpha} x$$

which can be written in the operator form as $\mathbf{R}^{*\alpha}_z \mathcal{O}_{E,T} = \mathcal{O}_{E,T} \mathbf{T}^{\alpha}$. Taking adjoints in the latter equality gives

$$\mathcal{O}^*_{E,T} \mathbf{R}^{\alpha^\top}_z = \mathbf{T}^{*\alpha^\top} \mathcal{O}^*_{E,T} : H^2_{\mathcal{Y}}(\mathbb{F}^+_d) \to \mathcal{X}. \tag{4.6}$$

We next apply the equality (4.6) to the vector $M_S h_\alpha \in H^2_{\mathcal{Y}}(\mathbb{F}^+_d)$ where S is the given contractive multiplier from $H^2_{\mathcal{U}}(\mathbb{F}^+_d)$ to $H^2_{\mathcal{Y}}(\mathbb{F}^+_d)$ and h_α is an arbitrary fixed vector in $\mathcal{U}$:

$$\mathbf{T}^{*\alpha^\top} \mathcal{O}^*_{E,T} M_S h_\alpha = \mathcal{O}^*_{E,T} \mathbf{R}^{\alpha^\top}_z M_S h_\alpha.$$

By the definition (1.3) of the right shift tuple $\mathbf{R}_z$ and due to (4.3), the latter equality can be written as

$$\mathcal{O}_{E,T}^* M_S : \ h_\alpha z^\alpha \mapsto \mathbf{T}^{*\alpha^\top} \mathcal{O}_{E,T}^* M_S h_\alpha = \mathbf{T}^{*\alpha^\top} N^* h_\alpha.$$

The latter formula extends by linearity to

$$\mathcal{O}_{E,T}^* M_S : \ \sum_{\alpha \in \mathbb{F}_d^+} h_\alpha z^\alpha \mapsto \sum_{\alpha \in \mathbb{F}_d^+} \mathbf{T}^{*\alpha^\top} N^* h_\alpha, \tag{4.7}$$

first to $\mathcal{U}$-valued polynomials, and then by continuity (since $\mathcal{O}_{E,T}^* M_S : \ H^2_{\mathcal{U}}(\mathbb{F}_d^+) \to \mathcal{X}$ is bounded) to all $\sum_{\alpha \in \mathbb{F}_d^+} h_\alpha z^\alpha \in H^2_{\mathcal{U}}(\mathbb{F}_d^+)$. The adjoint of this operator is a bounded operator from $\mathcal{X} \to H^2_{\mathcal{U}}(\mathbb{F}_d^+)$. Furthermore, for each $x \in \mathcal{X}$ and each $h(z) \in H^2_{\mathcal{U}}(\mathbb{F}_d^+)$, we have by (4.7),

$$\begin{aligned} \langle (\mathcal{O}_{E,T}^* M_S)^* x, \ h \rangle_{H^2_{\mathcal{U}}(\mathbb{F}_d^+)} &= \big\langle x, \ \mathcal{O}_{E,T}^* M_S h \big\rangle_{H^2_{\mathcal{U}}(\mathbb{F}_d^+)} \\ &= \big\langle x, \ \sum_{\alpha \in \mathbb{F}_d^+} \mathbf{T}^{*\alpha^\top} N^* h_\alpha \big\rangle_{\mathcal{X}} \\ &= \sum_{\alpha \in \mathbb{F}_d^+} \langle N \mathbf{T}^\alpha x, \ h_\alpha \rangle_{\mathcal{U}}. \end{aligned} \tag{4.8}$$

For a fixed integer $n > 0$, we introduce the $\mathcal{U}$-valued polynomial

$$h_{x,n}(z) = \sum_{\alpha \in \mathbb{F}_d^+ : |\alpha \le n} (N \mathbf{T}^\alpha x) z^\alpha$$

and observe that

$$\|h_{x,n}\|^2_{H^2_{\mathcal{U}}(\mathbb{F}_d^+)} = \sum_{\alpha \in \mathbb{F}_d^+ : |\alpha \le n} \|N \mathbf{T}^\alpha x\|^2_{\mathcal{U}}. \tag{4.9}$$

We next apply the equality (4.8) to the polynomial $h = h_{x,n}$:

$$\langle (\mathcal{O}_{E,T}^* M_S)^* x, \ h_{x,n} \rangle_{H^2_{\mathcal{U}}(\mathbb{F}_d^+)} = \sum_{\alpha \in \mathbb{F}_d^+ : |\alpha \le n} \langle N \mathbf{T}^\alpha x, \ N \mathbf{T}^\alpha x \rangle_{\mathcal{U}} = \|h_{x,n}\|^2_{H^2_{\mathcal{U}}(\mathbb{F}_d^+)}$$

and then conclude by the Cauchy inequality that

$$\|h_{x,n}\|^2_{H^2_{\mathcal{U}}(\mathbb{F}_d^+)} \le \|(\mathcal{O}_{E,T}^* M_S)^* x\|_{H^2_{\mathcal{U}}(\mathbb{F}_d^+)} \cdot \|h_{x,n}\|_{H^2_{\mathcal{U}}(\mathbb{F}_d^+)},$$

or equivalently,

$$\|h_{x,n}\|_{H^2_{\mathcal{U}}(\mathbb{F}^+_d)} \le \|(\mathcal{O}^*_{E,T}M_S)^*x\|_{H^2_{\mathcal{U}}(\mathbb{F}^+_d)}. \tag{4.10}$$

Letting $n \to \infty$ in (4.10) and making use of (4.9) we conclude that

$$\sum_{\alpha\in\mathbb{F}^+_d} \|N\mathbf{T}^\alpha x\|^2_{\mathcal{U}} = \lim_{n\to\infty} \sum_{\alpha\in\mathbb{F}^+_d:|\alpha\le n} \|N\mathbf{T}^\alpha x\|^2_{\mathcal{U}} \le \|(\mathcal{O}^*_{E,T}M_S)^*x\|^2_{H^2_{\mathcal{U}}(\mathbb{F}^+_d)} < \infty,$$

so that the pair $(N, \mathbf{T})$ is output stable. Now we get back to (4.8) and continue the calculation for an arbitrary $h \in H^2_{\mathcal{U}}(\mathbb{F}^+_d)$ as follows:

$$\langle(\mathcal{O}^*_{E,T}M_S)^*x,\ h\rangle_{H^2_{\mathcal{U}}(\mathbb{F}^+_d)} = \sum_{\alpha\in\mathbb{F}^+_d} \langle N\mathbf{T}^\alpha x,\ h_\alpha\rangle_{\mathcal{U}} = \langle\mathcal{O}_{N,\mathbf{T}}x,\ h\rangle_{H^2_{\mathcal{U}}(\mathbb{F}^+_d)}$$

verifying that $(\mathcal{O}^*_{E,T}M_S)^* = \mathcal{O}_{N,\mathbf{T}}$ which in turn is equivalent to (4.4).

If $S \in \mathcal{S}_{\mathrm{nc},d}(\mathcal{U},\mathcal{Y})$, then $I - M_SM_S^* \succeq 0$ and by (4.4) we have for all $x \in \mathcal{X}$,

$$\begin{aligned} 0 &\le \left\langle (I - M_SM_S^*)\mathcal{O}_{E,\mathbf{T}}x,\ \mathcal{O}_{E,\mathbf{T}}x\right\rangle_{H^2_{\mathcal{Y}}(\mathbb{F}^+_d)} \\ &= \|\mathcal{O}_{E,\mathbf{T}}x\|^2_{H^2_{\mathcal{Y}}(\mathbb{F}^+_d)} - \|M_S^*\mathcal{O}_{E,\mathbf{T}}x\|^2_{H^2_{\mathcal{U}}(\mathbb{F}^+_d)} \\ &= \|\mathcal{O}_{E,\mathbf{T}}x\|^2_{H^2_{\mathcal{Y}}(\mathbb{F}^+_d)} - \|\mathcal{O}_{N,\mathbf{T}}x\|^2_{H^2_{\mathcal{U}}(\mathbb{F}^+_d)} \end{aligned}$$

which proves the second statement and completes the proof of the proposition. □

Corollary 4.2 *Conditions* (4.3) *and* (4.4) *are equivalent.*

Proof Indeed, Proposition 4.1 shows that (4.3) implies (4.4). The converse implication follows upon restricting equality (4.4) to constant power series from $H^2_{\mathcal{U}}(\mathbb{F}^+_d)$:

$$(E^*S)^{\wedge L}(\mathbf{T}^*)u = \mathcal{O}^*_{E,\mathbf{T}}M_Su = \mathcal{O}^*_{N,\mathbf{T}}u$$

and taking into account that $\mathcal{O}^*_{N,\mathbf{T}}|_{\mathcal{U}} = N^*$. □

We have seen in Proposition 4.1 that a necessary condition for a problem **OAP**$(\mathbf{T}, E, N)$ to have a solution is that $P \succeq 0$ where $P\colon \mathcal{X} \to \mathcal{X}$ is given by (4.5). In the next section we shall see that this condition is also sufficient.

Now we will show that the problem **OAP** contains the commutant lifting problem as a particular case. Let us suppose that $\Psi \in \mathcal{M}_{\mathrm{nc},d}(\mathcal{U},\mathcal{Y})$ (with M_Ψ not necessarily contractive) satisfies the interpolation condition (4.3), i.e., $\mathcal{O}^*_{E,\mathbf{T}}M_\Psi|_{\mathcal{U}} = N^*$. Then any other multiplier S subject to (4.3) is necessarily of the form $S = \Psi + \Phi$ for some multiplier Φ with range space contained in $\operatorname{Ker}\mathcal{O}^*_{E,\mathbf{T}}$.

Let us now assume that the d-tuple $\mathbf{T} = (T_1, \dots, T_d)$ is *strongly stable* (see (2.9)). Then $\operatorname{Ker} \mathcal{O}^*_{E,\mathbf{T}}$ is a shift-invariant closed subspace of $H^2_{\mathcal{Y}}(\mathbb{F}^+_d)$ and hence, by the Beurling-Lax theorem for the Fock space (see [11, Theorem 5.2]), there is a strictly inner multiplier $\Theta \in \mathcal{S}_{\mathrm{nc},d}(\mathcal{F}, \mathcal{Y})$ (for some auxiliary Hilbert space $\mathcal{F}$) so that $\operatorname{Ker} \mathcal{O}^*_{E,\mathbf{T}} = \Theta H^2_{\mathcal{F}}(\mathbb{F}^+_d)$. The subspace $\Theta H^2_{\mathcal{F}}(\mathbb{F}^+_d)$ can be considered as a NFRKHS in its own right with reproducing kernel $\Theta(z)(k_{\mathrm{Sz}}(z,\zeta) \otimes I_{\mathcal{F}})\Theta(\zeta)^*$. If the bounded multiplier Ψ maps $H^2_{\mathcal{U}}(\mathbb{F}^+_d)$ into $\Theta H^2_{\mathcal{Y}}(\mathbb{F}^+_d)$ and as a normalization we suppose contractively, then a consequence of Proposition 2.2 is that the kernel

$$\Theta(z)(k_{\mathrm{Sz}}(z,\zeta) \otimes I_{\mathcal{F}})\Theta(\zeta)^* - \Psi(z)(k_{\mathrm{Sz}}(z,\zeta) \otimes I_{\mathcal{U}})\Psi(\zeta)^*$$

is a positive kernel. When we combine this observation with the Leech theorem (see Theorem 2.5), we conclude that any such multiplier Φ has the form ΘF for some contractive multiplier $F \in \mathcal{M}_{\mathrm{nc},d}(\mathcal{F}, \mathcal{U})$ (or more generally, with multiplier norm of F bounded by the multiplier norm of Ψ). We thus conclude that any $S \in \mathcal{M}_{\mathrm{nc},d}(\mathcal{U}, \mathcal{Y})$ subject to condition (4.3) is necessarily of the form $S = \Psi + \Theta F$ where $\Psi \in \mathcal{M}_{\mathrm{nc},d}(\mathcal{U}, \mathcal{Y})$ is some particular solution, Θ is the strictly inner multiplier equal to the Beurling-Lax representer of the shift invariant subspace $\operatorname{Ker} \mathcal{O}^*_{E,\mathbf{T}}$ of $H^2_{\mathcal{Y}}(\mathbb{F}^+_d)$, and $F \in \mathcal{M}_{\mathrm{nc},d}$ is a free parameter. The problem **OAP**$(\mathbf{T}, E, N)$ adds the constraint the multiplier norm of S be at most 1. We have thus shown that the problem **OAP**$(\mathbf{T}, E, N)$ with strongly stable $\mathbf{T}$ can be reformulated as a *Sarason interpolation problem* with the data set $\{\Psi, \Theta\}$:

Sarason Interpolation Problem (SIP) *Given* $\Psi \in \mathcal{M}_{\mathrm{nc},d}(\mathcal{U}, \mathcal{Y})$ *and a strictly inner* $\Theta \in \mathcal{S}_{\mathrm{nc},d}(\mathcal{F}, \mathcal{Y})$, find all $S \in \mathcal{S}_{\mathrm{nc},d}(\mathcal{U}, \mathcal{Y})$ *such that*

$$S(z) = \Psi(z) + \Theta(z)F(z) \quad \text{for some} \quad F \in \mathcal{M}_{\mathrm{nc},d}(\mathcal{U}, \mathcal{F}).$$

As was first done for the single-variable case in the classical paper [35] of Sarason and then in [27, 30] for both the noncommutative and commutative ball setting, the **SIP** can be put in more operator-theoretic form as follows. Given the data set $\{\Psi, \Theta\}$ for a Sarason interpolation problem, introduce the subspace $\mathcal{M}$ by

$$\mathcal{M} = H^2_{\mathcal{Y}}(\mathbb{F}^+_d) \ominus \Theta H^2_{\mathcal{F}}(\mathbb{F}^+_d)$$

and define the operator $\Phi \colon H^2_{\mathcal{U}}(\mathbb{F}^+_d) \to \mathcal{M}$ by $\Phi = P_{\mathcal{M}} M_{\Psi}$. Note that $\mathcal{M}$ is backward-shift-invariant and that Φ satisfies the intertwining property:

$$P_{\mathcal{M}} R_{z_j} \Phi = \Phi R_{z_j} \quad (j = 1, \dots, d). \tag{4.11}$$

Furthermore, a multiplier $S \in \mathcal{M}_{\mathrm{nc},d}(\mathcal{U}, \mathcal{Y})$ solves the **SIP**(Ψ, Θ) if and only if M_S satisfies the conditions

$$P_{\mathcal{M}} M_S = \Phi \quad \text{and} \quad \|M_S\| \le 1.$$

From these conditions we read off that if **SIP**(Ψ, Θ) has a solution, then necessarily

$$\|\Phi\| \leq 1. \tag{4.12}$$

As mentioned previously, multipliers M_S are characterized as those operators between $H^2_{\mathcal{U}}(\mathbb{F}_d^+)$ and $H^2_{\mathcal{Y}}(\mathbb{F}_d^+)$ which intertwine the respective shift operators on these spaces. It now follows that the **SIP** can be reformulated as the following *Commutant Lifting Problem*:

Commutant Lifting Problem (CLP) *Given an* $\mathbf{R}_z^*$*-invariant subspace* $\mathcal{M}$ *of* $H^2_{\mathcal{Y}}(\mathbb{F}_d^+)$ *and an operator* $\Phi : H^2_{\mathcal{U}}(\mathbb{F}_d^+) \to \mathcal{M}$ *subject to* (4.11)*, find an operator* $G : H^2_{\mathcal{U}}(\mathbb{F}_d^+) \to H^2_{\mathcal{Y}}(\mathbb{F}_d^+)$ *such that*

$$\|G\| \leq 1, \quad \mathcal{P}_{\mathcal{M}} G = \Phi \quad \text{and} \quad R_{z_j} G = G R_{z_j} \quad (j = 1, \ldots, d), \tag{4.13}$$

or equivalently, find an $S \in \mathcal{S}_{\text{nc},d}(\mathcal{U}, \mathcal{Y})$ *such that*

$$\mathcal{P}_{\mathcal{M}} M_S = \Phi. \tag{4.14}$$

If G is a solution of **CLP**, then it follows from (4.13) that

$$\|\Phi\| = \|\mathcal{P}_{\mathcal{M}} G\| \leq \|G\| \leq 1,$$

and also

$$\Phi R_{z_j} = \mathcal{P}_{\mathcal{M}} G R_{z_j} = \mathcal{P}_{\mathcal{M}} R_{z_j} G = \mathcal{P}_{\mathcal{M}} R_{z_j} \mathcal{P}_{\mathcal{M}} G = \mathcal{P}_{\mathcal{M}} R_{z_j} \Phi,$$

where the third equality holds due to the backward shift invariance of $\mathcal{M}$. Hence the conditions (4.11) and (4.12) are certainly necessary for the existence of a solution to **CLP**. That the converse holds is the assertion of the commutant lifting theorem in [14].

Given a Sarason interpolation problem **SIP**(Ψ, Θ), we have seen how to pass to a **CLP**$(\mathcal{M}, \Phi)$. Conversely, it is possible to pass from a **CLP**$(\mathcal{M}, \Phi)$ to a **SIP**(Ψ, Θ) as follows. Take any Beurling-Lax representer $\Theta \in \mathcal{S}_{\text{nc},d}(\mathcal{F}, \mathcal{Y})$ for $\mathcal{M} \subset H^2_{\mathcal{Y}}(\mathbb{F}_d^+)$ and choose any $\Psi \in \mathcal{M}_{\text{nc},d}(\mathcal{U}, \mathcal{Y})$ (not necessarily contractive) so that $\Phi = P_{\mathcal{M}} M_\Psi$. The fact that such a multiplier Ψ always exists is of course a consequence of Popescu's Commutant Lifting theorem [28].

To conclude this section we will show that the problem **CLP** is equivalent to a problem **OAP**$(\mathbf{T}, E, N)$ with strongly stable tuple $\mathbf{T}$.

Theorem 4.3 *Let* $\mathcal{M}$ *be a backward-shift-invariant subspace of* $H^2_{\mathcal{Y}}(\mathbb{F}_d^+)$*, let* $\Phi \in H^2_{\mathcal{U}}(\mathbb{F}_d^+) \to \mathcal{M}$ *satisfy conditions* (4.11) *and let* $\mathbf{T} = (T_1, \ldots, T_d) \in \mathcal{L}(\mathcal{M})^d$*,* $E \in \mathcal{L}(\mathcal{M}, \mathcal{Y})$ *and* $N \in \mathcal{L}(\mathcal{M}, \mathcal{U})$ *be defined by*

$$T_j = R_{z_j}^*|_{\mathcal{M}} \quad (j = 1, \ldots, d), \quad E \colon f \mapsto f_\emptyset \quad \textit{and} \quad N \colon h \mapsto (\Phi^* h)_\emptyset. \tag{4.15}$$

Then the tuple $\mathbf{T}$ *is strongly stable, and a contractive multiplier* S *solves* $\mathbf{CLP}(\mathcal{M}, \Phi)$ *if and only if* S *solves* $\mathbf{OAP}(\mathbf{T}, E, N)$.

Conversely, suppose that $\mathbf{T} = (T_1, \ldots, T_d)$, E *and* N *are such that* $\mathbf{T}$ *is strongly stable and the pairs* $(E, \mathbf{T})$ *and* $(N, \mathbf{T})$ *are output stable. Set* $\mathcal{M} = \operatorname{Ran} \mathcal{O}_{E,\mathbf{T}}$ *and define* $\Phi\colon H^2_{\mathcal{U}}(\mathbb{F}^+_d) \to \mathcal{M}$ *via its adjoint* Φ^*:

$$\Phi^*\colon \mathcal{O}_{E,\mathbf{T}}x \mapsto \mathcal{O}_{N,\mathbf{T}}x. \tag{4.16}$$

Then $\mathcal{M}$ *is a backward-shift-invariant subspace of* $H^2_{\mathcal{Y}}(\mathbb{F}^+_d)$, Φ *satisfies relations* (4.11), *and a contractive multiplier* $S \in \mathcal{S}_{\mathrm{nc},d}(\mathcal{U}, \mathcal{Y})$ *solves* $\mathbf{OAP}(\mathbf{T}, E, N)$ *if and only if* S *solves* $\mathbf{CLP}(\mathcal{M}, \Phi)$.

Proof Let $\mathbf{T}, E, N$ be defined as in (4.15). From the fact that the backward-shift d-tuple (1.4) is strongly stable on $H^2_{\mathcal{Y}}(\mathbb{F}^+_d)$ (see e.g., [10, Proposition 2.9]), its restriction to an invariant subspace is strongly stable as well. Iterating the formula for $R^*_{z_j}$ in (1.4) gives

$$\mathbf{T}^\beta = \mathbf{R}_z^{*\beta}|_{\mathcal{M}}: \sum_{\alpha \in \mathbb{F}^+_d} h_\alpha z^\alpha \mapsto \sum_{\alpha \in \mathbb{F}^+_d} h_{\alpha\beta} z^\alpha$$

for all $\beta \in \mathbb{F}^+_d$ and therefore,

$$E\mathbf{T}^\beta h = h_\beta \quad \text{for all} \quad \beta \in \mathbb{F}^+_d \quad \text{and} \quad h(z) = \sum_{\alpha \in \mathbb{F}^+_d} h_\alpha z^\alpha \in \mathcal{M}. \tag{4.17}$$

Hence,

$$\mathcal{O}_{E,\mathbf{T}}h = \sum_{\beta \in \mathbb{F}^+_d} (E\mathbf{T}^\beta h) z^\beta = \sum_{\beta \in \mathbb{F}^+_d} h_\beta z^\beta = h,$$

that is, the observability operator $\mathcal{O}_{E,\mathbf{T}}$ acting on an element $h \in \mathcal{M}$ simply reproduces h and hence can be viewed as the operator of inclusion of $\mathcal{M}$ in $H^2_{\mathcal{Y}}(\mathbb{F}^+_d)$; in particular, $(E, \mathbf{T})$ is also output-stable. Therefore we have

$$\mathcal{O}^*_{E,\mathbf{T}}\mathcal{O}_{E,\mathbf{T}} = I_{\mathcal{M}}, \quad \mathcal{O}^*_{E,\mathbf{T}}|_{\mathcal{M}} = I_{\mathcal{M}}$$

and furthermore,

$$\mathcal{O}^*_{E,\mathbf{T}}|_{\mathcal{M}^\perp} = 0 \quad \text{and} \quad \mathcal{P}_{\mathcal{M}} = \mathcal{O}^*_{E,\mathbf{T}}. \tag{4.18}$$

Next we show that, for operators T_j and N given by (4.15), we have

$$\mathcal{O}_{N,\mathbf{T}} = \Phi^*. \tag{4.19}$$

To this end, pick up an $h \in \mathcal{M}$ as in (4.17) and note that by (4.11),

$$\Phi^* T_j h = \Phi^* R_{z_j}^* h = R_{z_j}^* \Phi^* h \quad \text{for} \quad j = 1, \ldots, d.$$

Iterating the latter equalities gives $\Phi^* \mathbf{T}^\alpha h = (\mathbf{R}_z^*)^\alpha \Phi^* h$ for every $\alpha \in \mathbb{F}_d^+$, and hence,

$$N\mathbf{T}^\alpha h = (\Phi^* \mathbf{T}^\alpha h)_\emptyset = E(\mathbf{R}_z^*)^\alpha \Phi^* h = (\Phi^* h)_\alpha$$

for each $\alpha \in \mathbb{F}_d^+$ and any $h \in \mathcal{M}$. We now have

$$\mathcal{O}_{N,\mathbf{T}} h = \sum_{\alpha \in \mathbb{F}_d^+} (N\mathbf{T}^\alpha h) z^\alpha = \sum_{\alpha \in \mathbb{F}_d^+} (\Phi^* h)_\alpha z^\alpha = \Phi^* h,$$

and since h is arbitrary, (4.19) follows. In turn, (4.19) implies that the pair $(N, \mathbf{T})$ is output-stable. By Corollary 4.2, condition (4.3) is equivalent to

$$\mathcal{O}_{E,\mathbf{T}}^* M_S = \mathcal{O}_{N,\mathbf{T}}^* = \Phi$$

which coincides with (4.14) due to (4.18).

Conversely, if $\mathbf{T} = (T_1, \ldots, T_d)$ is strongly stable and the pairs $(E, \mathbf{T})$ and $(N, \mathbf{T})$ are output stable, then $\mathcal{M} = \operatorname{Ran} \mathcal{O}_{E,\mathbf{T}}$ is a closed backward-shift-invariant subspace of $H_{\mathcal{Y}}^2(\mathbb{F}_d^+)$ and $\mathcal{O}_{E,\mathbf{T}}$ is an isomorphism of $\mathcal{X}$ onto $\mathcal{M}$ (see [10]). We define $\Phi \colon H_{\mathcal{U}}^2(\mathbb{F}_d^+) \to \mathcal{M}$ via its adjoint Φ^* given by (4.16). Then, for $S \in \mathcal{S}_{\mathrm{nc},d}(\mathcal{U}, \mathcal{Y})$, from Proposition 4.1 we see that S solves $\mathbf{OAP}(\mathbf{T}, E, N)$ is equivalent to condition (4.4), or, after taking adjoints, to

$$M_S^* \mathcal{O}_{E,\mathbf{T}} = \mathcal{O}_{N,\mathbf{T}}. \tag{4.20}$$

By definition, this in turn is equivalent to $M_S^*|_{\mathcal{M}} = \Phi^*$, i.e., to S solving the problem $\mathbf{CLP}(\mathcal{M}, \Phi)$. □

5 Operator Argument Interpolation Problem: Solution

In this section we present a solution of the Operator Argument Interpolation Problem, including the parametrization of the set of all solutions for the case where the operator P (4.5) is invertible. The setting here is more general than the case handled by the Commutant Lifting Theorem discussed in the previous section in that we no longer insist that $\mathbf{T}$ be strongly stable. As a first step we present several useful reformulations of the problem. The main tool for this analysis is the following well-known Hilbert space result. In case the block A is invertible, the result can be seen as a consequence of a standard Schur-complement computation; the general result then follows by replacing A with $A + \epsilon I$ and then letting $\epsilon > 0$ tend to zero.

Proposition 5.1 *A Hilbert space operator*

$$\begin{bmatrix} P & B^* \\ B & A \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{H} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{H} \end{bmatrix}$$

is positive semidefinite if and only if A is positive semidefinite and for every $x \in \mathcal{X}$, there exists a vector $h_x \in \mathcal{H} \ominus \operatorname{Ker} A$ such that

$$A^{\frac{1}{2}} h_x = Bx \quad \text{and} \quad \|h_x\|_{\mathcal{H}} \le \|P^{\frac{1}{2}} x\|_{\mathcal{X}}.$$

Theorem 5.2 *Given the data set $\{\mathbf{T}, E, N\}$ such that the pairs $(E, \mathbf{T})$ and $(N, \mathbf{T})$ are output stable, let $P : \mathcal{X} \to \mathcal{X}$ be defined as in (4.5). Given $S \in \mathcal{L}(\mathcal{U}, \mathcal{Y})\langle\langle z \rangle\rangle$, let $F^S : \mathcal{X} \to H^2_{\mathcal{Y}}(\mathbb{F}^+_d)$ be the linear map defined by*

$$F^S : x \mapsto \left(\mathcal{O}_{E,\mathbf{T}} - M_S \mathcal{O}_{N,\mathbf{T}}\right) x. \tag{5.1}$$

The following are equivalent:

1. *S is a solution of the **OAP**$(\mathbf{T}, E, N)$, i.e., $S \in \mathcal{S}_{\mathrm{nc},d}(\mathcal{U}, \mathcal{Y})$ and S satisfies condition (4.3).*
2. *The operator*

$$\mathbf{P} := \begin{bmatrix} P & (F^S)^* \\ F^S & I - M_S M_S^* \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ H^2_{\mathcal{Y}}(\mathbb{F}^+_d) \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ H^2_{\mathcal{Y}}(\mathbb{F}^+_d) \end{bmatrix} \tag{5.2}$$

   is positive semidefinite.
3. *The following kernel is positive:*

$$\mathbf{K}(z, \zeta) = \begin{bmatrix} P & G(\zeta)^* \,(E^* - N^* S(\zeta)^*) \\ (E - S(z)N)\, G(z) & K_S(z, \zeta) \end{bmatrix} \succeq 0, \tag{5.3}$$

   where $K_S(z, \zeta)$ is given by (2.13) and

$$G(z) = (I - Z(z)T)^{-1}.$$

4. *$S \in \mathcal{S}_{\mathrm{nc},d}(\mathcal{U}, \mathcal{Y})$, and the formal power series $F^S x \in H^2_{\mathcal{Y}}(\mathbb{F}^+_d)$ belongs to the de Branges-Rovnyak space $\mathcal{H}(K_S)$ and satisfies*

$$\|F^S x\|_{\mathcal{H}(K_S)} \le \|P^{\frac{1}{2}} x\|_{\mathcal{X}} \quad \text{for every} \quad x \in \mathcal{X}. \tag{5.4}$$

5. *$S \in \mathcal{S}_{\mathrm{nc},d}(\mathcal{U},\mathcal{Y})$, and the power series $F^S x$ belongs to $\mathcal{H}(K_S)$ and satisfies*

$$\|F^S x\|_{\mathcal{H}(K_S)} = \|P^{\frac{1}{2}} x\|_{\mathcal{X}} \quad \textit{for every} \quad x \in \mathcal{X}. \tag{5.5}$$

Proof By taking the adjoint of the formulation (4.4) of the interpolation condition (4.3), we see that $S \in \mathcal{S}_{\mathrm{nc},d}(\mathcal{U},\mathcal{Y})$ solves the problem $\mathbf{OAP}(\mathbf{T}, E, N)$ if and only if

$$M_S^* \mathcal{O}_{E,\mathbf{T}} = \mathcal{O}_{N,\mathbf{T}} : \ \mathcal{X} \to H^2_{\mathcal{U}}(\mathbb{F}_d^+). \tag{5.6}$$

To prove Theorem 5.2, we shall show that (2) ⇔ (3) and that (1) ⇒ (5) ⇒ (4) ⇒ (2) ⇒ (1).

(2) ⇔ (3): This equivalence follows from the identity

$$\left\langle \mathbf{P} \begin{bmatrix} x \\ k_{\mathrm{Sz}}(\cdot\,,\,\zeta) y \end{bmatrix}, \begin{bmatrix} x' \\ k_{\mathrm{Sz}}(\cdot\,,\,z) y' \end{bmatrix} \right\rangle_{\mathcal{X} \oplus H^2_{\mathcal{Y}}(\mathbb{F}_d^+)}$$
$$= \left\langle \mathbf{K}(z,\zeta) \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x' \\ y' \end{bmatrix} \right\rangle_{\mathcal{X} \oplus \mathcal{Y}} \quad (x, x' \in \mathcal{X}, \ y, y' \in \mathcal{Y}).$$

(1) ⇒ (5): Assume that $S \in \mathcal{S}_{\mathrm{nc},d}(\mathcal{U},\mathcal{Y})$ solves $\mathbf{OAP}(\mathbf{T}, E, N)$. Then from (5.6) we see that

$$F^S = \mathcal{O}_{E,\mathbf{T}} - M_S \mathcal{O}_{N,\mathbf{T}} = \mathcal{O}_{E,\mathbf{T}} - M_S M_S^* \mathcal{O}_{E,\mathbf{T}} = (I - M_S M_S^*) \mathcal{O}_{E,T}.$$

Hence,

$$\begin{aligned} \|F^S x\|^2_{\mathcal{H}(K_S)} &= \langle (I - M_S M_S^*) \mathcal{O}_{E,\mathbf{T}} x, \mathcal{O}_{E,\mathbf{T}} x \rangle_{H^2_{\mathcal{Y}}(\mathbb{F}_d^+)} \\ &= \langle (\mathcal{O}^*_{E,\mathbf{T}} \mathcal{O}_{E,\mathbf{T}} - \mathcal{O}^*_{N,\mathbf{T}} \mathcal{O}_{N,\mathbf{T}}) x, x \rangle_{\mathcal{X}} \\ &= \langle P x, x \rangle_{\mathcal{X}} = \|P^{\frac{1}{2}} x\|^2_{\mathcal{X}} \end{aligned}$$

for all $x \in \mathcal{X}$ and (5) follows.

(5) ⇒ (4): This is trivial.

(4) ⇒ (2): Since S is in $\mathcal{S}_{\mathrm{nc},d}(\mathcal{U},\mathcal{Y})$, the operator $A := I - M_S M_S^*$ is positive semidefinite on $H^2_{\mathcal{Y}}(\mathbb{F}_d^+)$. Furthermore, $F^S x$ belongs to $\mathcal{H}(K_S)$ for every $x \in \mathcal{X}$ which means, due to (2.20), that $F^S x = (I - M_S M_S^*)^{\frac{1}{2}} h_x$ for some element $h_x \in H^2_{\mathcal{Y}}(\mathbb{F}_d^+)$ which can be chosen to be orthogonal to the Ker $(I - M_S M_S^*)$. Then the norm constraint (5.4) implies

$$\left\| (I - M_S M_S^*)^{\frac{1}{2}} h_x \right\|_{\mathcal{H}(K_S)} = \|h_x\|_{H^2_{\mathcal{Y}}(\mathbb{F}_d^+)} \le \|P^{\frac{1}{2}} x\|_{\mathcal{X}}$$

and positivity of the operator (5.2) follows by Proposition 5.1.

(2) ⇒ (1): Let the operator (5.2) be positive semidefinite. Then the operator $I - M_S M_S^*$ is positive semidefinite (equivalently, M_S is a contraction) which implies $S \in \mathcal{S}_{\mathrm{nc},d}(\mathcal{U}, \mathcal{Y})$. By definitions (4.5) and (5.1) we have

$$\mathbf{P} = \begin{bmatrix} \mathcal{O}_{E,\mathbf{T}}^* \mathcal{O}_{E,\mathbf{T}} - \mathcal{O}_{N,\mathbf{T}}^* \mathcal{O}_{N,\mathbf{T}} & \mathcal{O}_{E,\mathbf{T}}^* - \mathcal{O}_{N,\mathbf{T}}^* M_S^* \\ \mathcal{O}_{E,\mathbf{T}} - M_S \mathcal{O}_{N,\mathbf{T}} & I - M_S M_S^* \end{bmatrix} \succeq 0.$$

By the Schur complement argument, the latter inequality is equivalent to

$$\widehat{\mathbf{P}} := \begin{bmatrix} I_{H^2_{\mathcal{U}}(\mathbb{F}_d^+)} & \mathcal{O}_{N,\mathbf{T}} & M_S^* \\ \mathcal{O}_{N,\mathbf{T}}^* & \mathcal{O}_{E,\mathbf{T}}^* \mathcal{O}_{E,\mathbf{T}} & \mathcal{O}_{E,\mathbf{T}}^* \\ M_S & \mathcal{O}_{E,\mathbf{T}} & I_{H^2_{\mathcal{Y}}(\mathbb{F}_d^+)} \end{bmatrix} \succeq 0,$$

since $\mathbf{P}$ is the Schur complement of the block $I_{H^2_{\mathcal{U}}(\mathbb{F}_d^+)}$ in $\widehat{\mathbf{P}}$. On the other hand, the latter inequality holds if and only if the Schur complement of the block $I_{H^2_{\mathcal{Y}}(\mathbb{F}_d^+)}$ in $\widehat{\mathbf{P}}$ is positive semidefinite:

$$\begin{bmatrix} I_{H^2_{\mathcal{U}}(\mathbb{F}_d^+)} & \mathcal{O}_{N,\mathbf{T}} \\ \mathcal{O}_{N,\mathbf{T}}^* & \mathcal{O}_{E,\mathbf{T}}^* \mathcal{O}_{E,\mathbf{T}} \end{bmatrix} - \begin{bmatrix} M_S^* \\ \mathcal{O}_{E,\mathbf{T}}^* \end{bmatrix} \begin{bmatrix} M_S & \mathcal{O}_{E,\mathbf{T}} \end{bmatrix} \succeq 0. \tag{5.7}$$

Now we write (5.7) as

$$\begin{bmatrix} I_{H^2_{\mathcal{U}}(\mathbb{F}_d^+)} - M_S^* M_S & \mathcal{O}_{N,\mathbf{T}} - M_S^* \mathcal{O}_{E,\mathbf{T}} \\ \mathcal{O}_{N,\mathbf{T}}^* - \mathcal{O}_{E,\mathbf{T}}^* M_S & 0 \end{bmatrix} \succeq 0$$

and arrive at $\mathcal{O}_{E,\mathbf{T}}^* M_S = \mathcal{O}_{N,\mathbf{T}}^*$ which means that S solves $\mathbf{OAP}(\mathbf{T}, E, N)$. This completes the proof of the theorem. □

Reformulation of the problem $\mathbf{OAP}(\mathbf{T}, E, N)$ in terms of the operator

$$F^S = \begin{bmatrix} I & -M_S \end{bmatrix} \mathcal{O}_{\left[\begin{smallmatrix} E \\ N \end{smallmatrix}\right],\mathbf{T}} : \mathcal{X} \to \mathcal{H}(K_S) \tag{5.8}$$

mapping $(\mathcal{X}, P)$ contractively into the de Branges-Rovnyak space $\mathcal{H}(K_S)$ (condition (4) in Theorem 5.2), when combined with Theorems 3.7 and 3.8, leads to the a linear-fractional description of the set of all solutions in case the Pick operator P (see (4.5)) is strictly positive definite.

Theorem 5.3 *Given the data set* $\{\mathbf{T}, E, N\}$ *such that the pairs* $(E, \mathbf{T})$ *and* $(N, \mathbf{T})$ *are output stable, let us assume that the operator* P *defined as in* (4.5) *is strictly positive definite. Also let*

$$\mathfrak{A}(z) = \begin{bmatrix} \mathfrak{A}_{11}(z) & \mathfrak{A}_{12}(z) \\ \mathfrak{A}_{21}(z) & \mathfrak{A}_{22}(z) \end{bmatrix} = D + \begin{bmatrix} E \\ N \end{bmatrix} (I - Z(z)T)^{-1} Z(z) B$$

be the $(J_{\mathcal{F},\mathcal{U}}, J_{\mathcal{Y},\mathcal{U}})$*-inner power series constructed according to the recipe in Theorem* 3.7*. Then an* $S \in \mathcal{L}(\mathcal{U}, \mathcal{Y})\langle\langle z \rangle\rangle$ *is a solution of the problem* $\mathbf{OAP}(\mathbf{T}, E, N)$ *if and only if* S *can be written in the form*

$$S(z) = (\mathfrak{A}_{11}(z)\mathcal{E}(z) + \mathfrak{A}_{12}(z))\,(\mathfrak{A}_{21}(z)\mathcal{E}(z) + \mathfrak{A}_{22}(z))^{-1} \tag{5.9}$$

for some $\mathcal{E} \in \mathcal{S}_{\mathrm{nc},d}(\mathcal{U}, \mathcal{F})$*. Moreover, the condition* $P \succeq 0$ *is both necessary and sufficient for the problem* $\mathbf{OAP}(\mathbf{T}, E, N)$ *to have solutions.*

Proof By condition (4) in Theorem 5.2 we know that S solves $\mathbf{OAP}(\mathbf{T}, E, N)$ if and only if $S \in \mathcal{S}_{\mathrm{nc},d}(\mathcal{U}, \mathcal{Y})$ and the operator (5.8) maps $(\mathcal{X}, P)$ contractively into the de Branges-Rovnyak space $\mathcal{H}(K_S)$. On the other hand, by Theorem 3.7, we know that $\mathcal{O}_{\left[\begin{smallmatrix} E \\ N \end{smallmatrix}\right], \mathbf{T}}$ is a unitary identification between $(\mathcal{X}, P)$ and $\mathcal{H}(K^P_{\left[\begin{smallmatrix} E \\ N \end{smallmatrix}\right], \mathbf{T}}) = \mathcal{H}(K_{\mathfrak{A}}^{J_{\mathcal{F},\mathcal{U}}, J_{\mathcal{Y},\mathcal{U}}})$. Hence the condition for S to solve $\mathbf{OAP}(\mathbf{T}, E, N)$ translates to: $S \in \mathcal{S}_{\mathrm{nc},d}(\mathcal{U}, \mathcal{Y})$ *and the operator* $\begin{bmatrix} I & -M_S \end{bmatrix}$ *maps* $\mathcal{H}(K_{\mathfrak{A}}^{J_{\mathcal{F},\mathcal{U}}, J_{\mathcal{Y},\mathcal{U}}})$ *contractively into* $\mathcal{H}(K_S)$. By Theorem 3.8, this last condition is equivalent to $S = T_{\mathfrak{A}}[\mathcal{E}]$ for some $\mathcal{E} \in \mathcal{S}_{\mathrm{nc},d}(\mathcal{U}, \mathcal{F})$. In particular, if $P \succ 0$, it follows that the problem $\mathbf{OAP}(\mathbf{T}, E, N)$ has solutions.

If we only have $P \succeq 0$, then via a rescaling the result for the strictly positive definite case implies that, for each $\delta > 0$ there exist solutions $S_\delta \in \mathcal{S}_{\mathrm{nc},d}(\mathcal{U}, \mathcal{Y})$ for the interpolation conditions (4.3) with $\|M_{S_\delta}\| \le 1 + \delta$. The existence of a solution S of (4.3) with $\|M_S\| \le 1$ then follows by a standard weak-$*$ compactness argument which makes use of the fact that the operators $\mathcal{O}_{E,\mathbf{T}}$ and $\mathcal{O}_{N,\mathbf{T}}$ have ranges inside the Fock spaces $H^2_{\mathcal{Y}}(\mathbb{F}^+_d)$ and $H^2_{\mathcal{U}}(\mathbb{F}^+_d)$ respectively. The necessity of the condition $P \succeq 0$ for the existence of solutions is the content of part (2) of Proposition 4.1. □

6 The Analytic Abstract Interpolation Problem

Besides the left-tangential evaluation calculus (4.1), there is another way to evaluate a formal power series $f(z) = \sum_{\alpha \in \mathbb{F}^+_d} f_\alpha z^\alpha \in \mathcal{Y}\langle\langle z \rangle\rangle$ on a d-tuple $\mathbf{Z} = (Z_1, \ldots, Z_d) \in \mathcal{L}(\mathcal{X})^d$, namely,

$$f(\mathbf{Z}) = \sum_{\alpha \in \mathbb{F}^+_d} f_\alpha \otimes \mathbf{Z}^\alpha = \lim_{N \to \infty} \sum_{\alpha \in \mathbb{F}^+_d : |\alpha| \le N} f_\alpha \otimes \mathbf{Z}^\alpha \in \mathcal{Y} \otimes \mathcal{L}(\mathcal{X}), \tag{6.1}$$

provided the latter limit exists at least in the weak sense. The existence of the weak limit clearly depends on f and $\mathbf{Z}$. Let us denote by $\mathfrak{B}_d$ the set of all Hilbert space strict row contractions

$$\mathfrak{B}_d = \Big\{\mathbf{Z} = (Z_1, \ldots, Z_d) \in \mathcal{L}(\mathcal{X})^d : \sum_{j=1}^{d} Z_j Z_j^* \prec I_{\mathcal{X}}\Big\},$$

and let us introduce the space $\mathcal{H}_{\mathcal{Y}}(\mathfrak{B}_d)$ of formal power series $f \in \mathcal{Y}\langle\langle z\rangle\rangle$ such that the weak limit (6.1) exists for any d-tuple $\mathbf{Z} \in \mathfrak{B}_d$. Observe that by Cauchy inequality, $H^2_{\mathcal{Y}}(\mathbb{F}^+_d) \subset \mathcal{H}_{\mathcal{Y}}(\mathfrak{B}_d)$. In particular, for an output-stable pair $(C, \mathbf{T})$ and any $x \in \mathcal{X}$, the power series $\mathcal{O}_{C,\mathbf{T}}x$ belongs to $\mathcal{H}_{\mathcal{Y}}(\mathfrak{B}_d)$. Various spaces of such "free holomorphic functions" have been studied systematically in a series of papers by Popescu [31, 32]; when one restricts the Hilbert space $\mathcal{X}$ to be finite-dimensional $\mathcal{X} = \mathbb{C}^n$ and then defines $\mathfrak{B}_d$ to be the disjoint union of these unit balls of row contraction matrices over $n = 1, 2, 3, \ldots$, one comes into the setting of "free noncommutative functions" which is an area of active current interest (see [12, 13, 34]).

The very formulation of the problem $\mathbf{OAP}(\mathbf{T}, E, N)$ appears to require that the operators $\mathcal{O}_{E,\mathbf{T}}$ and $\mathcal{O}_{N,\mathbf{T}}$ be bounded operators from $\mathcal{X}$ into $H^2_{\mathcal{Y}}(\mathbb{F}^+_d)$ and $H^2_{\mathcal{U}}(\mathbb{F}^+_d)$ respectively. However, upon close inspection, one can see that conditions (2)–(5) in Theorem 5.2 make sense and moreover, conditions (2), (3), (4) are mutually equivalent if we only assume that

(a) $\mathbf{T} = (T_1, \ldots, T_d) \in \mathcal{L}(\mathcal{X})^d$, $E\colon \mathcal{X} \to \mathcal{Y}$ and $N\colon \mathcal{X} \to \mathcal{U}$ are such that

$$\mathcal{O}_{\left[\begin{smallmatrix} E \\ N \end{smallmatrix}\right],\mathbf{T}} : x \mapsto \begin{bmatrix} E \\ N \end{bmatrix} (I - Z(z)T)^{-1}x \quad \text{maps } \mathcal{X} \text{ into } \mathcal{H}_{\mathcal{Y}\oplus\mathcal{U}}(\mathfrak{B}_d),$$

(b) P is a positive semidefinite solution of the Stein equation (3.30).

This suggests that we use any of these conditions as the *definition* of a more general interpolation problem. This leads to the formulation of the *analytic Abstract Interpolation Problem*:

aAIP$(\mathbf{T}, E, N, P)$ *Given the data* $\{E, N, \mathbf{T}, P\}$ *subject to assumptions* (a), (b), *find all* $S \in \mathcal{S}_{\mathrm{nc},d}(\mathcal{U}, \mathcal{Y})$ *such that the formal power series* $F^S x$ *defined as in* (5.1) *belongs to the de Branges-Rovnyak space* $\mathcal{H}(K_S)$ *and satisfies the norm constraint* (5.4).

This problem always has a solution. Various characterizations of the solution set given below extend Theorems 5.1 and 5.3 to the present context.

Theorem 6.1 *Let* P, $\mathbf{T}$, E *and* N *satisfy assumptions* (a), (b). *The following are equivalent:*

1. *S is a solution of the* **aAIP**$(\mathbf{T}, E, N, P)$.
2. *The operator* $\mathbf{P}$ *of the form* (5.2) *is positive semidefinite.*
3. *The kernel* $\mathbf{K}(z, \zeta)$ *of the form* (5.3) *is positive.*

Moreover, if $P \succ 0$ *and if* $\mathfrak{A} \in \mathcal{S}_{\mathrm{nc},d}(J_{\mathcal{Y},\mathcal{U}}, J_{\mathcal{F},\mathcal{U}})$ *is constructed as in Theorem 3.7, then S solves* **aAIP(T**, $E, N, P)$ *if and only if S can be realized in the form* (5.9) *for a Schur-class power series* $\mathcal{E} \in \mathcal{S}_{\mathrm{nc},d}(\mathcal{U}, \mathcal{F})$.

Proof As we observed above, the equivalence (2)⇔(3) and the implication (4)⇒(2) in Theorem 5.2 go through even with the weaker hypotheses (a) and (b) in place; in particular this gives us the equivalence (2) ⇔ (3) and the implication (1) ⇒ (2) in Theorem 6.1. To complete the proof of the mutual equivalence of conditions (1), (2), (3) in Theorem 6.1, it remains to prove the implication (2)⇒(1).

To this end, we assume that the operator (5.2) is positive semidefinite. Then by Proposition 5.1 the operator $I - M_S M_S^*$ is positive semidefinite (i.e., M_S is a contraction which implies $S \in \mathcal{S}_{\mathrm{nc},d}(\mathcal{U}, \mathcal{Y})$) and for every $x \in \mathcal{X}$, there exists an element $h_x \in H^2_{\mathcal{Y}}(\mathbb{F}^+_d)$ which is orthogonal to the Ker $(I - M_S M_S^*)$ and such that

$$(I - M_S M_S^*)^{\frac{1}{2}} h_x = F^S x \quad \text{and} \quad \|h_x\|_{H^2_{\mathcal{Y}}(\mathbb{F}^+_d)} \le \|P^{\frac{1}{2}} x\|_{\mathcal{X}}. \tag{6.2}$$

The first relation in (6.2) implies that $F^S x$ belongs to $\mathrm{Ran}(I - M_S M_S^*)^{\frac{1}{2}}$ or equivalently, to $\mathcal{H}(K_S)$, due to characterization (2.20). Furthermore, since h_x is orthogonal to Ker $(I - M_S M_S^*)$, we conclude from (2.21) and (6.2) that

$$\begin{aligned}\|F^S x\|_{\mathcal{H}(K_S)} &= \|(I - M_S M_S^*)^{\frac{1}{2}} h_x\|_{\mathcal{H}(K_S)} \\ &= \|(I - Q)h_x\|_{H^2_{\mathcal{Y}}(\mathbb{F}^+_d)} = \|h_x\|_{H^2_{\mathcal{Y}}(\mathbb{F}^+_d)} \le \|P^{\frac{1}{2}} x\|_{\mathcal{X}},\end{aligned}$$

which proves (5.4), i.e., that S solves **aAIP(T**, $E, N, P)$ and hence condition (1) in Theorem 6.1 holds.

The proof of the second part of the theorem goes through in the same way as in Theorem 5.3. Note that the operator F^S being a contraction from $(\mathcal{X}, P)$ into the de Branges-Rovnyak space $\mathcal{H}(K_S)$ is now the interpolation condition while, on the other hand, if $P \succ 0$, the power series $\mathfrak{A}(z)$ constructed as in Theorem 3.7 satisfies the identity

$$K_{\mathfrak{A}}^{J_{\mathcal{F},\mathcal{U}}, J_{\mathcal{Y},\mathcal{U}}}(z, \zeta) = \begin{bmatrix} E \\ N \end{bmatrix} (I - Z(z)T)^{-1} P^{-1} (I - T^* Z(\zeta)^*)^{-1} \begin{bmatrix} E^* & N^* \end{bmatrix}$$

due to the Stein equation (3.30) (rather than the specific formula (4.5)).

In case we have only $P \succeq 0$, the same approximation argument as used in the proof of Theorem 5.3 then shows that the problem **aAIP(T**, $E, N, P)$ always has a solution. □

Remark 6.2 Note that a special feature of the problem **OAP(T**, $E, N)$ is expressed by the equivalence (4) ⇔ (5) in Theorem 5.2: for every solution S of the problem, inequality (5.4) implies equality (5.5) (in [21] such problems were called *possessing* *the Parseval equality*). As pointed out in [6] for the context of contractive multipliers on the Drury-Arveson space, concrete examples of interpolation problems which do not possess the Parseval property are *boundary Nevanlinna-Pick interpolation problems.* While there now are results on interior Nevanlinna-Pick interpolation on the noncommutative ball $\mathbb{B}^d$ (see [1, 13]), at this writing there does not appear to be any work on boundary interpolation in the noncommutative setting. An interesting area for future research is to work out some concrete noncommutative boundary interpolation problems as examples of **aOAP** problems which do not possess the Parseval equality.

7 The Abstract Interpolation Problem

We are now ready to formulate the Abstract Interpolation Problem **AIP** based on a data set $\{D, \mathfrak{T}, \mathbf{T}, E, N\}$ described as follows. We are given a linear space $\mathcal{X}$, a positive semidefinite Hermitian form D on $\mathcal{X}$, Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$, linear operators $\mathfrak{T}$, $\mathbf{T} = (T_1, \dots, T_d)$ on $\mathcal{X}$, and linear operators $N\colon \mathcal{X} \to \mathcal{U}$ and $E\colon \mathcal{X} \to \mathcal{Y}$. In addition we assume that

$$D(\mathfrak{T}x, \mathfrak{T}x) + \|Nx\|^2_{\mathcal{U}} = \sum_{j=1}^{d} D(T_jx, T_jx) + \|Ex\|^2_{\mathcal{Y}} \quad \text{for every } x \in \mathcal{X}. \tag{7.1}$$

Definition 7.1 Suppose that we are given the data set $\{D, \mathfrak{T}, \mathbf{T}, E, N\}$ for an **AIP** as in (7.1). We say that the pair (F, S) is a solution of the **AIP** if $S \in \mathcal{S}_{\mathrm{nc},d}(\mathcal{U}, \mathcal{Y})$ and F is a linear mapping from $\mathcal{X}$ into $\mathcal{H}(K_S)$ such that

$$\|Fx\|^2_{\mathcal{H}(K_S)} \le D(x, x) \quad \text{for all} \quad x \in \mathcal{X}, \tag{7.2}$$

$$(F\mathfrak{T}x)(z) - \sum_{j=1}^{d}(FT_jx)(z)z_j = (E - S(z)N)\,x. \tag{7.3}$$

Denote by $\mathcal{X}_0$ the Hilbert space equal to the completion of the space of equivalence classes of elements of $\mathcal{X}$ (where the zero equivalence class consists of elements x with $D(x, x) = 0$) in the D-inner product. Note that if (S, F) solves **AIP**, then condition (7.2) implies that F has a factorization $F_0 \circ \pi$ where π is the canonical projection operator $\pi : \mathcal{X} \to \mathcal{X}_0$ and where $F_0\colon \mathcal{X}_0 \to \mathcal{H}(K_S)$ has $\|F_0\| \le 1$. We abuse notation and denote also by $\mathfrak{T}$ and T_k the operators $\mathfrak{T}$ and T_k followed by the canonical projection into the equivalence class in $\mathcal{X}_0$; so $\mathfrak{T}$, $T_k\colon \mathcal{X} \to \mathcal{X}_0$. Let for short

$$T = \begin{bmatrix} T_1 \\ \vdots \\ T_d \end{bmatrix} : \mathcal{X} \to \mathcal{X}_0^d, \qquad Z(z) = \begin{bmatrix} z_1 I_{\mathcal{X}_0} & \cdots & z_d I_{\mathcal{X}_0} \end{bmatrix}.$$

If we further identify F_0 with the formal power series $F_0(z) \in \mathcal{L}(\mathcal{X}_0, \mathcal{Y})\langle\langle z\rangle\rangle$ defined by

$$F_0(z)x_0 = (F_0x_0)(z),$$

then we can rewrite (7.3) in the form

$$F_0(z)\mathfrak{T} - F_0(z)Z(z)T = E - S(z)N. \tag{7.4}$$

Note that the import of the hypothesis (7.1) is that there is a well-defined isometry $\mathbf{V}$ from

$$\mathcal{D}_{\mathbf{V}} = \overline{\text{Ran}} \begin{bmatrix} \mathfrak{T} \\ N \end{bmatrix} \subset \begin{bmatrix} \mathcal{X}_0 \\ \mathcal{U} \end{bmatrix} \quad \text{onto} \quad \mathcal{R}_{\mathbf{V}} = \overline{\text{Ran}} \begin{bmatrix} T \\ E \end{bmatrix} \subset \begin{bmatrix} \mathcal{X}_0^d \\ \mathcal{Y} \end{bmatrix} \tag{7.5}$$

such that

$$\mathbf{V}\colon \begin{bmatrix} \mathfrak{T} \\ N \end{bmatrix} x \to \begin{bmatrix} T \\ E \end{bmatrix} x \quad \text{for all } x \in \mathcal{X}. \tag{7.6}$$

Note also that the definition (7.5) and (7.6) of $\mathbf{V}$ is completely determined by the problem data $\{D, \mathfrak{T}, \mathbf{T}, E, N\}$.

If $\mathcal{X}$ is already a Hilbert space and there exists a bounded positive semidefinite operator $P \succeq 0$ such that $D(x, y) = \langle Px, \ y\rangle_{\mathcal{X}}$ for every $x, y \in \mathcal{X}$, then identity (7.1) can be written as

$$\mathfrak{T}^*P\mathfrak{T} - \sum_{k=1}^{d} T_j^*PT_j = E^*E - N^*N.$$

Furthermore, equality (7.4) can be written as

$$F_0(z)(\mathfrak{T} - Z(z)T)x = (E - S(z)N)\,x$$

and if the pencil $(\mathfrak{T} - Z(z)T)$ is in the space $\mathcal{H}_{\mathcal{X}}(\mathfrak{B}_d)$, then the latter equation defines F_0 uniquely by

$$F_0(z)x = (E - S(z)N)\,(\mathfrak{T} - Z(z)T)^{-1}x.$$

If furthermore, $\mathfrak{T} = I_{\mathcal{X}}$, then it is readily seen that the $\mathbf{AIP}(D, I_{\mathcal{X}}, \mathbf{T}, E, N)$ collapses to the $\mathbf{aAIP}(\mathbf{T}, E, N, P)$.

Our next goal is to establish a correspondence between solutions (F, S) of the problem $\mathbf{AIP}(D, \mathfrak{T}, \mathbf{T}, E, N)$ and unitary-colligation extensions of the partially defined isometric colligation $\mathbf{V}$ in (7.6).

Theorem 7.2 *Let* **V** *be the isometry defined by* (7.6) *associated with the data of a problem* **AIP** *and let*

$$\mathcal{C} = \left\{\mathcal{X}_0 \oplus \mathcal{X}_1, \mathcal{U}, \mathcal{Y}, \mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X}_0 \oplus \mathcal{X}_1 \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X}_0^d \oplus \mathcal{X}_1^d \\ \mathcal{Y} \end{bmatrix}\right\} \tag{7.7}$$

be a unitary-colligation extension of **V**. *Then the pair* (S, F_0) *defined by the formulas*

$$S(z) = D + C(I - Z(z)A)^{-1}Z(z)B, \tag{7.8}$$

$$F_0(z) = C(I - Z(z)A)^{-1}|_{\mathcal{X}_0}, \tag{7.9}$$

is a solution of **AIP**. *Moreover, every solution of* **AIP** *arises in this way.*

Proof Since the connecting matrix **U** in (7.7) is unitary, its characteristic formal power series (7.8) belongs to $\mathcal{S}_{\text{nc},d}(\mathcal{U}, \mathcal{Y})$ by Theorem 2.3. Furthermore, let $\mathcal{H}(K_S)$ be the de Branges-Rovnyak space associated with S. Making use of the power series (2.16) from the Kolmogorov decomposition (2.13) of the kernel K_S, we write (7.9) as

$$(F_0 x_0)(z) = H(z) \left[\begin{smallmatrix} x_0 \\ 0 \end{smallmatrix}\right] = C(I - Z(z)A)^{-1} \left[\begin{smallmatrix} x_0 \\ 0 \end{smallmatrix}\right] \quad \text{for} \quad x_0 \in \mathcal{X}_0, \tag{7.10}$$

and conclude by (2.22) that F_0 is a contraction from $\mathcal{X}_0$ into $\mathcal{H}(K_S)$.

It remains to check the identity (7.4). Due to (7.10), we see that (7.4) is the same as

$$H(z)\mathfrak{T}x = H(z)Z(z)Tx + Ex - S(z)Nx \tag{7.11}$$

(with $H(z)$ as in (2.16)). Using the unitary realization (7.8) of S written as

$$S(z) = D + H(z)Z(z)B,$$

we rewrite (7.11) as

$$H(z)\mathfrak{T} = H(z)Z(z)T + E - \big(D + H(z)Z(z)B\big)N. \tag{7.12}$$

To verify (7.12), we use the identity

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \mathfrak{T} \\ N \end{bmatrix} = \begin{bmatrix} T \\ E \end{bmatrix},$$

or, in more detail,

$$A\mathfrak{T} + BN = T, \qquad C\mathfrak{T} + DN = E,$$

which is true by the hypothesis that $\mathbf{U}$ extends $\mathbf{V}$, to see that the right-hand side of (7.12) is equal to

$$
\begin{aligned}
&H(z)Z(z)T + E - DN - H(z)Z(z)BN \\
&= H(z)Z(z)T + C\mathfrak{T} - H(z)Z(z)(T - A\mathfrak{T}) \\
&= C\mathfrak{T} + H(z)Z(z)A\mathfrak{T} \\
&= C\mathfrak{T} + C(I - Z(z)A)^{-1}Z(z)A\mathfrak{T} = H(z)\mathfrak{T}
\end{aligned}
$$

as wanted. We postpone the last statement to the proof of Theorem 7.3 where a more general result is proved. □

We next introduce the defect spaces

$$
\Delta = \begin{bmatrix} \mathcal{X}_0 \\ \mathcal{U} \end{bmatrix} \ominus \mathcal{D}_{\mathbf{V}} \quad \text{and} \quad \Delta_* = \begin{bmatrix} \mathcal{X}_0^d \\ \mathcal{Y} \end{bmatrix} \ominus \mathcal{R}_{\mathbf{V}}
$$

and let $\widetilde{\Delta}$ be another copy of Δ and $\widetilde{\Delta}_*$ another copy of Δ_* with unitary identification maps

$$
i : \Delta \to \widetilde{\Delta} \quad \text{and} \quad i_* : \ \Delta_* \to \widetilde{\Delta}_*.
$$

Following the ideas of Arov-Grossman [4, 5], define a unitary operator $\mathbf{U}_0$ from $\mathcal{D}_{\mathbf{V}} \oplus \Delta \oplus \widetilde{\Delta}_*$ onto $\mathcal{R}_{\mathbf{V}} \oplus \Delta_* \oplus \widetilde{\Delta}$ by the rule

$$
\mathbf{U}_0 x = \begin{cases} \mathbf{V}x, & \text{if } \ x \in \mathcal{D}_{\mathbf{V}}, \\ i(x) & \text{if } \ x \in \Delta, \\ i_*^{-1}(x) & \text{if } \ x \in \widetilde{\Delta}_*. \end{cases} \tag{7.13}
$$

Identifying $\begin{bmatrix} \mathcal{D}_{\mathbf{V}} \\ \Delta \end{bmatrix}$ with $\begin{bmatrix} \mathcal{X}_0 \\ \mathcal{U} \end{bmatrix}$ and $\begin{bmatrix} \mathcal{R}_{\mathbf{V}} \\ \Delta_* \end{bmatrix}$ with $\begin{bmatrix} \mathcal{X}_0^d \\ \mathcal{Y} \end{bmatrix}$, we decompose $\mathbf{U}_0$ defined by (7.13) according to

$$
\mathbf{U}_0 = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & 0 \end{bmatrix} : \quad \begin{bmatrix} \mathcal{X}_0 \\ \mathcal{U} \\ \widetilde{\Delta}_* \end{bmatrix} \to \begin{bmatrix} \mathcal{X}_0^d \\ \mathcal{Y} \\ \widetilde{\Delta} \end{bmatrix}. \tag{7.14}
$$

The (3, 3) block in this decomposition is zero, since (by definition (7.13)), for every $\widetilde{\delta}_* \in \widetilde{\Delta}_*$, the vector $\mathbf{U}_0\widetilde{\delta}_*$ belongs to Δ, which is a subspace of $\begin{bmatrix} \mathcal{X}_0^d \\ \mathcal{Y} \end{bmatrix}$ and therefore, is orthogonal to $\widetilde{\Delta}$; in other words $\mathcal{P}_{\widetilde{\Delta}}\mathbf{U}_0|_{\widetilde{\Delta}_*} = 0$ where $\mathcal{P}_{\widetilde{\Delta}}$ stands for the orthogonal projection of $\mathcal{R}_{\mathbf{V}} \oplus \Delta_* \oplus \widetilde{\Delta}$ onto $\widetilde{\Delta}$.

The unitary operator $\mathbf{U}_0$ is the connecting operator of the unitary colligation

$$\mathcal{C}_0 = \left\{ \mathcal{X}_0, \begin{bmatrix} \mathcal{U} \\ \widetilde{\Delta}_* \end{bmatrix}, \begin{bmatrix} \mathcal{Y} \\ \widetilde{\Delta} \end{bmatrix}, \mathbf{U}_0 \right\}, \tag{7.15}$$

which is called *the universal unitary colligation* associated with the **AIP**.

According to (2.17), the characteristic formal power series of the colligation $\mathcal{C}_0$ defined in (7.15) is given by

$$\begin{aligned} \Sigma(z) &= \begin{bmatrix} \Sigma_{11}(z) & \Sigma_{12}(z) \\ \Sigma_{21}(z) & \Sigma_{22}(z) \end{bmatrix} \\ &= \begin{bmatrix} U_{22} & U_{23} \\ U_{32} & 0 \end{bmatrix} + \begin{bmatrix} U_{21} \\ U_{31} \end{bmatrix} (I - Z(z)U_{11})^{-1} Z(z) \begin{bmatrix} U_{12} & U_{13} \end{bmatrix} \\ &= \mathcal{P}_{\mathcal{Y}\oplus\widetilde{\Delta}} \mathbf{U}_0 (I - \mathcal{P}_{\mathcal{X}_0}^* Z(z) \mathcal{P}_{\mathcal{X}_0^d} \mathbf{U}_0)^{-1}|_{\mathcal{U}\oplus\widetilde{\Delta}_*} \end{aligned} \tag{7.16}$$

and belongs to the class $\mathcal{S}_{\mathrm{nc},d}(\mathcal{U} \oplus \widetilde{\Delta}_*, \ \mathcal{Y} \oplus \widetilde{\Delta})$ by Theorem 2.3. The associated observability operator is given by

$$\begin{aligned} H_\Sigma(z) &= \begin{bmatrix} U_{21} \\ U_{31} \end{bmatrix} (I - Z(z)U_{11})^{-1} \\ &= \mathcal{P}_{\mathcal{Y}\oplus\widetilde{\Delta}} \mathbf{U}_0 (I - \mathcal{P}_{\mathcal{X}_0}^* Z(z) \mathcal{P}_{\mathcal{X}_0^d} \mathbf{U}_0)^{-1}|_{\mathcal{X}_0}. \end{aligned} \tag{7.17}$$

By another application of Theorem 2.3 we see that

$$\begin{aligned} K_\Sigma(z,\zeta) &:= k_{\mathrm{Sz}}(z,\zeta) \otimes I_{\mathcal{Y}\oplus\widetilde{\Delta}} - \Sigma(z)(k_{\mathrm{Sz}}(z,\zeta) \otimes I_{\mathcal{U}\oplus\widetilde{\Delta}^*})\Sigma(\zeta)^* \\ &= H_\Sigma(z) H_\Sigma(\zeta)^*. \end{aligned}$$

We shall also need an enlarged colligation

$$\mathcal{C}_{0,e} = \left\{ \mathcal{X}_0, \begin{bmatrix} \mathcal{X}_0 \\ \mathcal{U} \\ \widetilde{\Delta}_* \end{bmatrix}, \begin{bmatrix} \mathcal{Y} \\ \widetilde{\Delta} \end{bmatrix}, \mathbf{U}_{0,e} = \begin{bmatrix} U_{11} & U_{11} & U_{12} & U_{13} \\ U_{21} & U_{21} & U_{22} & U_{23} \\ U_{31} & U_{31} & U_{32} & 0 \end{bmatrix} \right\} \tag{7.18}$$

with associated characteristic formal power series

$$\begin{aligned} \Sigma_e(z) &= \mathcal{P}_{\mathcal{Y}\oplus\widetilde{\Delta}} \mathbf{U}_0 (I - \mathcal{P}_{\mathcal{X}_0}^* Z(z) \mathcal{P}_{\mathcal{X}_0^d} \mathbf{U}_0)^{-1} \\ &= \begin{bmatrix} U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & 0 \end{bmatrix} + \begin{bmatrix} U_{21} \\ U_{31} \end{bmatrix} (I - Z(z)U_{11})^{-1} Z(z) \begin{bmatrix} U_{11} & U_{12} & U_{13} \end{bmatrix} \\ &= \begin{bmatrix} U_{21}(I - Z(z)U_{11})^{-1} & \Sigma_{11}(z) & \Sigma_{12}(z) \\ U_{31}(I - Z(z)U_{11})^{-1} & \Sigma_{21}(z) & \Sigma_{22}(z) \end{bmatrix} = \begin{bmatrix} H_\Sigma(z) & \Sigma(z) \end{bmatrix}. \end{aligned} \tag{7.19}$$

These are the ingredients for the following parametrization for the set of all solutions of **AIP**. In particular, solutions of **AIP** exist for any data set $(D, \mathfrak{T}, \mathbf{T}, E, N)$ subject to condition (7.1).

Theorem 7.3 *Given the data set $(D, \mathfrak{T}, \mathbf{T}, E, N\}$ subject to condition (7.1), let $\mathbf{U}_0$ be the universal unitary-colligation extension of $\mathbf{V}$ given by (7.13) with characteristic formal power series (7.16) and let $\mathbf{U}_{0,e}$ be the connecting operator for the enlarged universal unitary colligation $\mathcal{C}_{0,e}$ given by (7.18). Then the pair $(S(z),\ F_0(z))$ solves the problem* **AIP** *if and only if there is a Schur-class multiplier $\mathcal{W} \in \mathcal{S}_{\mathrm{nc},d}(\widetilde{\Delta}, \widetilde{\Delta}_*)$ such that*

$$\begin{bmatrix} F_0(z) & S(z) \end{bmatrix} = \mathfrak{R}_{\Sigma_e}[\mathcal{W}](z), \tag{7.20}$$

i.e. (see the definition (3.45)), such that

$$S(z) = \Sigma_{11}(z) + \Sigma_{12}(z)(I - \mathcal{W}(z)\Sigma_{22}(z))^{-1}\mathcal{W}(z)\Sigma_{21}(z), \tag{7.21}$$

$$\begin{aligned} F_0(z) &= U_{21}(I - Z(z)U_{11})^{-1} \\ &\quad + \Sigma_{12}(z)(I - \mathcal{W}(z)\Sigma_{22}(z))^{-1}\mathcal{W}(z)U_{31}(I - Z(z)U_{11})^{-1}. \end{aligned} \tag{7.22}$$

Proof For the "only if" part we assume that the pair $(F_0(z), S(z))$ is a solution of **AIP** and show that necessarily $\begin{bmatrix} F_0(z) & S(z) \end{bmatrix}$ is in the range of the linear-fractional map $\mathfrak{R}_{\Sigma_e}$ acting on the Schur class $\mathcal{S}_{\mathrm{nc},d}(\widetilde{\Delta}, \widetilde{\Delta}_*)$. To verify that there is a $\mathcal{W} \in \mathcal{S}_{\mathrm{nc},d}(\widetilde{\Delta}, \widetilde{\Delta}_*)$ so that (7.20) holds, by Theorem 3.9 it suffices to produce a Schur pair $(\mathbf{a}, \mathbf{c})$ so that

$$\begin{bmatrix} I & \mathbf{c}(z) \end{bmatrix} \Sigma_e(z) = \begin{bmatrix} F_0(z) & S(z) & \mathbf{a}(z) \end{bmatrix}. \tag{7.23}$$

Using the last expression for $\Sigma_e(z)$ in (7.19), we may rewrite (7.23) as

$$\begin{bmatrix} I & \mathbf{c}(z) \end{bmatrix} P_{\mathcal{Y}\oplus\widetilde{\Delta}}\mathbf{U}_0(I - \mathcal{P}_{\mathcal{X}_0^*}Z(z)\mathcal{P}_{\mathcal{X}_0^d}\mathbf{U}_0)^{-1} = \begin{bmatrix} F_0(z) & S(z) & \mathbf{a}(z) \end{bmatrix}$$

which in turn can be converted to the more linear form

$$\begin{bmatrix} I & \mathbf{c}(z) \end{bmatrix} \mathcal{P}_{\mathcal{Y}\oplus\widetilde{\Delta}}\mathbf{U}_0 = \begin{bmatrix} F_0(z) & S(z) & \mathbf{a}(z) \end{bmatrix} (I - \mathcal{P}^*_{\mathcal{X}_0}Z(z)\mathcal{P}_{\mathcal{X}_0^d}\mathbf{U}_0). \tag{7.24}$$

Let us define formal power series $\mathbf{a} \in \mathcal{L}(\widetilde{\Delta}_*, \mathcal{Y})\langle\langle z\rangle\rangle$ and $\mathbf{c} \in \mathcal{L}(\widetilde{\Delta}, \mathcal{Y})\langle\langle z\rangle\rangle$ by the formulas

$$\mathbf{a}(z) = F_0(z)\left(Z(z)\mathcal{P}_{\mathcal{X}_0^d}\mathbf{U}_0\right)\Big|_{\widetilde{\Delta}_*} + \mathcal{P}_{\mathcal{Y}}\mathbf{U}_0|_{\widetilde{\Delta}_*}, \tag{7.25}$$

$$\mathbf{c}(z) = F_0(z)\mathcal{P}_{\mathcal{X}_0}\mathbf{U}_0^*|_{\widetilde{\Delta}} + S(z)\ \mathcal{P}_{\mathcal{U}}\mathbf{U}_0^*|_{\widetilde{\Delta}}. \tag{7.26}$$

Our goal is to show that $(\mathbf{a}, \mathbf{c})$ is a Schur-pair satisfying the condition (7.24). This will then complete the proof of the "only if" part of the theorem.

Note that the condition (7.24) must be verified on vectors from the space $\mathcal{X}_0 \oplus \mathcal{U} \oplus \widetilde{\Delta}_*$. Since $\mathcal{X}_0 \oplus \mathcal{U}$ has the alternative decomposition $\mathcal{X}_0 \oplus \mathcal{U} = \mathcal{D}_{\mathbf{V}} \oplus \Delta$, it suffices to verify the validity of (7.24) for three separate cases: (1) $y \in \mathcal{D}_{\mathbf{V}}$, (2) $y \in \Delta$, and (3) $y \in \widetilde{\Delta}_*$.

Case 1 $y \in \mathcal{D}_{\mathbf{V}}$. By construction, a dense subset of $\mathcal{D}_{\mathbf{V}}$ consists of vectors of the form $y = \mathfrak{T}x \oplus Nx \oplus 0$ where $x \in \mathcal{X}$. By definition (7.13) we then have $\mathbf{U}_0 y = Tx \oplus Ex \oplus 0$. Then condition (7.24) applied to the vector y for this case becomes simply

$$Ex = F_0(z)\mathfrak{T}x + S(z)Ex - F_0(z)Z(z)Tx$$

which holds true due to the data-admissibility condition (7.4). Note that this case holds automatically independently of the definition of $(\mathbf{a}, \mathbf{c})$.

Case 2 $y = \delta \in \Delta$. In this case, $\mathbf{U}_0\delta = i(\delta) \in \widetilde{\Delta} \perp \mathcal{X}^d$, and hence, the left and the right sides of (7.24) applied to a vector y of this form give us

$$\begin{bmatrix} I & \mathbf{c}(z) \end{bmatrix} \mathcal{P}_{\mathcal{Y}\oplus\widetilde{\Delta}} \mathbf{U}_0\delta = \begin{bmatrix} I & \mathbf{c}(z) \end{bmatrix} \mathcal{P}_{\mathcal{Y}\oplus\widetilde{\Delta}} \begin{bmatrix} 0 \\ 0 \\ i(\delta) \end{bmatrix} = \mathbf{c}(z)i(\delta),$$

$$\begin{aligned} \begin{bmatrix} F_0(z) & S(z) & \mathbf{a}(z) \end{bmatrix} (I - \mathcal{P}^*_{\mathcal{X}_0} Z(z) \mathcal{P}_{\mathcal{X}_0^d} \mathbf{U}_0)\delta &= \begin{bmatrix} F_0(z) & S(z) & \mathbf{a}(z) \end{bmatrix} \delta \\ &= F_0(z)\mathcal{P}_{\mathcal{X}_0}\delta + S(z)\mathcal{P}_{\mathcal{U}}\delta. \end{aligned}$$

Thus, equality (7.24) restricted to Δ amounts to

$$\mathbf{c}(z)i(\delta) = F_0(z)\mathcal{P}_{\mathcal{X}_0}\delta + S(z)\mathcal{P}_{\mathcal{U}}\delta,$$

which is equivalent, since $\mathbf{U}_0$ is unitary and hence $\delta = \mathbf{U}_0^* i(\delta)$, to

$$\mathbf{c}(z)i(\delta) = F_0(z)\mathcal{P}_{\mathcal{X}_0}\mathbf{U}_0^* i(\delta) + S(z)\mathcal{P}_{\mathcal{U}} = \mathbf{U}_0^* i(\delta),$$

which in turn, amounts to the definition of $\mathbf{c}(z)$ in (7.26).

Case 3 $y = \widetilde{\delta}_* \in \widetilde{\Delta}_*$. In this case, $\mathbf{U}_0 y = i_*^{-1}(\widetilde{\delta}_*) \in \mathcal{X}^d \oplus \mathcal{Y} \perp \Delta$. Then the left and the right sides of (7.24) applied to a vector y of this form give

$$\begin{bmatrix} I & \mathbf{c}(z) \end{bmatrix} \begin{bmatrix} \mathcal{P}_{\mathcal{Y}} i_*^{-1}(\widetilde{\delta}_*) \\ 0 \end{bmatrix} = \mathcal{P}_{\mathcal{Y}} i_*^{-1}(\widetilde{\delta}_*) = \mathcal{P}_{\mathcal{Y}} \mathbf{U}_0 \delta_*,$$

$$\begin{aligned} \begin{bmatrix} F_0(z) & S(z) & \mathbf{a}(z) \end{bmatrix} \begin{bmatrix} -Z(z)\mathcal{P}_{\mathcal{X}_0^d} i_*^{-1}(\widetilde{\delta}_*) \\ 0 \\ \widetilde{\delta}_* \end{bmatrix} &= -F_0(z)Z(z)\mathcal{P}_{\mathcal{X}_0^d} i_*^{-1}(\widetilde{\delta}_*) + \mathbf{a}(z)\widetilde{\delta}_* \\ &= -F_0(z)Z(z)\mathcal{P}_{\mathcal{X}_0^d} \mathbf{U}_0 \delta_* + \mathbf{a}(z)\widetilde{\delta}_* \end{aligned}$$

from which we see that equality (7.24) restricted to $\widetilde{\Delta}_*$ collapses to the definition (7.25) of $\mathbf{a}(z)$.

We thus verified the equality (7.24) which is equivalent to (7.23), which in turn, due to the last decomposition of Σ_e in (7.19), is equivalent to the following relations:

$$\begin{bmatrix} I_{\mathcal{Y}} & \mathbf{c}(z) \end{bmatrix} H_\Sigma(z)x_0 = F_0(z)x_0 \in \mathcal{H}(K_S) \tag{7.27}$$

for each $x_0 \in \mathcal{X}_0$ and

$$\begin{bmatrix} I_{\mathcal{Y}} & \mathbf{c}(z) \end{bmatrix} \Sigma(z) = \begin{bmatrix} S(z) & \mathbf{a}(z) \end{bmatrix}. \tag{7.28}$$

To verify that $(\mathbf{a}, \mathbf{c})$ defined via (7.25) and (7.26) is a Schur-pair, we will use the notation $H_\Sigma(z)$ for the observability operator (7.17) associated with the universal colligation $\mathcal{C}_0$ and $H(z)$ for any power series giving rise to a factorization of the kernel $K_S(z, \zeta)$ as in (2.13).

Note that since the power series on the left side of (7.27) belongs to $\mathcal{H}(K_S)$, it follows that for every $x \in \mathcal{X}_0$, there is a unique $g_x \in \mathcal{X}$ which is orthogonal to $\operatorname{Ker} M_H$ and such that

$$\begin{bmatrix} I_{\mathcal{Y}} & \mathbf{c}(z) \end{bmatrix} H_\Sigma(z)x = H(z)g_x.$$

Therefore we can define a linear operator $\Gamma : \ \mathcal{X} \to \mathcal{H}$ by the rule $\Gamma x = g_x$. Thus,

$$\begin{bmatrix} I_{\mathcal{Y}} & \mathbf{c}(z) \end{bmatrix} H_\Sigma(z) = H(z)\Gamma. \tag{7.29}$$

By the definition of the norm in $\mathcal{H}(K_S)$,

$$\|Fx\|_{\mathcal{H}(K_S)} = \|g_x\|_{\mathcal{X}} = \|\Gamma x\|_{\mathcal{X}}.$$

On the other hand, the operator $F\colon \ \mathcal{X}_0 \to \mathcal{H}(K_S)$ is contractive by assumption; hence $\|Fx\|_{\mathcal{H}(K_S)} \le \|x\|_{\mathcal{X}_0}$ and Γ is a contraction:

$$\|\Gamma x\|_{\mathcal{H}} = \|Fx\|_{\mathcal{H}(K_S)} \le \|x\|_{\mathcal{X}_0}.$$

We next show that the power series $\mathbf{a}$ and $\mathbf{c}$ defined in (7.25) and (7.26) satisfy

$$\mathbf{a}(z)(k_{\mathrm{Sz}}(z,\zeta) \otimes I_{\widetilde{\Delta}_*})\mathbf{a}(\zeta)^* - \mathbf{c}(z)(k_{\mathrm{Sz}}(z,\zeta) \otimes I_{\widetilde{\Delta}})\mathbf{c}(\zeta)^* = H(z)(I - \Gamma\Gamma^*)H(\zeta)^*. \tag{7.30}$$

To this end, let us rearrange the left-hand side expression in (7.30) as

$$\begin{aligned}
&\mathbf{a}(z)(k_{\rm Sz}(z,\zeta)\otimes I_{\widetilde{\Delta}_*})\mathbf{a}(\zeta)^* - \mathbf{c}(z)(k_{\rm Sz}(z,\zeta)\otimes I_{\widetilde{\Delta}})\mathbf{c}(\zeta)^* \\
&= k_{\rm Sz}(z,\zeta)\otimes I_{\mathcal{Y}} - S(z)(k_{\rm Sz}(z,\zeta)\otimes I_{\mathcal{U}})S(\zeta)^* \\
&\quad + \begin{bmatrix} S(z) & \mathbf{a}(z)\end{bmatrix}\left(k_{\rm Sz}(z,\zeta)\otimes I_{\mathcal{U}\oplus\widetilde{\Delta}_*}\right)\begin{bmatrix} S(\zeta)^* \\ \mathbf{a}(\zeta)^*\end{bmatrix} \\
&\quad - \begin{bmatrix} I_{\mathcal{Y}} & \mathbf{c}(z)\end{bmatrix}\left(k_{\rm Sz}(z,\zeta)\otimes I_{\mathcal{Y}\oplus\widetilde{\Delta}}\right)\begin{bmatrix} I_{\mathcal{Y}} \\ \mathbf{c}(\zeta)^*\end{bmatrix},
\end{aligned} \tag{7.31}$$

and then observe that the two first terms on the right side represent the kernel K_S, whereas the two last terms, due to (7.28), equal

$$\begin{bmatrix} I_{\mathcal{Y}} & \mathbf{c}(z)\end{bmatrix}\left(\Sigma(z)\big(k_{\rm Sz}(z,\zeta)\otimes I_{\mathcal{U}\oplus\widetilde{\Delta}_*}\big)\Sigma(\zeta)^* - k_{\rm Sz}(z,\zeta)\otimes I_{\mathcal{Y}\oplus\widetilde{\Delta}}\right)\begin{bmatrix} I_{\mathcal{Y}} \\ \mathbf{c}(\zeta)^*\end{bmatrix}.$$

Thus, the right side of (7.31) can be written as

$$\begin{aligned}
&K_S(z,\zeta) - \begin{bmatrix} I_{\mathcal{Y}} & \mathbf{c}(z)\end{bmatrix} K_\Sigma(z,\zeta)\begin{bmatrix} I_{\mathcal{Y}} \\ \mathbf{c}(\zeta)^*\end{bmatrix} \\
&= H(z)H(\zeta)^* - \begin{bmatrix} I_{\mathcal{Y}} & \mathbf{c}(z)\end{bmatrix} H_\Sigma(z)H_\Sigma(\zeta)^*\begin{bmatrix} I_{\mathcal{Y}} \\ \mathbf{c}(\zeta)^*\end{bmatrix} \\
&= H(z)\left(I - \Gamma\Gamma^*\right)H(\zeta)^*,
\end{aligned}$$

where we used the Kolmogorov decompositions for K_S and K_Σ for the first equality and the relation (7.29) for the second. This completes the verification of (7.30). Since the right side of (7.30) represents a positive kernel, the kernel on the left side is also positive. Thus, $(\mathbf{a},\mathbf{c})$ is a Schur-pair, which completes the proof of the "only if" part of the theorem.

To prove the "if" part, let us assume that the pair (S, F_0) is realized as in (7.21), (7.22) for some $\mathcal{W}\in\mathcal{S}_{{\rm nc},d}(\widetilde{\Delta},\widetilde{\Delta}_*)$. By Theorem 2.3, $\mathcal{W}(z)$ can be realized as the characteristic formal power series of some unitary colligation, i.e., $\mathcal{W}(z)$ is of the form

$$\mathcal{W}(z) = D' + C'(I - Z(z)A')^{-1}Z(z)B' \tag{7.32}$$

for some unitary connecting matrix

$$\mathbf{U}' = \begin{bmatrix} A' & B' \\ C' & D'\end{bmatrix} : \begin{bmatrix} \mathcal{X}' \\ \widetilde{\Delta}\end{bmatrix} \to \begin{bmatrix} \mathcal{X}'^d \\ \widetilde{\Delta}_*\end{bmatrix}. \tag{7.33}$$

It turns out that S and F_0 defined in (7.21), (7.22) admit realizations

$$S(z) = D + C(I - Z(z)A)^{-1}Z(z)B, \quad F_0(z) = C(I - Z(z)A)^{-1}|_{\mathcal{X}_0} \tag{7.34}$$

with the connecting matrix

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X}_0 \oplus \mathcal{X}' \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X}_0^d \oplus \mathcal{X}'^d \\ \mathcal{Y} \end{bmatrix}$$

which can be expressed in terms of connecting matrices $\mathbf{U}_0$ and $\mathbf{U}'$ as follows:

$$\begin{aligned} A &= \begin{bmatrix} U_{11} + U_{13}D'U_{31} & U_{13}C' \\ B'U_{31} & A' \end{bmatrix}, \; B = \begin{bmatrix} U_{12} + U_{13}D'U_{32} \\ B'U_{32} \end{bmatrix}, \\ C &= \begin{bmatrix} U_{21} + U_{23}D'U_{31} & U_{23}C' \end{bmatrix}, \;\; D = U_{22} + U_{23}D'U_{32}. \end{aligned} \tag{7.35}$$

In calculations below (which verify that the formulas in (7.34) define the same power series as in (7.21), (7.22)) we will often drop the argument z in $Z(z)$ and $\mathcal{W}(z)$. Applying the well known formula for the inverse of a 2×2 block matrix

$$I - ZA = \begin{bmatrix} I - Z(U_{11} + U_{13}D'U_{31}) & -ZU_{13}C' \\ -ZB'U_{31} & I - ZA' \end{bmatrix}$$

and taking into account that the Schur complement of the upper left block above equals

$$I - Z(U_{11} + U_{13}D'U_{31}) - ZU_{13}C'(I - ZA')^{-1}ZB' = I - ZU_{11} - ZU_{13}\mathcal{W}U_{31},$$

by (7.32), we get

$$\begin{aligned} (I - ZA)^{-1} &= \begin{bmatrix} I \\ (I - ZA')^{-1}ZB'U_{31} \end{bmatrix} (I - ZU_{11} - ZU_{13}\mathcal{W}U_{31})^{-1} \\ &\quad \times \begin{bmatrix} I & ZU_{13}C'(I - ZA')^{-1} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & (I - ZA')^{-1} \end{bmatrix}. \end{aligned} \tag{7.36}$$

Restricting (7.36) to $\mathcal{X}_0$ and multiplying the restricted equality on the left by the operator C defined in (7.35), we get, on account of (7.32),

$$C(I - ZA)^{-1}|_{\mathcal{X}_0} = (U_{21} + U_{23}\mathcal{W})(I - ZU_{11} - ZU_{13}\mathcal{W}U_{31})^{-1}.$$

We next observe from (7.16) that

$$
\begin{aligned}
&(I - ZU_{11} - ZU_{13}\mathcal{W}U_{31})^{-1} \qquad (7.37)\\
&= (I - ZU_{11})^{-1} + (I - ZU_{11})^{-1}ZU_{13}(I - \mathcal{W}\Sigma_{22}(z))^{-1}\mathcal{W}U_{31}(I - ZU_{11})^{-1}.
\end{aligned}
$$

Combining the two last relations and making use of the formula

$$
(U_{21} + U_{23}\mathcal{W}U_{31})(I - ZU_{11})^{-1}ZU_{13} = \Sigma_{12}(z) - U_{23}(I - \mathcal{W}\Sigma_{22}(z)) \qquad (7.38)
$$

which follows from the realization formulas (7.16) for Σ_{12} and Σ_{22}, we get

$$
\begin{aligned}
C(I - ZA)^{-1}|_{\mathcal{X}_0} &= (U_{21} + U_{23}\mathcal{W}U_{31})(I - ZU_{11})^{-1}\\
&\quad + \big(\Sigma_{12}(z)(I - \mathcal{W}\Sigma_{22}(z))^{-1} - U_{23}\big)\mathcal{W}U_{31}(I - ZU_{11})^{-1}\\
&= U_{21}(I - ZU_{11})^{-1}\\
&\quad + \Sigma_{12}(z)(I - \mathcal{W}\Sigma_{22}(z))^{-1}\mathcal{W}U_{31}(I - ZU_{11})^{-1}.
\end{aligned}
$$

Thus the second formula in (7.34) define the same power series as in (7.22).

We now turn to the first formula in (7.34). Multiplying (7.36) by C on the left and ZB on the right (where C and B are given in (7.35)), we get, again on account of (7.32),

$$
\begin{aligned}
S(z) &= C(I - ZA)^{-1}ZB + D\\
&= (U_{21} + U_{23}D'U_{31} + U_{23}C'(I - ZA')^{-1}ZB'U_{31})\\
&\qquad \times (I - ZU_{11} - ZU_{13}\mathcal{W}(z)U_{31})^{-1}\\
&\qquad \times Z(U_{12} + U_{13}D'U_{32} + U_{13}C'(I - ZA')^{-1}ZB'U_{32})\\
&\quad + U_{23}C'(I - ZA')^{-1}ZB'U_{32} + U_{22} + U_{23}D'U_{32}\\
&= (U_{21} + U_{23}\mathcal{W}U_{31})(I - ZU_{11} - ZU_{13}\mathcal{W}U_{31})^{-1}Z(U_{12} + U_{13}\mathcal{W}U_{32})\\
&\quad + U_{22} + U_{23}\mathcal{W}U_{32}. \qquad (7.39)
\end{aligned}
$$

To simplify the right side of (7.39), we first substitute (7.37) and then use the formula (7.38) and a similar formula

$$
U_{31}(I - ZU_{11})^{-1}Z(U_{12} + U_{13}\mathcal{W}U_{32}) = \Sigma_{21}(z) - (I - \Sigma_{22}(z)\mathcal{W})U_{32},
$$

which follows from the realization formulas (7.16) for Σ_{21} and Σ_{22}. We have

$$\begin{aligned} S(z) = &(U_{21} + U_{23}\mathcal{W}U_{31})(I - ZU_{11})^{-1}Z(U_{12} + U_{13}\mathcal{W}U_{32}) \\ &+ (\Sigma_{12}(z) - U_{23}(I - \mathcal{W}\Sigma_{22}(z)))(I - \mathcal{W}\Sigma_{22}(z))^{-1}\mathcal{W} \\ &\times (\Sigma_{21}(z) - (I - \Sigma_{22}(z)\mathcal{W})U_{32}) \\ &+ U_{22} + U_{23}\mathcal{W}U_{32}. \end{aligned} \tag{7.40}$$

By (7.16), the first term on the right side of (7.40) can be written as

$$\begin{aligned} ① &= \begin{bmatrix} I & U_{23}\mathcal{W} \end{bmatrix} \begin{bmatrix} U_{21} \\ U_{31} \end{bmatrix} (I - ZU_{11})^{-1} Z \begin{bmatrix} U_{12} & U_{13} \end{bmatrix} \begin{bmatrix} I \\ \mathcal{W}U_{32} \end{bmatrix} \\ &= \begin{bmatrix} I & U_{23}\mathcal{W} \end{bmatrix} \begin{bmatrix} \Sigma_{11}(z) - U_{22} & \Sigma_{12}(z) - U_{23} \\ \Sigma_{21}(z) - U_{32} & \Sigma_{22}(z) \end{bmatrix} \begin{bmatrix} I \\ \mathcal{W}U_{32} \end{bmatrix}, \end{aligned}$$

whereas the second term expands to

$$\begin{aligned} ② = &\Sigma_{12}(z)(I - \mathcal{W}\Sigma_{22}(z))^{-1}\mathcal{W}\Sigma_{21}(z) - U_{23}\mathcal{W}\Sigma_{21}(z) \\ &- \Sigma_{12}(z)\mathcal{W}U_{32} + U_{23}\mathcal{W}(I - \Sigma_{22}(z)\mathcal{W})U_{32}. \end{aligned}$$

Now it follows from (7.40) that

$$\begin{aligned} S(z) &= ① + ② + U_{22} + U_{23}\mathcal{W}U_{32} \\ &= \Sigma_{11}(z) + \Sigma_{12}(z)(I - \mathcal{W}\Sigma_{22}(z))^{-1}\mathcal{W}\Sigma_{21}(z), \end{aligned}$$

which is the same as (7.21).

We next verify that the connecting matrix $\mathbf{U}$ (7.35) is unitary. To this end, we pick up an arbitrary vector $g = x_0 \oplus x' \oplus u \in \mathcal{X}_0 \oplus \mathcal{X}' \oplus \mathcal{U}$ and compute

$$\mathbf{U}g = \mathbf{U}\begin{bmatrix} x_0 \\ x' \\ u \end{bmatrix} = \begin{bmatrix} (U_{11} + U_{13}D'U_{31})x_0 + U_{13}C'x' + (U_{12} + U_{13}D'U_{32})u \\ B'U_{31}x_0 + A'x' + B'U_{32}u \\ (U_{21} + U_{23}D'U_{31})x_0 + U_{23}C'x_1 + U_{22} + U_{23}D'U_{32})u \end{bmatrix}.$$

Note that by (7.14) and (7.33), the vectors

$$\widetilde{\delta} := U_{31}x_0 + U_{32}u, \tag{7.41}$$

$$\widetilde{\delta}_* := C'x' + D'\widetilde{\delta} = C'x' + D'(U_{31}x_0 + U_{32}u) \tag{7.42}$$

belong to the spaces $\widetilde{\Delta}$ and $\widetilde{\Delta}_*$, respectively. Making use of these vectors, one can write the formula for $\mathbf{U}g$ more compactly as

$$\mathbf{U}g = \mathbf{U}\begin{bmatrix} x_0 \\ x' \\ u \end{bmatrix} = \begin{bmatrix} U_{11}x_0 + U_{12}u + U_{13}\widetilde{\delta}_* \\ A'x' + B'\widetilde{\delta} \\ U_{21}x_0 + U_{22}u + U_{23}\widetilde{\delta}_* \end{bmatrix}. \tag{7.43}$$

We next observe from (7.14), (7.33) and (7.41), (7.42) that

$$\mathbf{U}_0\begin{bmatrix} x_0 \\ u \\ \widetilde{\delta}_* \end{bmatrix} = \begin{bmatrix} U_{11}x_0 + U_{12}u + U_{13}\widetilde{\delta}_* \\ U_{21}x_0 + U_{22}u + U_{23}\widetilde{\delta}_* \\ \widetilde{\delta} \end{bmatrix}, \qquad \mathbf{U}'\begin{bmatrix} x' \\ \widetilde{\delta} \end{bmatrix} = \begin{bmatrix} A'x' + B'\widetilde{\delta} \\ \widetilde{\delta}_* \end{bmatrix}. \tag{7.44}$$

We see from (7.43), (7.44) that

$$\|\mathbf{U}g\|^2 = \left\|\mathbf{U}_0\left[\begin{smallmatrix} x_0 \\ u \\ \widetilde{\delta}_* \end{smallmatrix}\right]\right\|^2 + \left\|\mathbf{U}'\left[\begin{smallmatrix} x' \\ \widetilde{\delta} \end{smallmatrix}\right]\right\|^2 - \|\widetilde{\delta}\|^2 - \|\widetilde{\delta}_*\|^2.$$

Since $\mathbf{U}_0$ and $\mathbf{U}'$ are unitary (and in particular, isometric), it follows from that latter equality that

$$\|\mathbf{U}g\|^2 = \|x_0\|^2 + \|u\|^2 + \|\widetilde{\delta}_*\|^2 + \|x'\|^2 + \|\widetilde{\delta}\|^2 - \|\widetilde{\delta}\|^2 - \|\widetilde{\delta}_*\|^2 = \|g\|^2,$$

and since g is arbitrary, we conclude that $\mathbf{U}$ is isometric. Applying similar arguments to $\mathbf{U}^*$ and an arbitrary vector $\widetilde{g} \in \mathcal{X}_0^d \oplus \mathcal{X}'^d \oplus \mathcal{Y}$, one can show that $\|\mathbf{U}^*\widetilde{g}\| = \|\widetilde{g}\|$ so that $\mathbf{U}^*$ is also isometric and hence $\mathbf{U}$ is unitary.

Our next step is to show that $\mathbf{U}$ is a colligation extension of the isometry $\mathbf{V}$ defined in (7.6). Note that by the definitions (7.13), (7.6) and the decomposition (7.14),

$$\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \\ U_{31} & U_{32} \end{bmatrix}\begin{bmatrix} \mathfrak{T}x \\ Nx \end{bmatrix} = \begin{bmatrix} Tx \\ Ex \\ 0 \end{bmatrix}. \tag{7.45}$$

Now we pick up an arbitrary vector $x \in \mathcal{X}$ and apply (7.43) to $x_0 = \mathfrak{T}x$, $x' = 0$ and $u = Nx$. Due to (7.45), for the present choice of x_0 and u, the formulas (7.41), (7.42) give $\widetilde{\delta} = 0$ and $\widetilde{\delta}_* = 0$. Then we get from (7.43)

$$\mathbf{U}\begin{bmatrix} \mathfrak{T}x \\ 0 \\ Nx \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ 0 & 0 \\ U_{21} & U_{22} \end{bmatrix}\begin{bmatrix} \mathfrak{T}x \\ Nx \end{bmatrix} = \begin{bmatrix} Tx \\ 0 \\ Ex \end{bmatrix},$$

where the last equality follows from (7.45) and tells us that $\mathbf{U}$ is a unitary colligation extension of the isometry $\mathbf{V}$ defined in (7.6). By Theorem 7.2, the pair (F_0, S)

defined in (7.34) (or, which is the same, defined in (7.21), (7.22)) is a solution of the **AIP**, which completes the proof of the "if" part of the theorem. □

Remark 7.4 We are now in position to complete the proof of Theorem 7.2. Indeed, if $(F_0(z), S(z))$ is a solution of the **AIP**, then Theorem 7.3 tells us that there is a Schur-class multiplier $\mathcal{W} \in \mathcal{S}_{\text{nc},d}(\widetilde{\Delta}, \widetilde{\Delta}_*)$ so that S and F_0 are of the form (7.21), (7.22). Then, as we have seen in the proof of the "if" part of Theorem 7.3 S and F_0 can be realized as in (7.34), that is, S is the characteristic formal power series of a unitary colligation $\mathbf{U}$ that extends the isometry $\mathbf{V}$, whereas $F_0(z)$ is the restriction of the associated observability operator

$$x \mapsto \mathcal{P}_{\mathcal{Y}}\mathbf{U}(I - P^*_{\mathcal{X}_0\oplus\mathcal{X}'}Z(z)P_{(\mathcal{X}_0\oplus\mathcal{X}')^d}\mathbf{U})^{-1}|_{\mathcal{X}_0\oplus\mathcal{X}_1}$$

to $\mathcal{X}_0$. Thus every solution of the **AIP** arises from the procedure given in the statement of Theorem 7.2.

Remark 7.5 The colligation $\mathbf{U}$ explicitly constructed from colligations $\mathbf{U}_0$ and $\mathbf{U}'$ via formulas (7.35) is the result of *feedback connection* of $\mathbf{U}'$ with $\mathbf{U}_0$. Less explicitly (but, perhaps, more suggestively) the feedback connection (or coupling) $\mathbf{U}$ of two colligations $\mathbf{U}'$ and $\mathbf{U}_0$ is defined by its action on vectors in $\mathcal{X}_0 \oplus \mathcal{X}' \oplus \mathcal{U}$ as follows:

$$\mathbf{U}\colon \begin{bmatrix} x_0 \\ x' \\ u \end{bmatrix} \mapsto \begin{bmatrix} \widetilde{x}_0 \\ \widetilde{x}' \\ y \end{bmatrix}$$

if and only if (compare with (7.43), (7.44)) there exist $\widetilde{\delta} \in \widetilde{\Delta}$, $\widetilde{\delta}_* \in \widetilde{\Delta}_*$ so that

$$\mathbf{U}_0\colon \begin{bmatrix} x_0 \\ u \\ \widetilde{\delta}_* \end{bmatrix} \mapsto \begin{bmatrix} \widetilde{x}_0 \\ y \\ \widetilde{\delta} \end{bmatrix} \quad \text{and} \quad \mathbf{U}'\colon \begin{bmatrix} x' \\ \widetilde{\delta} \end{bmatrix} \mapsto \begin{bmatrix} \widetilde{x}' \\ \widetilde{\delta}_* \end{bmatrix}.$$

Corollary 7.6 *Suppose that* $\{\mathbf{T}, E, N\}$ *is an admissible data set for a problem* $\mathbf{OAP}(\mathbf{T}, E, N)$ *and we set*

$$P = \mathcal{O}^*_{E,\mathbf{T}}\mathcal{O}_{E,\mathbf{T}} - \mathcal{O}^*_{N,\mathbf{T}}\mathcal{O}_{N,\mathbf{T}}.$$

Then P *is the minimal solution of the Stein equation* (3.30)*, i.e., if* $\widetilde{P}$ *is a solution of* (3.30) *with* $\widetilde{P} \preceq P$*, then* $\widetilde{P} = P$.

Proof Let $\widetilde{P} \succeq 0$ satisfy (3.30) and let us assume that

$$\widetilde{P} \preceq P := \mathcal{O}^*_{E,\mathbf{T}}\mathcal{O}_{E,\mathbf{T}} - \mathcal{O}^*_{N,\mathbf{T}}\mathcal{O}_{N,\mathbf{T}}.$$

Then any solution S of the problem **aAIP**$(\mathbf{T}, E, N, \widetilde{P})$ is also a solution of the problem **aAIP**$(\mathbf{T}, E, N, P)$. In other words, for every $x \in \mathcal{X}$, the power series $F^S x$ belongs to $\mathcal{H}(K_S)$ and

$$\|F^S x\|^2_{\mathcal{H}(K_S)} \leq \langle \widetilde{P} x, x \rangle_{\mathcal{X}} \leq \langle P x, x \rangle_{\mathcal{X}}.$$

But by Theorem 5.2, $\|F^S x\|^2_{\mathcal{H}(K_S)} = \langle P x, x \rangle_{\mathcal{X}}$, and hence, $P = \widetilde{P}$. □

Corollary 7.7 *For any* **aAIP**-*admissible data set* $\{\mathbf{T}, E, N, P\}$*, the problem* **aAIP** *has solutions.*

Proof We have already observed that the **aAIP** is a special form of the **AIP**. Hence the result of Theorem 2.5 that any admissible problem of the type **AIP** has solutions implies the same for **aAIP**. □

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References

1. J. Agler and J.E. McCarthy, *Pick interpolation for free holomorphic functions*, Amer. J. Math. **137** (6) (2015), 241–285.
2. A. Arias and G. Popescu, *Noncommutative interpolation and Poisson transforms*, Israel J. Math. **115** (2000), 205–234.
3. N. Aronszajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc., **68** (1950), 337–404.
4. D.Z. Arov and L.Z. Grossman, *Scattering matrices in the theory of unitary extensions of isometric operators*, Soviet Math. Dokl. **270** (1983), 17–20.
5. D.Z. Arov and L.Z. Grossman, *Scattering matrices in the theory of unitary extensions of isometric operators*, Math. Nachr. **157** (1992), 105–123.
6. J.A. Ball and V. Bolotnikov, *Interpolation problems for Schur multipliers on the Drury-Arveson space: from Nevanlinna-Pick to abstract interpolation problem*, Integral Equations Operator Theory **62** (2008), no. 3, 301–349.
7. J.A. Ball and V. Bolotnikov, *Nevanlinna-Pick interpolation for Schur-Agler-class functions on domains with matrix polynomial defining function in* C^n, New York J. Math. **11** (2005), 1–44.
8. J.A. Ball and V. Bolotnikov, *Interpolation in the noncommutative Schur-Agler class*, J. Operator Theory **58** (2007) no. 1, 83–126.
9. J.A. Ball and V. Bolotnikov, *Hardy-space function theory, operator model theory, and dissipative linear systems: the multivariable, free-noncommutative, weighted Bergman-space setting*, available at arXiv 1906.02814.
10. J.A. Ball, V. Bolotnikov and Q. Fang, *Multivariable backward-shift invariant subspaces and observability operators*, Multidimens. Syst. Signal Process. **18** (2007), no. 4, 191–248.
11. J.A. Ball, V. Bolotnikov, and Q. Fang, *Schur-class multipliers on the Fock space: de Branges-Rovnyak reproducing kernel spaces and transfer-function realizations*, in: Operator Theory, Structured Matrices, and Dilations: Tiberiu Constantinescu Memorial Volume (Ed. M. Bakonyi, A. Gheondea, M. Putinar, and J. Rovnyak), pp. 85–114, Theta Series in Advances Mathematics, Theta, Bucharest, 2007.

12. J.A. Ball, G. Marx, and V. Vinnikov, *Noncommutative reproducing kernel Hilbert spaces*, J. Func. Anal. **271** (2016), 1844–1920.
13. J.A. Ball, G. Marx, and V. Vinnikov, *Interpolation and transfer-function realization for the noncommutative Schur-Agler class*, in: *Operator Theory in Different Settings and Related Applications* (ed. R. Duduchava, M.A. Kaashoek, N. Vasilevski, V. Vinnikov) pp. 23–116, Oper. Theory Adv. Appl. **262**, Birkhäuser/Springer, Cham, 2018.
14. J.A. Ball, T.T. Trent and V. Vinnikov, *Interpolation and commutant lifting for multipliers on reproducing kernels Hilbert spaces*, in: Operator Theory and Analysis (Ed. H. Bart, I. Gohberg and A.C.M. Ran), Oper. Theory Adv. Appl. **122**, Birkhäuser, Basel, 2001, pp. 89–138.
15. J.A. Ball and V. Vinnikov, *Formal reproducing kernel Hilbert spaces: the commutative and noncommutative settings*, in: Reproducing Kernel Spaces and Applications (Ed. D. Alpay), pp. 77–134, **OT 143**, Birkhäuser, Basel, 2003.
16. L. de Branges and J. Rovnyak, *S*quare summable power series, Holt, Rinehart and Winston, New-York, 1966.
17. K.R. Davidson and D. R. Pitts, *Invariant subspaces and hyper-reflexivity for free semigroup algebras*, Proc. London Math. Soc. **78** (1999) no. 2, 401–430.
18. M. A. Dritschel and J. Rovnyak, *Extension theorems for contraction operators on Kreĭn spaces*, in: *Extension and interpolation of linear operators and matrix functions*, pp. 221–305, Oper. Theory Adv. Appl., **47**, Birkhäuser, Basel, 1990.
19. Y. P. Ginzburg, *On J-contractive operator functions*, Dokl. Akad. Nauk SSSR **117** (1957), 171–173.
20. V. Katsnelson, A. Kheifets and P. Yuditskii, *An abstract interpolation problem and extension theory of isometric operators*, in: *Operators in Spaces of Functions and Problems in Function Theory* (V.A. Marchenko, ed.), **146**, Naukova Dumka, Kiev, 1987, pp. 83–96. English transl. in: *Topics in Interpolation Theory* (H. Dym, B. Fritzsche, V. Katsnelson and B. Kirstein, eds.), Oper. Theory Adv. Appl. **95**, Birkhäuser, Basel, 1997, pp. 283–298.
21. A. Kheifets, *The Parseval equality in an abstract problem of interpolation, and the union of open systems*, Teor. Funktsii Funktsional. Anal. i Prilozhen., **49**, 1988, 112–120, and **50**, 1988, 98–103. English transl. in: *J. Soviet Math.*, **49(4)**, 1990, 114–1120 and **49(6)**, 1990, 1307–1310.
22. A. Kheifets *Scattering matrices and Parseval Equality in Abstract Interpolation Problem*, Ph.D. Thesis, 1990, Kharkov State University.
23. A. Kheifets, *The abstract interpolation problem and applications*, in: *Holomorphic spaces* (Ed. D. Sarason, S. Axler, J. McCarthy), pp. 351–379, Cambridge Univ. Press, Cambridge, 1998.
24. A. Kheifets and P. Yuditskii, *An analysis and extension of V.P. Potapov's approach to interpolation problems with applications to the generalized bitangential Schur-Nevanlinna-Pick problem and j-inner-outer factorization*, in: *Matrix and Operator Valued Functions* (Ed. I. Gohberg and L.A. Sakhnovich), Oper. Theory Adv. Appl. **72**, Birkhäuser, Basel, 1994, pp. 133–161.
25. R.B. Leech, *Factorization of analytic functions and operator inequalities*, Integral Equations and Operator Theory **78** (2014) no. 1, 71–73.
26. S. McCullough and T.T. Trent, *Invariant subspaces and Nevanlinna-Pick kernels*, J. Funct. Anal. **178** (2000), no. 1, 226–249.
27. G. Popescu, *Characteristic functions for infinite sequences of noncommuting operators*, J. Operator Theory **22** (1989), no. 1, 51–71.
28. G. Popescu, *On intertwining dilations for sequences of noncommuting operators*, J. Math. Anal. Appl. **167** (1992) no. 2, 382–402.
29. G. Popescu, *Multivariable Nehari problem and interpolation*, J. Funct. Anal. **200** (2003), no. 2, 536–581.
30. G. Popescu, *Entropy and multivariable interpolation*, Mem. Amer. Math. Soc. **184** (2006), no. 868.
31. G. Popescu, *Free Holomorphic functions on the unit ball of $B(\mathcal{H})^n$*, J. Funct. Anal. **241** (2006), 268–333.
32. G. Popescu, *Free Holomorphic functions on the unit ball of $B(\mathcal{H})^n$ II*, J. Funct. Anal. **258** (2010), 1513–1578.

33. V. P. Potapov, *The multiplicative structure of J-contractive matrix functions*, Amer. Math. Soc. Transl. **15** (1960) 131–243.
34. G. Salomon, O.M. Shalit, and E. Shamovich, *Algebras of bounded noncommutative analytic functions on subvarieties of the noncommutative unit ball*, Trans. Amer. Math. Soc. **370** no. 12 (2018), 8639–8690.
35. D. Sarason, *Generalized interpolation in H^∞*, Trans. Amer. Math. Soc. **127** (1967), 179–203.

Regular Extensions and Defect Functions of Contractive Measurable Operator-Valued Functions

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To V.E. Katsnelson with the deepest respect for his mathematical achievements, on the 75th birthday

Abstract The present paper is devoted to the exposition of a geometrical approach to the study of regular extensions within the class of contractive measurable operator functions on the unit circle for an arbitrary function from this class. The proposed approach is based on the methods of the theory of unitary couplings and regular factorizations of their scattering suboperators. The defect functions in the Schur class for a contractive measurable operator function are introduced. Using these concepts and considering the analogous problem within the Schur class of operator functions on the unit disk as a particular case, it is possible to obtain a description of the set of all regular extensions for a Schur operator function. It is done in terms of orthogonal internal unilateral input and output channels of the corresponding open system.

Keywords Contraction · Unitary coupling · Scattering suboperator · Open system · Internal channel · External channel · Regular extension · Defect function

Mathematics Subject Classification (2000) Primary: 47A48, 47A40, 47A56; Secondary: 47A20

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1 Introduction

Let $\mathfrak{G}$ and $\mathfrak{F}$ be Hilbert spaces (all Hilbert spaces considered in this paper are assumed to be complex and separable). By $[\mathfrak{G}, \mathfrak{F}]$ we denote the Banach space of bounded linear operators defined on $\mathfrak{G}$ and taking values in $\mathfrak{F}$. If $\mathfrak{F} = \mathfrak{G}$, we use the notation $[\mathfrak{G}] := [\mathfrak{G}, \mathfrak{G}]$.

Let $\mathbb{D} := \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ and $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. By $L^\infty[\mathfrak{G}, \mathfrak{F}]$ we denote the Banach space of measurable (indifferently in what sense, weak or strong, in view of the separability of the spaces $\mathfrak{G}$ and $\mathfrak{F}$) $[\mathfrak{G}, \mathfrak{F}]$-valued functions $\theta(z)$, $z \in \mathbb{T}$, such that

$$\|\theta\|_{L^\infty[\mathfrak{G},\mathfrak{F}]} := \operatorname*{ess\,sup}_z \|\theta(z)\|_{[\mathfrak{G},\mathfrak{F}]} < \infty.$$

Functions belonging to the closed unit ball

$$CM[\mathfrak{G}, \mathfrak{F}] := \{\theta(z) : \|\theta\|_{L^\infty[\mathfrak{G},\mathfrak{F}]} \le 1\}$$

of the space $L^\infty[\mathfrak{G}, \mathfrak{F}]$ are called *contractive measurable* $[\mathfrak{G}, \mathfrak{F}]$- *valued functions*.

If $H^\infty_+[\mathfrak{G}, \mathfrak{F}]$ is the Hardy space of bounded holomorphic $[\mathfrak{G}, \mathfrak{F}]$-valued functions $\theta(\zeta)$, $\zeta \in \mathbb{D}$, that is, such that

$$\|\theta\|_{H^\infty_+[\mathfrak{G},\mathfrak{F}]} := \sup_\zeta \|\theta(\zeta)\|_{[\mathfrak{G},\mathfrak{F}]} < \infty,$$

then by $L^\infty_+[\mathfrak{G}, \mathfrak{F}]$ we denote the subspace of $L^\infty[\mathfrak{G}, \mathfrak{F}]$ consisting of strong boundary value functions $\theta(z)$ for $\theta(\zeta) \in H^\infty_+[\mathfrak{G}, \mathfrak{F}]$. Moreover, the equality

$$\|\theta(z)\|_{L^\infty_+[\mathfrak{G},\mathfrak{F}]} = \|\theta(\zeta)\|_{H^\infty_+[\mathfrak{G},\mathfrak{F}]}$$

makes it possible to identify the spaces $H^\infty_+[\mathfrak{G}, \mathfrak{F}]$ and $L^\infty_+[\mathfrak{G}, \mathfrak{F}]$ up to the obvious isomorphism. Functions belonging to the closed unit ball

$$S[\mathfrak{G}, \mathfrak{F}] := \{\theta(\zeta) : \|\theta\|_{H^\infty_+[\mathfrak{G},\mathfrak{F}]} \le 1\}$$

of the space $H^\infty_+[\mathfrak{G}, \mathfrak{F}]$ are usually called *Schur* $[\mathfrak{G}, \mathfrak{F}]$-*valued functions*.

The main subjects of our considerations in this paper are the class $CM[\mathfrak{G}, \mathfrak{F}]$ of contractive measurable operator functions and the class $S[\mathfrak{G}, \mathfrak{F}]$ of Schur operator functions considered as its subclass.

The paper is devoted to the study of extensions of functions $\theta(z) \in CM[\mathfrak{G}, \mathfrak{F}]$. Hereinafter by *an extension of* $\theta(z)$ we mean a contractive operator function which can be represented in one of the following block forms:

$$\Omega(z) := \begin{bmatrix} \theta_{12}(z) \\ \theta(z) \end{bmatrix} \in CM[\mathfrak{G}, \mathfrak{F}^{(1)} \oplus \mathfrak{F}], \tag{1.1}$$

$$\Lambda(z) := [\theta_{21}(z), \theta(z)] \in CM[\mathfrak{G}^{(1)} \oplus \mathfrak{G}, \mathfrak{F}], \tag{1.2}$$

$$\Xi(z) := \begin{bmatrix} \theta_{11}(z) & \theta_{12}(z) \\ \theta_{21}(z) & \theta(z) \end{bmatrix} \in CM[\mathfrak{G}^{(1)} \oplus \mathfrak{G}, \mathfrak{F}^{(1)} \oplus \mathfrak{F}]. \tag{1.3}$$

Functions of type (1.1) and (1.2) are called by us *unidirectional upward and leftward extensions of* $\theta(\zeta)$, respectively, and functions of type (1.3) are called *bidirectional up-leftward extensions of it*. Extensions in other possible directions do not differ essentially from those defined above.

The proposed approach is based on the fundamental fact that any function $\theta(z) \in CM[\mathfrak{G}, \mathfrak{F}]$ is the scattering suboperator of some unitary coupling. If this coupling is minimal, then it is determined by $\theta(z)$ uniquely up to unitary equivalence (see [1, 11], and also Definitions 2.1, 2.7, and Theorem 2.9 in this paper). An important role is played by the fact that the study of extensions of forms (1.1)–(1.3) for $\theta(z)$ can be reduced to the study of special factorizations of $\theta(z)$, namely:

$$\theta(z) = [0, I_{\mathfrak{F}}]\Omega(z), \tag{1.4}$$

$$\theta(z) = \Lambda(z) \begin{bmatrix} 0 \\ I_{\mathfrak{G}} \end{bmatrix}, \tag{1.5}$$

$$\theta(z) = [0, I_{\mathfrak{F}}]\Xi(z) \begin{bmatrix} 0 \\ I_{\mathfrak{G}} \end{bmatrix}, \tag{1.6}$$

where $0 \in [\mathfrak{F}^{(1)}, \mathfrak{F}]$ in the block matrix $[0, I_{\mathfrak{F}}]$ and $0 \in [\mathfrak{G}, \mathfrak{G}^{(1)}]$ in the block matrix $\begin{bmatrix} 0 \\ I_{\mathfrak{G}} \end{bmatrix}$.

The concept of a unitary coupling was introduced in [1]. The notion of the product of unitary couplings was defined in [8] (see also [11] and Definition 2.1 in this paper). This enabled us to prove the statement that the scattering suboperator of the product of unitary couplings is equal to the product of their scattering suboperators (see [11] and also Theorem 2.18 in this paper). In turn, this multiplication theorem made it possible to generalize the main facts of the theory of unitary colligations (see [19]) to unitary couplings (see [11]) and to construct the geometrical theory of factorizations of functions $\theta(z) \in CM[\mathfrak{G}, \mathfrak{F}]$ (see [11–13]). Important in this theory are the notions of unilateral and bilateral channels of a unitary coupling introduced in [12] (see also Sect. 2.2 of this paper). In particular, those channels that generate factorizations of its scattering suboperator. The apparatus elaborated in these papers is the basic one in the proposed approach to the study of extensions of functions $\theta(z) \in CM[\mathfrak{G}, \mathfrak{F}]$.

In the theory of unitary couplings, as well as in the theory of unitary colligations (see [19]), a special place is occupied by regular factorizations of contractive operator functions (see [11], and also Sect. 2.3 of this paper). In this connection, in the paper the main attention is paid to the description of extensions of form (1.1)–

(1.3), for which factorizations of form (1.4)–(1.6) is regular. Such extensions $\Omega(z)$, $\Lambda(z)$ and $\Xi(z)$ of a function $\theta(z) \in CM[\mathfrak{G}, \mathfrak{F}]$ we also call regular. By similar methods one can describe arbitrary extensions, but this is beyond the scope of this work (see Remark 5.17).

The study of extensions of a contractive operator function was initiated by the following two problems which were interconnected.

Problem 1 Scattering through internal channels of an open system (of a scattering system).

Each function $\theta(\zeta) \in S[\mathfrak{G}, \mathfrak{F}]$ is *the transfer function* of some open system (see, e.g., [5, 27], §1 and also [14], Section 6). The fundamental operator of such a system is a contraction $T \in [\mathfrak{H}]$ where $\mathfrak{H}$ is the internal Hilbert space of the system. The function $\theta(\zeta)$ is often termed as *the characteristic operator function of the contraction* T (see, e.g., [19, 29]). If a contraction T is completely nonunitary (the basic definitions related to contractions can be found, e.g., in Sect. 6 of this paper), then its characteristic function $\theta(\zeta)$ determines T up to unitary equivalence.

Internal unilateral channels of an open system are closely related to *unilateral shifts and coshifts* (*backward shifts*) contained in the corresponding completely nonunitary contraction T. Internal channels can be partially brought out into the external spaces (see [14], Section 6). In this case, the internal unilateral channel generated by a coshift (shift) passes into the external input (output) space of the open system. This leads to some new open system whose transfer function takes the form

$$\begin{bmatrix} \theta_{12}(\zeta) \\ \theta(\zeta) \end{bmatrix} \in S[\mathfrak{G}, \mathfrak{F}^{(1)} \oplus \mathfrak{F}] \quad ([\theta_{12}(\zeta), \theta(\zeta)] \in S[\mathfrak{G}^{(1)} \oplus \mathfrak{G}, \mathfrak{F}]). \tag{1.7}$$

Here $\mathfrak{F}^{(1)}$ ($\mathfrak{G}^{(1)}$) is a Hilbert space related to the corresponding internal unilateral input (output) channel and the function $\theta_{12}(\zeta) \in S[\mathfrak{G}, \mathfrak{F}^{(1)}]$ ($\theta_{21}(\zeta) \in S[\mathfrak{G}^{(1)}, \mathfrak{F}]$) describes the scattering generated by this channel.

If the internal unilateral channel generated by the largest coshift (shift) is brought out into the external input (output) space of an open system with transfer function $\theta(\zeta) \in S[\mathfrak{G}, \mathfrak{F}]$, then in (1.7) we obtain

$$\theta_{12}(\zeta) = \varphi(\zeta) \quad (\theta_{21}(\zeta) = \psi(\zeta)),$$

where $\varphi(\zeta)$ ($\psi(\zeta)$) is so called *defect ($\star$-defect) function of* $\theta(\zeta)$. Here the essence of the defectiveness of the function $\theta(\zeta)$ is that there is an internal unilateral input (output) scattering channel which is not controlled by the transfer function $\theta(\zeta)$. This means that $\theta(\zeta)$ admits an extension of form (1.7) to another transfer function where $\theta(\zeta)$ is already its block controlling some part of the scattering of the new open system. Note that ideas about considering internal channels play an important role in [5–7, 9, 14–16, 23–25].

The holomorphy of a function $\theta(\zeta) \in S[\mathfrak{G}, \mathfrak{F}]$ corresponds to the orthogonality of *the external unilateral input and output scattering channels* (see, e.g. [4], §1 and also [11], Section 6). At the same time, an important point in the theory of Schur functions (and contractions) is that the subspaces of the internal space of the corresponding system, on which the largest internal shift and coshift act, are not necessarily orthogonal. In this case, the simultaneous bringing out the largest internal unilateral channels of the system into the external spaces leads to the bidirectional extension of the form

$$\begin{bmatrix} \chi(z) & \varphi(z) \\ \psi(z) & \theta(z) \end{bmatrix} \in CM[\mathfrak{K}_* \oplus \mathfrak{G}, \mathfrak{K} \oplus \mathfrak{F}] \tag{1.8}$$

for the boundary value function $\theta(z)$ of the function $\theta(\zeta)$. Here $\mathfrak{K}_*$ and $\mathfrak{K}$ are Hilbert spaces related to the largest internal unilateral output and input channels of the system, respectively, and, generally speaking, $\chi(z) \notin L^\infty_+[\mathfrak{K}_*, \mathfrak{K}]$. Thus, in the general case, the extension of form (1.8) is no longer the boundary value function of the transfer function of some open system. This forces us to use a more general interpretation. The function $\chi(z) \in CM[\mathfrak{K}_*, \mathfrak{K}]$ is now the scattering suboperator of the minimal unitary coupling generated by the largest internal unilateral input and output channels of the system. Therefore, it is called *the suboperator of internal scattering of the system*. It plays an important role in the scattering theory with losses (see, e.g., [5–7, 9, 14, 17, 18]).

Note that the methods of the present paper were elaborated mainly for the study of the function $\chi(z)$. The corresponding results were published without proof in [9, 10]. In the papers [17, 18] they were obtained by other methods. A number of open questions are connected with the scattering suboperator $\chi(z)$ (for example, a description of the corresponding class). We think that proposed methods will help to answer them.

If one passes to the strong boundary value functions in (1.7) setting $z := \zeta \in \mathbb{T}$, then extensions of form (1.1) and (1.2) for the boundary value function $\theta(z)$ of the function $\theta(\zeta) \in S[\mathfrak{G}, \mathfrak{F}]$ are obtained. These extensions are regular, as well as the extension of form (1.8) is. In this paper we generalize the proposed methods to the non-holomorphic case when $\theta(z) \in CM[\mathfrak{G}, \mathfrak{F}]$. The presence of the block $\chi(z) \notin L^\infty_+[\mathfrak{K}_*, \mathfrak{K}]$ in (1.8) makes these generalizations natural.

Problem 2 Completion of the Schur operator function to a two-sided inner function (the Darlington synthesis problem).

This problem consists in clarifying conditions under which the given Schur operator function $\theta(\zeta) \in S[\mathfrak{G}, \mathfrak{F}]$ can be a block of some two-sided inner function of the form

$$I(\zeta) := \begin{bmatrix} \theta_{11}(\zeta) & \theta_{12}(\zeta) \\ \theta_{21}(\zeta) & \theta(\zeta) \end{bmatrix} \in S[\mathfrak{G}^{(1)} \oplus \mathfrak{G}, \mathfrak{F}^{(1)} \oplus \mathfrak{F}], \tag{1.9}$$

and, in the case of the existence of $I(\zeta)$, in describing the set of all such functions $I(\zeta)$. This problem arose as a synthesis problem in the theory of electrical circuits (see, e.g., [20]). As shown in the papers [2, 3, 5, 21, 22, 26], this problem is closely related to the problem of the pseudocontinuation of the function $\theta(\zeta)$ across the unit circle $\mathbb{T}$.

If one passes to the strong boundary value functions in (1.9), setting $z := \zeta \in \mathbb{T}$, then, as shown in [5], §1, there is a direct connection between extensions (1.8) and (1.9). It is clear that the extension obtained in this way is a particular case of extensions of type (1.3). From the above mentioned connection between extensions (1.8) and (1.9) it follows that the existence of two-sided inner extensions of type (1.9) is associated with the special geometrical conditions on certain subspaces of the internal space of the corresponding open system. Namely, these are the subspaces on which the largest internal unilateral shift and coshift act.

Thus, the solvability of Problem 2 is closely related to the special conditions on the largest internal unilateral channels of the corresponding open system, what is the subject of Problem 1. For more details on this interrelations, see [5, 16, 25].

In Sect. 2 we adduce the results from [11–13] which are necessary for what follows.

In Sect. 3 the concepts of unidirectional and bidirectional regular extensions of contractive operator functions are introduced. Here we also define, as their particular case, completely regular extensions.

Descriptions of the sets of regular extensions for a contractive operator function are obtained in Sect. 4 by a parametrization in terms of isometric and coisometric operator functions. These results are proved in Theorem 4.2 (the unidirectional case) and Theorem 4.5 (the bidirectional case).

In Sect. 5, introducing the comparison relation, we study the sets of regular extensions for a contractive operator function as a partially preordered sets. The subsets of maximal extensions are described in Theorems 5.5 and 5.13 for the unidirectional and bidirectional cases, respectively. In Sect. 5.3 some extremal properties of the norm of regular extensions are considered.

The definitions and the results of Sect. 6 are key to the subsequent exposition. From now on we study unidirectional regular extensions of form (1.1) and (1.2) with $\theta_{12}(z) \in L_+^\infty[\mathfrak{G}, \mathfrak{F}^{(1)}]$ ($\theta_{21}(z) \in L_+^\infty[\mathfrak{G}^{(1)}, \mathfrak{F}]$). For this, generalizing the similar notion in the theory of unitary colligations, we introduce for a unitary coupling the notion of the fundamental contraction T (Definition 6.4). In turn, this makes it possible to define the notions of internal and external unilateral channels of the coupling.

In Theorem 6.12 we give descriptions of the sets of regular extensions considered here in terms of unilateral shifts and coshifts contained in the fundamental contraction T. This enables us to introduce the important concept of the defect ($\star$-defect) function $\varphi(\zeta) \in S[\mathfrak{G}, \mathfrak{K}]$ ($\psi(\zeta) \in S[\mathfrak{K}_*, \mathfrak{F}]$) in the Schur class for a function $\theta(z) \in CM[\mathfrak{G}, \mathfrak{F}]$ as the function generating the extension of form (1.1) and (1.2) that corresponds to the largest internal coshift (shift) (Definition 6.13).

In Sect. 7.1, using the concept of the defect functions, we obtain other descriptions of the sets of regular extensions, considered in Theorem 6.12, by a parametrization in terms of inner and $\star$-inner operator functions (Theorem 7.4). In Sect. 7.2 a refined comparison relation is defined on the sets of regular extensions introduced in Sect. 6 which turns them into complete lattices (Lemma 7.8). In Sect. 7.3 we prove that the defect and $\star$-defect functions in the Schur class for a function $\theta(z) \in CM[\mathfrak{G}, \mathfrak{F}]$ coincide with the largest minorant for its defect function $\Pi(z) := (I - \theta^*(z)\theta(z))^{1/2} \in CM[\mathfrak{G}]$ and with the largest $\star$-minorant for its $\star$-defect function $\Sigma(z) := (I - \theta(z)\theta^*(z))^{1/2} \in CM[\mathfrak{F}]$ in the class of contractive measurable operator functions, respectively.

Section 8 is devoted to the study of bidirectional regular extensions of form (1.3), where $\theta_{12}(z) \in L^\infty_+[\mathfrak{G}, \mathfrak{F}^{(1)}]$ and $\theta_{21}(z) \in L^\infty_+[\mathfrak{G}^{(1)}, \mathfrak{F}]$ are viewed as those that are generated by pairs of internal unilateral shifts and coshifts of corresponding minimal unitary coupling. In the case of the largest among such shifts and coshifts, it leads us to the regular extension of form (1.8), where the function $\chi(z) \in CM[\mathfrak{K}_*, \mathfrak{K}]$, mentioned above in the particular case of an open system, is now the suboperator of internal scattering of the coupling. As a corollary of Theorem 7.4, we obtain a description of the set of considered regular extensions by a parametrization in terms of inner and $\star$-inner operator functions (Theorem 8.4). In Sect. 8.2, introducing the refined comparison relation, we turn the set of regular extensions considered here in a complete lattice. The conditions on $\theta(z)$ under which $\chi(z) \in L^\infty_+[\mathfrak{K}_*, \mathfrak{K}]$ are discussed in Sect. 8.3. Requiring one more additional condition $\theta_{11}(z) \in L^\infty_+[\mathfrak{G}^{(1)}, \mathfrak{F}^{(1)}]$ for bidirectional regular extensions $\Xi(z)$ of form (1.3), we give a description of the subset of maximal extensions in the considered set (see Sect. 8.4). It makes possible to obtain the description of the set of all bidirectional regular extensions of form (1.3) for a function $\theta(z) \in CM[\mathfrak{G}, \mathfrak{F}]$, where $\theta_{12}(z)$, $\theta_{21}(z)$ and $\theta_{11}(z)$ are the boundary value functions of the operator functions of the Schur class (Theorem 8.22). In the case of $\theta(\zeta) \in S[\mathfrak{G}, \mathfrak{F}]$, this result can be considered as a description of all regular extensions of $\theta(\zeta)$ within the Schur classe.

2 Preliminaries

In this section we give information on unitary couplings and regular factorizations of contractive operator-valued functions that is necessary for what follows. A detailed exposition can be found in [11–13].

2.1 Unitary Couplings and Scattering Suboperators

Let U be a unitary operator acting on a Hilbert space $\mathfrak{H}$. A subspace $\mathfrak{N}$ in $\mathfrak{H}$ is called *wandering* with respect to U if $U^n\mathfrak{N} \perp U^m\mathfrak{N}$ for $n \neq m$ $(n, m = 0, \pm 1, \pm 2, \ldots)$.

Definition 2.1 A six-tuple

$$\sigma := (\mathfrak{H}, \mathfrak{F}, \mathfrak{G}; U, V_{\mathfrak{F}}, V_{\mathfrak{G}}) \tag{2.1}$$

is called a unitary coupling or simply a coupling if

(a) $\mathfrak{H}$, $\mathfrak{F}$, $\mathfrak{G}$ are Hilbert spaces;
(b) $U : \mathfrak{H} \to \mathfrak{H}$ is a unitary operator;
(c) $V_{\mathfrak{F}} : \mathfrak{F} \to \mathfrak{H}$, $V_{\mathfrak{G}} : \mathfrak{G} \to \mathfrak{H}$ are isometric operators, i.e.,

$$V_{\mathfrak{F}}^* V_{\mathfrak{F}} = I_{\mathfrak{F}}, \quad V_{\mathfrak{G}}^* V_{\mathfrak{G}} = I_{\mathfrak{G}};$$

(d) the subspaces $\overset{\circ}{\mathfrak{F}} := \operatorname{Ran} V_{\mathfrak{F}}$ and $\overset{\circ}{\mathfrak{G}} := \operatorname{Ran} V_{\mathfrak{G}}$ are wandering with respect to U.

The subspaces $\mathfrak{F}$ and $\mathfrak{G}$ are said to be *the input and output channelled subspaces of the unitary coupling* σ, respectively. The operator U is called *connecting* and the isometries $V_{\mathfrak{F}}$ and $V_{\mathfrak{G}}$ are termed *the embedding operators of the unitary coupling* σ.

Remark 2.2 In this paper $\overset{\circ}{\mathfrak{N}}$ will always denote $\operatorname{Ran} V$ for any isometry $V : \mathfrak{N} \to \mathfrak{H}$.

It should be noted that only in form Definition 2.1 differs from the definition of a unitary coupling for simple semi-unitary operators given in [1] and [2].

Any subspace $\mathfrak{N}$ wandering with respect to U generates subspaces

$$M(\mathfrak{N}) := \bigoplus_{-\infty}^{\infty} U^k \mathfrak{N}, \ \mathfrak{R}_{\mathfrak{N}} := \mathfrak{H} \ominus M(\mathfrak{N}),$$

$$M_+(\mathfrak{N}) := \bigoplus_{0}^{\infty} U^k \mathfrak{N}, \quad M_-(\mathfrak{N}) := \bigoplus_{-\infty}^{-1} U^k \mathfrak{N}.$$

We use the symbol

$$\bigvee_{\alpha \in A} \mathfrak{L}_\alpha ,$$

where $\mathfrak{L}_\alpha \subset \mathfrak{H}$, $\alpha \in A$, for denoting the smallest (closed) subspace of $\mathfrak{H}$ that contains all $\mathfrak{L}_\alpha$, $\alpha \in A$.

Definition 2.3 Let σ be a unitary coupling of form (2.1). We mean by the principal part of the coupling σ the coupling

$$\sigma^{(1)} := (\mathfrak{H}^{(1)}, \mathfrak{F}, \mathfrak{G}; U^{(1)}, V_{\mathfrak{F}}, V_{\mathfrak{G}}),$$

where $\mathfrak{H}^{(1)} := M(\overset{\circ}{\mathfrak{F}}) \vee M(\overset{\circ}{\mathfrak{G}})$, $U^{(1)} := U|_{\mathfrak{H}^{(1)}}$. The coupling σ is called minimal if $\sigma = \sigma^{(1)}$ and abundant otherwise.

In this definition and in what follows we retain the notation $V_{\mathfrak{F}}$ ($V_{\mathfrak{G}}$) when $\mathfrak{H}$ is replaced by its subspace that contains $\overset{\circ}{\mathfrak{F}}$ ($\overset{\circ}{\mathfrak{G}}$), or by any space that contains $\mathfrak{H}$.

In the sequel we will use the following simple assertion adduced without proof (here and henceforth $P_{\mathfrak{L}}$ is an orthogonal projection of $\mathfrak{H}$ onto a subspace $\mathfrak{L}$).

Lemma 2.4 ([11]) *A unitary coupling σ of form (2.1) is minimal iff any of the two following equivalent conditions*

$$\mathfrak{N}_{\overset{\circ}{\mathfrak{F}}} = \overline{P_{\mathfrak{N}_{\overset{\circ}{\mathfrak{F}}}} M(\overset{\circ}{\mathfrak{G}})} \quad or \quad \mathfrak{N}_{\overset{\circ}{\mathfrak{G}}} = \overline{P_{\mathfrak{N}_{\overset{\circ}{\mathfrak{G}}}} M(\overset{\circ}{\mathfrak{F}})}$$

is valid.

Definition 2.5 Unitary couplings

$$\sigma = (\mathfrak{H}, \mathfrak{F}, \mathfrak{G}; U, V_{\mathfrak{F}}, V_{\mathfrak{G}}), \quad \sigma' = (\mathfrak{H}', \mathfrak{F}, \mathfrak{G}; U', V'_{\mathfrak{F}}, V'_{\mathfrak{G}}) \tag{2.2}$$

are called unitarily equivalent if there exists a unitary operator $Z : \mathfrak{H} \to \mathfrak{H}'$ such that

$$U'Z = ZU, \quad V'_{\mathfrak{F}} = ZV_{\mathfrak{F}}, \quad V'_{\mathfrak{G}} = ZV_{\mathfrak{G}}. \tag{2.3}$$

We say about Z as an operator which establishes the unitary equivalence of σ and σ'.

Thereby, obviously, an equivalence relation in the set of unitary couplings is introduced. By $[\sigma]$ we denote the equivalence class to which a coupling σ belongs. Clearly, the minimality of a coupling σ persists when passing to a unitarily equivalent coupling σ'. We speak that a class $[\sigma]$ is minimal if it consists of minimal (unitarily equivalent) couplings.

Note that the operator Z which establishes the unitary equivalence of the minimal couplings σ and σ' is determined by conditions (2.3) uniquely (see [11], Section 1).

We recall the definitions of some spaces of vector and operator valued functions that will be used in what follows. A detailed treatment of this subject can be found, e.g., in [28, 29].

Let $\mathfrak{N}$ be a Hilbert space. By $L^2(\mathfrak{N})$ we will denote the Hilbert space of measurable (no matter in the weak or strong sense) functions $h(z), z \in \mathbb{T}$, with values in $\mathfrak{N}$ such that

$$||h||^2_{L^2(\mathfrak{N})} := \frac{1}{2\pi} \int_0^{2\pi} ||h(e^{it})||^2_{\mathfrak{N}} \, dt < \infty,$$

while the inner product in $L^2(\mathfrak{N})$ is defined by

$$\langle h_1, h_2 \rangle_{L^2(\mathfrak{N})} := \frac{1}{2\pi} \int_0^{2\pi} \langle h_1(e^{it}), h_2(e^{it}) \rangle_{\mathfrak{N}} \, dt.$$

It is known that a function $h(z)$ belongs to $L^2(\mathfrak{N})$ iff it admits the representation

$$h(e^{it}) = \sum_{-\infty}^{\infty} e^{ikt} h_k, \quad h_k \in \mathfrak{N} \, (k = 0, \pm 1, \pm 2, \ldots)$$

where the series convergence is understood as the convergence in the space $L^2(\mathfrak{N})$. Moreover, the condition

$$\sum_{-\infty}^{\infty} ||h_k||^2_{\mathfrak{N}} < \infty$$

is satisfied.

By $L^2_+(\mathfrak{N})$ we denote the important subspace of $L^2(\mathfrak{N})$ that consists of functions $h(z) \in L^2(\mathfrak{N})$ admitting the representation

$$h(e^{it}) = \sum_{0}^{\infty} e^{ikt} h_k, \quad h_k \in \mathfrak{N} \quad (k = 0, 1, 2, \ldots) .$$

Such functions can be considered as strong boundary value functions for ones of the Hardy class $H^2_+(\mathfrak{N})$ on the unit disk $\mathbb{D}$. The space $H^2_+(\mathfrak{N})$ is formed by holomorphic in $\mathbb{D}$ functions

$$h(\zeta) = \sum_{0}^{\infty} \zeta^k h_k, \quad h_k \in \mathfrak{N} \quad (k = 0, 1, 2, \ldots) ,$$

such that

$$||h||^2_{H^2_+(\mathfrak{N})} := \sup_{r<1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} ||h\left(re^{it}\right)||^2_{\mathfrak{N}} \, dt \right\} < \infty .$$

Furthermore, $||h(\zeta)||_{H^2_+(\mathfrak{N})} = ||h(z)||_{L^2_+(\mathfrak{N})}$. Denote also by $L^2_-(\mathfrak{N})$ the subspace $L^2(\mathfrak{N}) \ominus L^2_+(\mathfrak{N})$ that consists of the functions $h(z) \in L^2(\mathfrak{N})$ admitting the representation

$$h(e^{it}) = \sum_{-\infty}^{-1} e^{ikt} h_k, \quad h_k \in \mathfrak{N} \quad (k = -1, -2, \ldots) .$$

Let $\mathfrak{F}$ and $\mathfrak{G}$ be Hilbert spaces. Any function $\theta(z)$ from the space $L^\infty[\mathfrak{G}, \mathfrak{F}]$ introduced in Sect. 1 admits the following representation

$$\theta\left(e^{it}\right) = \sum_{-\infty}^{\infty} e^{ikt} \theta_k, \quad \theta_k \in [\mathfrak{G}, \mathfrak{F}] \quad (k = 0, \pm 1, \pm 2, \ldots) ,$$

where for any $g \in \mathfrak{G}$ the series $\sum\limits_{-\infty}^{\infty} e^{ikt}\theta_k g$ converges in the norm of the space $L^2(\mathfrak{F})$ and it is the Fourier expansion of the function $\theta\left(e^{it}\right) g$. Accordingly, any function $\theta(z)$ from the space $L^\infty_+[\mathfrak{G}, \mathfrak{F}]$ (see Sect. 1) admits the representation

$$\theta\left(e^{it}\right) = \sum_0^\infty e^{ikt}\theta_k, \quad \theta_k \in [\mathfrak{G}, \mathfrak{F}] \quad (k = 0, 1, 2, \ldots).$$

Moreover, the corresponding function $\theta(z) \in H^\infty_+[\mathfrak{G}, \mathfrak{F}]$ (see Sect. 1) admits the representation

$$\theta(\zeta) = \sum_0^\infty \zeta^k\theta_k, \quad \zeta \in \mathbb{D},$$

where the convergence of this power series does not depend on the choice of the weak, strong, or norm convergence in the space $H^\infty_+[\mathfrak{G}, \mathfrak{F}]$.

Together with $\theta\left(e^{it}\right) \in L^\infty[\mathfrak{G}, \mathfrak{F}]$ we consider the operator function

$$\theta^\sim\left(e^{it}\right) := \theta^*\left(e^{-it}\right) \in L^\infty[\mathfrak{F}, \mathfrak{G}],$$

which is called *associated* with respect to $\theta\left(e^{it}\right)$. In the case $\theta(\zeta) \in H^\infty_+[\mathfrak{G}, \mathfrak{F}]$ the *associated* function can be defined as $\theta^\sim(\zeta) := \theta^*(\bar{\zeta}) \in H^\infty_+[\mathfrak{F}, \mathfrak{G}]$.

Let $\theta : L^2(\mathfrak{G}) \mapsto L^2(\mathfrak{F})$ be a "multiplication" operator by an operator function $\theta\left(e^{it}\right) \in L^\infty[\mathfrak{G}, \mathfrak{F}]$. Hereafter the function $\theta\left(e^{it}\right)$ will be called *the suboperator* of the operator θ.

We call an operator function $\theta\left(e^{it}\right) \in L^\infty[\mathfrak{G}, \mathfrak{F}]$ *isometric (coisometric)* if the equality $\theta^*\left(e^{it}\right)\theta\left(e^{it}\right) = I_{\mathfrak{G}}$ ($\theta\left(e^{it}\right)\theta^*\left(e^{it}\right) = I_{\mathfrak{F}}$) holds almost everywhere. It is termed *unitary* if both of these equalities are valid almost everywhere. The corresponding "multiplication" operator $\theta \in [L^2(\mathfrak{G}), L^2(\mathfrak{F})]$ is called *the "multiplication" isometry, coisometry* or *unitary operator*, respectively. Moreover, if $\theta\left(e^{it}\right) \in L^\infty_+[\mathfrak{G}, \mathfrak{F}]$, then, in this case, it is the boundary value function of *an inner, ∗-inner or two-sided inner operator function*, respectively.

By *a "multiplication" orthoprojection* of $L^2(\mathfrak{N})$ we call any "multiplication" operator P whose suboperator $P\left(e^{it}\right) \in L^\infty[\mathfrak{N}](:= L^\infty[\mathfrak{N}, \mathfrak{N}])$ is an orthogonal projection of $\mathfrak{N}$ onto some subspace of $\mathfrak{N}$ at almost all t, i.e.,

$$P^2\left(e^{it}\right) = P\left(e^{it}\right), \quad P^*\left(e^{it}\right) = P\left(e^{it}\right) \quad \text{a.e.}$$

Let $U^\times_{\mathfrak{N}}$ be the "multiplication" operator by e^{it} on the space $L^2(\mathfrak{N})$. There exists a bijective correspondence between reducing subspaces $\mathfrak{M}$ of $L^2(\mathfrak{N})$ for the operator $U^\times_{\mathfrak{N}}$ and "multiplication" orthoprojections $P_{\mathfrak{M}}\left(e^{it}\right)$ of $L^2(\mathfrak{N})$ such that $\mathfrak{M} = P_{\mathfrak{M}}L^2(\mathfrak{N})$ (see [12], Theorem 7.14).

Let $\mathfrak{M} \subset L^2(\mathfrak{N})$ be a reducing subspace for $U_{\mathfrak{N}}^{\times}$. The function $\rho_{\mathfrak{M}}\left(e^{it}\right) := \dim P_{\mathfrak{M}}\left(e^{it}\right)\mathfrak{N}$ defined almost everywhere and taking on values in $\mathbb{N} \cup \{0, \infty\}$ will be called *the rank function* of the subspace $\mathfrak{M}$. We also set

$$\alpha_{\mathfrak{M}} := \operatorname{ess\,inf}_t \rho_{\mathfrak{M}}\left(e^{it}\right), \quad \beta_{\mathfrak{M}} := \operatorname{ess\,sup}_t \rho_{\mathfrak{M}}\left(e^{it}\right).$$

For an arbitrary operator function $\mu\left(e^{it}\right) \in L^{\infty}[\mathfrak{G}, \mathfrak{N}]$ we denote by $\rho_{\mu}\left(e^{it}\right)$ the rank function of the subspace $\mathfrak{M} := \overline{\mu L^2(\mathfrak{G})}$ reducing $U_{\mathfrak{N}}^{\times}$ and by P_{μ} we does the "multiplication" orthoprojection of $L^2(\mathfrak{N})$ onto $\mathfrak{M}$. We also set $\alpha_{\mu} := \alpha_{\mathfrak{M}}$, $\beta_{\mu} := \beta_{\mathfrak{M}}$.

Definition 2.6 Let $\mathfrak{H}$ and $\mathfrak{N}$ be Hilbert spaces and let $U : \mathfrak{H} \to \mathfrak{H}$ be a unitary operator. Let $V : \mathfrak{N} \to \mathfrak{H}$ be an isometric operator such that $\overset{\circ}{\mathfrak{N}} := \operatorname{Ran} V$ is a wandering subspace of $\mathfrak{H}$ with respect to U. The operator $\Phi_U^{\mathfrak{N}} : \mathfrak{H} \to L^2(\mathfrak{N})$ defined by

$$\left[\Phi_U^{\mathfrak{N}} h\right]\left(e^{it}\right) := \sum_{-\infty}^{\infty} e^{ikt} V^* U^{-k} h, \quad h \in \mathfrak{H},$$

is called the Fourier representation that corresponds to the space $\mathfrak{N}$ and the operators U and V (the dependence on V is not reflected in $\Phi_U^{\mathfrak{N}}$ to simplify notations).

Moreover, the equalities

$$\Phi_U^{\mathfrak{N}} M_{\pm}(\overset{\circ}{\mathfrak{N}}) = L_{\pm}^2(\mathfrak{N}).$$

are valid. In addition, $\Phi_U^{\mathfrak{N}} U = U_{\mathfrak{N}}^{\times} \Phi_U^{\mathfrak{N}}$.

Let σ be a unitary coupling of form (2.1). In the sequel the operator

$$S_{\sigma} := P_{M(\overset{\circ}{\mathfrak{F}})}\Big|_{M(\overset{\circ}{\mathfrak{G}})} \tag{2.4}$$

will play an important part. Obviously, S_{σ} is a contractive operator acting from $M(\overset{\circ}{\mathfrak{G}})$ into $M(\overset{\circ}{\mathfrak{F}})$. It satisfies the condition

$$U S_{\sigma} = S_{\sigma} U|_{M(\overset{\circ}{\mathfrak{G}})}.$$

In fact, since $M(\overset{\circ}{\mathfrak{F}})$ and $M(\overset{\circ}{\mathfrak{G}})$ reduce U,

$$S_{\sigma} U|_{M(\overset{\circ}{\mathfrak{G}})} = P_{M(\overset{\circ}{\mathfrak{F}})} U|_{M(\overset{\circ}{\mathfrak{G}})} = U P_{M(\overset{\circ}{\mathfrak{F}})}\Big|_{M(\overset{\circ}{\mathfrak{G}})} = U S_{\sigma}.$$

By means of the Fourier representations $\Phi_U^{\mathfrak{F}}$ and $\Phi_U^{\mathfrak{G}}$ we can assign to the operator S_σ the operator

$$\theta_\sigma \ := \ \Phi_U^{\mathfrak{F}} S_\sigma \left(\Phi_U^{\mathfrak{G}}\right)^* \ = \ \Phi_U^{\mathfrak{F}} \left(\Phi_U^{\mathfrak{G}}\right)^*, \tag{2.5}$$

acting from $L^2(\mathfrak{G})$ into $L^2(\mathfrak{F})$, which is said to be *the scattering operator of the coupling* σ. The operator θ_σ satisfies the condition

$$U_{\mathfrak{F}}^{\times} \theta_\sigma \ = \ \theta_\sigma U_{\mathfrak{G}}^{\times} .$$

As is well known [29], in this case there exists a unique (we do not distinguish operator functions from $L^\infty[\mathfrak{G}, \mathfrak{F}]$ that coincide pointwise almost everywhere) operator function $\theta_\sigma \left(e^{it}\right) \in L^\infty[\mathfrak{G}, \mathfrak{F}]$ such that

$$\theta_\sigma \left(e^{it}\right) g \left(e^{it}\right) := (\theta_\sigma g) \left(e^{it}\right), \ g \left(e^{it}\right) \in L^2(\mathfrak{G}). \tag{2.6}$$

Moreover,

$$\|\theta_\sigma \left(e^{it}\right) \|_{L^\infty[\mathfrak{G}, \mathfrak{F}]} = \ \|\theta_\sigma\|_{[L^2(\mathfrak{G}), L^2(\mathfrak{F})]} \leq \ 1 \, .$$

Thus, $\theta_\sigma(e^{it} \in CM[\mathfrak{G}, \mathfrak{F}]$ (see Sect. 1). Following [1], we formulate

Definition 2.7 The operator function $\theta_\sigma \left(e^{it}\right) \ \in \ CM[\mathfrak{G}, \mathfrak{F}]$ defined by equality (2.6) is called the scattering suboperator of a unitary coupling σ of form (2.1) .

It follows from (2.4)–(2.6) that $\theta_\sigma \left(e^{it}\right) = \theta_{\sigma^{(1)}} \left(e^{it}\right)$, where $\sigma^{(1)}$ is the principal part of the coupling σ.

Theorem 2.8 ([11]) *Let σ and σ' be unitarily equivalent couplings of form* (2.2)*. Then $\theta_\sigma \left(e^{it}\right) \ = \ \theta_{\sigma'} \left(e^{it}\right)$.*

The following theorem contains a complete description of the class of scattering suboperators and establishes a correspondence between contractive operator functions and minimal unitary couplings.

Theorem 2.9 ([1]) *An arbitrary function $\theta \left(e^{it}\right) \ \in \ CM[\mathfrak{G}, \mathfrak{F}]$ is the scattering suboperator $\theta_\sigma \left(e^{it}\right)$ of some minimal unitary coupling σ of form* (2.1)*. The coupling σ is determined by $\theta \left(e^{it}\right)$ up to unitary equivalence. Thereby a bijective correspondence between classes of minimal unitarily equivalent couplings and contractive operator functions of the class L^∞ is established.*

Remark 2.10 The bijective correspondence defined in Theorem 2.9 enables us to speak about a scattering suboperator $\theta_{[\sigma]} \left(e^{it}\right)$ of a class $[\sigma]$ of unitarily equivalent couplings.

It is convenient to consider together with the coupling σ of form (2.1) the ones

$$\sigma^* := (\mathfrak{H}, \mathfrak{G}, \mathfrak{F}; U, V_{\mathfrak{G}}, V_{\mathfrak{F}}), \tag{2.7}$$

$$\sigma^{\sim} := (\mathfrak{H}, \mathfrak{G}, \mathfrak{F}; U^*, V_{\mathfrak{G}}, V_{\mathfrak{F}}), \tag{2.8}$$

which are called *adjoint* and *associate* in relation to the coupling σ, respectively. Note (see [11], Section 1) that

$$\theta_{\sigma^*}\left(e^{it}\right) = \theta_\sigma^*\left(e^{it}\right), \ \theta_{\sigma^\sim}\left(e^{it}\right) = \theta_\sigma^\sim\left(e^{it}\right). \tag{2.9}$$

Using the ideas of [1], we construct (see [11], Section 2) for any function $\theta\left(e^{it}\right) \in CM[\mathfrak{G}, \mathfrak{F}]$ two unitarily equivalent functional models of unitary coupling

$$\hat{\sigma} := (\hat{\mathfrak{H}}, \mathfrak{F}, \mathfrak{G}; \hat{U}^\times, \hat{V}_{\mathfrak{F}}, \hat{V}_{\mathfrak{G}}), \quad \tilde{\sigma} := (\tilde{\mathfrak{H}}, \mathfrak{F}, \mathfrak{G}; \tilde{U}^\times, \tilde{V}_{\mathfrak{F}}, \tilde{V}_{\mathfrak{G}})$$

such that $\theta_{\hat{\sigma}}\left(e^{it}\right) = \theta_{\tilde{\sigma}}\left(e^{it}\right) = \theta\left(e^{it}\right)$.

So, let $\theta\left(e^{it}\right) \in CM[\mathfrak{G}, \mathfrak{F}]$ and σ be a minimal unitary coupling of form (2.1) such that $\theta_\sigma\left(e^{it}\right) = \theta\left(e^{it}\right)$. As follows from Lemma 2.4, in this case the equality

$$\mathfrak{H} = M(\mathring{\mathfrak{F}}) \oplus \overline{P_{\mathfrak{R}_{\mathring{\mathfrak{F}}}} M(\mathring{\mathfrak{G}})}$$

holds. Denote

$$\Pi\left(e^{it}\right) := \left(I_{\mathfrak{G}} - \theta^*\left(e^{it}\right)\theta\left(e^{it}\right)\right)^{1/2} \tag{2.10}$$

(all roots of nonnegative operators are supposed to be nonnegative) and let Π be the "multiplication" operator by the function $\Pi\left(e^{it}\right)$ on $L^2(\mathfrak{G})$. Since $\Pi \in [L^2(\mathfrak{G})]$ is the defect operator of the contraction $\theta \in [L^2(\mathfrak{G}), L^2(\mathfrak{F})]$ (see, e.g., [29], Ch.I, §3), we call the operator function $\Pi(e^{it}) \in CM[\mathfrak{G}]$ *the defect function of* $\theta(e^{it})$ *in the class of contractive measurable operator functions*. The subspace $\overline{\Pi L^2(\mathfrak{G})}$ is called *the defect subspace* of the operator θ.

Theorem 2.11 ([11]) *Let* $\theta\left(e^{it}\right)$ *belongs to* $CM[\mathfrak{G}, \mathfrak{F}]$ *and* $\Pi\left(e^{it}\right) \in CM[\mathfrak{G}]$ *be its defect function in the class of contractive measurable operator functions. If*

(a) $\hat{\mathfrak{H}} := L^2(\mathfrak{F}) \oplus \overline{\Pi L^2(\mathfrak{G})}$,
(b) $\hat{U}^\times$ *is the multiplication operator by* e^{it} *on* $\hat{\mathfrak{H}}$,
(c) $\hat{V}_{\mathfrak{F}} : \mathfrak{F} \to \hat{\mathfrak{H}}$ *is an inclusion operator of* $\mathfrak{F}$ *into* $\hat{\mathfrak{H}}$, *i.e.,* $\hat{V}_{\mathfrak{F}} := P_{\mathfrak{F}}\big|_{\mathfrak{F}}$,
(d) $\hat{V}_{\mathfrak{G}} : \mathfrak{G} \to \hat{\mathfrak{H}}$ *is an isometric operator defined by the formula*

$$(\hat{V}_{\mathfrak{G}} g)(e^{it}) := \left(\theta\left(e^{it}\right)g, \Pi\left(e^{it}\right)g\right), \quad g \in \mathfrak{G},$$

then

$$\hat{\sigma} \ := \ (\hat{\mathfrak{H}}, \mathfrak{F}, \mathfrak{G}; \hat{U}^{\times}, \hat{V}_{\mathfrak{F}}, \hat{V}_{\mathfrak{G}}) \tag{2.11}$$

is a minimal unitary coupling and, moreover, $\theta_{\hat{\sigma}}\left(e^{it}\right) = \theta\left(e^{it}\right)$.

Let us interchange the parts of the subspaces $\mathfrak{F}$ and $\mathfrak{G}$ in the above reasoning, i.e., instead of the decomposition $\mathfrak{H} = M(\overset{\circ}{\mathfrak{F}}) \oplus \mathfrak{N}_{\overset{\circ}{\mathfrak{F}}}$, we consider the decomposition $\mathfrak{H} = \mathfrak{N}_{\overset{\circ}{\mathfrak{G}}} \oplus M(\overset{\circ}{\mathfrak{G}})$. In this case, the part of the function $\theta\left(e^{it}\right)$ is played by $\theta^*\left(e^{it}\right)$, as it follows from (2.7) and the first equality (2.9) . Therefore, instead of the operator Π, the "multiplication" operator Σ by the operator function

$$\Sigma\left(e^{it}\right) \ := \ \left(I_{\mathfrak{F}} \ - \ \theta\left(e^{it}\right)\theta^*\left(e^{it}\right)\right)^{1/2}, \tag{2.12}$$

on $L^2(\mathfrak{F})$ is used. Since $\Sigma \in [L^2(\mathfrak{F})]$ is the defect operator of the contraction $\theta^* \in [L^2(\mathfrak{F}), L^2(\mathfrak{G})]$, we call the operator function $\Sigma(e^{it}) \in CM[\mathfrak{F}]$ *the* $*$*-defect function of* $\theta(e^{it})$ *in the class of contractive measurable operator functions*. The subspace $\overline{\Sigma L^2(\mathfrak{F})}$ is called *the* $*$*-defect subspace* of the operator θ. If $\theta(e^{it}) = \theta_j(e^{it})$, then we will write Π_j and Σ_j instead of Π and Σ, respectively.

So, we come to the assertion which is dual to Theorem 2.11.

Theorem 2.12 ([11]) *Let* $\theta\left(e^{it}\right)$ *belongs to* $CM[\mathfrak{G}, \mathfrak{F}]$ *and* $\Sigma\left(e^{it}\right) \in CM[\mathfrak{F}]$ *be its* $*$*-defect function in the class of contractive measurable operator functions. If*

(a) $\tilde{\mathfrak{H}} := \overline{\Sigma L^2(\mathfrak{F})} \oplus L^2(\mathfrak{G})$,
(b) $\tilde{U}^{\times}$ *is a multiplication operator by* e^{it} *on* $\tilde{\mathfrak{H}}$,
(c) $\tilde{V}_{\mathfrak{F}} : \mathfrak{F} \to \tilde{\mathfrak{H}}$ *is an isometric operator defined by the formula*

$$(\tilde{V}_{\mathfrak{F}} f)(e^{it}) := \left(\Sigma\left(e^{it}\right) f, \theta^*\left(e^{it}\right) f\right), \quad f \in \mathfrak{F},$$

(d) $\tilde{V}_{\mathfrak{G}} : \mathfrak{G} \to \tilde{\mathfrak{H}}$ *is an inclusion operator of* $\mathfrak{G}$ *into* $\tilde{\mathfrak{H}}$*, i.e.,* $\tilde{V}_{\mathfrak{G}} := P_{\mathfrak{G}}\big|_{\mathfrak{G}}$,

then

$$\tilde{\sigma} \ := \ (\tilde{\mathfrak{H}}, \mathfrak{F}, \mathfrak{G}; \tilde{U}^{\times}, \tilde{V}_{\mathfrak{F}}, \tilde{V}_{\mathfrak{G}}) \tag{2.13}$$

is a minimal unitary coupling and, moreover, $\theta_{\tilde{\sigma}}\left(e^{it}\right) = \theta\left(e^{it}\right)$.

Consider the unitary "multiplication" operator W by the operator function

$$W(e^{it}) := \begin{pmatrix} \Sigma(e^{it}) & -\theta(e^{it}) \\ \theta^*(e^{it}) & \Pi(e^{it}) \end{pmatrix} \tag{2.14}$$

on the space $L^2(\mathfrak{F}) \oplus L^2(\mathfrak{G})$.

Theorem 2.13 ([11]) *The restriction* $Z := W\big|_{\hat{\mathfrak{H}}}$ *of the operator* W *on* $\hat{\mathfrak{H}}$ *establishes the unitary equivalence of the minimal unitary couplings* $\hat{\sigma}$ *and* $\tilde{\sigma}$.

2.2 Product of Unitary Couplings and Factorizations of Contractive Operator Functions

Let σ be a unitary coupling of form (2.1). By *a bilateral* (*unilateral output*, *unilateral input*) *channel of the coupling* σ we mean ([12]) a triple

$$(\mathfrak{L}, \mathfrak{N}; V_{\mathfrak{N}}) \;\; ((\mathfrak{L}_+, \mathfrak{N}; V_{\mathfrak{N}}), (\mathfrak{L}_-, \mathfrak{N}; V_{\mathfrak{N}}))\,,$$

where $V_{\mathfrak{N}} \in [\mathfrak{N}, \mathfrak{H}]$ is a channeled isometry, that is, $V_{\mathfrak{N}}$ is an isometry such that $\overset{\circ}{\mathfrak{N}}(:= \operatorname{Ran} V_{\mathfrak{N}})$ is a wandering subspace of $\mathfrak{H}$ with respect to U and the equality $\mathfrak{L} = M(\overset{\circ}{\mathfrak{N}})$ $\left(\mathfrak{L}_+ = M_+(\overset{\circ}{\mathfrak{N}}),\ \mathfrak{L}_- = M_-(\overset{\circ}{\mathfrak{N}})\right)$ holds. The bilaterel channels $\left(M(\overset{\circ}{\mathfrak{G}}), \mathfrak{G}; V_{\mathfrak{G}}\right)$ and $\left(M(\overset{\circ}{\mathfrak{F}}), \mathfrak{F}; V_{\mathfrak{F}}\right)$ are called *the principal bilateral* channels of the coupling σ.

Now we consider the concept of the product of unitary couplings which was introduced in [8] and will be important in the sequel.

Definition 2.14 Unitary couplings

$$\sigma_2 := (\mathfrak{H}_2, \mathfrak{F}, \mathfrak{K}; U_2, V_{\mathfrak{F}}, V_{\mathfrak{K}})\,, \quad \sigma_1 := (\mathfrak{H}_1, \mathfrak{K}, \mathfrak{G}; U_1, V_{\mathfrak{K}}, V_{\mathfrak{G}}) \qquad (2.15)$$

are called concatenated if

(a) there exists a common subspace $\mathfrak{L}$ of the spaces $\mathfrak{H}_2$ and $\mathfrak{H}_1$ that reduces the operators U_2 and U_1;
(b) the operators U_2 and U_1 coincide on the subspace $\mathfrak{L}$, i.e.,

$$U_2|_{\mathfrak{L}} = U_1|_{\mathfrak{L}}; \qquad (2.16)$$

(c) the embedding isometry $V_{\mathfrak{K}}$ is common for the couplings σ_2 and σ_1 with $\operatorname{Ran} V_{\mathfrak{K}} \subset \mathfrak{L}$;
(d) the equality

$$\mathfrak{L} = \bigoplus_{k=-\infty}^{\infty} U_2^k \overset{\circ}{\mathfrak{K}} (= \bigoplus_{k=-\infty}^{\infty} U_1^k \overset{\circ}{\mathfrak{K}})$$

is valid.

In other words, the couplings σ_2 and σ_1 are concatenated if the principal bilateral channel $(M(\overset{\circ}{\mathfrak{K}}), \mathfrak{K}; V_{\mathfrak{K}})$ is common for them and the equality (2.16) holds on the subspace $\mathfrak{L} := M(\overset{\circ}{\mathfrak{K}})$.

Note that in [11] we called such couplings "coupled". However, now the term "concatenated couplings" seems to us more relevant to the essence of this concept.

Let σ_2 and σ_1 be concatenated unitary couplings of form (2.15). Consider the space

$$\mathfrak{H} := \mathfrak{N}^{(2)}_{\overset{\circ}{\mathfrak{K}}} \oplus M(\overset{\circ}{\mathfrak{K}}) \oplus \mathfrak{N}^{(1)}_{\overset{\circ}{\mathfrak{K}}} \tag{2.17}$$

where $\mathfrak{N}^{(j)}_{\overset{\circ}{\mathfrak{K}}} := \mathfrak{H}_j \ominus M(\overset{\circ}{\mathfrak{K}})$ $(j = 1, 2)$. If subspaces of $\mathfrak{H}_2$ and $\mathfrak{H}_1$ are identified with corresponding subspaces of $\mathfrak{H}$, then

$$\mathfrak{H} = \mathfrak{H}_2 \oplus \mathfrak{N}^{(1)}_{\overset{\circ}{\mathfrak{K}}} = \mathfrak{N}^{(2)}_{\overset{\circ}{\mathfrak{K}}} \oplus \mathfrak{H}_1 .$$

Define the unitary operator on $\mathfrak{H}$ by the equality

$$U := U_2 P_2 + U_1 Q_1, \tag{2.18}$$

where P_2 and Q_1 are the orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{H}_2$ and $\mathfrak{N}^{(1)}_{\overset{\circ}{\mathfrak{K}}}$, respectively. Obviously, U can also be represented in the form

$$U = U_2 Q_2 + U_1 P_1,$$

where Q_2 and P_1 are the orthogonal projections of $\mathfrak{H}$ onto $\mathfrak{N}^{(2)}_{\overset{\circ}{\mathfrak{K}}}$ and $\mathfrak{H}_1$, respectively.

Definition 2.15 The unitary coupling $\sigma := (\mathfrak{H}, \mathfrak{F}, \mathfrak{G}; U, V_{\mathfrak{F}}, V_{\mathfrak{G}})$, where $\mathfrak{H}$ and U are defined by (2.17) and (2.18), is called the product of the concatenated unitary couplings of form (2.15). In this case, we will write $\sigma = \sigma_2\sigma_1$.

It is easy to extend the notation of the product to any finite set of unitary couplings $\sigma_n, \ldots, \sigma_2, \sigma_1$ such that for each $j = 1, 2, \ldots, n-1$ the couplings σ_{j+1} and σ_j are concatenated.

Let $[\sigma_2]$, $[\sigma_1]$ be two classes of unitarily equivalent couplings such that they have a common channelled subspace which is the output one for each of $\sigma_2 \in [\sigma_2]$ and the input one for each of $\sigma_1 \in [\sigma_1]$. In such classes one can always choose unitary couplings $\sigma_2 \in [\sigma_2]$ and $\sigma_1 \in [\sigma_1]$ which are concatenated. In fact, as it follows from Theorems 2.11 and 2.12, for such classes $[\sigma_2]$ and $[\sigma_1]$ of minimal unitarily equivalent couplings, one can chose the couplings $\tilde{\sigma}_2 \in [\sigma_2]$ and $\hat{\sigma}_1 \in [\sigma_1]$ of types (2.13) and (2.11), respectively, which are obviously concatenated. The passage to classes of abundant couplings can be realized in an obvious way.

Definition 2.16 Let $[\sigma_2]$ and $[\sigma_1]$ be classes of unitarily equivalent couplings which have a common channelled subspace of the output for $[\sigma_2]$ and of the input for $[\sigma_1]$.

The class containing the product $\sigma_2\sigma_1$ of concatenated unitary couplings $\sigma_2 \in [\sigma_2]$ and $\sigma_1 \in [\sigma_1]$ is called the product of the classes $[\sigma_2]$ and $[\sigma_1]$ and is denoted by $[\sigma_2][\sigma_1]$.

Note that this definition is correct irrespectively of a choice of concatenated representatives $\sigma_2 \in [\sigma_2]$ and $\sigma_1 \in [\sigma_1]$ (see [11], Theorem 3.14).

Definition 2.17 By a factorization of a unitary coupling we mean any of its representations as the product of concatenated unitary couplings.

There exists the direct connection between factorizations of unitary couplings and factorizations of contractive operator functions of the class L^∞ such that factors are also contractive operator functions of the class L^∞.

Theorem 2.18 ([11]) *Let*

$$\sigma_2 := (\mathfrak{H}_2, \mathfrak{F}, \mathfrak{K}; U_2, V_{\mathfrak{F}}, V_{\mathfrak{K}}), \quad \sigma_1 := (\mathfrak{H}_1, \mathfrak{K}, \mathfrak{G}; U_1, V_{\mathfrak{K}}, V_{\mathfrak{G}}) \tag{2.19}$$

be concatenated unitary couplings. Then

$$\theta_{\sigma_2\sigma_1}\left(e^{it}\right) = \theta_{\sigma_2}\left(e^{it}\right)\theta_{\sigma_1}\left(e^{it}\right). \tag{2.20}$$

Conversely, if $\theta_2\left(e^{it}\right) \in CM[\mathfrak{K}, \mathfrak{F}]$, $\theta_1\left(e^{it}\right) \in CM[\mathfrak{G}, \mathfrak{K}]$, *then there exist concatenated unitary couplings* σ_2 *and* σ_1 *of form* (2.19) *such that*

$$\theta_2\left(e^{it}\right) = \theta_{\sigma_2}\left(e^{it}\right),\ \theta_1\left(e^{it}\right) = \theta_{\sigma_1}\left(e^{it}\right),\ \theta_2\left(e^{it}\right)\theta_1\left(e^{it}\right) = \theta_{\sigma_2\sigma_1}\left(e^{it}\right).$$

2.3 Regular Factorizations of Contractive Operator-Valued Functions

Definition 2.19 By a factorization of a function $\theta\left(e^{it}\right) \in CM[\mathfrak{G}, \mathfrak{F}]$ we mean any of its representations in the form

$$\theta\left(e^{it}\right) = \theta_2\left(e^{it}\right)\theta_1\left(e^{it}\right), \tag{2.21}$$

where $\theta_2\left(e^{it}\right) \in CM[\mathfrak{K}, \mathfrak{F}]$, $\theta_1\left(e^{it}\right) \in CM[\mathfrak{G}, \mathfrak{K}]$. A factorization of form (2.21) is called regular if the product $[\sigma_2][\sigma_1]$ of minimal classes $[\sigma_2]$ and $[\sigma_1]$ such that $\theta_{[\sigma_j]}\left(e^{it}\right) = \theta_j\left(e^{it}\right)$ $(j = 1, 2)$ is a minimal class as well.

Thus, the regularity of factorization (2.21) is equivalent to the minimality of the product $\sigma_2\sigma_1$ for arbitrary minimal concatenated unitary couplings σ_2 and σ_1 such that $\theta_{\sigma_j}\left(e^{it}\right) = \theta_j\left(e^{it}\right)$ $(j = 1, 2)$.

Theorem 2.20 ([11]) *Let σ be a minimal unitary coupling. For a given regular factorization $\theta_\sigma\left(e^{it}\right) = \theta_2\left(e^{it}\right)\theta_1\left(e^{it}\right)$ there exists one and only one factorization $\sigma = \sigma_2\sigma_1$ such that $\theta_{\sigma_j}\left(e^{it}\right) = \theta_j\left(e^{it}\right)$ $(j = 1, 2)$.*

Actually, Theorem 2.20 establishes a bijective correspondence between factorizations of a minimal unitary coupling σ and regular factorizations of its scattering suboperator $\theta_\sigma\left(e^{it}\right)$.

It is possible to introduce a comparison relation in the set of regular factorizations for a function $\theta\left(e^{it}\right) \in CM[\mathfrak{G}, \mathfrak{F}]$.

Definition 2.21 Let

$$\theta\left(e^{it}\right) = \theta_2\left(e^{it}\right)\theta_1\left(e^{it}\right), \theta_2\left(e^{it}\right) \in CM[\mathfrak{K}, \mathfrak{F}],\ \theta_1\left(e^{it}\right) \in CM[\mathfrak{G}, \mathfrak{K}], \tag{2.22}$$

$$\theta\left(e^{it}\right) = \theta_2'\left(e^{it}\right)\theta_1'\left(e^{it}\right), \theta_2'\left(e^{it}\right) \in CM[\mathfrak{K}', \mathfrak{F}],\ \theta_1'\left(e^{it}\right) \in CM[\mathfrak{G}, \mathfrak{K}'] \tag{2.23}$$

be two regular factorizations of a contractive operator function $\theta\left(e^{it}\right)$. We say that factorization (2.22) precedes factorization (2.23) if there exists a contractive operator function $\theta_0\left(e^{it}\right) \in L^\infty[\mathfrak{K}, \mathfrak{K}']$ such that

$$\theta_2\left(e^{it}\right) = \theta_2'\left(e^{it}\right)\theta_0\left(e^{it}\right), \quad \theta_0\left(e^{it}\right)\theta_1\left(e^{it}\right) = \theta_1'\left(e^{it}\right). \tag{2.24}$$

Theorem 2.22 ([11]) *Let σ be a minimal unitary coupling and $\sigma = \sigma_2\sigma_1 = \sigma_2'\sigma_1'$. The factorization*

$$\theta_\sigma\left(e^{it}\right) = \theta_{\sigma_2}\left(e^{it}\right)\theta_{\sigma_1}\left(e^{it}\right)$$

precedes the factorization

$$\theta_\sigma\left(e^{it}\right) = \theta_{\sigma_2'}\left(e^{it}\right)\theta_{\sigma_1'}\left(e^{it}\right)$$

iff there exists a unitary coupling σ_0 such that $\sigma_2 = \sigma_2'\sigma_0$, $\sigma_0\sigma_1 = \sigma_1'$.

It is easy to see that the introduced comparison relation for regular factorizations is reflexive and transitive, but it is not antisymmetric. In fact, if, for instance, in the definition of the comparison relation, we put $\theta_0\left(e^{it}\right) = X \neq I$ where X is a constant unitary factor, then the distinct factorizations (2.22) and (2.23) precede each other.

Thus, a partial preorder is established on the set of regular factorizations for any contractive operator function $\theta\left(e^{it}\right) \in L^\infty[\mathfrak{G}, \mathfrak{F}]$.

Theorem 2.23 ([11]) *Let* $\sigma := (\mathfrak{H}, \mathfrak{F}, \mathfrak{G}; U, V_{\mathfrak{F}}, V_{\mathfrak{G}})$ *be a minimal unitary coupling and let* $\sigma = \sigma_2\sigma_1 = \sigma_2'\sigma_1'$*, where*

$$\sigma_2 := (\mathfrak{H}_2, \mathfrak{F}, \mathfrak{K}; U_2, V_{\mathfrak{F}}, V_{\mathfrak{K}}),\ \sigma_1 := (\mathfrak{H}_1, \mathfrak{K}, \mathfrak{G}; U_1, V_{\mathfrak{K}}, V_{\mathfrak{G}}),$$

$$\sigma_2' := (\mathfrak{H}_2', \mathfrak{F}, \mathfrak{K}'; U_2', V_{\mathfrak{F}}, V_{\mathfrak{K}'}),\ \sigma_1' := (\mathfrak{H}_1', \mathfrak{K}', \mathfrak{G}; U_1', V_{\mathfrak{K}'}, V_{\mathfrak{G}}).$$

The factorizations

$$\theta_\sigma\left(e^{it}\right) = \theta_{\sigma_2}\left(e^{it}\right)\theta_{\sigma_1}\left(e^{it}\right), \quad \theta_\sigma\left(e^{it}\right) = \theta_{\sigma_2'}\left(e^{it}\right)\theta_{\sigma_1'}\left(e^{it}\right)$$

precede each other iff $M(\mathring{\mathfrak{K}}) = M(\mathring{\mathfrak{K}}')$.

Lemma 2.24 ([1]) *Let* $\sigma := (\mathfrak{H}, \mathfrak{F}, \mathfrak{G}; U, V_{\mathfrak{F}}, V_{\mathfrak{G}})$ *be a unitary coupling. The equality* $M(\mathring{\mathfrak{F}}) = M(\mathring{\mathfrak{G}})$ *is valid iff the scattering suboperator* $\theta_\sigma\left(e^{it}\right)$ *assumes unitary values at almost all* t.

Guided by ideas of [4], we introduce the following concept.

Definition 2.25 A unitary coupling σ is called lossless if its scattering suboperator takes on unitary values at almost all t. A minimal unitary coupling σ is called trivial if its scattering suboperator is a constant unitary operator $X \in [\mathfrak{G}, \mathfrak{F}]$. A trivial unitary coupling is called unity if $\mathfrak{F} = \mathfrak{G}$ and its scattering suboperator is $I_{\mathfrak{G}} \in [\mathfrak{G}]$.

It follows from Lemma 2.24 that $\sigma = (\mathfrak{H}, \mathfrak{F}, \mathfrak{G}; U, V_{\mathfrak{F}}, V_{\mathfrak{G}})$ is a lossless coupling iff $M(\mathring{\mathfrak{F}}) = M(\mathring{\mathfrak{G}})$. Note that a minimal coupling σ is trivial if $\mathring{\mathfrak{F}} = \mathring{\mathfrak{G}}$ and is unity if $\mathfrak{F} = \mathfrak{G}$, $V_{\mathfrak{F}} = V_{\mathfrak{G}}$.

Introduce an equivalence relation on the set of regular factorizations for a function $\theta\left(e^{it}\right) \in CM[\mathfrak{G}, \mathfrak{F}]$.

Definition 2.26 Two regular factorizations of forms (2.22) and (2.23) are called equivalent if they precede each other.

The following assertions easily follow from Theorems 2.22, 2.23, and Lemma 2.24.

Theorem 2.27 ([11])

(a) *Factorizations of forms* (2.22) *and* (2.23) *are equivalent iff there exists a unitary operator function* $\theta_0\left(e^{it}\right) \in CM[\mathfrak{K}, \mathfrak{K}']$ *such that equalities* (2.24) *hold.*
(b) *Let* σ *be a minimal unitary coupling and let* $\sigma = \sigma_2\sigma_1 = \sigma_2'\sigma_1'$. *The factorizations*

$$\theta_\sigma\left(e^{it}\right) = \theta_{\sigma_2}\left(e^{it}\right)\theta_{\sigma_1}\left(e^{it}\right), \quad \theta_\sigma\left(e^{it}\right) = \theta_{\sigma_2'}\left(e^{it}\right)\theta_{\sigma_1'}\left(e^{it}\right)$$

are equivalent iff there exists a lossless minimal coupling σ_0 such that

$$\sigma_2 = \sigma_2'\sigma_0 , \quad \sigma_0\sigma_1 = \sigma_1' . \tag{2.25}$$

In particular, the equalities

$$\theta_{\sigma_2}\left(e^{it}\right) = \theta_{\sigma_2'}\left(e^{it}\right) X , \quad X\theta_{\sigma_1}\left(e^{it}\right) = \theta_{\sigma_1'}\left(e^{it}\right) ,$$

where X is a constant unitary factor, are valid iff the coupling σ_0 in (2.25) is trivial.

Thus, partitioning the set of regular factorizations of a contractive operator function into equivalence classes, we can obviously extend the comparison relation from the set of regular factorizations to the set of classes. It is easy to verify that the relation defined in this way is a partial order on this set.

It is clear that the concept of regularity for a factorization of a contractive operator function is extended to the case of any finite number of factors in an obvious way.

In [11] the following important criteria of regular factorizations were proved.

Theorem 2.28 *Let $\theta_2\left(e^{it}\right) \in CM[\mathfrak{K}, \mathfrak{F}]$ and $\theta_1\left(e^{it}\right) \in CM[\mathfrak{G}, \mathfrak{K}]$. The following assertions are equivalent.*

(1) *The factorization $\theta\left(e^{it}\right) = \theta_2\left(e^{it}\right)\theta_1\left(e^{it}\right)$ is regular.*
(2) *The factorization $\theta^*\left(e^{it}\right) = \theta_1^*\left(e^{it}\right)\theta_2^*\left(e^{it}\right)$ is regular.*
(3) $\Pi_2 L^2(\mathfrak{K}) \cap \Sigma_1 L^2(\mathfrak{K}) = \{0\}$.
(4) $\Pi_2\left(e^{it}\right)\mathfrak{K} \cap \Sigma_1\left(e^{it}\right)\mathfrak{K} = \{0\}$ *almost everywhere.*

Corollary 2.29 *If at almost all t at least one of the operators $\theta_2\left(e^{it}\right)$ or $\theta_1^*\left(e^{it}\right)$ is isometric, then the factorization $\theta\left(e^{it}\right) = \theta_2\left(e^{it}\right)\theta_1\left(e^{it}\right)$ is regular.*

Remark 2.30 ([11]) In the case of finite dimensional spaces $\mathfrak{F}, \mathfrak{K}, \mathfrak{G}$, we obtain two more assertions that are equivalent to (1)–(4):

(5) $\operatorname{rank}\Pi\left(e^{it}\right) = \operatorname{rank}\Pi_2\left(e^{it}\right) + \operatorname{rank}\Pi_1\left(e^{it}\right)$ almost everywhere.
(6) $\operatorname{rank}\Sigma\left(e^{it}\right) = \operatorname{rank}\Sigma_2\left(e^{it}\right) + \operatorname{rank}\Sigma_1\left(e^{it}\right)$ almost everywhere.

Definition 2.31 Let $\theta_2(e^{it}) \in CM[\mathfrak{K}, \mathfrak{F}]$, $\theta_1(e^{it}) \in CM[\mathfrak{G}, \mathfrak{K}]$ and

$$\theta(e^{it}) := \theta_2(e^{it})\theta_1(e^{it}) \quad \text{a.e.}$$

This factorization is called completely regular if the condition

$$P_{\Pi_2}(e^{it})P_{\Sigma_1}(e^{it}) = 0 \quad \text{a.e.} \tag{2.26}$$

holds (cf. with assertion (4) from Theorem 2.28).

For more information about these factorizations see [13], Subsection 8.5.

Definition 2.32 The coupling σ of form (2.1) is called orthogonal if $M_-(\overset{\circ}{\mathfrak{F}}) \perp M_+(\overset{\circ}{\mathfrak{G}})$.

A characteristic property of orthogonal couplings σ is stated in the following assertion.

Theorem 2.33 ([4]) *A unitary coupling σ of form (2.1) is orthogonal iff its scattering suboperator $\theta_\sigma\left(e^{it}\right)$ belongs to the class $L_+^\infty[\mathfrak{G}, \mathfrak{F}]$.*

The theory of orthogonal couplings is equivalent to the theory of unitary colligations (see [4] and [11], Section 6).

Definition 2.34 A unitary coupling σ of form (2.1) is called non-degenerate if $M(\overset{\circ}{\mathfrak{F}}) \bigcap M(\overset{\circ}{\mathfrak{G}}) = \{0\}$ holds and degenerate otherwise.

3 Regular Extensions of Contractive Measurable Operator-Valued Functions

In this section, using some spacial cases of regular factorizations of contractive operator functions, we introduce for such functions the concepts of unidirectional and bidirectional regular extensions.

3.1 Unidirectional Regular Extensions

In the paper [13] (see Theorems 8.1 and 8.4) the connections between embeddings of the principal channels of a minimal coupling into its other ones and regular factorizations of its scattering suboperator was established. These results can be formulated in the following form.

Theorem 3.1 *Let σ be a minimal unitary coupling of form (2.1) and $\theta(e^{it}) := \theta_\sigma(e^{it})$. There exists a bijective correspondence between bilateral channels $(M(\overset{\circ}{\mathfrak{F}}_1), \mathfrak{F}_1; V_{\mathfrak{F}_1})$ $((M\ \overset{\circ}{\mathfrak{G}}_1), \mathfrak{G}_1; V_{\mathfrak{G}_1}))$ of the coupling σ satisfying the condition*

$$M(\overset{\circ}{\mathfrak{F}}_1) \supset M(\overset{\circ}{\mathfrak{F}}) \quad (M(\overset{\circ}{\mathfrak{G}}_1) \supset M(\overset{\circ}{\mathfrak{G}})) \tag{3.1}$$

and regular factorizations of form

$$\theta(e^{it}) = \theta_2(e^{it})\theta_1(e^{it}) \tag{3.2}$$

where $\theta_2(e^{it}) \in CM[\mathfrak{F}_1, \mathfrak{F}]$ $(\theta_1(e^{it}) \in CM[\mathfrak{G}, \mathfrak{G}_1])$ is a coisometric (isometric) operator function.This correspondence is established by the equalities $\theta_j(e^{it}) =$

$\theta_{\sigma_j}(e^{it})$ $(j = 1, 2)$, *where*

$$\sigma_2 := (M(\overset{\circ}{\mathfrak{F}}_1), \mathfrak{F}, \mathfrak{F}_1; U_2, V_{\mathfrak{F}}, V_{\mathfrak{F}_1}), U_2 := U|_{M(\overset{\circ}{\mathfrak{F}}_1)}; \ \sigma_1 := (\mathfrak{H}, \mathfrak{F}_1, \mathfrak{G}; U, V_{\mathfrak{F}_1}, V_{\mathfrak{G}}) \tag{3.3}$$

$$(\sigma_2 := (\mathfrak{H}, \mathfrak{F}, \mathfrak{G}_1; U, V_{\mathfrak{F}}, V_{\mathfrak{G}_1}); \ \sigma_1 := (M(\overset{\circ}{\mathfrak{G}}_1), \mathfrak{G}_1, \mathfrak{G}; U_1, V_{\mathfrak{G}_1}, V_{\mathfrak{G}}), U_1 := U|_{M(\overset{\circ}{\mathfrak{G}}_1)}). \tag{3.4}$$

In turn, of particular interest in studying the embeddings of the principal channel $(M(\overset{\circ}{\mathfrak{F}}), \mathfrak{F}; V_{\mathfrak{F}})$ into another one $(M(\overset{\circ}{\mathfrak{F}}_1), \mathfrak{F}_1; V_{\mathfrak{F}_1})$ is the case when $\mathfrak{F}_1 = \mathfrak{F}^{(1)} \oplus \mathfrak{F}$ and $V_{\mathfrak{F}_1}|_{\mathfrak{F}} = V_{\mathfrak{F}}$. This is tantamount to the existence of a bilateral channel $(M(\overset{\circ}{\mathfrak{F}}{}^{(1)}), \mathfrak{F}^{(1)}; V_{\mathfrak{F}^{(1)}})$ of the coupling σ such that

$$M(\overset{\circ}{\mathfrak{F}}_1) = M(\overset{\circ}{\mathfrak{F}}{}^{(1)}) \oplus M(\overset{\circ}{\mathfrak{F}}), \quad V_{\mathfrak{F}_1} = V_{\mathfrak{F}^{(1)}} P_{\mathfrak{F}^{(1)}} + V_{\mathfrak{F}} P_{\mathfrak{F}},$$

where $P_{\mathfrak{F}^{(1)}}$ and $P_{\mathfrak{F}}$ are the orthogonal projection of $\mathfrak{F}_1$ onto $\mathfrak{F}^{(1)}$ and $\mathfrak{F}$, respectively. Taking into account that up to the obvious unitary isomorphism the equality $L^2(\mathfrak{F}_1) = L^2(\mathfrak{F}^{(1)}) \oplus L^2(\mathfrak{F})$ holds, we can write for the Fourier representation $\Phi_U^{\mathfrak{F}_1}$ the equality

$$\Phi_U^{\mathfrak{F}_1}|_{M(\overset{\circ}{\mathfrak{F}}_1)} = \Phi_U^{\mathfrak{F}^{(1)}}|_{M(\overset{\circ}{\mathfrak{F}}{}^{(1)})} \oplus \Phi_U^{\mathfrak{F}}|_{M(\overset{\circ}{\mathfrak{F}})}.$$

This makes it possible to represent in the block form the operator functions $\theta_2(e^{it}) \in CM[\mathfrak{F}_1, \mathfrak{F}]$ and $\theta_1(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}_1]$ from factorization (3.2) corresponding to this special case of the inclusion of form (3.1). Namely,

$$\theta_2(e^{it}) = [0, I_{\mathfrak{F}}], \quad 0 \in [\mathfrak{F}^{(1)}, \mathfrak{F}]; \quad \theta_1(e^{it}) = \begin{bmatrix} \theta_{12}(e^{it}) \\ \theta(e^{it}) \end{bmatrix}, \tag{3.5}$$

where

$$\theta_2(e^{it}) := \theta_{\sigma_2}(e^{it}) \in CM[\mathfrak{F}^{(1)} \oplus \mathfrak{F}, \mathfrak{F}], \quad \theta_1(e^{it}) := \theta_{\sigma_1}(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}^{(1)} \oplus \mathfrak{F}],$$

$$\theta_{12}(e^{it}) := \theta_{\sigma_{12}}(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}^{(1)}],$$

σ_1, σ_2 are the couplings of form (3.3) under the conditions $\mathfrak{F}_1 = \mathfrak{F}^{(1)} \oplus \mathfrak{F}$, $V_{\mathfrak{F}_1}|_{\mathfrak{F}} = V_{\mathfrak{F}}$,

$$\sigma_{12} := (\mathfrak{H}, \mathfrak{F}^{(1)}, \mathfrak{G}; U, V_{\mathfrak{F}^{(1)}}, V_{\mathfrak{G}}) \tag{3.6}$$

(the coupling σ_{12} is not necessarily minimal).

Theorem 3.2 *Let σ be a minimal unitary coupling of form (2.1) and $\theta(e^{it}) := \theta_\sigma(e^{it})$.*

(a) There exists a bijective correspondence between bilateral channels $(M(\overset{\circ}{\mathfrak{F}}{}^{(1)}), \mathfrak{F}^{(1)}; V_{\mathfrak{F}^{(1)}})$ of the coupling σ satisfying the condition

$$M(\overset{\circ}{\mathfrak{F}}{}^{(1)}) \perp M(\overset{\circ}{\mathfrak{F}}) \tag{3.7}$$

and regular factorizations

$$\theta(e^{it}) = \theta_2(e^{it})\theta_1(e^{it}), \tag{3.8}$$

where $\theta_2(e^{it}) \in CM[\mathfrak{F}^{(1)} \oplus \mathfrak{F}, \mathfrak{F}]$, $\theta_1(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}^{(1)} \oplus \mathfrak{F}]$ are functions of form (3.5). This correspondence is established by the equality $\theta_{12}(e^{it}) = \theta_{\sigma_{12}}(e^{it})$, where σ_{12} is a unitary coupling of form (3.6).

(b) A regular factorization of the considered type is completely regular iff the corresponding channel $(M(\overset{\circ}{\mathfrak{F}}{}^{(1)}), \mathfrak{F}^{(1)}; V_{\mathfrak{F}^{(1)}})$, in addition to condition (3.7), also satisfies the condition

$$M(\overset{\circ}{\mathfrak{F}}{}^{(1)}) \subset M(\overset{\circ}{\mathfrak{G}}). \tag{3.9}$$

Proof

(a) This part is a particular case of Theorem 3.1.

(b) Since part (b) is a special case of part (a), we will confine ourselves only to the discussion of additional condition (3.9).

A factorization (3.8) of the considered type is completely regular iff the inclusion

$$\overline{\Pi_2 L^2(\overset{\circ}{\mathfrak{F}}{}^{(1)} \oplus \mathfrak{F})} \subset \mathrm{Ker}\Sigma_1$$

holds (see Definition 2.31). Taking into account form (3.5) of $\theta_2(e^{it})$, we obtain that $\Pi_2 L^2(\mathfrak{F}^{(1)} \oplus \mathfrak{F}) = L^2(\mathfrak{F}^{(1)})$ and, hence, $L^2(\mathfrak{F}^{(1)}) \subset \mathrm{Ker}\Sigma_1$. This inclusion is valid iff the operator

$$\theta_{12} := P_{L^2(\mathfrak{F}^{(1)})}\theta_1 \in [L^2(\mathfrak{G}), L^2(\mathfrak{F}^{(1)})]$$

is a coisometry. As is known (see [12], Theorem 7.6), the function $\theta_{12}(e^{it})$ ($:= \theta_{\sigma_{12}}(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}^{(1)}]$) is a coisometric operator function iff condition (3.9) is satisfied. □

Remark 3.3 It is seen from the proof of Theorem 3.2 that a factorization of form (3.8), where $\theta_2(e^{it}) \in CM[\mathfrak{F}^{(1)} \oplus \mathfrak{F}, \mathfrak{F}]$, $\theta_1(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}^{(1)} \oplus \mathfrak{F}]$ are functions of form (3.5), is completely regular iff the function $\theta_{12}(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}^{(1)}]$ is a coisometric operator function. Obviously, in this case the inclusion $\mathrm{Ran}\theta_{12}^* \subset \mathrm{Ker}\theta$ holds.

In the dual case, that is, when $\mathfrak{G}_1 = \mathfrak{G}^{(1)} \oplus \mathfrak{G}$, $V_{\mathfrak{G}_1}|_{\mathfrak{G}} = V_{\mathfrak{G}}$, we similarly obtain the block form of the operator functions $\theta_2(e^{it}) \in CM[\mathfrak{G}_1, \mathfrak{F}]$, $\theta_1(e^{it}) \in CM[\mathfrak{G}, \mathfrak{G}_1]$ from factorization (3.2). Namely,

$$\theta_2(e^{it}) = [\theta_{21}(e^{it}), \theta(e^{it})]; \quad \theta_1(e^{it}) = \begin{bmatrix} 0 \\ I_{\mathfrak{G}} \end{bmatrix}, \quad 0 \in [\mathfrak{G}, \mathfrak{G}^{(1)}], \tag{3.10}$$

where

$$\theta_2(e^{it}) := \theta_{\sigma_2}(e^{it}) \in CM[\mathfrak{G}^{(1)} \oplus \mathfrak{G}, \mathfrak{F}], \quad \theta_1(e^{it}) := \theta_{\sigma_1}(e^{it}) \in CM[\mathfrak{G}, \mathfrak{G}^{(1)} \oplus \mathfrak{G}],$$

$$\theta_{21}(e^{it}) := \theta_{\sigma_{21}}(e^{it}) \in CM[\mathfrak{G}^{(1)}, \mathfrak{F}],$$

σ_2, σ_1 are the couplings of form (3.4) under the conditions $\mathfrak{G} = \mathfrak{G}^{(1)} \oplus \mathfrak{G}$, $V_{\mathfrak{G}_1}|_{\mathfrak{G}} = V_{\mathfrak{G}}$,

$$\sigma_{21} := (\mathfrak{H}, \mathfrak{F}, \mathfrak{G}^{(1)}; U, V_{\mathfrak{F}}, V_{\mathfrak{G}^{(1)}}) \tag{3.11}$$

and $V_{\mathfrak{G}^{(1)}} := V_{\mathfrak{G}_1}|_{\mathfrak{G}^{(1)}}$ (the coupling σ_{12} is not necessarily minimal).

The following assertion is the dual analog of Theorem 3.2.

Theorem 3.4 *Let σ be a minimal unitary coupling of form (2.1) and $\theta(e^{it}) := \theta_\sigma(e^{it})$.*

(a) There exists a bijective correspondence between bilateral channels $(M(\overset{\circ}{\mathfrak{G}}{}^{(1)}), \mathfrak{G}^{(1)}; V_{\mathfrak{G}^{(1)}})$ of the coupling σ satisfying the condition

$$M(\overset{\circ}{\mathfrak{G}}{}^{(1)}) \perp M(\overset{\circ}{\mathfrak{G}}) \tag{3.12}$$

and regular factorizations

$$\theta(e^{it}) = \theta_2(e^{it})\theta_1(e^{it}), \tag{3.13}$$

where $\theta_2(e^{it}) \in CM[\mathfrak{G}^{(1)} \oplus \mathfrak{G}, \mathfrak{F}]$, $\theta_1(e^{it}) \in CM[\mathfrak{G}, \mathfrak{G}^{(1)} \oplus \mathfrak{G}]$ are functions of form (3.10). This correspondence is established by the equality $\theta_{21}(e^{it}) = \theta_{\sigma_{21}}(e^{it})$, where σ_{21} is a unitary coupling of form (3.11).

(b) A regular factorization of the considered type is completely regular iff the corresponding channel $(M(\overset{\circ}{\mathfrak{G}}{}^{(1)}), \mathfrak{G}^{(1)}; V_{\mathfrak{G}^{(1)}})$, in addition to condition (3.12), also satisfies the condition

$$M(\overset{\circ}{\mathfrak{G}}{}^{(1)}) \subset M(\overset{\circ}{\mathfrak{F}}). \tag{3.14}$$

Remark 3.5 A factorization of form (3.13), where $\theta_2(e^{it}) \in CM[\mathfrak{G}^{(1)} \oplus \mathfrak{G}, \mathfrak{F}]$, $\theta_1(e^{it}) \in CM[\mathfrak{G}, \mathfrak{G}^{(1)} \oplus \mathfrak{G}]$ are functions of form (3.10), is completely regular iff the function $\theta_{21}(e^{it}) \in CM[\mathfrak{G}^{(1)}, \mathfrak{F}]$ is an isometric operator function. Obviously, in this case the inclusion $\mathrm{Ran}\theta_{21} \subset \mathrm{Ker}\theta^*$ holds.

By analogy with the theory of open systems (see, e.g., [14], Section 6), the transition from a unitary coupling σ of form (2.1) to the unitary coupling σ_1 of form (3.3) (σ_2 of form (3.4)) with the input (output) channelled subspace $\mathfrak{F}_1 := \mathfrak{F}^{(1)} \oplus \mathfrak{F}(\mathfrak{G}_1 := \mathfrak{G}^{(1)} \oplus \mathfrak{G})$ and $V_{\mathfrak{F}_1}|_{\mathfrak{F}} = V_{\mathfrak{F}}(V_{\mathfrak{G}_1}|_{\mathfrak{G}} = V_{\mathfrak{G}})$ can be considered as the opening of an additional input (output) bilateral channel $(M(\overset{\circ}{\mathfrak{F}}{}^{(1)}), \mathfrak{F}^{(1)}; V_{\mathfrak{F}^{(1)}})$ $((M(\overset{\circ}{\mathfrak{G}}{}^{(1)}), \mathfrak{G}^{(1)}; V_{\mathfrak{G}^{(1)}}))$. Conversely, the transition from the coupling σ_1 (σ_2) to the coupling σ corresponds to the partial closing of the input (output) bilateral channel. The opening of an additional input (output) bilateral channel leads to the upward (leftward) extension $\theta_1(e^{it})$ ($\theta_2(e^{it})$) of form (3.5) and (3.10) for the scattering suboperator $\theta_\sigma(e^{it})$ ($= \theta(e^{it})$). The partial closing of the input (output) channel of the coupling σ_1 (σ_2) leads to the narrowing of its scattering suboperator $\theta_{\sigma_1}(e^{it})$ ($\theta_{\sigma_2}(e^{it})$) to the function $\theta(e^{it})$.

Theorems 3.2 and 3.4 enable us to formulate the following definition which is a generalization of its analog for Schur operator functions (see [23], Section 2 and [14], Definition 5.8).

Definition 3.6 A function

$$\Omega(e^{it}) := \begin{bmatrix} \theta_{12}(e^{it}) \\ \theta(e^{it}) \end{bmatrix} \in CM[\mathfrak{G}, \mathfrak{F}^{(1)} \oplus \mathfrak{F}]$$

$$(\Lambda(e^{it}) := [\theta_{21}(e^{it}), \theta(e^{it})] \in CM[\mathfrak{G}^{(1)} \oplus \mathfrak{G}, \mathfrak{F}]) \tag{3.15}$$

will be called a regular upward (leftward) extension of a function $\theta(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$ if the factorization

$$\theta(e^{it}) = \theta_2(e^{it})\theta_1(e^{it}), \tag{3.16}$$

where

$$\theta_2(e^{it}) := [0, I_{\mathfrak{F}}], \quad 0 \in [\mathfrak{F}^{(1)}, \mathfrak{F}]; \quad \theta_1(e^{it}) := \Omega(e^{it}) \tag{3.17}$$

$$(\theta_2(e^{it}) := \Lambda(e^{it})); \quad \theta_1(e^{it}) := \begin{bmatrix} 0 \\ I_{\mathfrak{G}} \end{bmatrix}, \quad 0 \in [\mathfrak{G}, \mathfrak{G}^{(1)}]), \tag{3.18}$$

is regular. It will be called a completely regular upward (leftward) extension if factorization (3.16) is completely regular. If $\mathfrak{F}^{(1)} = \{0\}(\mathfrak{G}^{(1)} = \{0\})$, we will call $\Omega(e^{it}) := \theta(e^{it})(\Lambda(e^{it}) := \theta(e^{it}))$ the trivial upward (leftward) extension of the function $\theta(e^{it})$. We will call the regular upward (leftward) extension

$\Omega(e^{it})(\Lambda(e^{it}))$ of form (3.15) isometric (coisometric) if $\Omega(e^{it})(\Lambda(e^{it}))$ is an isometric (coisometric) operator function.

Regular extensions of both types defined above, as well as downward and rightward regular extensions of contractive operator functions defined in an obvious way, will be called unidirectional regular extensions.

It is clear that a downward (rightward) regular extension of $\theta(e^{it})$ is simply the other form of a block matrix obtained from $\Omega(e^{it})$ $(\Lambda(e^{it}))$ by the permutation of its blocks.

In the case of finite-dimensional spaces $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{F}^{(1)}$ $(\mathfrak{G}^{(1)})$ the functions $\theta(e^{it})$ and $\Omega(e^{it})$ $(\Lambda(e^{it}))$ from Definition 3.6 may be considered as matrix-valued ones. If $\dim\mathfrak{F}^{(1)} = r$ $(\dim\mathfrak{G}^{(1)} = s)$, this enables us to call the matrix function $\Omega(e^{it})$ $(\Lambda(e^{it}))$ a regular upward (leftward) extension of the matrix function $\theta(e^{it})$ by r rows (s columns). Obviously, in this case the identity

$$\operatorname{rank}\Pi_2(e^{it}) \equiv r \quad (\operatorname{rank}\Sigma_1(e^{it}) \equiv s)$$

holds for the function $\theta_2(e^{it})$ $(\theta_1(e^{it}))$ of form (3.17) and (3.18). Taking into account that for a self-adjoint matrix A the equality $\operatorname{rank}A^2 = \operatorname{rank}A$ is valid, we can reformulate Remark 2.30 for the matrix case of regular extensions.

Remark 3.7 Let $\dim\mathfrak{F}^{(1)} = r < \infty(\dim\mathfrak{G}^{(1)} = s < \infty)$. A matrix function $\Omega(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}^{(1)} \oplus \mathfrak{F}](\Lambda(e^{it}) \in CM[\mathfrak{G}^{(1)} \oplus \mathfrak{G}, \mathfrak{F}])$ of form (3.15) is a regular upward (leftward) extension of a matrix function $\theta(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$ by r rows (s columns) iff the equality

$$\operatorname{rank}(I_{\mathfrak{G}} - \theta^*(e^{it})\theta(e^{it})) = r + \operatorname{rank}(I_{\mathfrak{G}} - \theta_{12}^*(e^{it})\theta_{12}(e^{it}) - \theta^*(e^{it})\theta(e^{it}))$$

$$(\operatorname{rank}(I_{\mathfrak{F}} - \theta(e^{it})\theta^*(e^{it})) = s + \operatorname{rank}(I_{\mathfrak{F}} - \theta_{21}(e^{it})\theta_{21}^*(e^{it}) - \theta(e^{it})\theta^*(e^{it})))$$

holds almost everywhere.

It should be pointed out that an operator function $\Omega(e^{it})$ $(\Lambda(e^{it}))$ of form (3.15), being a regular upward (leftward) extension of a contractive operator function $\theta(e^{it})$, does not need to be a regular downward (rightward) extension of the contractive operator function $\theta_{12}(e^{it})$ $(\theta_{21}(e^{it}))$. Indeed, the factorization

$$\theta_{12}(e^{it}) = \tau_2(e^{it})\tau_1(e^{it})(\theta_{21}(e^{it}) = \tau_2(e^{it})\tau_1(e^{it})), \tag{3.19}$$

where

$$\tau_2(e^{it}) := [I_{\mathfrak{F}^{(1)}}, 0], \quad 0 \in [\mathfrak{F}, \mathfrak{F}^{(1)}]; \quad \tau_1(e^{it}) := \Omega(e^{it})$$

$$(\tau_2(e^{it}) := \Lambda(e^{it}); \quad \tau_1(e^{it}) := \begin{bmatrix} I_{\mathfrak{G}^{(1)}} \\ 0 \end{bmatrix}, \quad 0 \in [\mathfrak{G}^{(1)}, \mathfrak{G}]),$$

is not necessarily regular. For example, let $\theta(e^{it}) := \theta_\sigma(e^{it})$, where σ is a coupling of form (2.1) such that $M(\overset{\circ}{\mathfrak{F}}) \not\subset M(\overset{\circ}{\mathfrak{G}})$, and let $(M(\overset{\circ}{\mathfrak{F}}{}^{(1)}), \mathfrak{F}^{(1)}; V_{\mathfrak{F}^{(1)}})$ be a bilateral channel of the coupling σ such that

$$M(\overset{\circ}{\mathfrak{F}}{}^{(1)}) \perp M(\overset{\circ}{\mathfrak{F}}), \quad M(\overset{\circ}{\mathfrak{F}}{}^{(1)}) \subset M(\overset{\circ}{\mathfrak{G}}).$$

Then, by Theorem 3.2 (part (*b*)) and Definition 3.6, the function $\Omega(e^{it})$ of form (3.15) is even completely regular extension of the function $\theta(e^{it})$. At the same time $\Omega(e^{it})$ is not regular download extension of the function $\theta_{12}(e^{it})$ since factorization (3.19) is not regular. The latter is valid by Theorem 3.2, because for the principal part

$$\sigma_{12}^{(1)} := (\mathfrak{H}^{(1)}, \mathfrak{F}^{(1)}, \mathfrak{G}; U^{(1)}, V_{\mathfrak{F}^{(1)}}, V_{\mathfrak{G}}),$$

$$\mathfrak{H}^{(1)} := M(\overset{\circ}{\mathfrak{F}}{}^{(1)}) \vee M(\overset{\circ}{\mathfrak{G}}) = M(\overset{\circ}{\mathfrak{G}}),\ U^{(1)} := U|_{\mathfrak{H}^{(1)}}$$

of the coupling σ_{12} (see Definition 2.3) the channel $(M(\overset{\circ}{\mathfrak{F}}), \mathfrak{F}; V_{\mathfrak{F}})$ is not a channel of the coupling $\sigma_{12}^{(1)}$.

3.2 Bidirectional Regular Extensions

The connection between simultaneous embeddings of both principal channels of a minimal coupling into its other ones and regular factorizations of its scattering suboperator is obtained in [13] (see Theorem 8.5) and can be formulated in the following form.

Theorem 3.8 *Let σ be a minimal unitary coupling of form (2.1) and $\theta(e^{it}) := \theta_\sigma(e^{it})$. There exists a bijective correspondence between pairs of bilateral channels $\{(M(\overset{\circ}{\mathfrak{F}}_1), \mathfrak{F}_1; V_{\mathfrak{F}_1}), (M(\overset{\circ}{\mathfrak{G}}_1), \mathfrak{G}_1; V_{\mathfrak{G}_1})\}$ of the coupling σ satisfying the conditions*

$$M(\overset{\circ}{\mathfrak{F}}_1) \supset M(\overset{\circ}{\mathfrak{F}}), \quad M(\overset{\circ}{\mathfrak{G}}_1) \supset M(\overset{\circ}{\mathfrak{G}}) \tag{3.20}$$

and regular factorizations

$$\theta(e^{it}) = \theta_3(e^{it})\theta_2(e^{it})\theta_1(e^{it}), \tag{3.21}$$

where $\theta_3(e^{it}) \in CM[\mathfrak{F}_1, \mathfrak{F}]$, $\theta_1(e^{it}) \in CM[\mathfrak{G}, \mathfrak{G}_1]$ are coisometric and isometric operator functions, respectively. This correspondence is established by the equali-

ties $\theta_j(e^{it}) = \theta_{\sigma_j}(e^{it})$ $(j = 1, 2, 3)$, *where*

$$\sigma_3 := (M(\overset{\circ}{\mathfrak{F}}_1), \mathfrak{F}, \mathfrak{F}_1; U_3, V_{\mathfrak{F}}, V_{\mathfrak{F}_1}), \quad U_3 := U|_{M(\overset{\circ}{\mathfrak{F}}_1)};$$

$$\sigma_2 := (\mathfrak{H}, \mathfrak{F}_1, \mathfrak{G}_1; U, V_{\mathfrak{F}_1}, V_{\mathfrak{G}_1});$$

$$\sigma_1 := (M(\overset{\circ}{\mathfrak{G}}_1), \mathfrak{G}_1, \mathfrak{G}; U_1, V_{\mathfrak{G}_1}, V_{\mathfrak{G}}), \quad U_1 := U|_{M(\overset{\circ}{\mathfrak{G}}_1)}. \tag{3.22}$$

In the particular case, when

$$\mathfrak{F}_1 = \mathfrak{F}^{(1)} \oplus \mathfrak{F}, \quad V_{\mathfrak{F}_1}|_{\mathfrak{F}} = V_{\mathfrak{F}}, \quad \mathfrak{G}_1 = \mathfrak{G}^{(1)} \oplus \mathfrak{G}, \quad V_{\mathfrak{G}_1}|_{\mathfrak{G}} = V_{\mathfrak{G}}, \tag{3.23}$$

we can represent in the block form the operator functions $\theta_j(e^{it})$ $(j = 1, 2, 3)$ from factorization (3.21) corresponding to this special case of the inclusions of form (3.20). Namely,

$$\theta_3(e^{it}) = [0, I_{\mathfrak{F}}], \quad 0 \in [\mathfrak{F}^{(1)}, \mathfrak{F}]; \quad \theta_2(e^{it}) = \begin{bmatrix} \theta_{11}(e^{it}) & \theta_{12}(e^{it}) \\ \theta_{21}(e^{it}) & \theta(e^{it}) \end{bmatrix};$$

$$\theta_3(e^{it}) = \begin{bmatrix} 0 \\ I_{\mathfrak{G}} \end{bmatrix}, \quad 0 \in [\mathfrak{G}, \mathfrak{G}^{(1)}], \tag{3.24}$$

where

$$\theta_3(e^{it}) := \theta_{\sigma_3}(e^{it}) \in CM[\mathfrak{F}^{(1)} \oplus \mathfrak{F}, \mathfrak{F}], \quad \theta_2(e^{it}) := \theta_{\sigma_2}(e^{it}) \in CM[\mathfrak{G}^{(1)} \oplus \mathfrak{G}, \mathfrak{F}^{(1)} \oplus \mathfrak{F}],$$

$$\theta_1(e^{it}) := \theta_{\sigma_1}(e^{it}) \in CM[\mathfrak{G}, \mathfrak{G}^{(1)} \oplus \mathfrak{G}],$$

σ_j $(j = 1, 2, 3)$ are the couplings of form (3.22) under conditions (3.23),

$$\theta_{12}(e^{it}) := \theta_{\sigma_{12}}(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}^{(1)}], \quad \theta_{21}(e^{it}) := \theta_{\sigma_{21}}(e^{it}) \in CM[\mathfrak{G}^{(1)}, \mathfrak{F}],$$

$$\theta_{11}(e^{it}) := \theta_{\sigma_{11}}(e^{it}) \in CM[\mathfrak{G}^{(1)}, \mathfrak{F}^{(1)}],$$

σ_{12}, σ_{21} are the couplings of forms (3.6) and (3.11), respectively, $V_{\mathfrak{F}^{(1)}} := V_{\mathfrak{F}_1}|_{\mathfrak{F}^{(1)}}$, $V_{\mathfrak{G}^{(1)}} := V_{\mathfrak{G}_1}|_{\mathfrak{G}^{(1)}}$,

$$\sigma_{11} := (\mathfrak{H}, \mathfrak{F}^{(1)}, \mathfrak{G}^{(1)}; U, V_{\mathfrak{F}^{(1)}}, V_{\mathfrak{G}^{(1)}}) \tag{3.25}$$

(the coupling σ_{11} is not necessarily minimal).

Theorem 3.9 *Let σ be a minimal unitary coupling of form* (2.1) *and* $\theta(e^{it}) := \theta_\sigma(e^{it})$.

(a) There exists a bijective correspondence between pairs

$$\{(M(\overset{\circ}{\mathfrak{F}}{}^{(1)}), \mathfrak{F}^{(1)}; V_{\mathfrak{F}^{(1)}}), (M(\overset{\circ}{\mathfrak{G}}{}^{(1)}), \mathfrak{G}^{(1)}; V_{\mathfrak{G}^{(1)}})\}$$

of bilateral channels of coupling σ *satisfying the conditions*

$$M(\overset{\circ}{\mathfrak{F}}{}^{(1)}) \perp M(\overset{\circ}{\mathfrak{F}}), \quad M(\overset{\circ}{\mathfrak{G}}{}^{(1)}) \perp M(\overset{\circ}{\mathfrak{G}}) \tag{3.26}$$

and regular factorizations of the form

$$\theta(e^{it}) = \theta_3(e^{it})\theta_2(e^{it})\theta_1(e^{it}), \tag{3.27}$$

where $\theta_3(e^{it}) \in CM[\mathfrak{F}^{(1)} \oplus \mathfrak{F}, \mathfrak{F}]$, $\theta_2(e^{it}) \in CM[\mathfrak{G}^{(1)} \oplus \mathfrak{G}, \mathfrak{F}^{(1)} \oplus \mathfrak{F}]$, $\theta_1(e^{it}) \in CM[\mathfrak{G}, \mathfrak{G}^{(1)} \oplus \mathfrak{G}]$ *are functions of form* (3.24)*. This correspondence is established by the equalities*

$$\theta_{12}(e^{it}) = \theta_{\sigma_{12}}(e^{it}), \quad \theta_{21}(e^{it}) = \theta_{\sigma_{21}}(e^{it}), \quad \theta_{11}(e^{it}) = \theta_{\sigma_{11}}(e^{it}),$$

where σ_{12}*,* σ_{21}*,* σ_{11} *are unitary couplings of forms* (3.6)*,* (3.11)*, and* (3.25)*, respectively.*

(b) A factorization of the considered type is completely regular iff the corresponding pair of channels $\{(M(\overset{\circ}{\mathfrak{F}}{}^{(1)}), \mathfrak{F}^{(1)}; V_{\mathfrak{F}^{(1)}}), (M(\overset{\circ}{\mathfrak{G}}{}^{(1)}), \mathfrak{G}^{(1)}; V_{\mathfrak{G}^{(1)}})\}$*, in addition to conditions* (3.26)*, also satisfies the conditions*

$$M(\overset{\circ}{\mathfrak{F}}{}^{(1)}) \subset M(\overset{\circ}{\mathfrak{G}}), \quad M(\overset{\circ}{\mathfrak{G}}{}^{(1)}) \subset M(\overset{\circ}{\mathfrak{F}}). \tag{3.28}$$

Proof

(a) The part (a) is a special case of Theorem 3.8.

(b) By definition (see [13], Definition 8.31) a factorization of form (3.27) is completely regular iff both factorizations

$$\Omega(e^{it}) := \theta_2(e^{it})\theta_1(e^{it}), \quad \theta(e^{it}) := \theta_3(e^{it})\Omega(e^{it})$$

are completely regular. Since $\theta_2(e^{it}) = [\Omega_{21}(e^{it}), \Omega(e^{it})]$ where

$$\Omega_{21}(e^{it}) := \begin{bmatrix} \theta_{11}(e^{it}) \\ \theta_{21}(e^{it}) \end{bmatrix} \in CM[\mathfrak{G}^{(1)}, \mathfrak{F}^{(1)} \oplus \mathfrak{F}],$$

$$\Omega(e^{it}) = \begin{bmatrix} \theta_{12}(e^{it}) \\ \theta(e^{it}) \end{bmatrix} \in CM[\mathfrak{G}, \mathfrak{F}^{(1)} \oplus \mathfrak{F}],$$

then, by Theorem 3.4 (part (*b*)), the first factorization is completely regular iff the conditions

$$M(\overset{\circ}{\mathfrak{G}}{}^{(1)}) \perp M(\overset{\circ}{\mathfrak{G}}), \quad M(\overset{\circ}{\mathfrak{G}}{}^{(1)}) \subset M(\overset{\circ}{\mathfrak{F}}{}^{(1)} \oplus \overset{\circ}{\mathfrak{F}})(= M(\overset{\circ}{\mathfrak{F}}{}^{(1)}) \oplus M(\overset{\circ}{\mathfrak{F}})) \tag{3.29}$$

are satisfied. By Theorem 3.2 (part (*b*)), the second factorizations is completely regular iff the conditions

$$M(\overset{\circ}{\mathfrak{F}}{}^{(1)}) \perp M(\overset{\circ}{\mathfrak{F}}), \quad M(\overset{\circ}{\mathfrak{F}}{}^{(1)}) \subset M(\overset{\circ}{\mathfrak{G}}) \tag{3.30}$$

are satisfied. Since the conditions

$$M(\overset{\circ}{\mathfrak{G}}{}^{(1)}) \perp M(\overset{\circ}{\mathfrak{G}}), \quad M(\overset{\circ}{\mathfrak{F}}{}^{(1)}) \subset M(\overset{\circ}{\mathfrak{G}})$$

imply the condition

$$M(\overset{\circ}{\mathfrak{F}}{}^{(1)}) \perp M(\overset{\circ}{\mathfrak{G}}{}^{(1)}), \tag{3.31}$$

we obtain that the inclusions $M(\overset{\circ}{\mathfrak{G}}{}^{(1)}) \subset M(\overset{\circ}{\mathfrak{F}})$ and $M(\overset{\circ}{\mathfrak{G}}{}^{(1)}) \subset M(\overset{\circ}{\mathfrak{F}}{}^{(1)} \oplus \overset{\circ}{\mathfrak{F}})$ are equivalent. Consequently, the conditions (3.29) and (3.30) are equivalent to the conditions (3.26) and (3.28). □

Remark 3.10 Taking into account Remarks 3.3 and 3.5, we conclude that a factorizations of form (3.27), where

$$\theta_3(e^{it}) \in CM[\mathfrak{F}^{(1)} \oplus \mathfrak{F}, \mathfrak{F}], \quad \theta_2(e^{it}) \in CM[\mathfrak{G}^{(1)} \oplus \mathfrak{G}, \mathfrak{F}^{(1)} \oplus \mathfrak{F}],$$

$$\theta_1(e^{it}) \in CM[\mathfrak{G}, \mathfrak{G}^{(1)} \oplus \mathfrak{G}]$$

are functions of forms (3.24), is completely regular iff $\theta_{12}(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}^{(1)}]$ is a coisometric operator function, $\theta_{21}(e^{it}) \in CM[\mathfrak{G}^{(1)}, \mathfrak{F}]$ is an isometric operator function, and $\theta_{11}(e^{it}) \equiv 0 \in [\mathfrak{G}^{(1)}, \mathfrak{F}^{(1)}]$. Moreover, in this case, the inclusions $\mathrm{Ran}\theta_{12}^* \subset \mathrm{Ker}\theta$ and $\mathrm{Ran}\theta_{21} \subset \mathrm{Ker}\theta^*$ hold.

Definition 3.11 A function

$$\Xi(e^{it}) := \begin{bmatrix} \theta_{11}(e^{it}) & \theta_{12}(e^{it}) \\ \theta_{21}(e^{it}) & \theta(e^{it}) \end{bmatrix} \in CM[\mathfrak{G}^{(1)} \oplus \mathfrak{G}, \mathfrak{F}^{(1)} \oplus \mathfrak{F}] \tag{3.32}$$

will be called a regular up-leftward extension of a function $\theta(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$ if the factorization

$$\theta(e^{it}) = \theta_3(e^{it})\theta_2(e^{it})\theta_1(e^{it}), \tag{3.33}$$

where

$$\theta_3(e^{it}) := [0, I_{\mathfrak{F}}], \quad 0 \in [\mathfrak{F}^{(1)}, \mathfrak{F}]; \quad \theta_2(e^{it}) := \Xi(e^{it});$$

$$\theta_1(e^{it}) := \begin{bmatrix} 0 \\ I_{\mathfrak{G}} \end{bmatrix}, \quad 0 \in [\mathfrak{G}, \mathfrak{G}^{(1)}],$$

is regular. It will be called a completely regular up-leftward extension if factorization (3.33) is completely regular. A regular upward (leftward) extension $\Omega(e^{it})(\Lambda(e^{it}))$ of form (3.15) will be considered as a special case of a regular up-leftward extension under the condition $\mathfrak{G}^{(1)} = \{0\}(\mathfrak{F}^{(1)} = \{0\})$. In the case of $\mathfrak{F}^{(1)} = \{0\}$ and $\mathfrak{G}^{(1)} = \{0\}$ we will call $\Xi(e^{it}) := \theta(e^{it})$ the trivial regular up-leftward extension.

A regular up-leftward extension $\Xi(e^{it})$ of form (3.32) will be called isometric (coisometric, unitary) if it is an isometric (coisometric, unitary) operator function.

Regular extensions of the just defined type, as well as the three more types (up-rightward, down-leftward and down-rightward extensions) that differ from the first only by permutatious of the matrix blocks, will be called bidirectional regular extensions.

As in the unidirectional case, it can be similarly showed that an operator function $\Xi(e^{it})$ of form (3.32), being a regular up-leftward extension of a contractive operator function $\theta(e^{it})$, does not need to be a regular extension (in the corresponding directions) for each of the other blocks $\theta_{12}(e^{it})$, $\theta_{21}(e^{it})$ and $\theta_{11}(e^{it})$ in representation (3.32).

Remark 3.12 As it follows from Definitions 3.6 and 3.11, Theorems 3.2, 3.4, and 3.9, there is a bijective correspondence between pairs of unidirectional regular extensions

$$\Omega(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}^{(1)} \oplus \mathfrak{F}], \quad \Lambda(e^{it}) \in CM[\mathfrak{G}^{(1)} \oplus \mathfrak{G}, \mathfrak{F}]$$

of form (3.15) for a function $\theta(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$ and bidirectional regular extensions $\Xi(e^{it}) \in CM[\mathfrak{G}^{(1)} \oplus \mathfrak{G}, \mathfrak{F}^{(1)} \oplus \mathfrak{F}]$ of form (3.32) for the same function $\theta(e^{it})$. This means that for a given function $\theta(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$ there exists a bijective correspondence between pairs of functions $\theta_{12}(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}^{(1)}]$, $\theta_{21}(e^{it}) \in CM[\mathfrak{G}^{(1)}, \mathfrak{F}]$ from representations (3.15) and functions $\theta_{11}(e^{it}) \in CM[\mathfrak{G}^{(1)}, \mathfrak{F}^{(1)}]$ from representation (3.32).

4 Descriptions of the Sets of Unidirectional and Bidirectional Regular Extensions

Denote by $\mathcal{U}_r(\theta)$ $(\mathcal{L}_r(\theta))$ or simply $\mathcal{U}_r$ $(\mathcal{L}_r)$ the set of regular upward (leftward) extensions $\Omega(e^{it}) \in L^\infty[\mathfrak{G}, \mathfrak{F}^{(1)} \oplus \mathfrak{F}]$ $(\Lambda(e^{it}) \in L^\infty[\mathfrak{G}^{(1)} \oplus \mathfrak{G}, \mathfrak{F}])$ of form (3.15) for a function $\theta(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$. Its subset of all completely regular upward (leftward) extensions will be denoted by $\mathcal{U}_{cr}(\theta)$ ($\mathcal{L}_{cr}(\theta)$) or simply $\mathcal{U}_{cr}$ ($\mathcal{L}_{cr}$). We also denote by $\mathcal{K}_r(\theta)$ or simply $\mathcal{K}_r$ the set of regular up-leftward extensions $\Xi(e^{it}) \in L^\infty[\mathfrak{G}^{(1)} \oplus \mathfrak{G}, \mathfrak{F}^{(1)} \oplus \mathfrak{F}]$ of form (3.32) for the function $\theta(e^{it})$ and by $\mathcal{K}_{cr}(\theta)$ or simply $\mathcal{K}_{cr}$ its subset of all completely regular up-leftward extensions. In this section we parameterize all these sets in both unidirectional and bidirectional cases.

4.1 Unidirectional Case

Realizing the unity coupling $\varepsilon := (L^2(\mathfrak{K}), \mathfrak{K}, \mathfrak{K}; U_{\mathfrak{K}}^\times, V_{\mathfrak{K}}, V_{\mathfrak{K}})$ in the Hilbert space $L^2(\mathfrak{K})$, where $V_{\mathfrak{K}}$ is the inclusion operator of $\mathfrak{K}$ into $L^2(\mathfrak{K})$, we can speak for simplicity about channels of the coupling ε as channels in $L^2(\mathfrak{K})$. A description of the set of all such channels was given in [12] (Theorem 7.10), and unitary couplings generated by pairs of such channels were also studied there (Theorem 7.26). Inclusions into each other of subspaces of $L^2(\mathfrak{K})$ reducing the operator $U_{\mathfrak{K}}^\times$ and subspaces $\mathfrak{L}$ of $L^2(\mathfrak{K})$, where a bilateral channel of the form $(\mathfrak{L}, \mathfrak{N}; V_{\mathfrak{N}})$ can be realized, were explored in [12] as well (Theorem 7.22 and Corollary 7.23). Some of these results can be formulated in the following form.

Theorem 4.1

(a) There exists a bijective correspondence between bilateral channels $(\mathfrak{L}, \mathfrak{N}; V_{\mathfrak{N}})$ *in* $L^2(\mathfrak{K})$ *and isometric operator functions* $\theta(e^{it}) \in CM[\mathfrak{N}, \mathfrak{K}]$. *This correspondence is established by the formulas*

$$\mathfrak{L} = \mathrm{Ran}\theta, \quad V_{\mathfrak{N}} = \theta|_{\mathfrak{N}}; \quad \theta = (\Phi_{U_{\mathfrak{K}}^\times}^{\mathfrak{N}})^*.$$

(b) Let $(\mathfrak{L}_2, \mathfrak{F}; V_{\mathfrak{F}})$ *and* $(\mathfrak{L}_1, \mathfrak{G}; V_{\mathfrak{G}})$ *be two channels in* $L^2(\mathfrak{K})$, $\theta_2(e^{it}) \in CM[\mathfrak{F}, \mathfrak{K}]$ *and* $\theta_1(e^{it}) \in CM[\mathfrak{G}, \mathfrak{K}]$ *be the corresponding pair of isometric operator functions, and let* $\sigma := (L^2(\mathfrak{K}), \mathfrak{F}, \mathfrak{G}; U_{\mathfrak{K}}^\times, V_{\mathfrak{F}}, V_{\mathfrak{G}})$. *Then*

$$\theta_\sigma(e^{it}) = \theta_2^*(e^{it})\theta_1(e^{it}).$$

The function $\theta_\sigma(e^{it})$ *is an isometric (coisometric, unitary) operator function iff* $\mathfrak{L}_1 \subset \mathfrak{L}_2 (\mathfrak{L}_1 \supset \mathfrak{L}_2, \mathfrak{L}_1 = \mathfrak{L}_2)$.

(c) Let $\mathfrak{M}$ *be a subspace of* $L^2(\mathfrak{K})$ *reducing the operator* $U_{\mathfrak{K}}^\times$. *There exists a channel* $(\mathfrak{L}, \mathfrak{N}; V_{\mathfrak{N}})$ *in* $L^2(\mathfrak{K})$ *such that*

$$\mathfrak{L} \subset \mathfrak{M} \quad \textit{and} \quad \dim\mathfrak{N} = \gamma$$

iff $0 \le \gamma \le \alpha_{\mathfrak{M}}$. *Moreover, the equality* $\mathfrak{L} = \mathfrak{M}$ *is possible iff* $\alpha_{\mathfrak{M}} = \beta_{\mathfrak{M}} = \gamma$, *that is, when* $\rho_{\mathfrak{M}}(e^{it}) = \gamma$ *almost everywhere.*

Theorem 4.2 *Let* $\theta(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$.

(a) *A nontrivial regular upward (leftward) extension of the function* $\theta(e^{it})$ *exists iff* $\alpha_\Pi > 0(\alpha_\Sigma > 0)$.

Let $\Omega(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}^{(1)} \oplus \mathfrak{F}](\Lambda(e^{it}) \in CM[\mathfrak{G}^{(1)} \oplus \mathfrak{G}, \mathfrak{F}])$ *be an operator function of form* (3.15).

(b) $\Omega(e^{it}) \in \mathcal{U}_r(\theta)(\Lambda(e^{it}) \in \mathcal{L}_r(\theta))$ *iff there exists a coisometric (isometric) operator function* $\omega(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}^{(1)}](\lambda(e^{it}) \in CM[\mathfrak{G}^{(1)}, \mathfrak{F}])$ *such that the inclusion*

$$\mathrm{Ran}\omega^* \subset \overline{\Pi L^2(\mathfrak{G})} \quad (\mathrm{Ran}\lambda \subset \overline{\Sigma L^2(\mathfrak{F})}) \tag{4.1}$$

holds and the function $\theta_{12}(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}^{(1)}]$ $(\theta_{21}(e^{it}) \in CM[\mathfrak{G}^{(1)}, \mathfrak{F}])$ *admits the representation of the form*

$$\theta_{12}(e^{it}) = \omega(e^{it})\Pi(e^{it}) \ a.e. \quad (\theta_{21}(e^{it}) = \Sigma(e^{it})\lambda(e^{it}) \ a.e.). \tag{4.2}$$

(c) $\Omega(e^{it}) \in \mathcal{U}_{cr}(\theta)$ $(\Lambda(e^{it}) \in \mathcal{L}_{cr}(\theta))$ *iff the corresponding coisometric (isometric) operator function* $\omega(e^{it})(\lambda(e^{it}))$ *satisfies the condition*

$$\mathrm{Ran}\omega^* \subset \mathrm{Ker}\theta \quad (\mathrm{Ran}\lambda \subset \mathrm{Ker}\theta^*). \tag{4.3}$$

In this case, equality (4.2) *takes the form*

$$\theta_{12}(e^{it}) = \omega(e^{it}) \ a.e. \quad (\theta_{21}(e^{it}) = \lambda(e^{it}) \ a.e.). \tag{4.4}$$

(d) *There exists an isometric (coisometric) extension* $\Omega(e^{it}) \in \mathcal{U}_r(\theta)(\Lambda(e^{it}) \in \mathcal{L}_r(\theta))$ *iff* $\rho_\Pi(e^{it})(\rho_\Sigma(e^{it}))$ *is constant almost everywhere. All such extensions are given in this case by formulas* (3.15) *and* (4.2)*, where for coisometric (isometric) functions* $\omega(e^{it})(\lambda(e^{it}))$ *inclusion* (4.1) *turns into the equality*

$$\mathrm{Ran}\omega^* = \overline{\Pi L^2(\mathfrak{G})} \quad (\mathrm{Ran}\lambda = \overline{\Sigma L^2(\mathfrak{F})}). \tag{4.5}$$

These extensions are unitary iff the function $\theta(e^{it})$ *is coisometric (isometric). In this case, they are completely regular, representation* (4.2) *is replaced by* (4.4) *and condition* (4.5) *turns into*

$$\mathrm{Ran}\omega^* = \mathrm{Ker}\theta \quad (\mathrm{Ran}\lambda = \mathrm{Ker}\theta^*).$$

Proof It suffices to prove the theorem for regular upward extensions.

(*a*) For the function $\theta(e^{it})$ we consider the functional model $\hat{\sigma}$ of a minimal unitary coupling of form (2.11) described in Theorem 2.11. For convenience, we use the other order of the components in the orthogonal decomposition of $\hat{\mathfrak{H}}$, namely, $\hat{\mathfrak{H}} := \overline{\Pi L^2(\mathfrak{G})} \oplus L^2(\mathfrak{F})$. Notice that the principal bilateral channel

$(\hat{M}(\overset{\circ}{\mathfrak{G}}), \mathfrak{G}; \hat{V}_{\mathfrak{G}})$ satisfies the conditions

$$\hat{M}(\overset{\circ}{\mathfrak{G}}) = \mathrm{Ran}\hat{X}, \quad \hat{V}_{\mathfrak{G}} = \hat{X}|_{\mathfrak{G}},$$

where, by Theorem 4.1 (part (a)), $\hat{X}(e^{it}) \in CM[\mathfrak{G}, \mathfrak{K}]$, $\mathfrak{K} := \mathfrak{G} \oplus \mathfrak{F}$, is an isometric operator function given in the block form by the formula

$$\hat{X}(e^{it}) := \begin{bmatrix} \Pi(e^{it}) \\ \theta(e^{it}) \end{bmatrix}. \tag{4.6}$$

By virtue of Definition 3.6 and Theorem 3.2 (part (a)), a nontrivial regular upward extension of the function $\theta(e^{it})$ exists iff there exists a nontrivial bilateral channel $(\hat{M}(\overset{\circ}{\mathfrak{F}}{}^{(1)}), \mathfrak{F}^{(1)}; \hat{V}_{\mathfrak{F}^{(1)}})$ of the coupling $\hat{\sigma}$ such that $\hat{M}(\overset{\circ}{\mathfrak{F}}{}^{(1)}) \perp \hat{M}(\overset{\circ}{\mathfrak{F}})$, that is, $\hat{M}(\overset{\circ}{\mathfrak{F}}{}^{(1)}) \subset \overline{\Pi L^2(\mathfrak{G})}$. By Theorem 4.1 (part (c)), applied to the defect subspace $\mathfrak{M} := \overline{\Pi L^2(\mathfrak{G})}$ of the operator θ, we infer that this is possible iff $\alpha_\Pi > 0$.

(b) By Theorem 4.1 (part (a)), there exists a bijective correspondence between bilateral channels $(\hat{M}(\overset{\circ}{\mathfrak{F}}{}^{(1)}), \mathfrak{F}^{(1)}; \hat{V}_{\mathfrak{F}^{(1)}})$ of the coupling $\hat{\sigma}$ satisfying the condition $\hat{M}(\overset{\circ}{\mathfrak{F}}{}^{(1)}) \subset \mathfrak{M}$ and isometric operator functions $\hat{Y}(e^{it}) \in CM[\mathfrak{F}^{(1)}, \mathfrak{K}]$ satisfying the conditions $\mathrm{Ran}\hat{Y} = \hat{M}(\overset{\circ}{\mathfrak{F}}{}^{(1)})$, $\hat{Y}|_{\mathfrak{F}^{(1)}} = \hat{V}_{\mathfrak{F}^{(1)}}$. It follows from the latter inclusion that the function $\hat{Y}(e^{it})$ can be represented in the following block form

$$\hat{Y}(e^{it}) = \begin{bmatrix} \omega^*(e^{it}) \\ 0 \end{bmatrix}, \quad 0 \in [\mathfrak{F}^{(1)}, \mathfrak{F}], \tag{4.7}$$

where $\omega(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}^{(1)}]$ is a coisometric operator function satisfying condition (4.1). Thereby, in view of Theorem 3.2 (part (a)), a bijective correspondence between $\Omega(e^{it}) \in \mathcal{U}_r$ and coisometric operator function $\omega(e^{it})$ satisfying condition (4.1) is obtained. On the other hand, by the same theorem, there exists a bijective correspondence between $\Omega(e^{it}) \in \mathcal{U}_r$ of form (3.15) and $\theta_{12}(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}^{(1)}]$ that is established by the equality $\theta_{12}(e^{it}) = \theta_{\hat{\sigma}_{12}}(e^{it})$, where the coupling $\hat{\sigma}_{12}$ is of type (3.6), namely,

$$\hat{\sigma}_{12} := (\hat{\mathfrak{H}}, \mathfrak{F}^{(1)}, \mathfrak{G}; \hat{U}^{\times}, \hat{V}_{\mathfrak{F}^{(1)}}, \hat{V}_{\mathfrak{G}}).$$

By Theorem 4.1 (part (b)), from (4.6) and (4.7) we obtain

$$\theta_{12}(e^{it}) = \hat{Y}^*(e^{it})\hat{X}(e^{it}) = \omega(e^{it})\Pi(e^{it}) \quad \text{a.e.}.$$

(*c*) Taking into account Remark 3.3 and part (*b*) of the present theorem, we see that the function $\theta_{12}(e^{it})$ is coisometric under conditions (4.1)–(4.2) iff condition (4.3) is satisfied. From this equality (4.4) follows.

(*d*) Recall that a function $\Omega(e^{it}) \in \mathcal{U}_r$ of form (3.15) is the scattering suboperator of the coupling of type (3.3), namely, $\hat{\sigma}_1 := (\hat{\mathfrak{H}}, \mathfrak{F}_1, \mathfrak{G}; \hat{U}^\times, \hat{V}_{\mathfrak{F}_1}, \hat{V}_{\mathfrak{G}})$, where $\hat{\mathfrak{H}}$, $\mathfrak{G}$, $\hat{U}^\times$ and $\hat{V}_{\mathfrak{G}}$ are the same as in $\hat{\sigma}$, $\mathfrak{F}_1 := \mathfrak{F}^{(1)} \oplus \mathfrak{F}$, $\hat{V}_{\mathfrak{F}_1} := \hat{V}_{\mathfrak{F}^{(1)}} P_{\mathfrak{F}^{(1)}} + \hat{V}_{\mathfrak{F}} P_{\mathfrak{F}}$, $P_{\mathfrak{F}^{(1)}}$ and $P_{\mathfrak{F}}$ are the orthoprojections of $\mathfrak{F}_1$ onto $\mathfrak{F}^{(1)}$ and $\mathfrak{F}$, respectively. By Theorem 4.1 (part (*b*)), $\Omega(e^{it})$ is an isometric function iff $\hat{M}(\mathring{\mathfrak{F}}_1) = \hat{\mathfrak{H}}$. In view of the inclusion $\hat{M}(\mathring{\mathfrak{F}}^{(1)}) \subset \overline{\Pi L^2(\mathfrak{G})}$, this equality is possible iff the equality $\hat{M}(\mathring{\mathfrak{F}}^{(1)}) = \overline{\Pi L^2(\mathfrak{G})}$ holds. By Theorem 4.1 (part (*c*)), the existence of such a channel $(\hat{M}(\mathring{\mathfrak{F}}^{(1)}), \mathfrak{F}^{(1)}; \hat{V}_{\mathfrak{F}^{(1)}})$ is equivalent to constancy of the rank function $\rho_\Pi(e^{it})$ almost everywhere. Taking into account that

$$\mathrm{Ran}\omega^* = \mathrm{Ran}Y^* = \hat{M}(\mathring{\mathfrak{F}}^{(1)}),$$

we see that $\Omega(e^{it}) \in \mathcal{U}_r$ is isometric iff inclusion (4.1) turns into equality (4.5).

The isometric extension $\Omega(e^{it}) \in \mathcal{U}_r$ is unitary iff, besides $\hat{M}(\mathring{\mathfrak{F}}_1) = \hat{\mathfrak{H}}$, the equality $\hat{M}(\mathring{\mathfrak{G}}) = \hat{\mathfrak{H}}$ is valid (see Theorem 4.1 (part (*b*)). But the latter is also equivalent, by the same theorem, to the coisometricity of the function $\theta(e^{it})$ $(= \theta_{\hat{\sigma}}(e^{it}))$. In this case, $\overline{\Pi L^2(\mathfrak{G})} = \mathrm{Ker}\theta$ and, by part (*c*) of the present theorem, all such extensions are completely regular, (4.2) turns into (4.4), and (4.5) takes the form $\mathrm{Ran}\omega^* = \mathrm{Ker}\theta$. □

Corollary 4.3 *Let* $\theta(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$ *and* $\Omega(e^{it}) \in \mathcal{U}_r(\theta)(\Lambda(e^{it}) \in \mathcal{L}_r(\theta))$ *be a regular extension of form* (3.15)*. Then*

$$\overline{\theta_{12} L^2(\mathfrak{G})} = L^2(\mathfrak{F}^{(1)}) \quad (\overline{\theta_{21}^* L^2(\mathfrak{F})} = L^2(\mathfrak{G}^{(1)})).$$

Proof This assertion is a direct corollary of representation (4.2), where $\omega(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}^{(1)}]$ $(\lambda(e^{it}) \in CM[\mathfrak{G}^{(1)}, \mathfrak{F}])$ is a coisometric (isometric) function satisfying condition (4.1) □

Remark 4.4 As is known (see [12], Subsection 7.7), in the case of $\theta(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$ that is strictly contractive almost everywhere (or, equivalently, in the case of a non-degenerate coupling $\hat{\sigma}$) both equalities

$$\overline{\Pi L^2(\mathfrak{G})} = L^2(\mathfrak{G}), \quad \overline{\Sigma L^2(\mathfrak{F})} = L^2(\mathfrak{F})$$

hold. This means that in this case an isometric extension $\Omega(e^{it}) \in \mathcal{U}_r$, as well as a coisometric one $\Lambda(e^{it}) \in \mathcal{L}_r$, exist. Note also that in this case, in view of the triviality of the conditions (4.1), the sets $\mathcal{U}_r(\theta)$ and $\mathcal{L}_r(\theta)$are parameterized by arbitrary coisometric functions $\omega(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}^{(1)}]$ and isometric ones $\lambda(e^{it}) \in CM[\mathfrak{G}^{(1)}, \mathfrak{F}]$ in equalities (4.2).

4.2 Bidirectional Case

Parameterizations of the sets $\mathcal{K}_r(\theta)$ and $\mathcal{K}_{cr}(\theta)$ are described in

Theorem 4.5 *Let $\theta(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$.*

(a) A nontrivial regular up-leftward extension of the function $\theta(e^{it})$ exists iff at least one of the two conditions $\alpha_\Pi > 0$ or $\alpha_\Sigma > 0$ is satisfied.

Let $\Xi(e^{it}) \in CM[\mathfrak{G}^{(1)} \oplus \mathfrak{G}, \mathfrak{F}^{(1)} \oplus \mathfrak{F}]$ be a function of form (3.32).

(b) $\Xi(e^{it}) \in \mathcal{K}_r(\theta)$ iff there exist a coisometric operator function $\omega(e^{it} \in CM[\mathfrak{G}, \mathfrak{F}^{(1)}]$ and an isometric operator function $\lambda(e^{it}) \in CM[\mathfrak{G}^{(1)}, \mathfrak{F}]$ such that the inclusions

$$\mathrm{Ran}\omega^* \subset \overline{\Pi L^2(\mathfrak{G})}, \quad \mathrm{Ran}\lambda \subset \overline{\Sigma L^2(\mathfrak{F})} \tag{4.8}$$

hold and the functions

$$\theta_{12}(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}^{(1)}],\ \theta_{21}(e^{it}) \in CM[\mathfrak{G}^{(1)}, \mathfrak{F}],\ \theta_{11}(e^{it}) \in CM[\mathfrak{G}^{(1)}, \mathfrak{F}^{(1)}]$$

admit the representations of the forms

$$\theta_{12}(e^{it}) = \omega(e^{it})\Pi(e^{it}), \theta_{21}(e^{it}) = \Sigma(e^{it})\lambda(e^{it}), \theta_{11}(e^{it}) = -\omega(e^{it})\theta^*(e^{it})\lambda(e^{it}). \tag{4.9}$$

(c) $\Xi(e^{it}) \in \mathcal{K}_{cr}(\theta)$ iff the corresponding pair $\{\omega(e^{it}), \lambda(e^{it})\}$ of coisometric and isometric operator functions satisfies the conditions

$$\mathrm{Ran}\omega^* \subset \mathrm{Ker}\theta, \quad \mathrm{Ran}\lambda \subset \mathrm{Ker}\theta^*. \tag{4.10}$$

In this case, equalities (4.9) take the forms

$$\theta_{12}(e^{it}) = \omega(e^{it}), \quad \theta_{21}(e^{it}) = \lambda(e^{it}), \quad \theta_{11}(e^{it}) \equiv 0 \quad (0 \in [\mathfrak{G}^{(1)}, \mathfrak{F}^{(1)}]). \tag{4.11}$$

(d) There exists an isometric (coisometric, unitary) extension $\Xi(e^{it}) \in \mathcal{K}_r(\theta)$ iff of the two rank functions $\rho_\Pi(e^{it})$ and $\rho_\Sigma(e^{it})$ the first (the second, both) is (is, are) constant almost everywhere. All such extensions are given in this case by formulas (3.32) and (4.9), where for the corresponding pair $\{\omega(e^{it}), \lambda(e^{it})\}$ of the two following conditions

$$\mathrm{Ran}\omega^* = \overline{\Pi L^2(\mathfrak{G})}, \quad \mathrm{Ran}\lambda = \overline{\Sigma L^2(\mathfrak{F})}$$

the first (the second, both) is (is, are) satisfied.

Proof

(a) This part follows from the definition of the trivial regular up-leftward extension (see Definition 3.11) and Theorem 4.2 (part (a)).

(b) As noted in Remark 3.12, there exists a bijective correspondence between $\Xi(e^{it}) \in \mathcal{K}_r$ and pairs $\Omega(e^{it}) \in \mathcal{U}_r$, $\Lambda(e^{it}) \in \mathcal{L}_r$. Consequently, by Theorem 4.2 (part (b)), there exists a bijective correspondence between extensions $\Xi(e^{it}) \in \mathcal{K}_r$ and pairs $\{\omega(e^{it}), \lambda(e^{it})\}$ of coisometric and isometric functions, respectively, satisfying conditions (4.8). Moreover, the first two equalities (4.9) are valid.

It remains to prove the third equality (4.9). For this, along with the functional model $\hat{\sigma}$ of a minimal unitary coupling of form (2.11) used in the proof of Theorem 4.2 (part (a)), we also considered the functional model $\tilde{\sigma}$ of form (2.13) described in Theorem 2.12. By Theorem 2.13, the couplings $\hat{\sigma}$ and $\tilde{\sigma}$ are unitarily equivalent. Taking into account that here the decompositions

$$\hat{\mathfrak{H}} := \overline{\Pi L^2(\mathfrak{G})} \oplus L^2(\mathfrak{F}), \quad \tilde{\mathfrak{H}} := \overline{\Sigma L^2(\mathfrak{F})} \oplus L^2(\mathfrak{G})$$

are used (see the proof of Theorem 4.2 (part (a))), this equivalence is established by the restriction $W|_{\hat{\mathfrak{H}}} \in [\hat{\mathfrak{H}}, \tilde{\mathfrak{H}}]$, where for $W(e^{it})$ formula (2.14) now takes the form

$$W(e^{it}) := \begin{bmatrix} -\theta(e^{it}) & \Sigma(e^{it}) \\ \Pi(e^{it}) & \theta^*(e^{it}) \end{bmatrix}.$$

According to Definition 3.11 and Theorem 3.9 (part (a)), the block $\theta_{11} \in L^\infty[\mathfrak{G}^{(1)}, \mathfrak{F}^{(1)}]$ of an extension $\Xi(e^{it}) \in \mathcal{K}_r$ of form (3.32) is the scattering suboperator of the coupling $\tilde{\sigma}_{11}$ of type (3.25), namely, $\tilde{\sigma}_{11} := (\tilde{\mathfrak{H}}, \mathfrak{F}^{(1)}, \mathfrak{G}^{(1)}; \tilde{U}^\times, \tilde{V}_{\mathfrak{F}^{(1)}}, \tilde{V}_{\mathfrak{G}^{(1)}})$. This coupling is generated by the pair

$$\{(\tilde{M}(\overset{\circ}{\mathfrak{F}}{}^{(1)}), \mathfrak{F}^{(1)}; \tilde{V}_{\mathfrak{F}^{(1)}}), \ (\tilde{M}(\overset{\circ}{\mathfrak{G}}{}^{(1)}), \mathfrak{G}^{(1)}; \tilde{V}_{\mathfrak{G}^{(1)}})\} \tag{4.12}$$

of bilateral channels of the coupling $\tilde{\sigma}$ satisfying the conditions of type (3.26), namely,

$$\tilde{M}(\overset{\circ}{\mathfrak{F}}{}^{(1)}) \perp \tilde{M}(\overset{\circ}{\mathfrak{F}}), \quad \tilde{M}(\overset{\circ}{\mathfrak{G}}{}^{(1)}) \perp \tilde{M}(\overset{\circ}{\mathfrak{G}}), \tag{4.13}$$

where $\{(\tilde{M}(\overset{\circ}{\mathfrak{F}}), \mathfrak{F}; \tilde{V}_{\mathfrak{F}}), (\tilde{M}(\overset{\circ}{\mathfrak{G}}), \mathfrak{G}; \tilde{V}_{\mathfrak{G}})\}$ is the pair of the principal bilateral channels of the coupling $\tilde{\sigma}$. By Theorem 4.1 (part (a)), there exist the unique isometric operator functions $\tilde{Y}(e^{it}) \in CM[\mathfrak{F}^{(1)}, \mathfrak{F} \oplus \mathfrak{G}]$ and $\tilde{Z}(e^{it}) \in CM[\mathfrak{G}^{(1)}, \mathfrak{F} \oplus \mathfrak{G}]$ corresponding to the bilateral channels of form (4.12), respectively. We express them in terms $\theta(e^{it})$, $\omega(e^{it})$ and $\lambda(e^{it})$.

Consider the bilateral channel $(\hat{M}(\overset{\circ}{\mathfrak{F}}^{(1)}), \mathfrak{F}^{(1)}; \hat{V}_{\mathfrak{F}^{(1)}})$ of the coupling $\hat{\sigma}$ that corresponds to the first channel (4.12) of the coupling $\tilde{\sigma}$ in view of the unitary equivalence of these couplings. As was shown in the proof of Theorem 4.2 (part (*b*)), the isometric function $\hat{Y}(e^{it}) \in CM[\mathfrak{F}^{(1)}, \mathfrak{G} \oplus \mathfrak{F}]$, generating this channel by Theorem 4.1 (part (*a*)), is given by the block matrix of form (4.7). From this it follows that

$$\tilde{Y}(e^{it}) = W(e^{it})\hat{Y}(e^{it}) = \begin{bmatrix} -\theta(e^{it}) & \Sigma(e^{it}) \\ \Pi(e^{it}) & \theta^*(e^{it}) \end{bmatrix} \begin{bmatrix} \omega^*(e^{it}) \\ 0 \end{bmatrix} = \begin{bmatrix} -\theta(e^{it})\omega^*(e^{it}) \\ \Pi(e^{it})\omega^*(e^{it}) \end{bmatrix}. \tag{4.14}$$

The function $\tilde{Z}(e^{it})$ corresponding to the second channel (4.12) can be obtained by the dual arguments in relation to those made in the proof of Theorem 4.2 (part (*b*)) for the function $\hat{Y}(e^{it})$. Thus, $\tilde{Z}(e^{it})$ can be given in the following block form

$$\tilde{Z}(e^{it}) := \begin{bmatrix} \lambda(e^{it}) \\ 0 \end{bmatrix}, \quad 0 \in [\mathfrak{G}^{(1)}, \mathfrak{G}], \tag{4.15}$$

where $\lambda(e^{it}) \in CM[\mathfrak{G}^{(1)}, \mathfrak{F}]$ is an isometric function corresponding to $\Lambda(e^{it}) \in \mathcal{L}_r$.

By Theorem 4.1 (part (*b*)), from (4.14) and (4.15) we obtain

$$\theta_{11}(e^{it}) = \theta_{\tilde{\sigma}_{11}}(e^{it}) = \tilde{Y}^*(e^{it})\tilde{Z}(e^{it}) = -\omega(e^{it})\theta^*(e^{it})\lambda(e^{it}) \quad \text{a.e..}$$

(*c*) Part (*c*) follows from part (*c*) of Theorem 4.2 and Remark 3.10.

(*d*) By Theorem 4.1 (part (*b*)), the isometricity (coisometricity) of an extension $\Xi(e^{it}) \in \mathcal{K}_r$ of form (3.32) is equivalent to the isometricity (coisometricity) of the corresponding extension $\Omega(e^{it}) \in \mathcal{U}_r$ $(\Lambda(e^{it}) \in \mathcal{L}_r)$ of form (3.15). Really, as was noted in the proof of part (*d*) of Theorem 4.2, the isometricity of $\Omega(e^{it})$ is tantamount to the equality $\hat{M}(\overset{\circ}{\mathfrak{F}}^{(1)} \oplus \mathfrak{F}) = \hat{\mathfrak{H}}$, where $\hat{\mathfrak{H}} = \overline{\Pi L^2(\mathfrak{G})} \oplus L^2(\mathfrak{F})$ is the space from $\hat{\sigma}_1$ of type (3.3) with $\mathfrak{F}_1 := \mathfrak{F}^{(1)} \oplus \mathfrak{F}$. At the same time the isometricity of $\Xi(e^{it})$ is tantamount to the same condition, where $\hat{\mathfrak{H}}$ is now the space from the coupling $\hat{\sigma}_2$ of type (3.22), namely,

$$\hat{\sigma}_2 := (\hat{\mathfrak{H}}, \mathfrak{F}_1\mathfrak{G}_1; \hat{U}^{\times}, \hat{V}_{\mathfrak{F}_1}, \hat{V}_{\mathfrak{G}_1}), \quad \mathfrak{F}_1 := \mathfrak{F}^{(1)} \oplus \mathfrak{F}, \quad \mathfrak{G}_1 := \mathfrak{G}^{(1)} \oplus \mathfrak{G}.$$

Similar arguments are valid in the dual case. Hence, part (*d*) of the present theorem follows from part (*d*) of Theorem 4.2. □

Remark 4.6 Part (*b*) of Theorem 4.5 can be reformulated in the following way.

The general form of a regular up-leftward extension

$$\Xi(e^{it}) \in CM[\mathfrak{G}^{(1)} \oplus \mathfrak{G}, \mathfrak{F}^{(1)} \oplus \mathfrak{F}]$$

of a function $\theta(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$ is given by the formula

$$\Xi(e^{it}) = \mathrm{A}(e^{it})\Xi_0(e^{it})\mathrm{B}(e^{it}), \tag{4.16}$$

where $\Xi_0(e^{it}) \in CM[\mathfrak{F} \oplus \mathfrak{G}, \mathfrak{G} \oplus \mathfrak{F}]$ is the unitary operator function of the form

$$\Xi_0(e^{it}) := \begin{bmatrix} -\theta^*(e^{it}) & \Pi(e^{it}) \\ \Sigma(e^{it}) & \theta(e^{it}) \end{bmatrix}, \tag{4.17}$$

$\mathrm{A}(e^{it}) \in CM[\mathfrak{G} \oplus \mathfrak{F}, \mathfrak{F}^{(1)} \oplus \mathfrak{F}]$, $\mathrm{B}(e^{it}) \in CM[\mathfrak{G}^{(1)} \oplus \mathfrak{G}, \mathfrak{F} \oplus \mathfrak{G}]$ are operator functions of the forms

$$\mathrm{A}(e^{it}) := \begin{bmatrix} \omega(e^{it}) & 0 \\ 0 & I_{\mathfrak{F}} \end{bmatrix}, \quad \mathrm{B}(e^{it}) := \begin{bmatrix} \lambda(e^{it}) & 0 \\ 0 & I_{\mathfrak{G}} \end{bmatrix}, \tag{4.18}$$

$\omega(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}^{(1)}]$ and $\lambda(e^{it}) \in CM[\mathfrak{G}^{(1)}, \mathfrak{F}]$ are coisometric and isometric operator functions, respectively, satisfying conditions (4.8).

Note that $\Xi_0(e^{it}) \in \mathcal{K}_r(\theta)$ iff the function $\theta(e^{it})$ is strictly contractive almost everywhere (see Remark 4.4 and Theorem 4.5 (part (*d*))). In this case, in representations (4.18) functions $\omega(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}^{(1)}]$, $\lambda(e^{it}) \in CM[\mathfrak{G}^{(1)}, \mathfrak{F}]$ are arbitrary coisometric and isometric operator functions, respectively.

Note also that in the case $\mathfrak{F}^{(1)} = \{0\}(\mathfrak{G}^{(1)} = \{0\})$ the functions $\omega(e^{it}) \equiv 0 \in [\mathfrak{G}, \mathfrak{F}^{(1)}](\lambda(e^{it}) \equiv 0 \in [\mathfrak{G}^{(1)}, \mathfrak{F}])$ can be viewed as a coisometric (isometric) function. Then representation (4.16) is transformed to the form

$$\Lambda(e^{it}) = \Lambda_0(e^{it})\mathrm{B}(e^{it}) \quad (\Omega(e^{it}) = \mathrm{A}(e^{it})\Omega_0(e^{it})),$$

where $\Lambda_0(e^{it}) \in CM[\mathfrak{F} \oplus \mathfrak{G}, \mathfrak{F}](\Omega_0(e^{it}) \in CM[\mathfrak{G}, \mathfrak{G} \oplus \mathfrak{F}])$ is the coicometric (isometric) operator function of the form

$$\Lambda_0(e^{it}) := [\Sigma(e^{it}), \theta(e^{it})] \quad (\Omega_0(e^{it}) := \begin{bmatrix} \Pi(e^{it}) \\ \theta(e^{it}) \end{bmatrix}),$$

and $\mathrm{B}(e^{it})(\mathrm{A}(e^{it}))$ has form (4.18). This is consistent with Definition 3.11, where unidirectional regular extensions of a contractive operator function are declared as a particular case of bidirectional ones.

5 Comparison Relations on the Sets of Unidirectional and Bidirectional Regular Extensions

In this section we introduce a natural generalization of the comparison relation that was considered in [14] (Definition 5.4) for unidirectional regular extensions of matrix and operator-valued Schur functions.

5.1 Unidirectional Case

Definition 5.1 Let $\theta(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$ and σ be a minimal unitary coupling of form (2.1) such that $\theta_\sigma(e^{it}) = \theta(e^{it})$. Let $\Omega_j(e^{it}) \in \mathcal{U}_r(\theta)(\Lambda_j(e^{it}) \in \mathcal{L}_r(\theta))$, $j = 1, 2$, be two regular extensions of the form

$$\Omega_j(e^{it}) := \begin{bmatrix} \theta_{12}^{(j)}(e^{it}) \\ \theta(e^{it}) \end{bmatrix} \in CM[\mathfrak{G}, \mathfrak{F}^{(j)} \oplus \mathfrak{F}]$$

$$(\Lambda_j(e^{it}) := [\theta_{21}^{(j)}(e^{it}), \theta(e^{it})] \in CM[\mathfrak{G}^{(j)} \oplus \mathfrak{G}, \mathfrak{F}]) \tag{5.1}$$

and $(M(\overset{\circ}{\mathfrak{F}}{}^{(j)}), \mathfrak{F}^{(j)}; V_{\mathfrak{F}^{(j)}})((M(\overset{\circ}{\mathfrak{G}}{}^{(j)}), \mathfrak{G}^{(j)}; V_{\mathfrak{G}^{(j)}}))$, $j = 1, 2$, be the two bilateral channels of the coupling σ that, according to Definition 3.6 and Theorem 3.2 (Theorem 3.4), correspond to the extensions $\Omega_j(e^{it})(\Lambda_j(e^{it}))$. We will say that the extension $\Omega_1(e^{it})(\Lambda_1(e^{it}))$ procedes the extension $\Omega_2(e^{it})(\Lambda_2(e^{it}))$ and write $\Omega_1(e^{it}) \prec \Omega_2(e^{it})(\Lambda_1(e^{it}) \prec \Lambda_2(e^{it}))$ if the inclusion

$$M(\overset{\circ}{\mathfrak{F}}{}^{(1)}) \subset M(\overset{\circ}{\mathfrak{F}}{}^{(2)}) \quad (M(\overset{\circ}{\mathfrak{G}}{}^{(1)}) \subset M(\overset{\circ}{\mathfrak{G}}{}^{(2)})) \tag{5.2}$$

holds. These extensions will be called equivalent if each of them precedes the other, that is, if inclusion (5.2) turns into the equality. In this case, we will use the notation

$$\Omega_1(e^{it}) \sim \Omega_2(e^{it}) \quad (\Lambda_1(e^{it}) \sim \Lambda_2(e^{it})).$$

Clearly, this definition does not depend on the choice of a minimal coupling σ such that $\theta_\sigma(e^{it}) = \theta(e^{it})$.

It is obvious that the formulated comparison relation on the set $\mathcal{U}_r(\theta)$ ($\mathcal{L}_r(\theta)$ is a partial preorder and the equivalence relation generates a partial order on the quotient set $\mathcal{U}_r(\theta)/\sim$ ($\mathcal{L}_r(\theta)/\sim$) of equivalence classes. The equivalence class generated by $\Omega(e^{it}) \in \mathcal{U}_r(\theta)(\Lambda(e^{it}) \in \mathcal{L}_r(\theta))$ will be denoted by $[\Omega]$ ($[\Lambda]$).

For what follows we need the following assertion (see [12], Theorems 7.3, 7.6 and 7.8), which can also be obtained from Theorems 3.1 or 4.1 (part (*b*)).

Theorem 5.2 *Let $\theta(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$ and σ be a unitary coupling of form (2.1) such that $\theta_\sigma(e^{it}) = \theta(e^{it})$. The function $\theta(e^{it})$ is coisometric (isometric, unitary) operator function iff the condition*

$$M(\overset{\circ}{\mathfrak{G}}) \supset M(\overset{\circ}{\mathfrak{F}}) \quad (M(\overset{\circ}{\mathfrak{G}}) \subset M(\overset{\circ}{\mathfrak{F}}), \quad M(\overset{\circ}{\mathfrak{G}}) = M(\overset{\circ}{\mathfrak{F}}))$$

is satisfied.

Definition 5.3 A function $\theta_1(e^{it}) \in CM[\mathfrak{G}, \mathfrak{K}](\theta_2(e^{it}) \in CM[\mathfrak{K}, \mathfrak{F}])$ is called a right (left) divisor of a function $\theta(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$ if there exists a factorization of form (2.21) according to Definition 2.19. In this case, we will also say that $\theta(e^{it})$ is divided by $\theta_1(e^{it})(\theta_2(e^{it}))$ on the right (on the left) or $\theta(e^{it})$ is a left (right) multiple of $\theta_1(e^{it})(\theta_2(e^{it}))$. For this we use the commonly accepted notation

$$\theta(e^{it}) \vdots \theta_1(e^{it}) \text{ (on the right)} \quad (\theta(e^{it}) \vdots \theta_2(e^{it}) \text{ (on the left)}).$$

In the case of regularity (complete regularity) of the factorization, the divisors and their multiple are called regular (completely regular).

Leaning on Theorem 5.2, it is easy to see that in the case of the left (right) coisometric (isometric) regular divisor of a coisometric (isometric) operator function the corresponding unique right (left) divisor is also coisometric (isometric). Moreover, by Corollary 2.29 and Definition 2.31, the divisors are even completely regular.

In the other case, the similar result is valid without the requirement of regularity of the right (left) divisor.

Lemma 5.4

(a) Let $\omega(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$ and $\omega_1(e^{it}) \in CM[\mathfrak{G}, \mathfrak{K}]$ be coisometric operator functions. If $\omega_1(e^{it})$ is the right divisor of $\omega(e^{it})$, then the corresponding unique left divisor $\omega_2(e^{it})$ is also coisometric and both of them are completely regular.
(b) Let $\lambda(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$ and $\lambda_2(e^{it}) \in CM[\mathfrak{K}, \mathfrak{F}]$ be isometric operator functions. If $\lambda_2(e^{it})$ is the left divisor of $\lambda(e^{it})$, then the corresponding unique right divisor $\lambda_1(e^{it})$ is also isometric and both of them are completely regular.

Proof It suffices to prove only part (a). Let $\omega(e^{it}) = \omega_2(e^{it})\omega_1(e^{it})$ almost everywhere, where $\omega_2(e^{it}) \in CM[\mathfrak{K}, \mathfrak{F}]$. By Corollary 2.29 and Definition 2.31, this factorization is completely regular.

Let σ be a minimal coupling of form (2.1) such that $\theta_\sigma(e^{it}) = \omega(e^{it})$ and $\sigma = \sigma_2\sigma_1$ be the unique factorization corresponding, by Theorem 2.20, to the considered factorization of the function $\omega(e^{it})$. Here σ_2, σ_1 are the couplings of form (2.15) such that $\theta_{\sigma_j}(e^{it}) = \omega_j(e^{it})$, $j = 1, 2$. In view of the coisometricity of $\omega(e^{it})$ and $\omega_1(e^{it})$, by Theorem 5.2, we obtain

$$M(\overset{\circ}{\mathfrak{F}}) \subset M(\overset{\circ}{\mathfrak{G}}), \quad M(\overset{\circ}{\mathfrak{K}}) \subset M(\overset{\circ}{\mathfrak{G}}).$$

Since $\mathfrak{H} = M(\overset{\circ}{\mathfrak{G}})$, from representation (2.17) we infer that $\mathfrak{N}^{(2)}_{\overset{\circ}{\mathfrak{K}}} \perp \mathfrak{N}^{(1)}_{\overset{\circ}{\mathfrak{K}}}$, where

$$\mathfrak{N}^{(2)}_{\overset{\circ}{\mathfrak{K}}} := (M(\overset{\circ}{\mathfrak{F}}) \vee M(\overset{\circ}{\mathfrak{K}})) \ominus M(\overset{\circ}{\mathfrak{K}}), \quad \mathfrak{N}^{(1)}_{\overset{\circ}{\mathfrak{K}}} := \mathfrak{H} \ominus M(\overset{\circ}{\mathfrak{K}}).$$

From this it follows that $M(\overset{\circ}{\mathfrak{F}}) \subset M(\overset{\circ}{\mathfrak{K}})$, that is, by Theorem 5.2, $\omega_2(e^{it})$ is a coisometric operator function.

The uniqueness of $\omega_2(e^{it})$ follows from the uniqueness of σ_2. □

Thus, the latter lemma enables us to speak of right (left) divisibility within the class of coisometric (isometric) functions without specifying its regularity or complete regularity.

Theorem 5.5 *Let $\theta(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$ and $\Omega_j(e^{it}) \in \mathcal{U}_r(\theta)$ $(\Lambda_j(e^{it}) \in \mathcal{L}_r(\theta))$, $j = 1, 2$, be its regular extensions of form (5.1). Let*

$$\theta^{(j)}_{12}(e^{it}) = \omega_j(e^{it})\Pi(e^{it}) \quad (\theta^{(j)}_{21}(e^{it}) = \Sigma(e^{it})\lambda_j(e^{it})), \quad j = 1, 2, \tag{5.3}$$

where $\omega_j(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}^{(j)}](\lambda_j(e^{it}) \in CM[\mathfrak{G}^{(j)}, \mathfrak{F}])$, $j = 1, 2$, are coisometric (isometric) operator functions corresponding to $\Omega_j(e^{it})$ $(\Lambda_j(e^{it}))$ by Theorem 4.2 (part (b)) and satisfying the conditions

$$\mathrm{Ran}\omega^*_j \subset \overline{\Pi L^2(\mathfrak{G})} \quad (\mathrm{Ran}\lambda_j \subset \overline{\Sigma L^2(\mathfrak{F})}), \quad j = 1, 2. \tag{5.4}$$

The following statement are equivalent:

(a) $\Omega_1(e^{it}) \prec \Omega_2(e^{it}) \quad (\Lambda_1(e^{it}) \prec \Lambda_2(e^{it}))$;
(b) $(\theta^{(1)}_{12}(e^{it}))^*\theta^{(1)}_{12}(e^{it}) \le (\theta^{(2)}_{12}(e^{it}))^*\theta^{(2)}_{12}(e^{it})$ *a.e.*
$(\theta^{(1)}_{21}(e^{it})(\theta^{(1)}_{21}(e^{it}))^* \le \theta^{(2)}_{21}(e^{it})(\theta^{(2)}_{21}(e^{it}))^*$ *a.e.*);
(c) $\omega_1(e^{it}) \vdots \omega_2(e^{it})$*(on the right)*$(\lambda_1(e^{it}) \vdots \lambda_2(e^{it})$*(on the left)*).

Proof Again we consider only the case of regular upward extensions.

First of all, note that, in view of conditions (5.3)–(5.4), inequality (*b*) is equivalent to condition

(b') $\omega^*_1(e^{it})\omega_1(e^{it}) \le \omega^*_2(e^{it})\omega_2(e^{it})$ a.e.

In turn, this is obviously equivalent to the existence of a factorization of the form $\omega_1(e^{it}) = \omega_{12}(e^{it})\omega_2(e^{it})$, where $\omega_{12}(e^{it}) \in CM[\mathfrak{F}^{(2)}, \mathfrak{F}^{(1)}]$ is a contractive operator function. By Lemma 5.4 this is equivalent to the statement (*c*).

It remains to prove, for example, the equivalence of (*a*) and (*b'*). Consider again the functional model $\hat{\sigma}$ of a minimal coupling of form (2.11). As was shown in the proof of Theorem 4.2 (part (*b*)), the bilateral channels $(\hat{M}(\overset{\circ}{\mathfrak{F}}{}^{(j)}), \mathfrak{F}^{(j)}; \hat{V}_{\mathfrak{F}^{(j)}})$, $j = 1, 2$, of the coupling $\hat{\sigma}$ that, by Definition 3.6 and Theorem 3.2 (part (*a*)), correspond to the extensions $\Omega_j(e^{it}) \in \mathcal{U}_r(\theta)$, $j = 1, 2$, satisfy the conditions

$$\mathrm{Ran}\omega_j^* = \hat{M}(\overset{\circ}{\mathfrak{F}}{}^{(j)}),\ j = 1, 2.$$

The statement (b') is equivalent to the validity of the inclusion $\mathrm{Ran}\omega_1^* \subset \mathrm{Ran}\omega_2^*$, that is, of the inclusion $\hat{M}(\overset{\circ}{\mathfrak{F}}{}^{(1)}) \subset \hat{M}(\overset{\circ}{\mathfrak{F}}{}^{(2)})$. Thus, by Definition 5.1, the statement (b') is equivalent to the statement (a). □

Corollary 5.6 *Let the requirements of Theorem 5.5 be valid. The following statements are equivalent:*

(a) $\Omega_1(e^{it}) \sim \Omega_2(e^{it})(\Lambda_1(e^{it}) \sim \Lambda_2(e^{it})))$;
(b) $(\theta_{12}^{(1)}(e^{it}))^*\theta_{12}^{(1)}(e^{it}) = (\theta_{12}^{(2)}(e^{it}))^*\theta_{12}^{(2)}(e^{it})$ *a.e.*
$(\theta_{21}^{(1)}(e^{it})(\theta_{21}^{(1)}(e^{it}))^* = \theta_{21}^{(2)}(e^{it})(\theta_{21}^{(2)}(e^{it}))^*$ *a.e.);*
(c) $\omega_1(e^{it}) = \omega_{12}(e^{it})\omega_2(e^{it})$ *a.e.* $(\lambda_1(e^{it}) = \lambda_2(e^{it})\lambda_{21}(e^{it})$ *a.e.), where* $\omega_{12}(e^{it}) \in CM[\mathfrak{F}^{(2)}, \mathfrak{F}^{(1)}](\lambda_{21}(e^{it}) \in CM[\mathfrak{G}^{(1)}, \mathfrak{G}^{(2)}])$ *is a unitary operator function.*

For any function $\theta(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$ we will denote by $\mathcal{J}_*(\theta)$ ($\mathcal{J}(\theta)$) or simply $\mathcal{J}_*$ ($\mathcal{J}$) the set of coisometric (isometric) operator functions $\omega(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}^{(1)}]$ ($\lambda(e^{it}) \in CM[\mathfrak{G}^{(1)}, \mathfrak{F}]$) satisfying condition (4.1). By Theorems 4.2 (part (*b*)) and 5.5, we have established an isomorphism between the set $\mathcal{U}_r(\theta)$ ($\mathcal{L}_r(\theta)$), partially preordered by the relation $\prec$, and the set $\mathcal{J}_*(\theta)$ ($\mathcal{J}(\theta)$), partially preordered by the relation $\vdots$ on the right (on the left). Consequently, their quotient sets with respect to the corresponding equivalence relations are also isomorphic partially ordered sets.

It should be pointed out that the comparison relation for regular extensions introduced by Definition 5.1 agrees with the inverse comparison relation for regular factorizations introduced by Definition 2.21.

Really, taking into account Definition 3.6, for two extensions $\Omega_j(e^{it}) \in \mathcal{U}_r(\theta)$, $j = 1, 2$, of form (5.1) such that $\Omega_1(e^{it}) \prec \Omega_2(e^{it})$ we consider the corresponding factorizations of forms (2.22) and (2.23), where

$$\theta_2(e^{it}) := [0, I_{\mathfrak{F}}], \quad 0 \in [\mathfrak{F}^{(2)}, \mathfrak{F}]; \quad \theta_1(e^{it}) := \Omega_2(e^{it});$$

$$\theta_2'(e^{it}) := [0, I_{\mathfrak{F}}], \quad 0 \in [\mathfrak{F}^{(1)}, \mathfrak{F}]; \quad \theta_1'(e^{it}) := \Omega_1(e^{it}).$$

By Theorem 4.2 (part (*b*)), the functions $\theta_{12}^{(j)}(e^{it})$, $j = 1, 2$, admit the representations of form (5.3), where, by Theorem 5.5, the coisometric functions $\omega_j(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}^{(j)}]$, $j = 1, 2$, satisfy the condition $\omega_1(e^{it}) \vdots \omega_2(e^{it})$ (on the right). Thus, there exists a unique coisometric function $\omega_{12}(e^{it}) \in CM[\mathfrak{F}^{(2)}, \mathfrak{F}^{(1)}]$ such that $\omega_1(e^{it}) = \omega_{12}(e^{it})\omega_2(e^{it})$ almost everywhere. Setting

$$\theta_0(e^{it}) := \begin{bmatrix} \omega_{12}(e^{it}) & 0 \\ 0 & I_{\mathfrak{F}} \end{bmatrix} \in CM[\mathfrak{F}^{(2)} \oplus \mathfrak{F}, \mathfrak{F}^{(1)} \oplus \mathfrak{F}],$$

we obtain the equalities of form (2.24). Hence, by Definition 2.21, factorization (2.22) procedes factorization (2.23) in contrast to the corresponding extensions $\Omega_j(e^{it})$, $j = 1, 2$.

Definition 5.7 Let $\theta(e^{it})$ belongs to $CM[\mathfrak{G}, \mathfrak{F}]$.

(a) An element $[\Omega] \in \mathcal{U}_r(\theta)/\sim$ $([\Lambda] \in \mathcal{L}_r(\theta)/\sim)$ is called maximal if for any element $[\Omega_1] \in \mathcal{U}_r(\theta)/\sim$ $([\Lambda_1] \in \mathcal{L}_r(\theta/\sim)$ from the condition $[\Omega] \prec [\Omega_1]$ the equality $[\Omega] = [\Omega_1]$ follows. We will also call maximal any extension $\Omega(e^{it}) \in \mathcal{U}_r(\theta)(\Lambda(e^{it}) \in \mathcal{L}_r(\theta))$ such that $[\Omega]([\Lambda])$ is a maximal element of the quotient set $\mathcal{U}_r(\theta)/\sim$ $(\mathcal{L}_r(\theta)/\sim)$. The subset of $\mathcal{U}_r(\theta)(\mathcal{L}_r(\theta))$ consisting of all maximal extensions will be denoted by $\mathcal{U}_{r,\max}(\theta)(\mathcal{L}_{r,\max}(\theta))$ or simply $\mathcal{U}_{r,\max}(\mathcal{L}_{r,\max})$.
(b) An element $[\Omega_0] \in \mathcal{U}_r/\sim$ $([\Lambda_0] \in \mathcal{L}_r/\sim)$ is called the largest if for any $[\Omega] \in \mathcal{U}_r/\sim$ $([\Lambda] \in \mathcal{L}_r/\sim)$ the condition $[\Omega] \prec [\Omega_0]([\Lambda] \prec [\Lambda_0])$ is satisfied.

It is clear that the largest element, if it exists, is unique.

The analogous concepts for the partially ordered set $\mathcal{U}_{cr}(\theta)/\sim$ $(\mathcal{L}_{cr}(\theta)/\sim)$ and the partially preordered set $\mathcal{U}_{cr}(\theta)$ $(\mathcal{L}_{cr}(\theta))$ can be defined in a similar way. The subset of $\mathcal{U}_{cr}(\theta)$ $(\mathcal{L}_{cr}(\theta))$ consisting of all maximal extensions from $\mathcal{U}_{cr}(\theta)$ $(\mathcal{L}_{cr}(\theta))$ will be denoted by $\mathcal{U}_{cr,\max}(\theta)$ $(\mathcal{L}_{cr,\max}(\theta))$ or simply $\mathcal{U}_{cr,\max}$ $(\mathcal{L}_{cr,\max})$.

To describe the set $\mathcal{U}_{r,\max}$ $(\mathcal{L}_{r,\max})$ we need some result from [12] (Lemma 7.20).

Theorem 5.8 *Let $\mathfrak{M}$ be a nontrivial subspace of $L^2(\mathfrak{K})$ reducing the operator $U_{\mathfrak{K}}^{\times}$ and $E := \{t : \rho_{\mathfrak{M}}(e^{it}) > 0\}$. There exists a set of measurable $\mathfrak{K}$-valued functions $\{k_j(e^{it})\}_{j=1}^{\beta_{\mathfrak{M}}}$ such that for all $t \in E$ the set of vectors $\{k_j(e^{it})\}_{j=1}^{\rho_t}$, $\rho_t := \rho_{\mathfrak{M}}(e^{it})$, is an orthonormal basis for the space $P_{\mathfrak{M}}(e^{it})\mathfrak{K}$.*

Theorem 5.9 *Let $\theta(e^{it})$ belongs to $CM[\mathfrak{G}, \mathfrak{F}]$.*

(a) Let $\Omega(e^{it}) \in \mathcal{U}_r(\theta)(\Lambda(e^{it}) \in \mathcal{L}_r(\theta))$ be a regular extension of form (3.15), $\omega(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}^{(1)}]$ $(\lambda(e^{it}) \in CM[\mathfrak{G}^{(1)}, \mathfrak{F}])$ be the corresponding coisometric (isometric) operator function from representation (4.2) satisfying condition (4.1) and let

$$\mathfrak{L}^{\perp} := \overline{\Pi L^2(\mathfrak{G})} \ominus \mathrm{Ran}\omega^* \quad (\mathfrak{L}_*^{\perp} := \overline{\Sigma L^2(\mathfrak{F})} \ominus \mathrm{Ran}\lambda). \tag{5.5}$$

Then $\Omega(e^{it}) \in \mathcal{U}_{r,\max}(\theta)(\Lambda(e^{it}) \in \mathcal{L}_{r,\max}(\theta))$ iff the conditions

$$\alpha_{\Pi} < \infty, \alpha_{\mathfrak{L}^{\perp}} = 0 \quad (\alpha_{\Sigma} < \infty, \alpha_{\mathfrak{L}_*^{\perp}} = 0) \tag{5.6}$$

or

$$\alpha_{\Pi} = \infty, \mathfrak{L}^{\perp} = \{0\} \quad (\alpha_{\Sigma} = \infty, \mathfrak{L}_*^{\perp} = \{0\}) \tag{5.7}$$

are satisfied. The element $[\Omega] \in \mathcal{U}_r(\theta)/\sim$ $([\Lambda] \in \mathcal{L}_r(\theta)/\sim)$ is the largest iff the extension $\Omega(e^{it})(\Lambda(e^{it}))$ is coisometric (isometric).

(b) *Let* $\Omega(e^{it}) \in \mathcal{U}_{cr}(\theta)(\Lambda(e^{it}) \in \mathcal{L}_{cr}(\theta))$ *be a completely regular extension of form* (3.15)*,* $\omega(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}^{(1)}](\lambda(e^{it}) \in CM[\mathfrak{G}^{(1)}, \mathfrak{F}])$ *be the corresponding coisometric* (*isometric*) *operator function from representation* (4.4) *satisfying condition* (4.3) *and*

$$\mathfrak{N} := \mathrm{Ker}\theta \quad (\mathfrak{N}_* := \mathrm{Ker}\theta^*), \quad \mathfrak{L}^\perp := \mathfrak{N} \ominus \mathrm{Ran}\omega^* \quad (\mathfrak{L}_*^\perp := \mathfrak{N}_* \ominus \mathrm{Ran}\lambda).$$

Then $\Omega(e^{it}) \in \mathcal{U}_{cr,\max}(\theta)(\Lambda(e^{it}) \in \mathcal{L}_{cr,\max}(\theta))$ *iff the conditions*

$$\alpha_{\mathfrak{N}} < \infty, \alpha_{\mathfrak{L}^\perp} = 0 \quad (\alpha_{\mathfrak{N}_*} < \infty, \alpha_{\mathfrak{L}_*^\perp} = 0)$$

or

$$\alpha_{\mathfrak{N}} = \infty, \mathfrak{L}^\perp = \{0\} \quad (\alpha_{\mathfrak{N}_*} = \infty, \mathfrak{L}_*^\perp = \{0\})$$

are satisfied. The element $[\Omega] \in \mathcal{U}_{cr}(\theta)/\sim$ $([\Lambda] \in \mathcal{L}_{cr}(\theta)/\sim)$ *is the largest iff the equality*

$$\mathrm{Ran}\omega^* = \mathfrak{N} \quad (\mathrm{Ran}\lambda = \mathfrak{N}_*)$$

is valid.

Proof We again confine ourselves to the case of regular upward extensions.

(a) As before, consider the functional model $\hat{\sigma}$ of a minimal unitary coupling of form (2.11) described in Theorem 2.11.

By Definition 5.1, the extension $\Omega(e^{it})$ is maximal iff the corresponding bilateral channel $(\hat{M}(\overset{\circ}{\mathfrak{F}}{}^{(1)}), \mathfrak{F}^{(1)}; \hat{V}_{\mathfrak{F}^{(1)}})$ of the coupling $\hat{\sigma}$ is also maximal regarding the inclusion relation. The latter means that there is no other bilateral channel $(\hat{M}(\overset{\circ}{\mathfrak{F}}{}^{(2)}), \mathfrak{F}^{(2)}; \hat{V}_{\mathfrak{F}^{(2)}})$ of the coupling $\hat{\sigma}$ such that

$$\hat{M}(\overset{\circ}{\mathfrak{F}}{}^{(2)}) \neq \hat{M}(\overset{\circ}{\mathfrak{F}}{}^{(1)}) \subset \hat{M}(\overset{\circ}{\mathfrak{F}}{}^{(2)}).$$

As was shown in the proof of Theorem 4.2 (part (b)), $\mathrm{Ran}\omega^* = \hat{M}(\overset{\circ}{\mathfrak{F}}{}^{(1)})$ and, by Theorem 4.1 (part (c)), the equality $\rho_{\omega^*}(e^{it}) = \alpha_{\omega^*}$ holds almost everywhere.

If $\Omega(e^{it}) \in \mathcal{U}_{r,\max}$, then $\alpha_{\mathfrak{L}^\perp} = 0$. Indeed, if the inequality $\alpha_{\mathfrak{L}^\perp} > 0$ were true, then by Theorem 4.1 (part (c)), there would be a nontrivial channel $(\hat{M}(\overset{\circ}{\mathfrak{F}}{}^{(0)}), \mathfrak{F}^{(0)}; \hat{V}_{\mathfrak{F}^{(0)}})$ of the coupling $\hat{\sigma}$ such that

$$\hat{M}(\overset{\circ}{\mathfrak{F}}{}^{(0)}) \subset \mathfrak{L}^\perp \subset \overline{\Pi L^2(\mathfrak{G})}.$$

Let $\mathfrak{F}^{(2)} := \mathfrak{F}^{(0)} \oplus \mathfrak{F}^{(1)}$. Denote by P_j the orthoprojection of $\mathfrak{F}^{(2)}$ onto $\mathfrak{F}^{(j)}$, $j = 0, 1$, and let $\hat{V}_{\mathfrak{F}^{(2)}} := \hat{V}_{\mathfrak{F}^{(0)}} P_0 + \hat{V}_{\mathfrak{F}^{(1)}} P_1$. Then for the channel $(\hat{M}(\overset{\circ}{\mathfrak{F}}{}^{(2)}), \mathfrak{F}^{(2)}; \hat{V}_{\mathfrak{F}^{(2)}})$ of the coupling $\hat{\sigma}$ the strict inclusion $\hat{M}(\overset{\circ}{\mathfrak{F}}{}^{(2)}) \supset \hat{M}(\overset{\circ}{\mathfrak{F}}{}^{(1)})$ is valid, what contradicts the maximality of $\Omega(e^{it})$. Moreover, if $\alpha_\Pi = \infty$, then $\beta_\Pi = \infty$, that is, $\rho_\Pi(e^{it}) = \infty$ almost everywhere. By Theorem 4.2 (part (d)), in this case there exists an isometric extension $\Omega_0(e^{it}) \in \mathcal{U}_r$. It is obvious that $[\Omega_0]$ is the largest element in the quotient set $\mathcal{U}_r/\sim$. Therefore, the extension $\Omega(e^{it})$ can be maximal only if $\Omega \in [\Omega_0]$. Hence, $\Omega(e^{it})$ is also isometric, whence the equality $\mathrm{Ran}\omega^* = \overline{\Pi L^2(\mathfrak{G})}$ follows. Thus, $\mathfrak{L}^\perp = \{0\}$.

Conversely, if conditions (5.7) are satisfied, then condition (4.5) is valid. Hence, by Theorem 4.2 (part (d)), $\Omega(e^{it})$ is an isometric extension and, as noted above, $[\Omega]$ is the largest element in $\mathcal{U}_r/\sim$.

If conditions (5.6) are satisfied, then the equality $\alpha_{\omega^*} = \alpha_\Pi$ holds, that is, $\rho_{\omega^*}(e^{it}) = \alpha_\Pi$ almost everywhere. Let $(\hat{M}(\overset{\circ}{\mathfrak{F}}{}^{(2)}), \mathfrak{F}^{(2)}; \hat{V}_{\mathfrak{F}^{(2)}})$ be an arbitrary channel of the coupling $\hat{\sigma}$ such that $\hat{M}(\overset{\circ}{\mathfrak{F}}{}^{(2)}) \supset \hat{M}(\overset{\circ}{\mathfrak{F}}{}^{(1)})$ and $\omega_2(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}^{(2)}]$ be a coisometric operator function corresponding, by Theorem 4.2 (part (b)), to this channel. Hence, $\mathrm{Ran}\omega_2^* = \hat{M}(\overset{\circ}{\mathfrak{F}}{}^{(2)}) \subset \overline{\Pi L^2(\mathfrak{G})}$ and the inequalities

$$P_{\omega^*}(e^{it}) \le P_{\omega_2^*}(e^{it}) \le P_\Pi(e^{it}) \tag{5.8}$$

hold almost everywhere. From this and the equality $\rho_{\omega^*}(e^{it}) = \alpha_\Pi$ a.e. we obtain

$$\rho_{\omega^*}(e^{it}) = \rho_{\omega_2^*}(e^{it}) = \alpha_\Pi \quad \text{a.e.} \tag{5.9}$$

Since $\alpha_\Pi < \infty$, (5.8) and (5.9) imply the equality $P_{\omega^*}(e^{it}) = P_{\omega_2^*}(e^{it})$ a.e., that is, $\hat{M}(\overset{\circ}{\mathfrak{F}}{}^{(1)}) = \hat{M}(\overset{\circ}{\mathfrak{F}}{}^{(2)})$. Thus, $\Omega(e^{it}) \in \mathcal{U}_{r,\max}$.

It remains to prove that in the case $0 < \alpha_\Pi < \infty$ $[\Omega] \in \mathcal{U}_r/\sim$ is the largest element only if $\Omega(e^{it}) \in \mathcal{U}_r$ is an isometric extension. For this it suffices to show that for every non-isometric maximal extension $\Omega(e^{it}) \in \mathcal{U}_{r,\max}$ there exists an extension $\Omega_1(e^{it}) \in \mathcal{U}_r$ that is incomparable with $\Omega(e^{it})$.

In view of $\alpha_\Pi < \infty$, for the coisometric operator function $\omega(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}^{(1)}]$ corresponding to $\Omega(e^{it})$ of form (3.15) the conditions

$$\alpha_{\omega^*} = \alpha_\Pi < \beta_\Pi \le \infty$$

are satisfied. By Theorem 5.8, for the subspace $\mathfrak{L} := \mathrm{Ran}\omega^*$ there exists a set $\{k_j(e^{it})\}_{j=1}^{\alpha_\Pi}$ of measurable $\mathfrak{G}$-valued functions such that for almost all t the set $\{k_j(e^{it})\}_{j=1}^{\alpha_\Pi}$ of vectors is an orthonormal basis of the subspace $P_{\omega^*}(e^{it})\mathfrak{G}$.

Similarly, for the subspace $\mathfrak{L}^{\perp}$ of form (5.5) there exists a set $\{k_j^{\perp}(e^{it})\}_{j=1}^{\beta_{\mathfrak{L}^{\perp}}}$ of measurable $\mathfrak{G}$-valued functions such that for all $t \in E := \{t : \rho_{\mathfrak{L}^{\perp}}(e^{it}) > 0\}$ the set $\{k_j^{\perp}(e^{it})\}_{j=1}^{\rho_t}$, $\rho_t := \rho_{\mathfrak{L}^{\perp}}(e^{it})$, of vectors is an orthonormal basis of the subspace $P_{\mathfrak{L}^{\perp}}(e^{it})\mathfrak{G}$. Let $E^c := \{t : \rho_{\mathfrak{L}^{\perp}}(e^{it}) = 0\}$. Note that $E^c = \{t : \rho_{\Pi}(e^{it}) = \alpha_{\Pi}\}$ and it is a set of positive Lebesgue measure, just as the set E. Note also that $k_1^{\perp}(e^{it}) \neq 0$ for all $t \in E$, while $k_j(e^{it}) \neq 0$ a.e. ($j = 1, 2, \ldots, \alpha_{\Pi}$). Consider the measurable $\mathfrak{G}$-valued function

$$m(e^{it}) := \begin{cases} k_1(e^{it}), & t \in E^c, \\ k_1^{\perp}(e^{it}), & t \in E, \end{cases}$$

which is defined almost everywhere. Moreover, $\|m(e^{it})\|_{\mathfrak{G}} = 1$ a.e. Thus, $m(e^{it})$ determines the subspace $\mathfrak{M} \subset \overline{\Pi L^2(\mathfrak{G})}$ that reduces $U_{\mathfrak{G}}^{\times}$ and the "multiplication" ortoprojection of which is defined by the formula

$$P_{\mathfrak{M}}(e^{it})g(e^{it}) := \langle g(e^{it}), m(e^{it})\rangle_{\mathfrak{G}}\, m(e^{it}), \quad g(e^{it}) \in L^2(\mathfrak{G}).$$

Since $\rho_{\mathfrak{M}}(e^{it}) = 1$ a.e., by Theorem 4.1 (part (c)), there exists a bilateral channel $(\mathfrak{M}, \mathfrak{F}^{(0)}; \hat{V}_{\mathfrak{F}^{(0)}})$ of the coupling $\hat{\sigma}$. As $\mathfrak{M} \not\subset \mathfrak{L}$, the extension $\Omega_1(e^{it}) \in \mathcal{U}_r$ corresponding to this channel is incomparable with $\Omega(e^{it})$.

(b) The proof of part (b) does not essentially differ from the proof of part (a). □

5.2 Bidirectional Case

Taking into account Remark 3.12, we can spread to the set $\mathcal{K}_r(\theta)$ the comparison relation introduced by Definition 5.1.

Definition 5.10 Let $\theta(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$ and $\Xi_j(e^{it}) \in \mathcal{K}_r(\theta)$, $j = 1, 2$, be two its regular extensions of the form

$$\Xi_j(e^{it}) := \begin{bmatrix} \theta_{11}^{(j)}(e^{it}) & \theta_{12}^{(j)}(e^{it}) \\ \theta_{21}^{(j)}(e^{it}) & \theta(e^{it}) \end{bmatrix} \in CM[\mathfrak{G}^{(j)} \oplus \mathfrak{G}, \mathfrak{F}^{(j)} \oplus \mathfrak{F}] \tag{5.10}$$

and let $\Omega_j(e^{it}) \in \mathcal{U}_r(\theta)$, $\Lambda_j(e^{it}) \in \mathcal{L}_r(\theta)$, $j = 1, 2$, be two pairs of regular extensions of the form

$$\Omega_j(e^{it}) := \begin{bmatrix} \theta_{12}^{(j)}(e^{it}) \\ \theta(e^{it}) \end{bmatrix} \in CM[\mathfrak{G}, \mathfrak{F}^{(j)} \oplus \mathfrak{F}], \tag{5.11}$$

$$\Lambda_j(e^{it}) := [\theta_{21}^{(j)}(e^{it}), \theta(e^{it})] \in CM[\mathfrak{G}^{(j)} \oplus \mathfrak{G}, \mathfrak{F}] \tag{5.12}$$

corresponding to $\Xi_j(e^{it})$. We will say that the extensions $\Xi_1(e^{it})$ precedes the extension $\Xi_2(e^{it})$ if the extension $\Omega_1(e^{it})$ precedes the extension $\Omega_2(e^{it})$ and the extension $\Lambda_1(e^{it})$ precedes the extension $\Lambda_2(e^{it})$. In this case, we will write $\Xi_1(e^{it}) \prec \Xi_2(e^{it})$. These extensions will be called equivalent if each of them precedes the other and this relation will be denoted by $\Xi_1(e^{it}) \sim \Xi_2(e^{it})$.

Let σ be a minimal unitary coupling of form (2.1) such that $\theta_\sigma(e^{it}) = \theta(e^{it})$ and

$$\{(M(\overset{\circ}{\mathfrak{F}}{}^{(j)}), \mathfrak{F}^{(j)}; V_{\mathfrak{F}^{(j)}}), (M(\overset{\circ}{\mathfrak{G}}{}^{(j)}), \mathfrak{G}^{(j)}; V_{\mathfrak{G}^{(j)}})\}, \quad j = 1, 2,$$

be two pairs of bilateral channels of the coupling σ corresponding, by Definition 3.11 and Theorem 3.8, to the extensions $\Xi_j(e^{it}) \in \mathcal{K}_r(\theta)$ of form (5.10). It follows from Definitions 5.1 and 5.10 that $\Xi_1(e^{it}) \prec \Xi_2(e^{it})$ iff the inclusions

$$M(\overset{\circ}{\mathfrak{F}}{}^{(1)}) \subset M(\overset{\circ}{\mathfrak{F}}{}^{(2)}), \quad M(\overset{\circ}{\mathfrak{G}}{}^{(1)}) \subset M(\overset{\circ}{\mathfrak{G}}{}^{(2)})$$

hold. Clearly, this conclusion does not depend on the choice of such a minimal coupling σ.

It is obvious that, by Definition 5.10, the set $\mathcal{K}_r(\theta)$ becomes partially preordered and the quotient set $\mathcal{K}_r/\sim$ does partially ordered. The equivalence class generated by $\Xi(e^{it}) \in \mathcal{K}_r(\theta)$ will be denoted by $[\Xi]$.

The following theorem is a direct corollary of Theorem 5.5 and Definition 5.10.

Theorem 5.11 *Let $\theta(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$ and $\Xi_j(e^{it}) \in \mathcal{K}_r(\theta)$, $j = 1, 2$, be its regular extensions of form (5.10). Let*

$$\theta_{12}^{(j)}(e^{it}) = \omega_j(e^{it})\Pi(e^{it}), \quad \theta_{21}^{(j)}(e^{it}) = \Sigma(e^{it})\lambda_j(e^{it}),$$

$$\theta_{11}^{(j)}(e^{it}) = -\omega_j(e^{it})\theta^*(e^{it})\lambda_j(e^{it}), \quad j = 1, 2,$$

where $\omega_j(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}^{(j)}]$ and $\lambda_j(e^{it}) \in CM[\mathfrak{G}^{(j)}, \mathfrak{F}]$ are coisometric and isometric operator functions, respectively, corresponding, by Theorem 4.5 (part (b)), to $\Xi_j(e^{it})$ and satisfying the conditions of type (4.8), namely,

$$\mathrm{Ran}\omega_j^* \subset \overline{\Pi L^2(\mathfrak{G})}, \quad \mathrm{Ran}\lambda_j \subset \overline{\Sigma L^2(\mathfrak{F})}, \quad j = 1, 2.$$

The following statements are equivalent:

(a) $\Xi_1(e^{it}) \prec \Xi_2(e^{it})$;
(b) $(\theta_{12}^{(1)}(e^{it}))^*\theta_{12}^{(1)}(e^{it}) \le (\theta_{12}^{(2)}(e^{it}))^*\theta_{12}^{(2)}(e^{it})$ *a.e. and*
$\theta_{21}^{(1)}(e^{it})(\theta_{21}^{(1)}(e^{it}))^* \le \theta_{21}^{(2)}(e^{it})(\theta_{21}^{(2)}(e^{it}))^*$ *a.e.;*
(c) $\omega_1(e^{it}) \vdots \omega_2(e^{it})$ *(on the right) and* $\lambda_1(e^{it}) \vdots \lambda_2(e^{it})$ *(on the left).*

Corollary 5.12 *Let the requirements of Theorem 5.11 be valid. The following statements are equivalent:*

(a) $\Xi_1(e^{it}) \sim \Xi_2(e^{it})$;
(b) $(\theta_{12}^{(1)}(e^{it}))^* \theta_{12}^{(1)}(e^{it}) = (\theta_{12}^{(2)}(e^{it}))^* \theta_{12}^{(2)}(e^{it})$ *a.e. and*
$\theta_{21}^{(1)}(e^{it})(\theta_{21}^{(1)}(e^{it}))^* = \theta_{21}^{(2)}(e^{it})(\theta_{21}^{(2)}(e^{it}))^*$ *a.e.;*
(c) $\omega_1(e^{it}) = \omega_{12}(e^{it})\omega_2(e^{it})$ *a.e. and* $\lambda_1(e^{it}) = \lambda_2(e^{it})\lambda_{21}(e^{it})$ *a.e., where* $\omega_{12}(e^{it}) \in CM[\mathfrak{F}^{(2)}, \mathfrak{F}^{(1)}]$ *and* $\lambda_{21}(e^{it}) \in CM[\mathfrak{G}^{(1)}, \mathfrak{G}^{(2)}]$ *are unitary operator functions.*

Similarly, as it was done in Definition 5.7 for the set $\mathcal{U}_r(\theta)/\sim$ and $\mathcal{L}_r(\theta)/\sim$, we can consider the concepts of a maximal element and the largest one for the partially ordered set $\mathcal{K}_r(\theta)/\sim$, as well as the concept of a maximal extension for the partially preordered set $\mathcal{K}_r(\theta)$. The subset of $\mathcal{K}_r(\theta)$ consisting of all maximal extensions will be denoted by $\mathcal{K}_{r,\max}(\theta)$ or simply $\mathcal{K}_{r,\max}$. The analogous concepts for the sets $\mathcal{K}_{cr}(\theta)$ and $\mathcal{K}_{cr}(\theta)/\sim$ can be introduced in a similar way. The subset of $\mathcal{K}_{cr}(\theta)$ consisting of all maximal extensions from $\mathcal{K}_{cr}(\theta)$ will be denoted by $\mathcal{K}_{cr,\max}(\theta)$ or simply $\mathcal{K}_{cr,\max}$.

A description of the sets $\mathcal{K}_{r,\max}(\theta)$ and $\mathcal{K}_{cr,\max}(\theta)$ is given by the following theorem, which follows directly from Definition 5.10 and Theorem 5.9.

Theorem 5.13 *Let* $\theta(e^{it})$ *belongs to* $CM[\mathfrak{G}, \mathfrak{F}]$.

(a) Let $\Xi(e^{it}) \in \mathcal{K}_r(\theta)$ *be a regular extension of form* (3.32)*,* $\omega(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}^{(1)}]$ *and* $\lambda(e^{it}) \in CM[\mathfrak{G}^{(1)}, \mathfrak{F}]$ *be the corresponding coisometric and isometric operator functions, respectively, from representation* (4.9) *satisfying conditions* (4.8) *and let*

$$\mathfrak{L}^{\perp} := \overline{\Pi L^2(\mathfrak{G})} \ominus \mathrm{Ran}\omega^*, \quad \mathfrak{L}_*^{\perp} := \overline{\Sigma L^2(\mathfrak{F})} \ominus \mathrm{Ran}\lambda.$$

Then $\Xi(e^{it}) \in \mathcal{U}_{r,\max}(\theta)$ *iff the conditions*

$$\alpha_\Pi < \infty, \quad \alpha_{\mathfrak{L}^\perp} = 0 \quad \textit{or} \quad \alpha_\Pi = \infty, \quad \mathfrak{L}^\perp = \{0\}$$

and

$$\alpha_\Sigma < \infty, \quad \alpha_{\mathfrak{L}_*^\perp} = 0 \quad \textit{or} \quad \alpha_\Sigma = \infty, \quad \mathfrak{L}_*^\perp = \{0\}$$

are satisfied. The element $[\Xi] \in \mathcal{K}_r(\theta/\sim$ *is the largest iff the extension* $\Xi(e^{it})$ *is unitary.*

(b) Let $\Xi(e^{it}) \in \mathcal{K}_{cr}(\theta)$ *be a completely regular extension of form* (3.32)*,* $\omega(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}^{(1)}]$ *and* $\lambda(e^{it}) \in CM[\mathfrak{G}^{(1)}, \mathfrak{F}]$ *be the corresponding coisometric and isometric operator functions, respectively, from representations* (4.11) *satisfying condition* (4.10) *and let*

$$\mathfrak{N} := \mathrm{Ker}\theta, \quad \mathfrak{N}_* := \mathrm{Ker}\theta^*, \quad \mathfrak{L}^\perp := \mathfrak{N} \ominus \mathrm{Ran}\omega^*, \quad \mathfrak{L}_*^\perp := \mathfrak{N}_* \ominus \mathrm{Ran}\lambda.$$

Then $\Xi(e^{it}) \in \mathcal{K}_{cr,\max}(\theta)$ *iff the conditions*

$$\alpha_{\mathfrak{N}} < \infty, \quad \alpha_{\mathfrak{L}^\perp} = 0 \quad or \quad \alpha_{\mathfrak{N}} = \infty, \quad \mathfrak{L}^\perp = \{0\}$$

and

$$\alpha_{\mathfrak{N}_*} < \infty, \quad \alpha_{\mathfrak{L}_*^\perp = 0} \quad or \quad \alpha_{\mathfrak{N}_*} = \infty, \quad \mathfrak{L}_*^\perp = \{0\}$$

are satisfied. The element $[\Xi] \in \mathcal{K}_{cr}(\theta)/\sim$ *is the largest iff the equalities*

$$\mathrm{Ran}\omega^* = \mathfrak{N}, \quad \mathrm{Ran}\lambda = \mathfrak{N}_*$$

are valid.

5.3 On Some Extremal Properties of the Norm of Regular Extensions

The following assertion can be regarded as a special case of Theorem 8.9 ([13]), but with somewhat weakened requirements.

Theorem 5.14 *Let* $\theta(e^{it})$ *belongs to* $CM[\mathfrak{G}, \mathfrak{F}]$. *If* $\mathfrak{F} \neq \{0\}(\mathfrak{G} \neq \{0\})$ *and there exists only trivial extension* $\Omega(e^{it}) \in \mathcal{U}_r(\theta)(\Lambda(e^{it}) \in \mathcal{L}_r(\theta))$, *then* $\|\theta\|_{L^\infty[\mathfrak{G},\mathfrak{F}]} = 1$.

Proof In both cases, we infer that $\mathfrak{F} \neq \{0\}$ and $\mathfrak{G} \neq \{0\}$. If $\|\theta\|_{L^\infty[\mathfrak{G},\mathfrak{F}]} < 1$, then $\theta(e^{it})$ is a strictly contractive operator at almost all t. But in this case, as was noted in Remark 4.4, there exist non-trivial regular extensions both upwards and leftwards. This contradicts the prerequisites of the theorem. □

The following assertion was proved in [13] (see Theorem 8.8).

Theorem 5.15 *Let* σ *be a minimal unitary coupling of form* (2.1) *and* $\theta(e^{it}) := \theta_\sigma(e^{it})$. *If for a pair of bilateral channels*

$$\{(M(\overset{\circ}{\mathfrak{F}}_1), \mathfrak{F}_1; V_{\mathfrak{F}_1}), (M(\overset{\circ}{\mathfrak{G}}_1), \mathfrak{G}_1; V_{\mathfrak{G}_1})\}$$

of the coupling σ *at least one of the inclusions* (3.20) *is strict, then for the function* $\theta_2(e^{it}) \in CM[\mathfrak{G}_1, \mathfrak{F}_1]$ *from the corresponding factorization* (3.21) *the equality* $\|\theta_2\|_{L^\infty[\mathfrak{G}_1,\mathfrak{F}_1]} = 1$ *holds.*

From this theorem, by Theorem 3.8 and Definition 3.11, we obtain

Corollary 5.16 *If* $\theta(e^{it}) \in CM[\mathfrak{G},\mathfrak{F}]$ *and* $\Xi(e^{it}) \in \mathcal{K}_r(\theta)$ *is a non-trivial extension of form* (3.32)*, then*

$$\|\Xi\|_{L^\infty[\mathfrak{G}^{(1)}\oplus\mathfrak{G},\mathfrak{F}^{(1)}\oplus\mathfrak{F}]} = 1.$$

Remark 5.17 In conclusion of this section, we note that, considering extensions $\Omega(e^{it})(\Lambda(e^{it}))$ of form (3.15) and $\Xi(e^{it})$ of form (3.22), but not requiring their regularity, we could constract a similar theory. For example, in an obvious way, we could give a description of the sets $\mathcal{U}(\theta)(\mathcal{L}(\theta))$ and $\mathcal{K}(\theta)$ of all (not necessarily regular) extensions, using representations (4.2) and (4.9), but without conditions (4.1) and (4.8). Or, introducing the comparison relation on $\mathcal{U}(\theta)(\mathcal{L}(\theta))$ by condition (b) from Theorem 5.5, we could study its properties. But this is beyond the scope of the present work.

6 Defect Functions in the Schur Class for Contractive Measurable Operator Functions

In this section we consider regular extensions of form (3.15), where $\theta_{12}(e^{it}) \in L^\infty_+[\mathfrak{G},\mathfrak{F}^{(1)}]$ $(\theta_{21}(e^{it}) \in L^\infty_+[\mathfrak{G}^{(1)},\mathfrak{F}])$.

6.1 Fundamental Contraction, Internal and External Unilateral Channels of Unitary Couplings

We begin with some necessary information on properties of contractions acting on Hilbert spaces. Recall that a contraction $T \in [\mathfrak{H}]$, where $\mathfrak{H}$ is a Hilbert space, is called completely nonunitary if there is no non-zero subspace of $\mathfrak{H}$ that reduces T and on which the restriction of T is unitary.

Theorem 6.1 (The Canonical Decomposition of a Contraction, [29]) *Let* $\mathfrak{H}$ *be a Hilbert space and* $T \in [\mathfrak{H}]$ *be a contraction. There exists the unique decomposition* $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1$ *such that*

(1) $\mathfrak{H}_j$ *reduces* T $(j = 0, 1)$*;*
(2) $U_0 := T|_{\mathfrak{H}_0}$ *is a unitary operator;*
(3) $T_1 := T|_{\mathfrak{H}_1}$ *is a completely nonunitary contraction.*

Certainly, it is possible that $\mathfrak{H}_0 = \{0\}$ or $\mathfrak{H}_1 = \{0\}$ is valid. The operators $U_0 \in [\mathfrak{H}_0]$ and $T_1 \in [\mathfrak{H}_1]$ are called *the unitary and completely nonunitary parts of the contraction* T, respectively. The decomposition $T = U_0 \oplus T_1$ is called *the canonical decomposition* of T.

Corollary 6.2 (The Wold Decomposition, [29]) *Let $\mathfrak{H}$ be a Hilbert space and a contraction $V \in [\mathfrak{H}]$ be an isometry. Then the subspace $\mathfrak{N} := \operatorname{Ker} V^*$ of $\mathfrak{H}$ is wandering with respect to V and the subspaces $\mathfrak{H}_0$ and $\mathfrak{H}_1$ corresponding to the canonical decomposition of V are given by the formulas*

$$\mathfrak{H}_0 := \bigcap_{n=1}^{\infty} V^n \mathfrak{H}, \quad \mathfrak{H}_1 := M_+(\mathfrak{N}).$$

Thus, in this case, the completely nonunitary contraction $V_1 := V|_{\mathfrak{H}_1}$ is a unilateral shift, that is,

$$\mathfrak{H}_1 = \bigoplus_{k=0}^{\infty} V_1^k \mathfrak{N}.$$

If $V \in [\mathfrak{H}]$ is a unilateral shift, then $\tilde{V} := V^* \in [\mathfrak{H}]$ is termed the backward shift or, for simplicity, the coshift associated with the shift V.

If $T \in [\mathfrak{H}]$ is a contraction, $\mathfrak{L}$ is a subspace of $\mathfrak{H}$ invariant for T, and $V := T|_{\mathfrak{L}}$ is an isometry, then we say that $V \in [\mathfrak{L}]$ is contained in T and denote this by $V \subset T$. If $\mathfrak{L}_*$ is a subspace of $\mathfrak{H}$ invariant for T^*, then we say that $\tilde{V} \in [\mathfrak{L}_*]$ is a coisometry contained in T if $\tilde{V}^* \in [\mathfrak{L}_*]$ is an isometry contained in T^*. We also denote this by $\tilde{V} \subset T$. Further, considering the set $\mathcal{V}_T$ ($\tilde{\mathcal{V}}_T$) of isometries (coisometries) contained in T with the inclusion relation on it, we turn it into a partially ordered set. From the arguments carried out in [29] (Chapter I, §3, Theorem 3.2) we can formulate

Theorem 6.3 *Let $T \in [\mathfrak{H}]$ be a contraction.*

(a) In the set $\mathcal{V}_T(\tilde{\mathcal{V}}_T)$ there exists the largest isometry $V_T \in [\mathfrak{H}_{V_T}]$(coisometry $\tilde{V}_T \in [\mathfrak{H}_{\tilde{V}_T}]$), where

$$\mathfrak{H}_{V_T} := \{h \in \mathfrak{H} : \|T^n h\| = \|h\|, n = 1, 2, \ldots\}$$

$$(\mathfrak{H}_{\tilde{V}_T} := \{h \in \mathfrak{H} : \|(T^*)^n h\| = \|h\|, n = 1, 2, \ldots\}),$$

$$V_T := T|_{\mathfrak{H}_{V_T}} \quad (\tilde{V}_T := (T^*|_{\mathfrak{H}_{\tilde{V}_T}})^*).$$

(b) If $U_0 \in [\mathfrak{H}_0]$ is the unitary part of T, then

$$\mathfrak{H}_0 = \mathfrak{H}_{V_T} \cap \mathfrak{H}_{\tilde{V}_T}.$$

In the case of a completely nonunitary contraction T, the set $\mathcal{V}_T$ ($\tilde{\mathcal{V}}_T$) consists of all unilateral shifts (coshifts) contained in T and its largest element V_T ($\tilde{V}_T$) will be called *the largest shift (coshift) contained in T*.

Now let σ be a unitary coupling of form (2.1). Note that any unilateral output (input) channel $(\mathfrak{L}_+, \mathfrak{N}; V_{\mathfrak{N}})$ $((\mathfrak{L}_-, \mathfrak{N}; V_{\mathfrak{N}}))$ of the coupling σ determines uniquely the unilateral shift V (coshift $\tilde{V}$) defined by the formula $V := U|_{\mathfrak{L}_+} \in [\mathfrak{L}_+](\tilde{V} := (U^*|_{\mathfrak{L}_-})^* \in [\mathfrak{L}_-])$ and contained in U. Conversely, any unilateral shift V (coshift $\tilde{V}$) contained in U and defined on a subspace $\mathfrak{L}_+$ ($\mathfrak{L}_-$) of $\mathfrak{H}$ generates the unilateral output (input) channel $(\mathfrak{L}_+, \mathfrak{N}; V_{\mathfrak{N}})$ $((\mathfrak{L}_-, \mathfrak{N}; V_{\mathfrak{N}}))$ of σ. It is determined up to the choice of a Hilbert space $\mathfrak{N}$ such that

$$\dim\mathfrak{N} = \dim(\mathrm{Ker}V^*) \quad (\dim\mathfrak{N} = \dim(\mathrm{Ker}\tilde{V}))$$

and of an embedding isometry $V_{\mathfrak{N}} \in [\mathfrak{N}, \mathfrak{H}]$ such that

$$\mathrm{Ran}\, V_{\mathfrak{N}} = \mathrm{Ker}V^* \quad (\mathrm{Ran}\, V_{\mathfrak{N}} = U(\mathrm{Ker}\tilde{V})).$$

In addition to the notation $\mathfrak{R}_{\mathfrak{N}} := \mathfrak{H} \ominus M(\mathfrak{N})$ for any wandering subspace $\mathfrak{N}$ of $\mathfrak{H}$ introduced in Sect. 2.1, we denote by $\mathfrak{R}_{\mathfrak{N}}^+ := \mathfrak{H} \ominus M_-(\mathfrak{N})(\mathfrak{R}_{\mathfrak{N}}^- := \mathfrak{H} \ominus M_+(\mathfrak{N}))$ the subspace of $\mathfrak{H}$ that is invariant with respect to $U(U^*)$. It is clear that $\mathfrak{R}_{\mathfrak{N}} = \mathfrak{R}_{\mathfrak{N}}^+ \cap \mathfrak{R}_{\mathfrak{N}}^-$.

Definition 6.4 Let σ be a unitary coupling of form (2.1) and

$$\mathfrak{H}_T := \mathfrak{H} \ominus (M_-(\overset{\circ}{\mathfrak{F}}) \vee M_+(\overset{\circ}{\mathfrak{G}}))(= \mathfrak{R}^+_{\overset{\circ}{\mathfrak{F}}} \cap \mathfrak{R}^-_{\overset{\circ}{\mathfrak{G}}}), \quad T := P_{\mathfrak{H}_T} U|_{\mathfrak{H}_T}. \tag{6.1}$$

The contraction $T \in [\mathfrak{H}_T]$ will be called the fundamental contraction of the coupling σ. To emphasize its dependence on σ we sometimes denote it by T_σ. The subspace $\mathfrak{H}_T$ will be termed the internal subspace of the coupling σ. A unilateral output (input) channel $(\mathfrak{L}_+, \mathfrak{N}; V_{\mathfrak{N}})((\mathfrak{L}_-, \mathfrak{N}; V_{\mathfrak{N}}))$ such that $\mathfrak{L}_+ \subset \mathfrak{H}_T(\mathfrak{L}_- \subset \mathfrak{H}_T)$, as well as the corresponding unilateral shift $V := U|_{\mathfrak{L}_+} \in [\mathfrak{L}_+]$(coshift $\tilde{V} := (U^*|_{\mathfrak{L}_-})^* \in [\mathfrak{L}_-]$), will be called internal for the coupling σ.

At the same time, the unilateral output (input) channel $(\mathfrak{L}_+, \mathfrak{N}; V_{\mathfrak{N}})$ $((\mathfrak{L}_-, \mathfrak{N}; V_{\mathfrak{N}}))$, as well as the corresponding unilateral shift $V($ coshift $\tilde{V})$, will be called external for the coupling σ if

$$\mathfrak{L}_+ \subset M_-(\overset{\circ}{\mathfrak{F}}) \vee M_+(\overset{\circ}{\mathfrak{G}}) \quad (\mathfrak{L}_- \subset M_-(\overset{\circ}{\mathfrak{F}}) \vee M_+(\overset{\circ}{\mathfrak{G}})).$$

In particular, the shift $V_\sigma := U|_{M_+(\overset{\circ}{\mathfrak{G}})} \in [M_+(\overset{\circ}{\mathfrak{G}})]$ and the coshift $\widetilde{V}_\sigma := (U^*|_{M_-(\overset{\circ}{\mathfrak{F}})})^* \in [M_-(\overset{\circ}{\mathfrak{F}})]$ will be referred as the principal external unilateral shift and coshift of σ.

The concept of the fundamental contraction for unitary couplings is a generalization of its particular case for orthogonal unitary couplings (see [11], Section 6) and, hence, of the concept of the basic contraction for unitary colligations (see [19]).

Theorem 6.5 *A unitary coupling σ is minimal iff the fundamental contraction T_σ is completely nonunitary.*

Proof Let σ be a coupling of form (2.1) and $T := T_\sigma$. If $U_0 \in [\mathfrak{H}_0]$ is the unitary part of the contraction T, then $U_0 = U|_{\mathfrak{H}_0}$ follows from $U_0 = T|_{\mathfrak{H}_0}$. Hence, in view of the reducibility of $\mathfrak{H}_0$ for U, the condition $\mathfrak{H}_0 \perp M_-(\mathring{\mathfrak{F}})$ $(\mathfrak{H}_0 \perp M_+(\mathring{\mathfrak{G}}))$ is equivalent to the condition $\mathfrak{H}_0 \perp M(\mathring{\mathfrak{F}})$ $(\mathfrak{H}_0 \perp M(\mathring{\mathfrak{G}}))$. Thus, from $\mathfrak{H}_0 \subset \mathfrak{H}_T$ we obtain

$$\mathfrak{H}_0 \subset \mathfrak{H}^{(0)} := \mathfrak{H} \ominus (M(\mathring{\mathfrak{F}}) \vee M(\mathring{\mathfrak{G}})).$$

Similarly, we infer that $\mathfrak{H}^{(0)} \subset \mathfrak{H}_0$, whence $\mathfrak{H}^{(0)} = \mathfrak{H}_0$ follows. $\square$

Remark 6.6 Let σ^* be the adjoint unitary coupling of form (2.7) in relation to the coupling σ and $T_* := T_{\sigma^*} \in [\mathfrak{H}_{T_*}]$. Then for the abundant subspace $\mathfrak{H}^{(0)}$ of σ (and, hence, of σ^*) we obtain

$$\mathfrak{H}^{(0)} = \mathfrak{R}_{\mathring{\mathfrak{F}}} \cap \mathfrak{R}_{\mathring{\mathfrak{G}}} = (\mathfrak{R}^+_{\mathring{\mathfrak{F}}} \cap \mathfrak{R}^-_{\mathring{\mathfrak{G}}}) \cap (\mathfrak{R}^+_{\mathring{\mathfrak{G}}} \cap \mathfrak{R}^-_{\mathring{\mathfrak{F}}}) = \mathfrak{H}_T \cap \mathfrak{H}_{T_*}.$$

For the fundamental contraction of a minimal unitary coupling we can refine the information formulated in Theorem 6.3.

Theorem 6.7 *Let σ be a minimal unitary coupling of form* (2.1) *and $T := T_\sigma$. Then*

$$\mathfrak{H}_{V_T} = \mathfrak{R}^+_{\mathring{\mathfrak{F}}} \cap \mathfrak{R}_{\mathring{\mathfrak{G}}} \quad (\mathfrak{H}_{\tilde{V}_T} = \mathfrak{R}_{\mathring{\mathfrak{F}}} \cap \mathfrak{R}^-_{\mathring{\mathfrak{G}}}) \tag{6.2}$$

and for the largest internal unilateral shift V_T (coshift $\tilde{V}_T$) the equality

$$V_T = U|_{\mathfrak{H}_{V_T}} \quad (\tilde{V}_T = (U^*|_{\mathfrak{H}_{\tilde{V}_T}})^*)$$

holds.

Proof It suffices to prove the theorem for V_T.

The subspace $\mathfrak{L}^{(0)}_+ := \mathfrak{R}^+_{\mathring{\mathfrak{F}}} \cap \mathfrak{R}_{\mathring{\mathfrak{G}}}$ is obviously invariant with respect to U and, in addition, we obtain that

$$\mathfrak{L}^{(0)}_+ = (\mathfrak{H} \ominus M_-(\mathring{\mathfrak{F}})) \cap (\mathfrak{H} \ominus M(\mathring{\mathfrak{G}})) \subset (\mathfrak{H} \ominus M_-(\mathring{\mathfrak{F}})) \cap (\mathfrak{H} \ominus M_+(\mathring{\mathfrak{G}})) = \mathfrak{H}_T.$$

Hence,

$$U|_{\mathfrak{L}^{(0)}_+} = (P_{\mathfrak{H}_T} U|_{\mathfrak{H}_T})|_{\mathfrak{L}^{(0)}_+} = T|_{\mathfrak{L}^{(0)}_+}$$

and, in view of the complete nonunitarity of T, the isometry $V^{(0)} := U|_{\mathfrak{L}^{(0)}_+}$ is an internal unilateral shift of the coupling σ.

If $V \in [\mathfrak{L}_+]$ is an internal unilateral shift of the coupling σ, then $\mathfrak{L}_+$ is invariant for T and

$$V = T|_{\mathfrak{L}_+} = P_{\mathfrak{L}_+} T|_{\mathfrak{L}_+} = P_{\mathfrak{L}_+}(P_{\mathfrak{H}_T} U|_{\mathfrak{H}_T})|_{\mathfrak{L}_+} = P_{\mathfrak{L}_+} U|_{\mathfrak{L}_+} = U|_{\mathfrak{L}_+}.$$

Thus, $\mathfrak{L}_+$ is invariant for U and, hence, from $\mathfrak{L}_+ \perp M_+(\overset{\circ}{\mathfrak{G}})$ it follows that $\mathfrak{L}_+ \perp M(\overset{\circ}{\mathfrak{G}})$, whence we obtain the inclusion

$$\mathfrak{L}_+ \subset \mathfrak{R}^+_{\overset{\circ}{\mathfrak{F}}} \cap \mathfrak{R}_{\overset{\circ}{\mathfrak{G}}} = \mathfrak{L}_+^{(0)}$$

Thus, $\mathfrak{L}_+^{(0)} = \mathfrak{H}_{V_T}$ and $V^{(0)}$ is the largest internal unilateral shift V_T of the coupling σ. □

In this connection, in the case of a minimal unitary coupling σ, the largest internal shift $V_T \in [\mathfrak{H}_{V_T}]$ and coshift $\tilde{V}_T \in [\mathfrak{H}_{\tilde{V}_T}]$ will be referred as *the principal internal shift and coshift* of σ. The corresponding unilateral channels will also be called *the principal internal unilateral output and input channels of* σ, respectively.

Important for the study of completely regular extensions is the subset

$$\mathcal{W}_T := \{V \in \mathcal{V}_T : V \subset V_{\sigma^*}\} \quad (\tilde{\mathcal{W}}_T := \{\tilde{V} \in \tilde{\mathcal{V}}_T : \tilde{V} \subset \tilde{V}_{\sigma^*}\})$$

of the set $\mathcal{V}_T(\tilde{\mathcal{V}}_T)$. Here $V_{\sigma^*}(\tilde{V}_{\sigma^*})$ is the principal external unilateral shift (coshift) of the coupling σ^* of form (2.7).

Theorem 6.8 *Let σ be a minimal unitary coupling and $T := T_\sigma$. Among shifts (coshifts) from $\mathcal{W}_T$ $(\tilde{\mathcal{W}}_T)$ there exists the largest shift $W_T \in [\mathfrak{H}_{W_T}]$ (coshift $\tilde{W}_T \in [\mathfrak{H}_{\tilde{W}_T}]$). Moreover,*

$$\mathfrak{H}_{W_T} := \mathfrak{H}_{V_T} \cap M_+(\overset{\circ}{\mathfrak{F}}) \quad (\mathfrak{H}_{\tilde{W}_T} := \mathfrak{H}_{\tilde{V}_T} \cap M_-(\overset{\circ}{\mathfrak{G}})), \tag{6.3}$$

$$W_T := U|_{\mathfrak{H}_{W_T}} \quad (\tilde{W}_T := (U^*|_{\mathfrak{H}_{\tilde{W}_T}})^*).$$

Proof This assertion for W_T is a direct corollary of the equalities

$$V_T = U|_{\mathfrak{H}_{V_T}}, \quad V_{\sigma^*} = U|_{M_+(\overset{\circ}{\mathfrak{F}})}.$$

The assertion for $\tilde{W}_T$ is dual in relation to it. □

6.2 Extensions from $\mathcal{U}_r(\theta)$ ($\mathcal{L}_r(\theta)$) Generated by Internal Unilateral Coshifts (Shifts) of a Coupling

Let $\theta(e^{it})$ belong to $CM[\mathfrak{G},\mathfrak{F}]$. Denote by $\mathcal{U}_r^+(\theta)$ ($\mathcal{L}_r^+(\theta)$) or simply $\mathcal{U}_r^+$ ($\mathcal{L}_r^+$) the subset of $\mathcal{U}_r(\theta)$ ($\mathcal{L}_r(\theta)$) consisting of regular upward (leftward) extensions of $\Omega(e^{it})$ ($\Lambda(e^{it})$) of form (3.15) such that

$$\theta_{12}(e^{it}) \in L_+^\infty[\mathfrak{G},\mathfrak{F}^{(1)}] \quad (\theta_{21}(e^{it}) \in L_+^\infty[\mathfrak{G}^{(1)},\mathfrak{F}]). \tag{6.4}$$

Its subset of completely regular upward (leftward) extensions will be denoted by $\mathcal{U}_{cr}^+(\theta)$ ($\mathcal{L}_{cr}^+(\theta)$) or simply $\mathcal{U}_{cr}^+$ ($\mathcal{L}_{cr}^+$).

Let σ be a minimal unitary coupling of form (2.1) and $\theta(e^{it}) := \theta_\sigma(e^{it})$. As it follows from Definition 3.6 and Theorem 3.2 (Theorem 3.4), $\theta_{12}(e^{it}) = \theta_{\sigma_{12}}(e^{it})$ ($\theta_{21}(e^{it}) = \theta_{\sigma_{21}}(e^{it})$), where σ_{12} (σ_{21}) is the coupling of form (3.6) and (3.11). Hence, by Theorem 2.33, condition (6.4) is satisfied iff the coupling σ_{12} (σ_{21}) is orthogonal, that is, the condition

$$M_-(\mathring{\mathfrak{F}}^{(1)}) \perp M_+(\mathring{\mathfrak{G}}) \quad (M_-(\mathring{\mathfrak{F}}) \perp M_+(\mathring{\mathfrak{G}}^{(1)})) \tag{6.5}$$

is fulfilled. Taking into account (3.7) (3.12), (6.1), and (6.5), we see that condition (6.4) is satisfied iff the inclusion

$$M_-(\mathring{\mathfrak{F}}^{(1)}) \subset \mathfrak{H}_T \quad (M_+(\mathring{\mathfrak{G}}^{(1)}) \subset \mathfrak{H}_T) \tag{6.6}$$

is satisfied.

In the case of completely regular extensions, taking into account additional condition (3.9) and (3.14), we obtain that $\Omega(e^{it}) \in \mathcal{U}_{cr}^+$ ($\Lambda(e^{it}) \in \mathcal{L}_{cr}^+$) iff, besides (6.6), the inclusion

$$M_-(\mathring{\mathfrak{F}}^{(1)}) \subset M_-(\mathring{\mathfrak{G}}) \quad (M_+(\mathring{\mathfrak{G}}^{(1)}) \subset M_+(\mathring{\mathfrak{F}})) \tag{6.7}$$

holds.

Thus, we can reformulate Theorem 3.2 (Theorem 3.4) for $\mathcal{U}_r^+$ ($\mathcal{L}_r^+$) in terms of internal unilateral channels.

Theorem 6.9 *Let $\theta(e^{it}) \in CM[\mathfrak{G},\mathfrak{F}]$ and σ be a minimal unitary coupling of form (2.1) such that $\theta_\sigma(e^{it}) = \theta(e^{it})$.*

(a) There exists a bijective correspondence between extensions $\Omega(e^{it}) \in \mathcal{U}_r^+(\theta)$ ($\Lambda(e^{it}) \in \mathcal{L}_r^+(\theta)$) of form (3.15) and internal unilateral input (output) channels $(M_-(\mathring{\mathfrak{F}}^{(1)}), \mathfrak{F}^{(1)}; V_{\mathfrak{F}^{(1)}})$ ($(M_+(\mathring{\mathfrak{G}}^{(1)}), \mathfrak{G}^{(1)}; V_{\mathfrak{G}^{(1)}})$) of the coupling σ. This correspondence is established by the equality $\theta_{12}(e^{it}) = \theta_{\sigma_{12}}(e^{it})$ ($\theta_{21}(e^{it}) = \theta_{\sigma_{21}}(e^{it})$), where σ_{12} (σ_{21}) is a unitary coupling of form (3.6) and (3.11).

(b) An extension $\Omega(e^{it}) \in \mathcal{U}_r^+(\theta)$ $(\Lambda(e^{it}) \in \mathcal{L}_r^+(\theta))$ is completely regular iff the corresponding internal unilateral input (output) channel satisfies additionally condition (6.7).

Parts (*b*) and (*c*) of Theorem 4.2 can be refine for $\mathcal{U}_r^+$ $(\mathcal{L}_r^+)$ in the following way.

Theorem 6.10 *Let $\theta(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$ and $\Omega(e^{it}) \in \mathcal{U}_r(\theta)$ $(\Lambda(e^{it}) \in \mathcal{L}_r(\theta))$ be its extension of form (3.15).*

(a) $\Omega(e^{it}) \in \mathcal{U}_r^+(\theta)(\Lambda(e^{it}) \in \mathcal{L}_r^+(\theta))$ iff there exists a coisometric (isometric) operator function $\omega(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}^{(1)}](\lambda(e^{it}) \in CM[\mathfrak{G}^{(1)}, \mathfrak{F}])$ such that for the function $\omega(e^{it})$ $(\lambda(e^{it}))$ inclusion (4.1) and the condition

$$\omega(e^{it})\Pi(e^{it}) \in L_+^\infty[\mathfrak{G}, \mathfrak{F}^{(1)}] \quad (\Sigma(e^{it})\lambda(e^{it}) \in L_+^\infty[\mathfrak{G}^{(1)}, \mathfrak{F}])$$

are satisfied and representation (4.2) is valid.

(b) $\Omega(e^{it}) \in \mathcal{U}_{cr}^+(\theta)(\Lambda(e^{it}) \in \mathcal{L}_{cr}^+(\theta))$ iff the corresponding function $\omega(e^{it})(\lambda(e^{it}))$ is the boundary value function of some $*$-inner (inner) operator function $\omega(\zeta) \in S[\mathfrak{G}, \mathfrak{F}^{(1)}](\lambda(\zeta) \in S[\mathfrak{G}^{(1)}, \mathfrak{F}])$ and inclusion (4.3) is valid. In this case, representation (4.2) takes form (4.4).

For what follows we need some refinement of Theorem 4.1 (part (*a*)) for unilateral channels (see [12], Theorem 7.10).

Theorem 6.11 *There exists a bijective correspondence between unilateral input (output) channels $(\mathcal{L}_-, \mathfrak{N}; V_{\mathfrak{N}})$ $((\mathcal{L}_+, \mathfrak{N}; V_{\mathfrak{N}}))$ in $L^2(\mathfrak{K})$ and isometric operator functions $\theta(e^{it}) \in CM[\mathfrak{N}, \mathfrak{K}]$. This correspondence is established by the formulas*

$$V_{\mathfrak{N}} = \theta|_{\mathfrak{N}}, \quad \mathfrak{L}_- = M_-(\overset{\circ}{\mathfrak{N}}) \quad (\mathfrak{L}_+ = M_+(\overset{\circ}{\mathfrak{N}})); \quad \theta = (\Phi^{\mathfrak{N}}_{U^\times_{\mathfrak{K}}})^*.$$

Moreover, the subspace $M_-(\overset{\circ}{\mathfrak{N}})(M_+(\overset{\circ}{\mathfrak{N}}))$ determines the function $\theta(e^{it})$ uniquely up to a right constant unitary factor.

From Theorems 6.9–6.11 we obtain

Theorem 6.12 *Let $\theta(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$, σ be a minimal unitary coupling of form (2.1) such that $\theta_\sigma(e^{it}) = \theta(e^{it})$, and $T := T_\sigma$.*

(a) There exists a bijective correspondence between internal coshifts $\tilde{V} \in \tilde{\mathcal{V}}_T$ (shifts $V \in \mathcal{V}_T$) of the coupling σ and extensions $\Omega(e^{it}) \in \mathcal{U}_r^+(\theta)$ $(\Lambda(e^{it}) \in \mathcal{L}_r^+(\theta))$ of form (3.15) if the block $\theta_{12}(e^{it})$ $(\theta_{21}(e^{it}))$ is considered up to a left (right) constant unitary factor. This correspondence is obtained from the correspondence established in Theorem 6.9 (part (a)) if we take into account the equality

$$\tilde{V} = (U^*|_{M_-(\overset{\circ}{\mathfrak{F}}{}^{(1)})})^* \quad (V = U|_{M_+(\overset{\circ}{\mathfrak{G}}{}^{(1)})}).$$

(b) *An extension $\Omega(e^{it}) \in \mathcal{U}_r^+(\theta)$ $(\Lambda(e^{it}) \in \mathcal{L}_r^+(\theta))$ is completely regular iff the corresponding coshift $\tilde{V} \in \tilde{\mathcal{V}}_T$ (shift $V \in \mathcal{V}_T$) belongs to the subset $\tilde{\mathcal{W}}_T$ $(\mathcal{W}_T)$.*

Proof

(*a*) As was shown in the proof of Theorem 4.2 (part (*b*)), the isometric operator function $\hat{Y}(e^{it}) \in CM[\mathfrak{F}^{(1)}, \mathfrak{K}]$, $\mathfrak{K} := \mathfrak{G} \oplus \mathfrak{F}$, corresponding to a bilateral channel $(\hat{M}(\overset{\circ}{\mathfrak{F}}{}^{(1)}), \mathfrak{F}^{(1)}; \hat{V}_{\mathfrak{F}^{(1)}})$ (and, hence, to a unilateral input channel $(\hat{M}_-(\overset{\circ}{\mathfrak{F}}{}^{(1)}),$ $\mathfrak{F}^{(1)}; \hat{V}_{\mathfrak{F}^{(1)}}))$ of the coupling σ, is given by formula (4.7). By Theorem 6.11, the function $\hat{Y}(e^{it})$ is determined by the subspace $\hat{M}_-(\overset{\circ}{\mathfrak{F}}{}^{(1)})$ (and, hence, by the coshift $\tilde{V} := (\hat{U}^*|_{\hat{M}_-(\overset{\circ}{\mathfrak{F}}{}^{(1)})})^*$) up to a right constant unitary factor. It follows from this that the functions $\omega(e^{it})$ and, in view of (4.2), $\theta_{12}(e^{it})$ are determined up to left constant unitary factor.

(*b*) This part follows from part (*a*) of this theorem, Theorems 6.9 (part (*b*)), 6.10 (part (*b*)), and the definition of $\tilde{\mathcal{W}}_T$.

The dual assertion is proved in a similar way. □

Theorem 6.12 enables us to say that extensions $\Omega(e^{it}) \in \mathcal{U}_r^+(\theta)$ $(\Lambda(e^{it}) \in \mathcal{L}_r^+(\theta))$ of form (3.15) are generated by internal unilateral coshifts (shifts) of a minimal unitary coupling σ such that $\theta_\sigma(e^{it}) = \theta(e^{it})$.

6.3 Defect Functions in the Schur Class

Let $\theta(e^{it})$ belong to $CM[\mathfrak{G}, \mathfrak{F}]$. From now on we will study extensions $\Omega(e^{it}) \in \mathcal{U}_r^+(\theta)$ $(\Lambda(e^{it}) \in \mathcal{L}_r^+(\theta))$ of form (3.15) considering both the block $\theta_{12}(e^{it})$ $(\theta_{21}(e^{it}))$ and the coisometric (isometric) operator function $\omega(e^{it})$ $(\lambda(e^{it}))$ from representation (4.2) up to a left (right) constant unitary factor.

Let $\theta(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$, σ be a minimal unitary coupling of form (2.1) such that $\theta_\sigma(e^{it}) = \theta(e^{it})$, and $T := T_\sigma$.

Leaning on Theorem 6.12 (part (*a*)), we denote by

$$\Omega_0^+(e^{it}) := \begin{bmatrix} \varphi_+(e^{it}) \\ \theta(e^{it}) \end{bmatrix} \in CM[\mathfrak{G}, \mathfrak{K} \oplus \mathfrak{F}](\Lambda_0^+(e^{it}) := [\psi_+(e^{it}), \theta(e^{it})] \in CM[\mathfrak{K}_* \oplus \mathfrak{G}, \mathfrak{F}]) \tag{6.8}$$

the extension from $\mathcal{U}_r^+(\theta)$ $(\mathcal{L}_r^+(\theta))$ corresponding to the principal internal coshift $\tilde{V}_T \in \tilde{\mathcal{V}}_T$ (shift $V_T \in \mathcal{V}_T$) of the coupling σ. Thus, $\varphi_+(e^{it})$ $(\psi_+(e^{it})$ is the boundary value function of some Schur function $\varphi_+(\zeta) \in S[\mathfrak{G}, \mathfrak{K}]$ $(\psi_+(\zeta) \in S[\mathfrak{K}_*, \mathfrak{F}])$. It admits the representation of type (4.2), that is,

$$\varphi_+(e^{it}) = \omega_0^+(e^{it})\Pi(e^{it}) \text{ a.e.} \quad (\psi_+(e^{it}) = \Sigma(e^{it})\lambda_0^+(e^{it}) \text{ a.e.}), \tag{6.9}$$

where $\omega_0^+(e^{it}) \in CM[\mathfrak{G}, \mathfrak{K}]$ ($\lambda_0^+(e^{it}) \in CM[\mathfrak{K}_*, \mathfrak{G}]$) is the coisometric (isometric) operator function corresponding to the largest internal coshift $\tilde{V}_T$ (shift V_T) of the coupling σ and satisfying the condition of type (4.1), that is,

$$\mathrm{Ran}(\omega_0^+)^* \subset \overline{\Pi L^2(\mathfrak{G})} \quad (\mathrm{Ran}\lambda_0^+ \subset \overline{\Sigma L^2(\mathfrak{F})}).$$

Definition 6.13 Let $\theta(e^{it})$ belong to $CM[\mathfrak{G}, \mathfrak{F}]$. The Schur function $\varphi_+(\zeta) \in S[\mathfrak{G}, \mathfrak{K}]$ ($\psi_+(\zeta) \in S[\mathfrak{K}_*, \mathfrak{F}]$) generating the regular extension $\Omega_0^+(e^{it}) \in \mathcal{U}_r^+(\theta)$ ($\Lambda_0^+(e^{it}) \in \mathcal{L}_r^+(\theta)$) of form (6.8) will be called the defect ($\star$-defect) function of $\theta(e^{it})$ in the Schur class.

Note that these concepts are generalizations of their analogs for a Schur operator function $\theta(\zeta) \in S[\mathfrak{G}, \mathfrak{F}]$ studied in [14] and [23], where they were called the right and left defect functions, respectively.

Recall that a Schur operator function $\theta(\zeta) \in S[\mathfrak{G}, \mathfrak{F}]$ is termed outer ($*$-outer) if

$$\overline{\theta H_+^2(\mathfrak{G})} = H_+^2(\mathfrak{F}) \quad (\overline{\theta^{\sim} H_+^2(\mathfrak{F})} = H_+^2(\mathfrak{G})).$$

It is said to be two-sided outer if it is outer and $*$-outer.

Theorem 6.14 *Let $\theta(e^{it})$ belong to $\in$ $CM[\mathfrak{G}, \mathfrak{F}]$. Then its defect ($*$-defect) function $\varphi_+(\zeta) \in S[\mathfrak{G}, \mathfrak{K}]$ ($\psi_+(\zeta) \in S[\mathfrak{K}_*, \mathfrak{F}]$) is outer ($*$-outer).*

Proof It suffices to prove the theorem for $\varphi_+(\zeta)$.

Let σ be a minimal unitary coupling of form (2.1) such that $\theta_\sigma(e^{it}) = \theta(e^{it})$, and $T := T_\sigma$.

Since $\Omega_0^+(e^{it}) \in CM[\mathfrak{G}, \mathfrak{K} \oplus \mathfrak{F}]$ of form (6.8) corresponds to the largest coshift $\tilde{V}_T \in [M_-(\mathring{\mathfrak{K}})]$ in the set $\tilde{\mathcal{V}}_T$, then, by Theorem 6.9 (part (a)), $\varphi_+(e^{it}) = \theta_{\sigma_{12}^+}(e^{it})$. Here

$$\sigma_{12}^+ := (\mathfrak{H}, \mathfrak{K}, \mathfrak{G}; U, V_{\mathfrak{K}}, V_{\mathfrak{G}}) \tag{6.10}$$

is a coupling of type (3.6) generated by the principal internal unilateral input channel $(M_-(\mathring{\mathfrak{K}}), \mathfrak{K}; V_{\mathfrak{K}})$. In view of (6.2), $M(\mathring{\mathfrak{K}}) \subset \mathfrak{N}_{\mathring{\mathfrak{F}}}$ and, hence, we obtain

$$M_-(\mathring{\mathfrak{K}}) = \mathfrak{N}_{\mathring{\mathfrak{F}}} \cap \mathfrak{N}_{\mathring{\mathfrak{G}}}^- = M(\mathring{\mathfrak{K}}) \cap (M_+(\mathring{\mathfrak{G}}))^\perp. \tag{6.11}$$

Since for any subspace $\mathfrak{M} \subset \mathfrak{H}$ the equality

$$M(\mathring{\mathfrak{K}}) = (M(\mathring{\mathfrak{K}}) \cap \mathfrak{M}^\perp) \oplus \overline{P_{M(\mathring{\mathfrak{K}})} \mathfrak{M}}$$

holds, then, setting $\mathfrak{M} := M_+(\overset{\circ}{\mathfrak{G}})$, we infer from (6.11) that

$$M_+(\overset{\circ}{\mathfrak{K}}) = \overline{P_{M(\overset{\circ}{\mathfrak{K}})} M_+(\overset{\circ}{\mathfrak{G}})} = \overline{S_{\sigma_{12}^+} M_+(\overset{\circ}{\mathfrak{G}})}$$

where $S_{\sigma_{12}^+} := P_{M(\overset{\circ}{\mathfrak{K}})}|_{M(\overset{\circ}{\mathfrak{G}})}$ (see (2.4)). From this, taking into account (2.5) and the properties of the Fourier representations, we obtain

$$\overline{\varphi_+ L_+^2(\mathfrak{G})} = \overline{\Phi_U^{\mathfrak{K}} S_{\sigma_{12}^+} (\Phi_U^{\mathfrak{G}})^* L_+^2(\mathfrak{G})} = \Phi_U^{\mathfrak{K}} \overline{S_{\sigma_{12}^+} M_+(\overset{\circ}{\mathfrak{G}})} = \Phi_U^{\mathfrak{K}} M_+(\overset{\circ}{\mathfrak{K}}) = L_+^2(\mathfrak{K}).$$

□

Similarly, taking into account Theorem 6.12 (part (*b*)), we denote by

$$\Omega_{c0}^+(e^{it}) := \begin{bmatrix} \gamma_+(e^{it}) \\ \theta(e^{it}) \end{bmatrix} \in CM[\mathfrak{G}, \mathfrak{N} \oplus \mathfrak{F}]$$

$$(\Lambda_{c0}^+(e^{it}) := [\delta_+(e^{it}), \theta(e^{it})] \in CM[\mathfrak{N}_* \oplus \mathfrak{G}, \mathfrak{F}]) \tag{6.12}$$

the extension from $\mathcal{U}_{cr}^+(\theta)$ $(\mathcal{L}_{cr}^+(\theta))$ corresponding to the largest coshift $\tilde{W}_T \in \tilde{\mathcal{W}}_T$ (shift $W_T \in \mathcal{W}_T$). Thus, by virtue of Theorem 6.10 (part (*b*)), $\gamma_+(e^{it})(\delta_+(e^{it}))$ is the boundary value function of some $*$-inner (inner) operator function $\gamma_+(\zeta) \in S[\mathfrak{G}, \mathfrak{N}](\delta_+(\zeta) \in S[\mathfrak{N}_*, \mathfrak{F}])$ which satisfies the condition of type (4.3), that is,

$$\mathrm{Ran}\gamma_+^* \subset \mathrm{Ker}\theta \quad (\mathrm{Ran}\delta_+ \subset \mathrm{Ker}\theta^*).$$

Definition 6.15 Let $\theta(e^{it})$ belong to $CM[\mathfrak{G}, \mathfrak{F}]$. The $*$-inner (inner) operator function $\gamma_+(\zeta) \in S[\mathfrak{G}, \mathfrak{N}]$ $(\delta_+(\zeta) \in S[\mathfrak{N}_*, \mathfrak{F}])$ generating the completely regular extension $\Omega_{c0}^+(e^{it}) \in \mathcal{U}_{cr}^+(\theta)$ $(\Lambda_{c0}^+(e^{it}) \in \mathcal{L}_{cr}^+(\theta))$ of form (6.12) will be called the defect ($*$-defect) function of $\theta(e^{it})$ in the class of $*$-inner (inner) operator functions.

Similarly, for any function $\theta(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$ we can consider the set $\mathcal{U}_r^-(\theta)(\mathcal{L}_r^-(\theta))$ of all regular upward (leftward) extensions $\Omega(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}^{(1)} \oplus \mathfrak{F}](\Lambda(e^{it}) \in CM[\mathfrak{G}^{(1)} \oplus \mathfrak{G}, \mathfrak{F}])$ of form (3.15) where the block $\theta_{12}(e^{it})$ $(\theta_{21}(e^{it}))$ is the boundary value function of some contractive antiholomorphic operator function $\theta_{12}(\zeta)(\theta_{21}(\zeta))$ on $\mathbb{D}$. The latter means that $\theta_{12}^*(\zeta) \in S[\mathfrak{F}^{(1)}, \mathfrak{G}](\theta_{21}^*(\zeta) \in S[\mathfrak{F}, \mathfrak{G}^{(1)}])$ is a Schur function. This leads us to the necessity of considering unilateral shifts (coshifts) contained in the fundamental contraction $T_* := T_{\sigma^*}$. It is clear that

$$\mathcal{U}_r^-(\theta) = \{\Omega(e^{it}) \in \mathcal{U}_r(\theta) : \Omega^*(e^{it}) \in \mathcal{L}_r^+(\theta^*)\}$$

$$(\mathcal{L}_r^-(\theta) = \{\Lambda(e^{it}) \in \mathcal{L}_r(\theta) : \Lambda^*(e^{it}) \in \mathcal{U}_r^+(\theta^*)\}). \tag{6.13}$$

The largest shift $V_{T_*} \in [\mathfrak{H}_{V_{T_*}}]$ (coshift $\tilde{V}_{T_*} \in [\mathfrak{H}_{\tilde{V}_{T_*}}]$) acts on the subspace $\mathfrak{H}_{V_{T_*}} := \mathfrak{R}_{\overset{\circ}{\mathfrak{F}}} \cap \mathfrak{H}_{T_*}$ ($\mathfrak{H}_{\tilde{V}_{T_*}} := \mathfrak{H}_{T_*} \cap \mathfrak{R}_{\overset{\circ}{\mathfrak{G}}}$) of the space $\mathfrak{H}_{T_*}$.

It generates the regular upward (leftward) extension

$$\Omega_0^-(e^{it}) := \begin{bmatrix} \varphi_-(e^{it}) \\ \theta(e^{it}) \end{bmatrix} \in CM[\mathfrak{G}, \mathfrak{K}^- \oplus \mathfrak{F}]$$

$$(\Lambda_0^-(e^{it}) := [\psi_-(e^{it}), \theta(e^{it})] \in CM[\mathfrak{K}_*^- \oplus \mathfrak{G}, \mathfrak{F}])$$

from $\mathcal{U}_r^-(\theta)$ ($\mathcal{L}_r^-(\theta)$), where $\varphi_-^*(\zeta) \in S[\mathfrak{K}^-, \mathfrak{G}]$ ($\psi_-^*(\zeta) \in S[\mathfrak{G}, \mathfrak{K}_*^-]$) is some $*$-outer (outer) operator function.

The function $\varphi_-(\zeta)$ ($\psi_-(\zeta)$) will be called *the defect ($*$-defect) function of $\theta(e^{it})$ in the class of contractive antiholomorphic operator functions*. It is clear that, denoting $\varphi_\pm(\zeta;\theta)$ and $\psi_\pm(\zeta;\theta)$ the corresponding defect function of the function $\theta(e^{it})$, we come to the equalities

$$\varphi_\pm^*(\zeta;\theta) = \psi_\mp(\zeta;\theta^*), \quad \varphi_\pm^\sim(\zeta;\theta) = \psi_\pm(\zeta;\theta^\sim), \quad \zeta \in \mathbb{D}, \tag{6.14}$$

(see Sect. 2.1, where the associated function $\theta^\sim(e^{it})$ is defined).

Similarly, one can introduce the set $\mathcal{U}_{cr}^-(\theta)(\mathcal{L}_{cr}^-(\theta))$, the largest shift W_{T_*} (coshift $\tilde{W}_{T_*}$) contained simultaneously in T_{σ^*} and V_σ ($\tilde{V}_\sigma$). Analogously, the defect function $\gamma_-(\zeta)$ ($\delta_-(\zeta)$, where $\gamma_-^*(\zeta) \in S[\mathfrak{N}^-, \mathfrak{G}]$ ($\delta_-^*(\zeta) \in S[\mathfrak{F}, \mathfrak{N}_*^-]$) is an inner ($*$-inner) operator function, is determined by the extension

$$\Omega_{c0}^-(e^{it}) := \begin{bmatrix} \gamma_-(e^{it}) \\ \theta(e^{it}) \end{bmatrix} \in \mathcal{U}_{cr}^-(\theta) \quad (\Lambda_{c0}^-(e^{it}) := [\delta_-(e^{it}), \theta(e^{it})] \in \mathcal{L}_{cr}^-(\theta))$$

that corresponds to W_{T_*} ($\tilde{W}_{T_*}$).

We will call an antiholomorphic operator function inner ($*$-inner) if its boundary value function is isometric (coisometric). The function $\gamma_-(\zeta)$ ($\delta_-(\zeta)$) will be called *the defect ($*$-defect) function of $\theta(e^{it})$ in the class of an antiholomorphic $*$-inner (inner) operator functions*. It is obvious that

$$\gamma_\pm^*(\zeta;\theta) = \delta_\mp(\zeta;\theta^*), \quad \gamma_\pm^\sim(\zeta;\theta) = \delta_\pm(\zeta;\theta^\sim). \tag{6.15}$$

Established connections (6.13)–(6.15) allow us to reduce the study of regular extensions from $\mathcal{U}_r^-$ ($\mathcal{L}_r^-$) to the investigation of regular extensions from $\mathcal{L}_r^+$ ($\mathcal{U}_r^+$). Therefore, all subsequent results will be formulated only for extensions from the sets $\mathcal{U}_r^+$ and $\mathcal{L}_r^+$.

7 Applications of Defect Functions in the Schur Class to the Study of Regular Extensions

7.1 Descriptions of the Sets $\mathcal{U}_r^+(\theta)$ and $\mathcal{L}_r^+(\theta)$

First of all, we need some refinements of Theorems 3.1 and 5.2 for the cases of the boundary value functions of $*$-inner and inner operator functions (see [12], Corollaries 7.5 and 7.7; [13], Corollaries 8.2 and 8.4).

Theorem 7.1 *Let $\theta(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$ and σ be a minimal unitary coupling of form* (2.1) *such that $\theta_\sigma(e^{it}) = \theta(e^{it})$.*

(a) $\theta(e^{it})$ is the boundary value function of some $*$-inner (inner) operator function $\theta(\zeta) \in S[\mathfrak{G}, \mathfrak{F}]$ iff

$$M_-(\overset{\circ}{\mathfrak{F}}) \subset M_-(\overset{\circ}{\mathfrak{G}}) \quad (M_+(\overset{\circ}{\mathfrak{G}}) \subset M_+(\overset{\circ}{\mathfrak{F}})).$$

The function $\theta(\zeta)$ is two-sided inner iff both these conditions are satisfied.

(b) There exists a bijective correspondence between unilateral input (output) channels $(M_-(\overset{\circ}{\mathfrak{F}}_1), \mathfrak{F}_1; V_{\mathfrak{F}_1})$ $((M_+(\overset{\circ}{\mathfrak{G}}_1), \mathfrak{G}_1; V_{\mathfrak{G}_1}))$ of the coupling σ satisfying the condition

$$M_-(\overset{\circ}{\mathfrak{F}}) \subset M_-(\overset{\circ}{\mathfrak{F}}_1) \quad (M_+(\overset{\circ}{\mathfrak{G}}) \subset M_+(\overset{\circ}{\mathfrak{G}}_1)) \tag{7.1}$$

and regular factorizations of the form

$$\theta(e^{it}) = \theta_2(e^{it})\theta_1(e^{it}), \tag{7.2}$$

where $\theta_2(e^{it}) \in CM[\mathfrak{F}_1, \mathfrak{F}]$ $(\theta_1(e^{it}) \in CM[\mathfrak{G}, \mathfrak{G}_1])$ is the boundary value function of some $*$-inner (inner) operator function. This correspondence is established by the equality $\theta_2(e^{it}) = \theta_{\sigma_2}(e^{it})(\theta_1(e^{it}) = \theta_{\sigma_1}(e^{it}))$, where $\sigma_2(\sigma_1)$ is unitary coupling of form (3.3) *and* (3.4).

Remark 7.2 Note that in the case of a non-minimal coupling σ one can only state that for any unilateral input (output) channel of σ satisfying condition (7.1) there exists a unique factorization of form (7.2) with the required properties, but not necessarily regular. This factorization is regular iff $(M_-(\overset{\circ}{\mathfrak{F}}_1), \mathfrak{F}_1; V_{\mathfrak{F}_1})$ $((M_+(\overset{\circ}{\mathfrak{G}}_1), \mathfrak{G}_1; V_{\mathfrak{G}_1}))$ is a unilateral channel of the principal part $\sigma^{(1)}$ of the coupling σ (see Definition 2.3).

We also need the following refinement of Lemma 5.4 for the cases of $*$-inner and inner operator functions.

Lemma 7.3

(a) *Let* $\omega(\zeta) \in S[\mathfrak{G}, \mathfrak{F}]$ *and* $\omega_1(\zeta) \in S[\mathfrak{G}, \mathfrak{K}]$ *be* $*$*-inner operator functions. If* $\omega_1(e^{it})$ *is a right divisor of* $\omega(e^{it})$ *and the corresponding unique left divisor* $\omega_2(e^{it})$ *belongs to* $L^\infty_+[\mathfrak{K}, \mathfrak{F}]$*, then* $\omega_2(\zeta) \in S[\mathfrak{K}, \mathfrak{F}]$ *is also* $*$*-inner and both divisors* $\omega_2(e^{it})$ *and* $\omega_1(e^{it})$ *are completely regular.*

(b) *Let* $\lambda(\zeta) \in S[\mathfrak{G}, \mathfrak{F}]$ *and* $\lambda_2(\zeta) \in S[\mathfrak{K}, \mathfrak{F}]$ *be inner operator functions. If* $\lambda_2(e^{it})$ *is the left divisor of* $\lambda(e^{it})$ *and the corresponding unique right divisor* $\lambda_1(e^{it})$ *belongs to* $L^\infty_+[\mathfrak{G}, \mathfrak{K}]$*, then* $\lambda_1(\zeta) \in S[\mathfrak{G}, \mathfrak{K}]$ *is also inner and both divisors* $\lambda_2(e^{it})$ *and* $\lambda_1(e^{it})$ *are completely regular.*

Proof It suffices to prove the part (a). By Lemma 5.4 (part (a)), $\omega_2(e^{it})$ is a coisometric operator function and both divisors are completely regular. In view of $\omega_2(e^{it}) \in L^\infty_+[\mathfrak{K}, \mathfrak{F}]$, it is the boundary value function of some $*$-inner operator function $\omega_2(\zeta) \in S[\mathfrak{K}, \mathfrak{F}]$. □

Thus, we can consider the concept of the divisibility within the class of $*$-inner (inner) operator functions, denoting it by $\omega(\zeta)\dot{:}\omega_1(\zeta)$ (on the right) ($\lambda(\zeta)\dot{:}\lambda_2(\zeta)$) (on the left)), if the conditions of part (a) (part (b)) of Lemma 7.3 are valid.

Theorem 7.4 *Let* $\theta(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$, $\varphi_+(\zeta) \in S[\mathfrak{G}, \mathfrak{K}]$, ($\psi_+(\zeta) \in S[\mathfrak{K}_*, \mathfrak{F}]$) *be its defect* ($*$*-defect*) *function in the Schur class, and* $\gamma_+(\zeta) \in S[\mathfrak{G}, \mathfrak{N}]$($\delta_+(\zeta) \in S[\mathfrak{N}_*, \mathfrak{F}]$) *be its defect* ($*$*-defect*) *function in the class of* $*$*-inner* (*inner*) *operator functions.*

(a) *A nontrivial extension* $\Omega(e^{it}) \in \mathcal{U}_r^+(\theta)$ ($\Lambda(e^{it}) \in \mathcal{L}_r^+(\theta)$) *exists iff* $\varphi_+(\zeta) \not\equiv 0$ ($\psi_+(\zeta) \not\equiv 0$).

Let $\Omega(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}^{(1)} \oplus \mathfrak{F}]$($\Lambda(e^{it}) \in CM[\mathfrak{G}^{(1)} \oplus \mathfrak{G}, \mathfrak{F}]$) *be an operator function of form* (3.15).

(b) $\Omega(e^{it}) \in \mathcal{U}_r^+(\theta)$($\Lambda(e^{it}) \in \mathcal{L}_r^+(\theta)$) *iff there exists a* $*$*-inner* (*inner*) *operator function* $\omega(\zeta) \in S[\mathfrak{K}, \mathfrak{F}^{(1)}]$($\lambda(\zeta) \in S[\mathfrak{G}^{(1)}, \mathfrak{K}_*]$) *such that the function* $\theta_{12}(e^{it})$ ($\theta_{21}(e^{it})$) *admits the representation of the form*

$$\theta_{12}(e^{it}) = \omega(e^{it})\varphi_+(e^{it}) \ a.e. \quad (\theta_{21}(e^{it}) = \psi_+(e^{it})\lambda(e^{it}) \ a.e.). \tag{7.3}$$

(c) $\Omega(e^{it}) \in \mathcal{U}_{cr}^+(\theta)$ ($\Lambda(e^{it}) \in \mathcal{L}_{cr}^+(\theta)$) *iff there exists a* $*$*-inner* (*inner*) *operator function* $\omega(\zeta) \in S[\mathfrak{N}, \mathfrak{F}^{(1)}]$($\lambda(\zeta) \in S[\mathfrak{G}^{(1)}, \mathfrak{N}_*]$) *such that the function* $\theta_{12}(e^{it})$($\theta_{21}(e^{it})$) *admits the representation of the form*

$$\theta_{12}(e^{it}) = \omega(e^{it})\gamma_+(e^{it}) \ a.e. \quad (\theta_{21}(e^{it}) = \delta_+(e^{it})\lambda(e^{it}) \ a.e.). \tag{7.4}$$

(d) *There exists an isometric* (*coisometric*) *extension* $\Omega(e^{it}) \in \mathcal{U}_r^+(\theta)$ ($\Lambda(e^{it}) \in \mathcal{L}_r^+(\theta)$) *iff*

$$\overline{\varphi_+^* L^2(\mathfrak{K})} = \overline{\Pi L^2(\mathfrak{G})} \quad (\overline{\psi_+ L^2(\mathfrak{K}_*)} = \overline{\Sigma L^2(\mathfrak{F})}). \tag{7.5}$$

All such extensions in this case are given by formulas (3.15) *and* (7.3)*, where* $\omega(\zeta) \in S[\mathfrak{K}, \mathfrak{F}^{(1)}](\lambda(\zeta) \in S[\mathfrak{G}^{(1)}, \mathfrak{K}_*])$ *is an arbitrary two-sided inner operator function. These extensions are unitary iff* $\theta(e^{it})$ *is a coisometric* (*isometric*) *operator function. In this case, they are completely regular,*

$$\varphi_+(\zeta) = \gamma_+(\zeta) \quad (\psi_+(\zeta) = \delta_+(\zeta)), \quad \zeta \in \mathbb{D},$$

and condition (7.5) *is equivalent to the equality*

$$\mathrm{Ran}\varphi_+^* \oplus \mathrm{Ran}\theta^* = L^2(\mathfrak{G}) \quad (\mathrm{Ran}\psi_+ \oplus \mathrm{Ran}\theta = L^2(\mathfrak{F})). \tag{7.6}$$

Proof

(*a*) This part follows from part (*b*).
(*b*) As before, we again restrict ourselves to the consideration of regular upward extensions. Let $\Omega(e^{it}) \in \mathcal{U}_r^+$ and $\hat{\sigma}$ be the functional model of a minimal coupling of form (2.11) described in Theorem 2.11. We again consider $\hat{\mathfrak{H}}$ with the other order of its orthogonal components, as in the proof of Theorem 4.2. It was shown there that $\theta_{12}(e^{it}) = \theta_{\hat{\sigma}_{12}}$, where

$$\hat{\sigma}_{12} := (\hat{\mathfrak{H}}, \mathfrak{F}^{(1)}, \mathfrak{G}; \hat{U}^\times, \hat{V}_{\mathfrak{F}^{(1)}}, \hat{V}_{\mathfrak{G}})$$

is not necessarily minimal coupling. By Theorem 6.12 (part (*a*)), $\Omega(e^{it})$ corresponds to some internal coshift $\tilde{V} \in \tilde{\mathcal{V}}_{\hat{T}}$ of the coupling $\hat{\sigma}$ where $\hat{T} := T_{\hat{\sigma}}$. The coshift $\tilde{V}$ is also the principal external coshift for $\hat{\sigma}_{12}$, that is, $\tilde{V} = \tilde{V}_{\hat{\sigma}_{12}} \in [M_-(\mathring{\mathfrak{F}}^{(1)})]$.

Consider the unilateral input channel $(\hat{M}_-(\mathring{\mathfrak{K}}), \mathfrak{K}; \hat{V}_{\mathfrak{K}})$ of the coupling $\hat{\sigma}$ (and, hence, of $\hat{\sigma}_{12}$) corresponding, by Theorem 6.9 (part (*a*)), to the extension $\Omega_0^+(e^{it}) \in \mathcal{U}_r^+$ of form (6.8) and generating, by Definition 6.13, the defect function $\varphi_+(\zeta)$. It corresponds to the largest internal coshift $\tilde{V}_{\hat{T}} \in \tilde{\mathcal{V}}_{\hat{T}}$ of $\hat{\sigma}$ and, hence, $\tilde{V} \subset \tilde{V}_T$, whence the inclusion $\hat{M}_-(\mathring{\mathfrak{F}}^{(1)}) \subset \hat{M}_-(\mathring{\mathfrak{K}})$ holds. By Theorem 7.1 (part (*b*)) applied to the function $\theta_{12}(e^{it})$ and the coupling $\hat{\sigma}_{12}$, taking into account Remark 7.2, we obtain the factorization

$$\theta_{12}(e^{it}) = \theta_{\hat{\tau}}(e^{it})\theta_{\hat{\sigma}_{12}^+}(e^{it}) \ a.e.,$$

where

$$\hat{\tau} := (\hat{M}(\mathring{\mathfrak{K}}), \mathfrak{F}^{(1)}, \mathfrak{K}; \hat{U}_{\hat{\tau}}, \hat{V}_{\mathfrak{F}^{(1)}}, \hat{V}_{\mathfrak{K}}), \quad \hat{U}_{\hat{\tau}} := \hat{U}^\times|_{\hat{M}(\mathring{\mathfrak{K}})},$$

and $\hat{\sigma}_{12}^{+}$ is the coupling of type (6.10), that is,

$$\hat{\sigma}_{12}^{+} := (\hat{\mathfrak{H}}, \mathfrak{K}, \mathfrak{G}; \hat{U}^{\times}, \hat{V}_{\mathfrak{K}}, \hat{V}_{\mathfrak{G}}).$$

Moreover, $\omega(e^{it}) := \theta_{\hat{\tau}}(e^{it})$ is the boundary value function of some $*$-inner operator function $\omega(\zeta) \in S[\mathfrak{K}, \mathfrak{F}^{(1)}]$ and, by Definition 6.13, $\theta_{\hat{\sigma}_{12}^{+}}(e^{it}) = \varphi_{+}(e^{it})$.

Conversely, let representation (7.3) be valid. Consequently, $\theta_{12}(e^{it}) \in L_{+}^{\infty}[\mathfrak{G}, \mathfrak{F}^{(1)}]$. In view of (6.9),

$$\theta_{12}(e^{it}) = \alpha(e^{it})\Pi(e^{it}) \text{ a.e.}, \tag{7.7}$$

where $\alpha(e^{it}) = \omega(e^{it})\omega_{0}^{+}(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}^{(1)}]$ is a coisometric operator function satisfying the condition

$$\mathrm{Ran}\alpha^{*} \subset \mathrm{Ran}(\omega_{0}^{+})^{*} \subset \overline{\Pi L^{2}(\mathfrak{G})}. \tag{7.8}$$

By Theorem 4.2 (part(b)), $\Omega(e^{it}) \in \mathcal{U}_{r}$, whence, in view of $\theta_{12}(e^{it}) \in L_{+}^{\infty}[\mathfrak{G}, \mathfrak{F}^{(1)}]$, $\Omega(e^{it}) \in \mathcal{U}_{r}^{+}$

(c) The proof of part (c) does not differ in principal from the proof of part (b) if we replace the function $\varphi_{+}(\zeta)$ by the function $\gamma_{+}(\zeta)$, the subspace $\overline{\Pi L^{2}(\mathfrak{G})}$ by the subspace Kerθ, and representation (4.2) by representation (4.4).

(d) As was shown in the proof of Theorem 4.2 (part (d)), an extension $\Omega(e^{it}) \in \mathcal{U}_{r}$ of form (3.15) is isometric iff the equality $\mathrm{Ran}\omega^{*} = \overline{\Pi L^{2}(\mathfrak{G})}$ holds, where $\omega(e^{it})$ is a coisometric function from representation (4.2). By part (b) of the present theorem, for $\Omega(e^{it}) \in \mathcal{U}_{r}^{+}$ of form (3.15) the function $\theta_{12}(e^{it})$ admits representation (7.7), where $\alpha(e^{it}) := \omega(e^{it})\omega_{0}^{+}(e^{it})$ a.e., $\omega(\zeta) \in S[\mathfrak{K}, \mathfrak{F}^{(1)}]$ is a $*$-inner function, $\omega_{0}^{+}(e^{it})$ is a coisometric operator function from equality (6.9). Hence, inclusions (7.8) are valid. Thus, $\Omega(e^{it})$ is isometric iff

$$\mathrm{Ran}\alpha^{*} = \mathrm{Ran}(\omega_{0}^{+})^{*} = \overline{\Pi L^{2}(\mathfrak{G})}.$$

In view of (6.9), these equalities are equivalent to the conditions

$$\mathrm{Ran}\omega^{*} = \overline{\varphi_{+}^{*} L^{2}(\mathfrak{K})} = \overline{\Pi L^{2}(\mathfrak{G})}.$$

Consequently, $\Omega(e^{it})$ is isometric iff condition (7.5) is satisfied and $\omega(\zeta)$ is two-sided inner functions.

As was shown in Theorem 4.2 (part (d)), an isometric extension $\Omega(e^{it}) \in \mathcal{U}_{r}$ is unitary iff $\theta(e^{it})$ is a coisometric function and, in this case, $\overline{\Pi L^{2}(\mathfrak{G})} = \mathrm{Ker}\theta$, $\Omega(e^{it}) \in \mathcal{U}_{cr}$, $\theta_{12}(e^{it}) = \omega(e^{it})$ a.e., and $\mathrm{Ran}\omega^{*} = \mathrm{Ker}\theta$. In view of $\overline{\varphi_{+}^{*} L^{2}(\mathfrak{K})} = \mathrm{Ker}\theta$, from parts ($b$) and ($c$) we obtain that $\varphi_{+}(\zeta) = \gamma_{+}(\zeta)$, $\zeta \in \mathbb{D}$.

Since the defect function $\varphi_+(\zeta)$ is $*$-inner in this case, we see that (7.5) is equivalent to (7.6).

Similarly, the dual assertion can be proved. □

Remark 7.5 Note that constancy of $\rho_\Pi(e^{it})$ ($\rho_\Sigma(e^{it})$) almost everywhere is a necessary condition for the existence of an isometric (coisometric) extensions $\Omega(e^{it}) \in \mathcal{U}_r^+(\theta)$ ($\Lambda(e^{it}) \in \mathcal{L}_r^+(\theta)$) (see Theorem 4.2 (part (d))). But it is not sufficient for this, since the existence of such an extension is obviously equivalent to the validity of the equality

$$\varphi_+^*(e^{it})\varphi_+(e^{it}) = \Pi^2(e^{it})\ a.e. \quad (\psi_+(e^{it})\psi_+^*(e^{it}) = \Sigma^2(e^{it})\ a.e.),$$

what can be untruly under such a condition (in this regard see [13], Remark 8.46).

7.2 Refined Comparison Relations on the Sets $\mathcal{U}_r^+(\theta)$ and $\mathcal{L}_r^+(\theta)$

Theorem 6.12 gives us a possibility to introduce a stronger comparison relation on $\mathcal{U}_r^+$ ($\mathcal{L}_r^+$) than the relation induced on it by the comparison relation $\prec$ on $\mathcal{U}_r$ ($\mathcal{L}_r$). As agreed at the beginning of Sect. 6.3, we identify extensions from $\mathcal{U}_r^+$ ($\mathcal{L}_r^+$) of form (3.15) that differ by a left (right) constant unitary factor at the block $\theta_{12}(e^{it})$ ($\theta_{21}(e^{it})$).

Definition 7.6 Let $\theta(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$ and σ be a minimal unitary coupling of form (2.1) such that $\theta_\sigma(e^{it}) = \theta(e^{it})$. Let $\Omega_j(e^{it}) \in \mathcal{U}_r^+(\theta)(\Lambda_j(e^{it}) \in \mathcal{L}_r^+(\theta))$, $j = 1, 2$, be two extensions of form (5.1) and $\tilde{V}_j \in [M_-(\mathring{\mathfrak{F}}^{(j)})](V_j \in [M_+(\mathring{\mathfrak{G}}^{(j)})])$, $j = 1, 2$, be the corresponding two internal unilateral coshifts (shifts) of the coupling σ (see Theorem 6.12). We will write $\Omega_1(e^{it}) \lll \Omega_2(e^{it})$ ($\Lambda_1(e^{it}) \lll \Lambda_2(e^{it})$) if $\tilde{V}_1 \subset \tilde{V}_2$ ($V_1 \subset V_2$), and call the relation $\lll$ the refined comparison relation on $\mathcal{U}_r^+(\theta)$ ($\mathcal{L}_r^+(\theta)$).

Clearly, this definition does not depend on the choice of a minimal unitary coupling σ such that $\theta_\sigma(e^{it}) = \theta(e^{it})$.

Since $\tilde{\mathcal{V}}_T$ ($\mathcal{V}_T$), where $T := T_\sigma$, is a partially ordered set, then Definition 7.6 turns the $\mathcal{U}_r^+$ ($\mathcal{L}_r^+$) into a partially ordered set too and the sets $\tilde{\mathcal{V}}_T$ ($\mathcal{V}_T$) and $\mathcal{U}_r^+$ ($\mathcal{L}_r^+$) become isomorphic to each other. Since $\tilde{V}_T(V_T)$ is the largest coshift (shift) in $\tilde{\mathcal{V}}_T$ ($\mathcal{V}_T$), then, in view of Definitions 6.13 and 7.6, $\Omega_0^+(e^{it})$ ($\Lambda_0^+(e^{it})$ is the largest extension in $\mathcal{U}_r^+$ ($\mathcal{L}_r^+$) in the refined sense. Similarly, by Definitions 6.15 and 7.6, $\Omega_{c0}^+(e^{it})$ ($\Lambda_{c0}^+(e^{it})$) is the largest extension in $\mathcal{U}_{cr}^+$ ($\mathcal{L}_{cr}^+$) in the same sense.

The introduced so refined comparison relation $\lll$ on $\mathcal{U}_r^+$ ($\mathcal{L}_r^+$) is indeed stronger than the relation induced on it by the comparison relation $\prec$ on $\mathcal{U}_r$ ($\mathcal{L}_r$). Really, from the inclusion

$$M_-(\overset{\circ}{\mathfrak{F}}^{(1)}) \subset M_-(\overset{\circ}{\mathfrak{F}}^{(2)}) \quad (M_+(\overset{\circ}{\mathfrak{G}}^{(1)}) \subset M_+(\overset{\circ}{\mathfrak{G}}^{(2)}))$$

the inclusion

$$M(\overset{\circ}{\mathfrak{F}}^{(1)}) \subset M(\overset{\circ}{\mathfrak{F}}^{(2)}) \quad (M(\overset{\circ}{\mathfrak{G}}^{(1)}) \subset M(\overset{\circ}{\mathfrak{G}}^{(2)}))$$

follows, but the converse is not always true.

Theorem 7.7 *Let* $\theta(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$ *and* $\Omega_j(e^{it}) \in \mathcal{U}_r^+(\theta)$ $(\Lambda_j(e^{it}) \in \mathcal{L}_r^+(\theta))$, $j = 1, 2$, *be its extensions of form* (5.1)*. Let*

$$\theta_{12}^{(j)}(e^{it}) = \omega_j(e^{it})\varphi_+(e^{it}) \ a.e. \quad (\theta_{21}^{(j)}(e^{it}) = \psi_+(e^{it})\lambda_j(e^{it}) \ a.e.), \quad j = 1, 2, \tag{7.9}$$

where $\omega_j(\zeta) \in S[\mathfrak{K}, \mathfrak{F}^{(j)}](\lambda_j(\zeta) \in S[\mathfrak{G}^{(j)}, \mathfrak{K}_*])$, $j = 1, 2$, *are* $*$*-inner* (*inner*) *operator functions corresponding, by Theorem* 7.4 (*part* (*b*)), *to* $\Omega_j(e^{it})$ $(\Lambda_j(e^{it}))$. *The following statements are equivalent:*

(a) $\Omega_1(e^{it}) \lll \Omega_2(e^{it})$ $(\Lambda_1(e^{it}) \lll \Lambda_2(e^{it}))$*;*

(b) $\omega_1(\zeta) \vdots \omega_2(\zeta)$*(on the right)*$(\lambda_1(\zeta) \vdots \lambda_2(\zeta)$*(on the left))*.

Proof As usual, we prove the theorem only for regular upward extensions from $\mathcal{U}_r^+$.

As was shown in the proof of Theorem 7.4 (part (*b*)), $\omega_j(e^{it}) := \theta_{\hat{\tau}_j}(e^{it})$ $(j = 1, 2)$, where

$$\hat{\tau}_j := (\hat{M}(\overset{\circ}{\mathfrak{K}}), \mathfrak{F}^{(j)}, \mathfrak{K}; \hat{U}_{\hat{\tau}_j}, \hat{V}_{\mathfrak{F}^{(j)}}, \hat{V}_{\mathfrak{K}}) \ (j = 1, 2), \quad \hat{U}_{\hat{\tau}_1} = \hat{U}_{\hat{\tau}_2} := \hat{U}^\times|_{\hat{M}(\overset{\circ}{\mathfrak{K}})},$$

$\hat{U}^\times \in [\hat{\mathfrak{F}}]$ is the unitary operator of the coupling $\hat{\sigma}$ of form (2.11), $\hat{V}_{\mathfrak{F}^{(j)}}$ $(j = 1, 2)$ and $\hat{V}_{\mathfrak{K}}$ are the embedding isometries from the internal unilateral input channels corresponding, by Theorem 6.9 (part (*a*)), to the extensions $\Omega_j(e^{it})$ $(j = 1, 2)$ and $\Omega_0^+(e^{it})$, respectively. By Definition 7.6, the statement (*a*) is equivalent to the validity of the inclusions

$$\hat{M}_-(\overset{\circ}{\mathfrak{F}}^{(1)}) \subset \hat{M}_-(\overset{\circ}{\mathfrak{F}}^{(2)}) \subset \hat{M}_-(\overset{\circ}{\mathfrak{K}}). \tag{7.10}$$

Applying Theorem 7.1 to the function $\omega_1(e^{it})$, we see that (7.10) is equivalent to the existence of the factorization $\omega_1(\zeta) = \omega_{12}(\zeta)\omega_2(\zeta)$, $\zeta \in \mathbb{D}$, where $\omega_{12}(\zeta) \in S[\mathfrak{F}^{(2)}, \mathfrak{F}^{(1)}]$ is also $*$-inner function. Thus, the statements (*a*) and (*b*) are equivalent. □

For any function $\theta(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$ we will denote by $\mathcal{J}_*^+(\theta)$ ($\mathcal{J}^+(\theta)$) or simply $\mathcal{J}_*^+$ ($\mathcal{J}^+$) the set of $*$-inner (inner) operator functions $\omega(\zeta) \in S[\mathfrak{K}, \mathfrak{F}^{(1)}]$ ($\lambda(\zeta) \in S[\mathfrak{G}^{(1)}, \mathfrak{K}_*]$) identifying them up to a left (right) constant unitary factor. By Theorem 7.7, we establish one more isomorphism between the set $\mathcal{U}_r^+(\theta)$ ($\Lambda_r^+(\theta)$), partially ordered by the relation $\lll$, and the set $\mathcal{J}_*^+(\theta)$ ($\mathcal{J}^+(\theta)$, partially ordered by the right (left) divisibility relation. It is obvious that the largest extension $\Omega_0^+(e^{it}) \in \mathcal{U}_r^+(\theta)$ ($\Lambda_0^+(e^{it}) \in \mathcal{L}_r^+(\theta)$) corresponds to the greatest common right (left) divisor $\alpha(\zeta) \equiv I_{\mathfrak{K}}(\beta(\zeta) \equiv I_{\mathfrak{K}_*})$ of all function from $\mathcal{J}_*^+(\theta)$ ($\mathcal{J}^+(\theta)$) within the class of $*$-inner (inner) operator functions.

In fact, we can say more about the pairwise isomorphic partially ordered sets $\tilde{\mathcal{V}}_T$ ($\mathcal{V}_T$), $\mathcal{J}_*^+(\theta)$ ($\mathcal{J}^+(\theta)$), and $\mathcal{U}_r^+(\theta)$ ($\mathcal{L}_r^+(\theta)$). Recall that a partially ordered set X is called *a complete lattice* if for any subset Y of the set X there exist $\inf Y$ and $\sup Y$.

Lemma 7.8 *Let σ be a minimal unitary coupling of form* (2.1) *and $T := T_\sigma$. The partially ordered set $\tilde{\mathcal{V}}_T$ ($\mathcal{V}_T$) is a complete lattice.*

Proof It suffices to prove the lemma for the set $\mathcal{V}_T$. If $V_T \in [M_+(\overset{\circ}{\mathfrak{K}}_*)]$, then, by the definition of the partial order on $\mathcal{V}_T$ (see Sect. 6.1) and Corollary 6.2, the partially ordered set $\mathcal{V}_T$ is isomorphic to the partially ordered (by inclusion) set $\mathrm{Lat}V_T$ of subspaces of the space $M_+(\overset{\circ}{\mathfrak{K}}_*)$ that are invariant for the shift V_T. The set $\mathrm{Lat}V_T$ is a complete lattice, since for any nonempty set $\{\mathfrak{L}_\gamma : \gamma \in \Gamma\} \subset \mathrm{Lat}V_T$ there exist

$$\inf_{\gamma\in\Gamma} \mathfrak{L}_\gamma (= \bigcap_{\gamma\in\Gamma} \mathfrak{L}_\gamma), \quad \sup_{\gamma\in\Gamma} \mathfrak{L}_\gamma (= \overline{\operatorname*{span}_{\gamma\in\Gamma} \mathfrak{L}_\gamma}).$$

□

Corollary 7.9 *Let $\theta(e^{it})$ belong to $CM[\mathfrak{G}, \mathfrak{F}]$. The sets $\mathcal{J}_*^+(\theta)$ ($\mathcal{J}^+(\theta)$) and $\mathcal{U}_r^+(\theta)$ ($\mathcal{L}_r^+(\theta)$) are isomorphic complete lattices. Thus, for any subset $\{\omega_\gamma(\zeta) : \gamma \in \Gamma\} \subset \mathcal{J}_*^+(\theta)$ ($\{\lambda_\gamma(\zeta) : \gamma \in \Gamma\} \subset \mathcal{J}^+(\theta)$) there exists the greatest common right* (*left*) *divisor and the least common left* (*right*) *multiple within the set $\mathcal{J}_*^+(\theta)$ ($\mathcal{J}^+(\theta)$).*

Similarly, one can introduce the set of $*$-inner (inner) operator functions $\omega(\zeta) \in S[\mathfrak{N}, \mathfrak{F}^{(1)}](\lambda(\zeta) \in S[\mathfrak{G}^{(1)}, \mathfrak{N}_*])$ from representation (7.4) for the block $\theta_{12}(e^{it})(\theta_{21}(e^{it}))$ of $\Omega(e^{it}) \in \mathcal{U}_{cr}^+(\theta)(\Lambda(e^{it}) \in \mathcal{L}_{cr}^+(\theta))$ and partially order it by divisibility on the right (on the left). On the other hand, taking into account the inclusion $\mathcal{U}_{cr}^+(\theta) \subset \mathcal{U}_r^+(\theta)$ ($\mathcal{L}_{cr}^+(\theta) \subset \mathcal{L}_r^+(\theta)$), one can consider the corresponding subset of $\mathcal{J}_*^+(\theta)$ ($\mathcal{J}^+(\theta)$) partially ordered by divisibility on the right (on the left) as well. Thirdly, one can partially order by divisibility on the right (on the left) the set of $*$-inner (inner) operator function $\theta_{12}(\zeta) \in S[\mathfrak{G}, \mathfrak{F}^{(1)}](\theta_{21}(\zeta) \in S[\mathfrak{G}^{(1)}, \mathfrak{F}])$ satisfying the condition $\mathrm{Ran}\theta_{12}^* \subset \mathrm{Ker}\theta$ ($\mathrm{Ran}\theta_{21} \subset \mathrm{Ker}\theta^*$). It is easy to understand that all three approaches lead to the three complete lattices of parameters that are also isomorphic to the complete lattices $\tilde{\mathcal{W}}_T$ ($\mathcal{W}_T$) and $\mathcal{U}_{cr}^+(\theta)$ ($\mathcal{L}_{cr}^+(\theta)$).

7.3 Defect Functions as the Largest Minorants

Theorem 7.4 (part (*b*)) enables us to establish a connection between the defect ($*$-defect) function $\varphi_+(\zeta) \in S[\mathfrak{G}, \mathfrak{K}](\psi_+(\zeta) \in S[\mathfrak{K}_*, \mathfrak{F}])$ in the Schur class for a function $\theta(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$ and the largest minorant ($*$-minorant) in this class for its defect function $\Pi(e^{it}) \in CM[\mathfrak{G}]$ $(\Sigma(e^{it}) \in CM[\mathfrak{F}])$ in the class of contractive measurable operator functions (see, e.g. [29], Chapter V, Subsection 4.2; [14], Section 4; [13], Subsection 8.8).

Definition 7.10 Let $\nu(e^{it}) \in CM[\mathfrak{G}]$ be a function that is nonnegative at almost all t. A Schur function $\alpha_+(\zeta) \in S[\mathfrak{G}, \mathfrak{C}](\beta_+(\zeta) \in S[\mathfrak{C}_*, \mathfrak{G}])$ is called the largest holomorphic minorant ($*$-minorant) for the function $\nu(e^{it})$ if

(a)

$$\alpha_+^*(e^{it})\alpha_+(e^{it}) \le \nu^2(e^{it}) \ a.e. \quad (\beta_+(e^{it})\beta_+^*(e^{it}) \le \nu^2(e^{it}) \ a.e.); \tag{7.11}$$

(b) for any function $\tau(\zeta) \in S[\mathfrak{G}, \mathfrak{M}]$ $(\upsilon(\zeta) \in S[\mathfrak{M}_*, \mathfrak{G}])$ such that

$$\tau^*(e^{it})\tau(e^{it}) \le \nu^2(e^{it}) \ a.e. \quad (\upsilon(e^{it})\upsilon^*(e^{it}) \le \nu^2(e^{it}) \ a.e.) \tag{7.12}$$

the inequality

$$\tau^*(e^{it})\tau(e^{it}) \le \alpha_+^*(e^{it})\alpha_+(e^{it}) \ a.e. \quad (\upsilon(e^{it})\upsilon^*(e^{it}) \le \beta_+(e^{it})\beta_+^*(e^{it}) \ a.e.) \tag{7.13}$$

holds.

Theorem 7.11 ([13, 29]) *Let $\nu(e^{it}) \in CM[\mathfrak{G}]$ be nonnegative at almost all t. There exists the largest minorant ($*$-minorant)$\alpha_+(\zeta) \in S[\mathfrak{G}, \mathfrak{C}](\beta_+(\zeta) \in S[\mathfrak{C}_*, \mathfrak{G}])$ for the function $\nu(e^{it})$, $\alpha_+(\zeta)(\beta_+(\zeta))$ is outer ($*$-outer), and it is unique up to a left (right) constant unitary factor.*

Moreover, inequality (7.11) turns into equality almost everywhere iff the condition

$$\bigcap_{n=0}^{\infty}(U_{\mathfrak{G}}^{\times})^n\overline{\nu L_+^2(\mathfrak{G})} = \{0\} \quad \left(\bigcap_{n=0}^{\infty}(U_{\mathfrak{G}}^{\times})^{-n}\overline{\nu L_-^2(\mathfrak{G})} = \{0\}\right)$$

is satisfied.

We need the following well-known fact (see, e.g., [29], Chapter V, §4, Theorem 4.1; [13], Theorem 8.30).

Lemma 7.12 *Let $\mu(\zeta) \in S[\mathfrak{G}, \mathfrak{F}]$ and $\varkappa(\zeta) \in S[\mathfrak{G}, \mathfrak{M}](\tau(\zeta) \in S[\mathfrak{L}, \mathfrak{F}])$ be two outer (∗-outer) operator function such that the equality*

$$\mu^*(e^{it})\mu(e^{it}) = \varkappa^*(e^{it})\varkappa(e^{it}) \ a.e. \quad (\mu(e^{it})\mu^*(e^{it}) = \tau(e^{it})\tau^*(e^{it}) \ a.e.)$$

holds. Then the functions $\mu(\zeta)$ and $\varkappa(\zeta)$ $(\tau(\zeta))$ coincide up to a left (right) constant unitary factor.

The following assertion follows directly from the proof of Theorem 8.54 in [13].

Lemma 7.13 *Let $\nu(e^{it}) \in CM[\mathfrak{G}]$ be nonnegative almost everywhere and $\alpha_+(\zeta) \in S[\mathfrak{G}, \mathfrak{C}](\beta_+(\zeta) \in S[\mathfrak{C}_*, \mathfrak{G}])$ be the largest minorant (∗-minorant) for the function $\nu(e^{it})$. Let $\sigma_\nu := (\mathfrak{H}, \mathfrak{G}, \mathfrak{G}; U, V_{\mathfrak{G}}^{(i)}, V_{\mathfrak{G}}^{(o)})$ be a minimal unitary coupling such that $\theta_{\sigma_\nu}(e^{it}) = \nu(e^{it})$. Then $\alpha_+(e^{it}) = \theta_{\sigma_{\alpha_+}}(e^{it})$ $(\beta_+(e^{it}) = \theta_{\sigma_{\beta_+}}(e^{it}))$, where the coupling*

$$\sigma_{\alpha_+} := (\mathfrak{H}, \mathfrak{C}, \mathfrak{G}; U, V_{\mathfrak{C}}, V_{\mathfrak{G}}^{(o)}) \quad (\sigma_{\beta_+} := (\mathfrak{H}, \mathfrak{G}, \mathfrak{C}_*; U, V_{\mathfrak{G}}^{(i)}, V_{\mathfrak{C}_*})) \tag{7.14}$$

is generated by the bilateral channel $(M(\overset{\circ}{\mathfrak{C}}), \mathfrak{C}; V_{\mathfrak{C}})$ $((M(\overset{\circ}{\mathfrak{C}}_*), \mathfrak{C}_*; V_{\mathfrak{C}_*}))$ of the coupling σ_ν, the subspace $\overset{\circ}{\mathfrak{C}}$ $(\overset{\circ}{\mathfrak{C}}_*)$ is determined by the Wold decomposition

$$\overline{P_{M(\overset{\circ}{\mathfrak{G}}_i)} M_+(\overset{\circ}{\mathfrak{G}}_o)} = M_+(\overset{\circ}{\mathfrak{C}}) \oplus \mathfrak{N}_0^+ \quad (\overline{P_{M(\overset{\circ}{\mathfrak{G}}_o)} M_-(\overset{\circ}{\mathfrak{G}}_i)} = M_-(\overset{\circ}{\mathfrak{C}}_*) \oplus \mathfrak{N}_0^-), \tag{7.15}$$

$$\overset{\circ}{\mathfrak{G}}_i := \operatorname{Ran} V_{\mathfrak{G}}^{(i)}, \quad \overset{\circ}{\mathfrak{G}}_o := \operatorname{Ran} V_{\mathfrak{G}}^{(o)},$$

$$\mathfrak{N}_0^+ := \bigcap_{n=0}^{\infty} U^n \overline{P_{M(\overset{\circ}{\mathfrak{G}}_i)} M_+(\overset{\circ}{\mathfrak{G}}_o)} \quad (\mathfrak{N}_o^- := \bigcap_{n=0}^{\infty} U^{-n} \overline{P_{M(\overset{\circ}{\mathfrak{G}}_o)} M_-(\overset{\circ}{\mathfrak{G}}_i)}).$$

The next theorem is a generalization of the result obtained in [23] and [14] for matrix- and operator-valued Schur functions $\theta(\zeta) \in S[\mathfrak{G}, \mathfrak{F}]$.

Theorem 7.14 *Let $\theta(e^{it})$ belong to $CM[\mathfrak{G}, \mathfrak{F}]$. The defect (∗-defect) function $\varphi_+(\zeta) \in S[\mathfrak{G}, \mathfrak{K}]$ $(\psi_+(\zeta) \in S[\mathfrak{K}_*, \mathfrak{F}])$ of $\theta(e^{it})$ in the Schur class is the largest minorant (∗-minorant) for its defect (∗-defect) function $\Pi(e^{it}) \in CM[\mathfrak{G}](\Sigma(e^{it}) \in CM[\mathfrak{F}])$ in the class of contractive measurable operator function.*

Proof We consider only the case of the defect function $\varphi_+(\zeta)$.

Let $\alpha_+(\zeta) \in S[\mathfrak{G}, \mathfrak{C}]$ be the largest minorant for the function $\Pi(e^{it}) \in CM[\mathfrak{G}]$ and $\sigma_\Pi := (\mathfrak{H}, \mathfrak{G}, \mathfrak{G}; U, V_{\mathfrak{G}}^{(i)}, V_{\mathfrak{G}}^{(o)})$ be a minimal unitary coupling such that $\theta_{\sigma_\Pi}(e^{it}) = \Pi(e^{it})$. Let σ_{α_+} be unitary coupling of type (7.14) generated, as in Lemma 7.13, by the channel $(M(\overset{\circ}{\mathfrak{C}}), \mathfrak{C}; V_{\mathfrak{C}})$ of σ_Π that is determined by the Wold decomposition (7.15). By the same Lemma 7.13, $\theta_{\sigma_{\alpha_+}}(e^{it}) = \alpha_+(e^{it})$.

Consider the coupling

$$\sigma_\omega := (M(\overset{\circ}{\mathfrak{G}}_i), \mathfrak{C}, \mathfrak{G}; U_\omega, V_{\mathfrak{C}}, V_{\mathfrak{G}}^{(i)}), \quad U_\omega := U|_{M(\overset{\circ}{\mathfrak{G}}_i)},$$

and the function $\omega(e^{it}) := \theta_{\sigma_\omega}(e^{it}) \in CM[\mathfrak{G}, \mathfrak{C}]$. In view of the inclusion $M(\overset{\circ}{\mathfrak{C}}) \subset M(\overset{\circ}{\mathfrak{G}}_i)$ and Definition 2.15, we obtain $\sigma_{\alpha_+} = \sigma_\omega \sigma_\Pi$ and, hence,

$$\alpha_+(e^{it}) = \omega(e^{it})\Pi(e^{it}) \text{ a.e.} \tag{7.16}$$

By Theorem 5.2, the function $\omega(e^{it})$ is coisometric. From equality (7.15) it follows that

$$M(\overset{\circ}{\mathfrak{C}}) \subset \overline{P_{M(\overset{\circ}{\mathfrak{G}}_i)} M(\overset{\circ}{\mathfrak{G}}_o)} \tag{7.17}$$

(see, e.g., [13], Lemma 8.32). Taking into account (2.4), we can rewrite (7.17) in the form

$$S_{\sigma_\omega}^* M(\overset{\circ}{\mathfrak{G}}_i) \subset \overline{S_{\sigma_\Pi} M(\overset{\circ}{\mathfrak{G}}_o)},$$

which, by virtue of (2.5), is equivalent to

$$\operatorname{Ran}\omega^* \subset \overline{\Pi L^2(\mathfrak{G})}. \tag{7.18}$$

According to Theorem 4.2 (part (*b*)), from (7.16) and (7.18) it follows that

$$\Omega_{\alpha_+}(e^{it}) := \begin{bmatrix} \alpha_+(e^{it}) \\ \theta(e^{it}) \end{bmatrix} \in \mathcal{U}_r(\theta).$$

In view of $\alpha_+(e^{it}) \in L_+^\infty[\mathfrak{G}, \mathfrak{C}]$, we see that $\Omega_{\alpha_+}(e^{it}) \in \mathcal{U}_r^+(\theta)$. Consequently, dy Definition 7.6, $\Omega_{\alpha_+}(e^{it}) \lll \Omega_0^+(e^{it})$, where $\Omega_0^+(e^{it})$ is the largest extension in $\mathcal{U}_r^+(\theta)$ of form (6.8). From this we obtain that $\Omega_{\alpha_+}(e^{it}) \prec \Omega_0^+(e^{it})$ and, by Theorem 5.5, the inequality

$$\alpha_+^*(e^{it})\alpha_+(e^{it}) \leq \varphi_+^*(e^{it})\varphi_+(e^{it}) \text{ a.e.} \tag{7.19}$$

holds.

On the other hand, the contractivity of $\Omega_0^+(e^{it})$ is equivalent to the inequality

$$\varphi_+^*(e^{it})\varphi_+(e^{it}) \leq \Pi^2(e^{it}) \text{ a.e.},$$

whence, by Definition 7.10, the inequality

$$\varphi_+^*(e^{it})\varphi_+(e^{it}) \le \alpha_+^*(e^{it})\alpha_+(e^{it}) \text{ a.e.} \tag{7.20}$$

follows. From (7.19) and (7.20), taking into account Lemma 7.12, we infer that $\varphi_+(\zeta)$ and $\alpha_+(\zeta)$ coincide up to a left constant unitary factor. □

Taking into account Theorem 7.4 (part (d)) and Theorem 7.11, from Theorem 7.14 we obtain

Corollary 7.15 *Let $\theta(e^{it})$ belong to $CM[\mathfrak{G}]$. There exists an isometric (coisometric) extension $\Omega(e^{it}) \in \mathcal{U}_r^+(\theta)$ $(\Lambda(e^{it}) \in \mathcal{L}_r^+(\theta)$ iff the condition*

$$\bigcap_{n=0}^{\infty}(U_{\mathfrak{G}}^{\times})^n\overline{\Pi L_+^2(\mathfrak{G})} = \{0\} \quad (\bigcap_{n=0}^{\infty}(U_{\mathfrak{F}}^{\times})^{-n}\overline{\Sigma L_-^2(\mathfrak{F})} = \{0\})$$

is satisfied.

Definition 7.10 and Theorem 7.14 enable us to refine the fact from Sect. 7.2 that $\Omega_0^+(e^{it})$ $(\Lambda_0^+(e^{it}))$ is the largest extension in $\mathcal{U}_r^+(\theta)$ $(\mathcal{L}_r^+(\theta))$.

Remark 7.16 Considering the set $\mathcal{U}^+(\theta)$ $(\mathcal{L}^+(\theta))$ of all contractive extension $\Omega(e^{it})$ $(\Lambda(e^{it}))$ of form (3.15) with $\theta_{12}(e^{it}) \in L_+^\infty[\mathfrak{G}, \mathfrak{F}^{(1)}]$ $(\theta_{21}(e^{it}) \in L_+^\infty[\mathfrak{G}^{(1)}, \mathfrak{F}])$, we can spread, as was already noted in Remark 5.17, the comparison relation $\prec$ on it using condition (b) from Theorem 5.5. Thus, the set $\mathcal{U}^+(\theta)$ $(\mathcal{L}^+(\theta))$ becomes partially preordered and $[\Omega_0^+]$ $([\Lambda_0^+])$ is the greatest element in the quotient set $\mathcal{U}^+/\sim$ $(\mathcal{L}^+/\sim)$. Note also that for $\Omega_j(e^{it}) \in \mathcal{U}^+(\theta)$ $(\Lambda_j(e^{it}) \in \mathcal{L}^+(\theta))$, $j = 1, 2$, of form (5.1)

$$\Omega_1(e^{it}) \sim \Omega_2(e^{it}) \quad (\Lambda_1(e^{it}) \sim \Lambda_2(e^{it}))$$

iff the function $\theta_{12}^{(j)}(\zeta) \in S[\mathfrak{G}, \mathfrak{F}^{(1)}]$ $(\theta_{21}^{(j)}(\zeta) \in S[\mathfrak{G}^{(1)}, \mathfrak{F}])$, $j = 1, 2$, have the common outer ($*$-outer) factor in their canonical ($*$-canonical) inner-outer factorizations up to a left (right) constant unitary factor (see, e.g., [29], Chapter V, Subsection 4.3; [13], Corollaries 8.41 and 8.44).

Theorem 7.11 gives us a possibility to strengthen Theorem 5.14 in the case when $\varphi_+(\zeta) \equiv 0$ or $\psi_+(\zeta) \equiv 0$, $\zeta \in \mathbb{D}$.

Theorem 7.17 *Let $\theta(e^{it})$ belong to $CM[\mathfrak{G}]$. If $\mathfrak{F} \neq \{0\}$ $(\mathfrak{G} \neq \{0\})$ and there is only a trivial extension $\Omega(e^{it}) := \theta(e^{it})$ $(\Lambda(e^{it}) := \theta(e^{it}))$ in $\mathcal{U}_r^+(\theta)$ $(\mathcal{L}_r^+(\theta))$, then $\|\theta\|_{L^\infty[\mathfrak{G},\mathfrak{F}]} = 1$.*

Proof It is obvious that in both cases $\mathfrak{F} \neq \{0\}$ and $\mathfrak{G} \neq \{0\}$. It suffices to consider only the case when, according to Theorem 7.4 (part (a)), $\varphi_+(\zeta) \equiv 0 \in [\mathfrak{G}, \mathfrak{K}]$, where $\mathfrak{K} = \{0\}$.

Suppose that $\|\theta\|_{L^\infty[\mathfrak{G},\mathfrak{F}]} = a < 1$. Then the inequality

$$\Pi^2(e^{it}) \geq b^2 I_{\mathfrak{G}} > 0,$$

where $b := \sqrt{1-a^2} > 0$, holds almost everywhere. Then, by Definition 7.10 and Theorem 7.14, setting $\tau(\zeta) := bI_{\mathfrak{G}} \in S[\mathfrak{G}]$, $\nu(e^{it}) := \Pi(e^{it}) \in CM[\mathfrak{G}]$ in (7.12), from (7.13) we obtain that $\varphi_+^*(e^{it})\varphi_+(e^{it}) \geq b^2 I_{\mathfrak{G}} > 0$ is valid almost everywhere. Thus, $\varphi(\zeta) \not\equiv 0$, $\zeta \in \mathbb{D}$, which contradicts the assumption. □

8 Bidirectional Extensions Generated by Extensions from $\mathcal{U}_r^+(\theta)$ and $\mathcal{L}_r^+(\theta)$

We will denote by $\mathcal{K}_r^+(\theta)$ or simply $\mathcal{K}_r^+$ the set of extensions $\Xi(e^{it}) \in \mathcal{K}_r(\theta)$ of form (3.32) such that the corresponding unidirectional extensions $\Omega(e^{it})$ and $\Lambda(e^{it})$ of form (3.15) belong to the sets $\mathcal{U}_r^+(\theta)$ and $\mathcal{L}_r^+(\theta)$, respectively. In other words, $\Xi(e^{it}) \in \mathcal{K}_r^+(\theta)$ iff $\theta_{12}(e^{it}) \in L_+^\infty[\mathfrak{G},\mathfrak{F}^{(1)}]$ and $\theta_{21}(e^{it}) \in L_+^\infty[\mathfrak{G}^{(1)},\mathfrak{F}]$. By $\mathcal{K}_{cr}^+(\theta)$ or simply $\mathcal{K}_{cr}^+$ we will denote the subset of $\mathcal{K}_r^+(\theta)$ consisting of completely regular extensions. The subset of extensions $\Xi(e^{it}) \in \mathcal{K}_r^+(\theta)$ such that $\theta_{11}(e^{it}) \in L_+^\infty[\mathfrak{G}^{(1)},\mathfrak{F}^{(1)}]$ will be denoted by $\mathcal{K}_r^{++}(\theta)$ or simply $\mathcal{K}_r^{++}$. It is not necessary to introduce the notation $\mathcal{K}_{cr}^{++}(\theta)$, since the subset of completely regular extensions $\Xi(e^{it}) \in \mathcal{K}_r^{++}(\theta)$ coincides with $\mathcal{K}_{cr}^+(\theta)$ in view of $\theta_{11}(e^{it}) \equiv 0 \in [\mathfrak{G}^{(1)},\mathfrak{F}^{(1)}]$ (see Remark 3.10).

8.1 Description of the Set $\mathcal{K}_r^+(\theta)$

Let $\theta(e^{it}) \in CM[\mathfrak{G},\mathfrak{F}]$ and $\Omega_0^+(e^{it}) \in CM[\mathfrak{G},\mathfrak{K}\oplus\mathfrak{F}]$, $\Lambda_0^+(e^{it}) \in CM[\mathfrak{K}_*\oplus\mathfrak{G},\mathfrak{F}]$ be its largest unidirectional extensions of form (6.8) in the sets $\mathcal{U}_r^+(\theta)$ and $\mathcal{L}_r^+(\theta)$, respectively. Denote by

$$\Xi_0^+(e^{it}) := \begin{bmatrix} \chi(e^{it}) & \varphi_+(e^{it}) \\ \psi_+(e^{it}) & \theta(e^{it}) \end{bmatrix} \in CM[\mathfrak{K}_*\oplus\mathfrak{G},\mathfrak{K}\oplus\mathfrak{F}] \tag{8.1}$$

the regular up-leftward extension from $\mathcal{K}_r^+(\theta)$ corresponding to $\Omega_0^+(e^{it})$ and $\Lambda_0^+(e^{it})$. The function $\chi(e^{it}) \in CM[\mathfrak{K}_*,\mathfrak{K}]$ is determined uniquely by the defect functions $\varphi_+(\zeta) \in S[\mathfrak{G},\mathfrak{K}]$ and $\psi_+(\zeta) \in S[\mathfrak{K}_*,\mathfrak{F}]$ of the function $\theta(e^{it})$ (see Remark 3.12). Since $\varphi_+(\zeta)$ and $\psi_+(\zeta)$ are considered up to left and right constant unitary factors, respectively (see Sect. 6.3), the function $\chi(e^{it})$ is also viewed only up to constant unitary factors on both sides.

If σ is a minimal unitary coupling of form (2.1), then, by Theorem 3.9 and Definition 3.11, $\chi(e^{it}) = \theta_{\sigma_{11}^+}(e^{it})$, where σ_{11}^+ is the coupling of type (3.25), namely,

$$\sigma_{11}^+ := (\mathfrak{H}, \mathfrak{K}, \mathfrak{K}_*; U, V_{\mathfrak{K}}, V_{\mathfrak{K}_*}). \tag{8.2}$$

Notice that the coupling σ_{11}^+, unlike the couplings

$$\sigma_{12}^+ := (U, \mathfrak{K}, \mathfrak{G}; U, V_{\mathfrak{K}}, V_{\mathfrak{G}}), \quad \sigma_{21}^+ := (\mathfrak{H}, \mathfrak{F}, \mathfrak{K}_*; U, V_{\mathfrak{F}}, V_{\mathfrak{K}_*}),$$

need not to be orthogonal and, hence, the condition $\chi(e^{it}) \in L_+^\infty[\mathfrak{K}_*, \mathfrak{K}]$ is not necessarily satisfied (see [11], Theorem 6.2).

The following theorem is a direct corollary of Theorem 6.9.

Theorem 8.1 *Let $\theta(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$ and σ be a minimal unitary coupling of form* (2.1) *such that $\theta_\sigma(e^{it}) = \theta(e^{it})$.*

(a) There exists a bijective correspondence between pairs

$$\{(M_-(\overset{\circ}{\mathfrak{F}}{}^{(1)}), \mathfrak{F}^{(1)}; V_{\mathfrak{F}^{(1)}}), \ (M_+(\overset{\circ}{\mathfrak{G}}{}^{(1)}), \mathfrak{G}^{(1)}; V_{\mathfrak{G}^{(1)}})\} \tag{8.3}$$

of internal unilateral input and output channels of the coupling σ and extensions $\Xi(e^{it}) \in \mathcal{K}_r^+(\theta)$ of form (3.32)*. This correspondence is established by the equalities*

$$\theta_{12}(e^{it}) = \theta_{\sigma_{12}}(e^{it}), \quad \theta_{21}(e^{it}) = \theta_{\sigma_{21}}(e^{it}), \quad \theta_{11}(e^{it}) = \theta_{\sigma_{11}}(e^{it}),$$

where σ_{12}, σ_{21}, σ_{11} are unitary couplings of form (3.6)*,* (3.11)*,* (3.25)*, respectively.*

(b) An extension $\Xi(e^{it}) \in \mathcal{K}_r^+(\theta)$ is completely regular iff the corresponding pair of internal unilateral channels additionally satisfies the conditions

$$M_-(\overset{\circ}{\mathfrak{F}}{}^{(1)}) \subset M_-(\overset{\circ}{\mathfrak{G}}), \quad M_+(\overset{\circ}{\mathfrak{G}}{}^{(1)}) \subset M_+(\overset{\circ}{\mathfrak{F}}).$$

Taking into account the obvious interrelations between unilateral input (output) channels of σ and unilateral coshifts (shifts) contained in U (see Sect. 6.1), we can reformulate Theorem 8.1 in terms of unilateral coshifts (shifts).

Theorem 8.2 *Suppose that the conditions of the preceding theorem are satisfied and $T := T_\sigma$.*

(a) There exists a bijective correspondence between pairs $\{\tilde{V}, V\} \in \tilde{\mathcal{V}}_T \times \mathcal{V}_T$ of internal unilateral coshifts and shifts of σ and extensions $\Xi(e^{it}) \in \mathcal{K}_r^+(\theta)$ of form (3.32) *if the pairs of blocks of the upper row and the left column are considered up to left and right common constant unitary factors, respectively. This correspondence is established as in Theorem 8.1.*

(b) An extension $\Xi(e^{it}) \in \mathcal{K}_r^+(\theta)$ is completely regular iff the corresponding pair $\{\tilde{V}, V\}$ belongs to $\tilde{\mathcal{W}}_T \times \mathcal{W}_T$.

Definition 8.3 Let $\theta(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$, σ be a minimal unitary coupling of form (2.1) such that $\theta_\sigma(e^{it}) = \theta(e^{it})$, and $T := T_\sigma$. If $\Xi(e^{it}) \in \mathcal{K}_r^+(\theta)$ of form (3.32) corresponds to a pair of form (8.3) consisting of internal unilateral input and output channels of the coupling σ (in other words, corresponds to a pair of internal coshift $\tilde{V} := (U^*|_{M_-(\overset{\circ}{\mathfrak{F}}^{(1)})})^* \in \tilde{\mathcal{V}}_T$ and shift $V := U|_{M_+(\overset{\circ}{\mathfrak{G}}^{(1)})} \in \mathcal{V}_T$), then the function $\theta_{11}(e^{it}) \in CM[\mathfrak{G}^{(1)}, \mathfrak{F}^{(1)}]$ will be called the scattering suboperator generated by the internal unilateral channels (8.3) of the coupling σ. In particular, the function $\chi(e^{it}) \in CM[\mathfrak{K}_*, \mathfrak{K}]$ from the representation (8.1) of $\Xi_0^+(e^{it})$ generated by the pair

$$(M_-(\overset{\circ}{\mathfrak{K}}), \mathfrak{K}; V_{\mathfrak{K}}),\ (M_+(\overset{\circ}{\mathfrak{K}}_*), \mathfrak{K}_*; V_{\mathfrak{K}_*})$$

of the principal internal unilateral channels of σ will be called the suboperator of internal scattering of the coupling σ.

In the case of the completely regular extension $\Xi(e^{it}) \in \mathcal{K}_{cr}^+(\theta)$ of form (3.32), the function $\theta_{11}(e^{it})$ vanishes identically. This fact can be interpreted as the absence of the scattering generated by the corresponding pair of internal unilateral channels of form (8.3).

As it follows from Theorem 4.5 (part (*b*)), the function $\chi(e^{it})$ admits the representation

$$\chi(e^{it}) = -\omega_0^+(e^{it})\theta^*(e^{it})\lambda_0^+(e^{it}), \tag{8.4}$$

where $\omega_0^+(e^{it}) \in CM[\mathfrak{G}, \mathfrak{K}]$ and $\lambda_0^+(e^{it}) \in CM[\mathfrak{K}_*, \mathfrak{F}]$ are coisometric and isometric operator functions, respectively, from representation (6.9).

From Theorem 7.4 we obtain

Theorem 8.4 *Let $\theta(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$ and $\varphi_+(\zeta) \in S[\mathfrak{G}, \mathfrak{K}]$, $\psi_+(\zeta) \in S[\mathfrak{K}_*, \mathfrak{F}]$ be its defect and $\star$-defect functions in the Schur class, $\gamma_+(\zeta) \in S[\mathfrak{G}, \mathfrak{N}]$ and $\delta_+(\zeta) \in S[\mathfrak{N}_*, \mathfrak{F}]$ be its defect and $\star$-defect functions in the classes of $\star$-inner and inner operator functions, respectively.*

(a) A nontrivial extension $\Xi(e^{it}) \in \mathcal{K}_r^+(\theta)$ exists iff $\varphi_+(\zeta) \not\equiv 0$ or $\psi_+(\zeta) \not\equiv 0$.

Let $\Xi(e^{it}) \in CM[\mathfrak{G}^{(1)} \oplus \mathfrak{G}, \mathfrak{F}^{(1)} \oplus \mathfrak{F}]$ be an operator function of form (3.32).

(b) $\Xi(e^{it}) \in \mathcal{K}_r^+(\theta)$ iff there exists a $\star$-inner operator function $\omega(\zeta) \in S[\mathfrak{K}, \mathfrak{F}^{(1)}]$ and an inner operator function $\lambda(\zeta) \in S[\mathfrak{G}^{(1)}, \mathfrak{K}_*]$ such that the functions $\theta_{12}(e^{it}), \theta_{21}(e^{it})$ and $\theta_{11}(e^{it})$ admit the representations

$$\theta_{12}(e^{it}) = \omega(e^{it})\varphi_+(e^{it}), \theta_{21}(e^{it}) = \psi_+(e^{it})\lambda(e^{it}), \theta_{11}(e^{it}) = \omega(e^{it})\chi(e^{it})\lambda(e^{it}) \tag{8.5}$$

almost everywhere.

(c) $\Xi(e^{it}) \in \mathcal{K}_{cr}^{+}(\theta)$ *iff there exist a* $\star$*-inner operator function* $\omega(\zeta) \in S[\mathfrak{N}, \mathfrak{F}^{(1)}]$ *and an inner operator function* $\lambda(\zeta) \in S[\mathfrak{G}^{(1)}, \mathfrak{N}_*]$ *such that the function* $\theta_{12}(e^{it})$ *and* $\theta_{21}(e^{it})$ *admit the representations*

$$\theta_{12}(e^{it}) = \omega(e^{it})\gamma_+(e^{it}), \quad \theta_{21}(e^{it}) = \delta_+(e^{it})\lambda(e^{it})$$

almost everywhere and $\theta_{11}(e^{it}) \equiv 0 \in [\mathfrak{G}^{(1)}, \mathfrak{F}^{(1)}]$.

(d) There exists an isometric (coisometric) extension $\Xi(e^{it}) \in \mathcal{K}_r^{+}(\theta)$ *iff the condition*

$$\overline{\varphi_+^* L^2(\mathfrak{K})} = \overline{\Pi L^2(\mathfrak{G})} \quad (\overline{\psi_+ L^2(\mathfrak{K}_*)} = \overline{\Sigma L^2(\mathfrak{F})}) \tag{8.6}$$

is satisfied. If equality (8.6) is valid, then all isometric (coisometric) extensions are given by formulas (3.32) and (8.5), where $\omega(\zeta)$ $(\lambda(\zeta))$ *is an arbitrary two-sided inner operator function.*

A unitary extension $\Xi(e^{it}) \in \mathcal{K}_r^{+}(\theta)$ *exists iff both dual equalities (8.6) are valid. In this case, all such extensions are given by the same formulas (3.32) and (8.5), where both functions* $\omega(\zeta)$ *and* $\lambda(\zeta)$ *are arbitrary two-sided inner operator functions.*

Proof It is necessary to prove only the third equality (8.5).

From the first two equalities (8.5) and equalities (6.9) we obtain

$$\theta_{12}(e^{it}) = \alpha(e^{it})\Pi(e^{it}) \ a.e., \quad \theta_{21}(e^{it}) = \Sigma(e^{it})\beta(e^{it}) \ a.e.,$$

where $\alpha(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}^{(1)}]$ and $\beta(e^{it}) \in CM[\mathfrak{G}^{(1)}, \mathfrak{F}]$ are coisometric and isometric functions, respectively, that are defined by the equalities

$$\alpha(e^{it}) := \omega(e^{it})\omega_0^{+}(e^{it}), \quad \beta(e^{it}) := \lambda_0^{+}(e^{it})\lambda(e^{it}).$$

Then, by Theorem 4.5 (part (*b*)), we obtain $\theta_{11}(e^{it}) = -\alpha(e^{it})\theta^*(e^{it})\beta(e^{it})$, whence, taking into account (8.4), the third equality (8.5) follows. □

Remark 8.5 Recall that the condition (8.6) is equivalent to the condition

$$\bigcap_{n=0}^{\infty}(U_{\mathfrak{G}}^{\times})^n \overline{\Pi L_+^2(\mathfrak{G})} = \{0\} \quad (\bigcap_{n=0}^{\infty}(U_{\mathfrak{F}}^{\times})^{-n} \overline{\Sigma L_-^2(\mathfrak{F})} = \{0\})$$

(see Corollary 7.15).

Remark 8.6 As it was already done for $\mathcal{K}_r(\theta)$ in Remark 4.6, we can also reformulate the part (b) of Theorem 8.4 in the following way.

The general form of $\Xi(e^{it}) \in \mathcal{K}_r^+(\theta)$ for any function $\theta(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$ is given by the formula

$$\Xi(e^{it}) = A(e^{it})\Xi_0^+(e^{it})B(e^{it}),$$

where $\Xi_0^+(e^{it}) \in CM[\mathfrak{K}_* \oplus \mathfrak{G}, \mathfrak{K} \oplus \mathfrak{F}]$ is the extension from $\mathcal{K}_r^+(\theta)$ of form (8.1), $A(e^{it}) \in L_+^\infty[\mathfrak{K} \oplus \mathfrak{F}, \mathfrak{F}^{(1)} \oplus \mathfrak{F}]$, $B(e^{it}) \in L_+^\infty[\mathfrak{G}^{(1)} \oplus \mathfrak{G}, \mathfrak{K}_* \oplus \mathfrak{G}]$ are operator functions of form (4.18), $\omega(\zeta) \in S[\mathfrak{K}, \mathfrak{F}^{(1)}]$ and $\lambda(\zeta) \in S[\mathfrak{G}^{(1)}, \mathfrak{K}_*]$ are arbitrary $\star$-inner and inner operator functions, respectively.

8.2 Refined Comparison Relation on the Set $\mathcal{K}_r^+(\theta)$

Using the refined comparison relations $\lll$ on $\mathcal{U}_r^+(\theta)$ and $\mathcal{L}_r^+(\theta)$, we can introduce the relation $\lll$ on $\mathcal{K}_r^+(\theta)$ as well. As before, we identify extensions $\Xi(e^{it}) \in \mathcal{K}_r^+(\theta)$ of form (3.32) for which the pairs of blocks of the upper row and the left column differ by left and right common constant unitary factors, respectively.

Definition 8.7 Let $\theta(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$ and $\Xi_j(e^{it}) \in \mathcal{K}_r^+(\theta)$, $j = 1, 2$, be two its extensions of form (5.10). Let $\Omega_j(e^{it}) \in \mathcal{U}_r^+(\theta)$, $\Lambda_j(e^{it}) \in \mathcal{L}_r^+(\theta)$, $j = 1, 2$, be two pairs of extensions of form (5.11) and (5.12), respectively, that correspond to $\Xi_j(e^{it})$, $j = 1, 2$. We will write $\Xi_1(e^{it}) \lll \Xi_2(e^{it})$ if $\Omega_1(e^{it}) \lll \Omega_2(e^{it})$ and $\Lambda_1(e^{it}) \lll \Lambda_2(e^{it})$. The relation $\lll$ will be called the refined comparison relation on $\mathcal{K}_r^+(\theta)$.

Let σ be a minimal unitary coupling of form (2.1), $\theta(e^{it}) := \theta_\sigma(e^{it})$, and $T := T_\sigma$. If $\Xi_j(e^{it}) \in \mathcal{K}_r^+(\theta)$, $j = 1, 2$, and $\{\tilde{V}_j, V_j\} \in \tilde{\mathcal{V}}_T \times \mathcal{V}_T$, $j = 1, 2$, are the two corresponding pairs of internal unilateral coshifts and shifts of σ (see Theorem 8.2), then $\Xi_1(e^{it}) \lll \Xi_2(e^{it})$ iff $\tilde{V}_1 \subset \tilde{V}_2$ and $V_1 \subset V_2$ (see Definitions 7.6 and 8.7). Note that the latter does not depend on the choice of a minimal coupling σ such that $\theta_\sigma(e^{it}) = \theta(e^{it})$. Thus, introducing the partial order on $\tilde{\mathcal{V}}_T \times \mathcal{V}_T$ in a natural way, we obtain that the partially ordered sets $\mathcal{K}_r^+(\theta)$ and $\tilde{\mathcal{V}}_T \times \mathcal{V}_T$ are isomorphic to each other.

As it follows from the arguments similar to those given in Sect. 7.2, the relation $\lll$ on $\mathcal{K}_r^+$ is stronger than the relation induced on it by the relation $\prec$ on $\mathcal{K}_r$.

Since $\Omega_0^+(e^{it}) \in \mathcal{U}_r^+$ and $\Lambda_0^+(e^{it}) \in \mathcal{L}_r^+$ of form (6.8) are the largest extensions in the sets $\mathcal{U}_r^+(\theta)$ and $\mathcal{L}_r^+(\theta)$, respectively, then, by Definition 8.7, $\Xi_0^+(e^{it}) \in \mathcal{K}_r^+$ of form (8.1) is the largest extension in the set $\mathcal{K}_r^+$. Similarly, since $\Omega_{c0}^+(e^{it}) \in \mathcal{U}_{cr}^+$ and $\Lambda_{c0}^+(e^{it}) \in \mathcal{L}_{cr}^+$ of form (6.12) are the largest extensions in the sets $\mathcal{U}_{cr}^+$ and $\mathcal{L}_{cr}^+$, respectively, then $\Xi_{c0}^+(e^{it}) \in \mathcal{K}_{cr}^+$, where

$$\Xi_{c0}^+(e^{it}) := \begin{bmatrix} 0 & \gamma_+(e^{it}) \\ \delta_+(e^{it}) & \theta(e^{it}) \end{bmatrix} \in CM[\mathfrak{N}_* \oplus \mathfrak{G}, \mathfrak{N} \oplus \mathfrak{F}],$$

is the largest extension in the set $\mathcal{K}_{cr}^+$.

Using the sets $\mathcal{J}_*^+(\theta)$ and $\mathcal{J}^+(\theta)$, introduced in Sect. 7.2 and partially ordered by the relation $\vdots$ of divisibility on the right and on the left, respectively, from Theorem 7.7, Theorem 8.4 (part(b)) and Definition 8.7 we obtain

Theorem 8.8 *Let $\theta(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$, $\Xi_j(e^{it}) \in \mathcal{K}_r^+(\theta)$, $j = 1, 2$, be its extensions of form (5.10) and*

$$\theta_{12}^{(j)}(e^{it}) = \omega_j(e^{it})\varphi_+(e^{it}), \quad \theta_{21}^{(j)}(e^{it}) = \psi_+(e^{it})\lambda_j(e^{it}),$$

$$\theta_{11}^{(j)}(e^{it}) = \omega_j(e^{it})\chi(e^{it})\lambda_j(e^{it}),$$

where $\omega_j(\zeta) \in S[\mathfrak{K}, \mathfrak{F}^{(j)}]$ and $\lambda_j(\zeta) \in S[\mathfrak{G}^{(j)}, \mathfrak{K}_*]$, $j = 1, 2$, are the two pairs of the $\star$-inner and inner functions that, by Theorem 8.4 (part (b)), correspond to the pair $\Xi_j(e^{it})$, $j = 1, 2$. The following statements are equivalent:

(a) $\Xi_1(e^{it}) \lll \Xi_2(e^{it})$*;*

(b) $\omega_1(\zeta) \vdots \omega_2(\zeta)$ *(on the right) and* $\lambda_1(\zeta) \vdots \lambda_2(\zeta)$ *(on the left).*

In this way we establish an isomorphism between the set $\mathcal{K}_r^+(\theta)$, partially ordered by the relation $\lll$, and the set $\mathcal{J}_*^+(\theta) \times \mathcal{J}^+(\theta)$, partially ordered in a natural way (as the Cartasian product), by the statement (b).

In the same way, as it was noted at the end of Sect. 7.2, we can remark that the pairwise isomorphic partially ordered sets $\mathcal{K}_r^+$, $\tilde{\mathcal{V}}_T \times \mathcal{V}_T$ and $\mathcal{J}_*^+(\theta) \times \mathcal{J}^+(\theta)$ are complete lattices.

Similarly, using the part (c) of Theorem 8.4 instead of the part (b), we could introduce the refined comparison relation on the set $\mathcal{K}_{cr}^+(\theta)$.

8.3 The Case of $\chi(e^{it}) \in L_+^\infty[\mathfrak{K}_*, \mathfrak{K}]$

If the suboperator $\chi(e^{it}) \in CM[\mathfrak{K}_*, \mathfrak{K}]$ of internal scattering of the coupling from (8.1) is the boundary value function of some Schur operator function $\chi(\zeta) \in S[\mathfrak{K}_*, \mathfrak{K}]$, then, obviously, $\mathcal{K}_r^{++}(\theta) = \mathcal{K}_r^+(\theta)$. Hence, in this case, the required description of $\mathcal{K}_r^{++}(\theta)$ was already obtained in Theorem 8.4. We find conditions on a function $\theta(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$ under which $\chi(e^{it}) \in L_+^\infty[\mathfrak{K}_*, \mathfrak{K}]$.

Theorem 8.9 *Let $\theta(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$ and σ be a minimal unitary coupling of form (2.1) such that $\theta_\sigma(e^{it}) = \theta(e^{it})$. Let $\chi(e^{it}) \in CM[\mathfrak{K}_*, \mathfrak{K}]$ be the suboperator of internal scattering of the coupling σ. Then the function $\chi(e^{it})$ belongs to $L_+^\infty[\mathfrak{K}_*, \mathfrak{K}]$ iff any of the two following equivalent conditions*

(a) θg *belongs to* $\overline{\Sigma L_-^2(\mathfrak{F})}$ *for any* $g \in \overline{\Pi L^2(\mathfrak{G})}$ *such that* $\Pi g \in L_-^2(\mathfrak{G})$*;*

(b) $\theta^* f$ *belongs to* $\overline{\Pi L_+^2(\mathfrak{G})}$ *for any* $f \in \overline{\Sigma L^2(\mathfrak{F})}$ *such that* $\Sigma f \in L_+^2(\mathfrak{F})$

is valid.

Proof Consider the functional model $\hat{\sigma}$ of a minimal unitary coupling of form (2.11) described in Theorem 2.11. Let

$$\hat{\sigma}_{11}^{+} := (\hat{\mathfrak{H}}, \mathfrak{K}, \mathfrak{K}_*; \hat{U}, \hat{V}_{\mathfrak{K}}, \hat{V}_{\mathfrak{K}_*}) \tag{8.7}$$

be the coupling of type (8.2) generated by the largest internal unilateral input and output channels $(\hat{M}_-(\overset{\circ}{\mathfrak{K}}), \mathfrak{K}; \hat{V}_{\mathfrak{K}})$ and $(\hat{M}_+(\overset{\circ}{\mathfrak{K}}_*), \mathfrak{K}_*; \hat{V}_{\mathfrak{K}_*})$, respectively. If $\hat{T} \in [\hat{\mathfrak{H}}_{\hat{T}}]$ is the fundamental contraction of $\hat{\sigma}$ and $\tilde{V}_{\hat{T}} \in [\hat{\mathfrak{H}}_{\tilde{V}_{\hat{T}}}]$ is the largest internal unilateral coshift of $\hat{\sigma}$, then, by Theorem 6.7,

$$\hat{M}_-(\overset{\circ}{\mathfrak{K}}) = \hat{\mathfrak{H}}_{\tilde{V}_{\hat{T}}} = \hat{\mathfrak{R}}_{\overset{\circ}{\mathfrak{F}}} \cap \hat{\mathfrak{R}}^{-}_{\overset{\circ}{\mathfrak{G}}}, \tag{8.8}$$

where $\hat{\mathfrak{R}}_{\overset{\circ}{\mathfrak{F}}} := \hat{\mathfrak{H}} \ominus \hat{M}(\overset{\circ}{\mathfrak{F}})$, $\hat{\mathfrak{R}}^{-}_{\overset{\circ}{\mathfrak{G}}} := \hat{\mathfrak{H}} \ominus \hat{M}_+(\overset{\circ}{\mathfrak{G}})$. Since $\hat{\mathfrak{H}} := L^2(\mathfrak{F}) \oplus \overline{\Pi L^2(\mathfrak{G})}$ and $\hat{M}(\overset{\circ}{\mathfrak{F}}) = L^2(\mathfrak{F}) \oplus \{0\}$, we obtain $\hat{\mathfrak{R}}_{\overset{\circ}{\mathfrak{F}}} = \{0\} \oplus \overline{\Pi L^2(\mathfrak{G})}$. Hence, in view of

$$\hat{M}_+(\overset{\circ}{\mathfrak{G}}) = \{(\theta g, \Pi g) : g \in L^2_+(\mathfrak{G})\},$$

from (8.8) we see that $(0, g) \in \hat{M}_-(\overset{\circ}{\mathfrak{K}})$ iff $g \in \overline{\Pi L^2(\mathfrak{G})} \ominus \overline{\Pi L^2_+(\mathfrak{G})}$, that is, for any $g_1 \in L^2_+(\mathfrak{G})$ the equality $\langle g, \Pi g_1\rangle = 0$ holds. Thus,

$$\hat{M}_-(\overset{\circ}{\mathfrak{K}}) = \{(0, g) : g \in \overline{\Pi L^2(\mathfrak{G})}, \Pi g \in L^2_-(\mathfrak{G})\}. \tag{8.9}$$

In the dual way, considering the functional model $\tilde{\sigma}$ of a minimal unitary coupling of form (2.13) described in Theorem 2.12 and the coupling

$$\tilde{\sigma}_{11}^{+} := (\hat{\mathfrak{H}}, \mathfrak{K}, \mathfrak{K}_*; \tilde{U}, \tilde{V}_{\mathfrak{K}}, \tilde{V}_{\mathfrak{K}_*})$$

of type (8.2), we obtain

$$\tilde{M}_+(\overset{\circ}{\mathfrak{K}}_*) = \{(f, 0) : f \in \overline{\Sigma L^2(\mathfrak{F})}, \Sigma f \in L^2_+(\mathfrak{F})\}. \tag{8.10}$$

In view of the unitary equivalence of the coupling $\hat{\sigma}$ and $\tilde{\sigma}$, the couplings $\hat{\sigma}_{11}^{+}$ and $\tilde{\sigma}_{11}^{+}$ are also unitarily equivalent. Consequently, $\hat{M}_+(\overset{\circ}{\mathfrak{K}}_*) = W^*\tilde{M}_+(\overset{\circ}{\mathfrak{K}}_*)$, where $W \in L^\infty[L^2(\mathfrak{F}) \oplus L^2(\mathfrak{G})]$ is "multiplication" unitary operator by the operator function $W(e^{it}) \in L^\infty[\mathfrak{F} \oplus \mathfrak{G}]$ of form (2.14) (see Theorem 2.13). Thus, from (2.14) and (8.10) we infer that

$$\hat{M}_+(\overset{\circ}{\mathfrak{K}}_*) = \{(\Sigma f, -\theta^* f) : f \in \overline{\Sigma L^2(\mathfrak{F})}, \Sigma f \in L^2_+(\mathfrak{F})\}. \tag{8.11}$$

Since $\chi(e^{it}) = \theta_{\hat{\sigma}_{11}^{+}}(e^{it})$, the condition $\chi(e^{it}) \in L_{+}^{\infty}[\mathfrak{K}_*, \mathfrak{K}]$ is equivalent to the orthogonality of the coupling $\hat{\sigma}_{11}^{+}$, that is, to the condition

$$\hat{M}_{-}(\overset{\circ}{\mathfrak{K}}) \perp \hat{M}_{+}(\overset{\circ}{\mathfrak{K}}_*) \tag{8.12}$$

(see Theorem 2.33). The latter, in view of (8.9) and (8.11), is tantamount to the validity of the equality $\langle \theta^* f, g\rangle = 0$ for any $f \in \overline{\Sigma L^2(\mathfrak{F})}$ and $g \in \overline{\Pi L^2(\mathfrak{G})}$ such that $\Sigma f \in L_{+}^{2}(\mathfrak{F})$ and $\Pi g \in L_{-}^{2}(\mathfrak{G})$. Taking into account that $\theta^* f \in \overline{\Pi L^2(\mathfrak{G})}$ for any $f \in \overline{\Sigma L^2(\mathfrak{F})}$, we see that (8.12) is valid iff $\theta^* f \in \overline{\Pi L_{+}^{2}(\mathfrak{G})}$ for any $f \in \overline{\Sigma L^2(\mathfrak{F})}$ such that $\Sigma f \in L_{+}^{2}(\mathfrak{F})$, that is, (8.12) is equivalent to the condition (b).

Similarly, since $\langle \theta^* f, g\rangle = \langle f, \theta g\rangle$, we obtain that (8.12) is equivalent to the condition (a). □

In the general case, when the condition $\chi(e^{it}) \in L_{+}^{\infty}[\mathfrak{K}_*, \mathfrak{K}]$ is not necessarily valid, maximal extensions in the set $\mathcal{K}_r^{++}(\theta)$ with respect to the relation $\lll$ are of particular interest. Denote by $\mathcal{K}_{r,\max}^{++}(\theta)$ or simply $\mathcal{K}_{r,\max}^{++}$ the subset of $\mathcal{K}_r^{++}(\theta)$ consisting of all such extensions.

8.4 Description of the Set $\mathcal{K}_{r,\max}^{++}(\theta)$

Let $\theta(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$, σ be a minimal unitary coupling of form (2.1) such that $\theta_\sigma(e^{it}) = \theta(e^{it})$, and $T := T_\sigma$. Taking into account that the complete lattices $\mathcal{K}_r^{+}(\theta)$, $\tilde{\mathcal{V}}_T \times \mathcal{V}_T$ and $\mathcal{J}_*^{+}(\theta) \times \mathcal{J}^{+}(\theta)$ are pairwise isomorphic (see Definition 8.7 and Theorem 8.8), we can study the subsets $\mathcal{K}_r^{++}(\theta)$ and $\mathcal{K}_{r,\max}^{++}(\theta)$ using the corresponding subsets of the sets $\tilde{\mathcal{V}}_T \times \mathcal{V}_T$ and $\mathcal{J}_*^{+}(\theta) \times \mathcal{J}^{+}(\theta)$.

For the convenience of the further exposition, a pair $\{\tilde{V}, V\} \in \tilde{\mathcal{V}}_T \times \mathcal{V}_T$ will be called *orthogonal* and this will be denoted by $\tilde{V} \perp V$ if $\tilde{V} \in [M_{-}(\overset{\circ}{\mathfrak{F}}{}^{(1)})]$, $V \in [M_{+}(\overset{\circ}{\mathfrak{G}}{}^{(1)})]$ and $M_{-}(\overset{\circ}{\mathfrak{F}}{}^{(1)}) \perp M_{+}(\overset{\circ}{\mathfrak{G}}{}^{(1)})$.

The next theorem is a direct corollary of Theorems 2.33, 8.2, 8.4, 8.8, and Definition 8.7.

Theorem 8.10 *Let* $\theta(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$*,* σ *be a minimal unitary coupling of form* (2.1) *such that* $\theta_\sigma(e^{it}) = \theta(e^{it})$*, and* $T := T_\sigma$*. Let* $\Xi(e^{it}) \in \mathcal{K}_r^{+}(\theta)$ *be extension of form* (3.32)*,* $\{\tilde{V}, V\} \in \tilde{\mathcal{V}}_T \times \mathcal{V}_T$ *be the corresponding, by Theorem* 8.2 *(part* (*a*))*, pair of internal unilateral coshift and shift of the coupling* σ*,* $\{\omega(\zeta), \lambda(\zeta)\} \in \mathcal{J}_*^{+}(\theta) \times \mathcal{J}^{+}(\theta)$ *be the corresponding, by Theorem* 8.4 *(part* (*b*))*, pair of* $\star$*-inner and inner functions.*

(a) The following statements are equivalent:

(1) $\Xi(e^{it})$ *belongs to* $\mathcal{K}_r^{++}(\theta)$*;*
(2) $\{\tilde{V}, V\}$ *is an orthogonal pair;*

(3) *$\omega(\zeta)$ and $\lambda(\zeta)$ satisfy the condition*

$$\omega(e^{it})\chi(e^{it})\lambda(e^{it}) \in L_+^\infty[\mathfrak{G}^{(1)}, \mathfrak{F}^{(1)}], \tag{8.13}$$

(b) The following statements are equivalent:

(1) *$\Xi(e^{it})$ belongs to $\mathcal{K}_{r,\max}^{++}(\theta)$;*
(2) *$\{\tilde{V}, V\}$ is a maximal pair in the subset of orthogonal pairs of the set $\tilde{\mathcal{V}}_T \times \mathcal{V}_T$;*
(3) *$\{\omega(\zeta), \lambda(\zeta)\}$ is a maximal pair in the subset of the set $\mathcal{J}_*^+(\theta) \times \mathcal{J}^+(\theta)$ that consists of all pairs satisfying the condition (8.13).*

Theorem 8.11 *Let $\theta(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$, $\Xi(e^{it}) \in \mathcal{K}_r^{++}(\theta)$ be an extension of form (3.32), and $\Omega(e^{it}) \in \mathcal{U}_r^+(\theta)$, $\Lambda(e^{it}) \in \mathcal{L}_r^+(\theta)$ be the corresponding pair of unidirectional extensions of form (3.15). The extension $\Xi(e^{it})$ belongs to $\mathcal{K}_{r,\max}^{++}(\theta)$ iff*

$$[\theta_{11}(\zeta), \theta_{12}(\zeta)] = \varphi_+(\zeta; \Lambda), \quad \mathrm{col}[\theta_{11}(\zeta), \theta_{21}(\zeta)] = \psi_+(\zeta; \Omega). \tag{8.14}$$

Proof Since the bidirectional extension $\Xi(e^{it}) \in \mathcal{K}_r^{++}(\theta)$ can be simultaneously considered as unidirectional extensions from $\mathcal{U}_r^+(\Lambda)$ and $\mathcal{L}_r^+(\Omega)$, we infer that $\Xi(e^{it}) \in \mathcal{K}_{r,\max}^{++}(\theta)$ iff $\Xi(e^{it})$ is simultaneously the largest extension in the sets $\mathcal{U}_r^+(\Lambda)$ and $\mathcal{L}_r^+(\Omega)$. The latter is equivalent to the simultaneous fulfillment of conditions (8.14) (see Definition 6.13). □

Corollary 8.12 *Let $\Xi(e^{it}) \in \mathcal{K}_{r,\max}^{++}(\theta)$ be an extension of form (3.32). Then the functions*

$$[\theta_{11}(\zeta), \theta_{12}(\zeta)] \in S[\mathfrak{G}^{(1)} \oplus \mathfrak{G}, \mathfrak{F}^{(1)}], \quad \mathrm{col}[\theta_{11}(\zeta), \theta_{21}(\zeta)] \in S[\mathfrak{G}^{(1)}, \mathfrak{F}^{(1)} \oplus \mathfrak{F}]$$

are outer and $\star$-outer, respectively.

Proof The assertion follows from Theorems 6.14 and 8.11. □

Now we consider in more detail the subset of maximal orthogonal pairs of the set $\tilde{\mathcal{V}}_T \times \mathcal{V}_T$. For convenience, we denote by

$$\tilde{V}_0' := \tilde{V}_T; \quad V_0' := \sup\{V \in \mathcal{V}_T : \tilde{V}_0' \perp V\}, \tag{8.15}$$

$$V_0'' := V_T; \quad \tilde{V}_0'' := \sup\{\tilde{V} \in \tilde{\mathcal{V}}_T : \tilde{V} \perp V_0''\}, \tag{8.16}$$

Note that $\tilde{V}_0' \perp V_0'$ and $\tilde{V}_0'' \perp V_0''$, since, for any set $\{\mathfrak{L}_\gamma\}_{\gamma\in\Gamma}$ of subspaces of a Hilbert space $\mathfrak{H}$ and any subspace $\mathfrak{M}$ of $\mathfrak{H}$, the condition $\mathfrak{L}_\gamma \perp \mathfrak{M}$, $\gamma \in \Gamma$, implies the condition $\bigvee_{\gamma\in\Gamma} \mathfrak{L}_\gamma \perp \mathfrak{M}$. It is obvious that the pairs $\{\tilde{V}_0', V_0'\}$ and $\{\tilde{V}_0'', V_0''\}$ are maximal in the subset of orthogonal pairs of the set $\tilde{\mathcal{V}}_T \times \mathcal{V}_T$.

Lemma 8.13 *Let σ be a minimal unitary coupling of form (2.1) and $T := T_\sigma$.*

(a) If $\{\tilde{V}, V\} \in \tilde{\mathcal{V}}_T \times \mathcal{V}_T$ is a maximal pair in the subset of orthogonal pairs, then the inclusions

$$\tilde{V}_0'' \subset \tilde{V} \subset \tilde{V}_0', \quad V_0' \subset V \subset V_0'' \tag{8.17}$$

are valid.

(b) For any coshift $\tilde{V} \in \tilde{\mathcal{V}}_T$ (shift $V \in \mathcal{V}_T$) satisfying the first (second) inclusion (8.17) there exists the unique shift $V \in \mathcal{V}_T$ (coshift $\tilde{V} \in \tilde{\mathcal{V}}_T$) such that the pair $\{\tilde{V}, V\}$ is the maximal in the subset of orthogonal pairs of the set $\tilde{\mathcal{V}}_T \times \mathcal{V}_T$.

Proof

(*a*) The inclusions $\tilde{V} \subset \tilde{V}_0'$, $V \subset V_0''$ follow from the definitions of $\tilde{V}_0'$ and V_0'' (see (8.15) and (8.16), respectively). Since the pair $\{\tilde{V}, V\}$ is maximal in the subset of orthogonal pairs of the set $\tilde{\mathcal{V}}_T \times \mathcal{V}_T$, then

$$\tilde{V} = \sup\{\tilde{V}_1 \in \tilde{\mathcal{V}}_T : \tilde{V}_1 \perp V\}, \quad V = \sup\{V_1 \in \mathcal{V}_T : \tilde{V} \perp V_1\}. \tag{8.18}$$

Taking into account (8.16), (8.18) and the inclusions $\tilde{V} \subset \tilde{V}_0'$, $V \subset V_0''$, we obtain the inclusions $\tilde{V}_0'' \subset \tilde{V}$, $V_0' \subset V$.

(*b*) Let $\tilde{V} \in \tilde{\mathcal{V}}_T$ be a coshift satisfying the first inclusion (8.17). Let

$$V := \sup\{V_1 \in \mathcal{V}_T : V_1 \perp \tilde{V}\}. \tag{8.19}$$

As was noted above, the shift $V \in \mathcal{V}_T$ defined in this way is orthogonal to the coshift $\tilde{V}$. For the maximality of the pair $\{\tilde{V}, V\}$ in the subset of orthogonal pairs of the set $\tilde{\mathcal{V}}_T \times \mathcal{V}$, it remains to prove the equality

$$\tilde{V} = \sup\{\tilde{V}_1 \in \tilde{\mathcal{V}}_T : \tilde{V}_1 \perp V\}. \tag{8.20}$$

If $\tilde{V} \in [M_-(\mathring{\mathfrak{F}}^{(1)})]$, $V \in [M_+(\mathring{\mathfrak{G}}^{(1)})]$, $\tilde{V}_0' \in [M_-(\mathring{\mathfrak{K}})]$, $V_0'' \in [M_+(\mathring{\mathfrak{K}}_*)]$, then (8.19) is equivalent to the equality

$$M_+(\mathring{\mathfrak{G}}^{(1)}) = M_+(\mathring{\mathfrak{K}}_*) \cap \mathfrak{R}^+_{\mathring{\mathfrak{F}}^{(1)}}. \tag{8.21}$$

In view of (8.16), the inclusions $\tilde{V}_0'' \subset \tilde{V} \subset \tilde{V}_0'$ means that the inclusions

$$M_-(\mathring{\mathfrak{K}}) \cap \mathfrak{R}^-_{\mathring{\mathfrak{K}}_*} \subset M_-(\mathring{\mathfrak{F}}^{(1)}) \subset M_-(\mathring{\mathfrak{K}}) \tag{8.22}$$

hold. Hence, by virtue of (8.21) and (8.22), we come to

$$M_-(\overset{\circ}{\mathfrak{K}}) \cap \mathfrak{R}^-_{\overset{\circ}{\mathfrak{G}}^{(1)}} = M_-(\overset{\circ}{\mathfrak{K}}) \cap (\mathfrak{R}_{\overset{\circ}{\mathfrak{K}}_*} \vee M_-(\overset{\circ}{\mathfrak{F}}^{(1)})) = M_-(\overset{\circ}{\mathfrak{K}}) \cap M_-(\overset{\circ}{\mathfrak{F}}^{(1)}) = M_-(\overset{\circ}{\mathfrak{F}}^{(1)}),$$

that is, (8.20) is valid.

The uniqueness of V follows from (8.19).

The dual assertion can be proved in a similar way.

□

Leaning on the isomorphicity of the complete lattices $\tilde{\mathcal{V}}_T \times \mathcal{V}_T$ and $\mathcal{J}_*^+(\theta) \times \mathcal{J}^+(\theta)$, we can reformulate Lemma 8.13. Let $\omega_0'(\zeta) := I_{\mathfrak{K}}$ ($\lambda_0''(\zeta) := I_{\mathfrak{K}_*}$) and $\omega_0''(\zeta) \in S[\mathfrak{K}, \mathfrak{K}'']$ ($\lambda_0'(\zeta) \in S[\mathfrak{K}_*', \mathfrak{K}_*]$) be the greatest right (left) common divisor of all $\star$-inner (inner) functions $\omega(\zeta) \in S[\mathfrak{K}, \mathfrak{F}^{(1)}]$ ($\lambda(\zeta) \in S[\mathfrak{G}^{(1)}, \mathfrak{K}_*]$ satisfying the condition

$$\omega(e^{it})\chi(e^{it}) \in L_+^\infty[\mathfrak{K}_*, \mathfrak{F}^{(1)}] \quad (\chi(e^{it})\lambda(e^{it}) \in L_+^\infty[\mathfrak{G}^{(1)}, \mathfrak{K}])$$

(see Corollary 7.9). The pairs $\{\omega'(\zeta), \lambda'(\zeta)\}$ and $\{\omega''(\zeta), \lambda''(\zeta)\}$ are maximal in the subset of all pairs $\{\omega(\zeta), \lambda(\zeta)\} \in \mathcal{J}_*^+(\theta) \times \mathcal{J}^+(\theta)$ satisfying condition (8.13).

Lemma 8.14 *Let $\theta(e^{it})$ belong to $CM[\mathfrak{G}, \mathfrak{F}]$.*

(a) If $\{\omega(\zeta), \lambda(\zeta)\} \in \mathcal{J}_*^+(\theta) \times \mathcal{J}^+(\theta)$ is a maximal pair in the subset of pairs from $\mathcal{J}_*^+(\theta) \times \mathcal{J}^+(\theta)$ satisfying condition (8.13), then $\omega(\zeta)$ and $\lambda(\zeta)$ are right and left divisors of the function $\omega_0''(\zeta)$ and $\lambda_0'(\zeta)$, respectively.

(b) For any right (left) divisor $\omega(\zeta)$ ($\lambda(\zeta)$) of the $\omega_0''(\zeta)$ ($\lambda_0'(\zeta)$) within the class of $\star$-inner (inner) operator functions there exists a unique left (right) divisor $\lambda(\zeta)$ ($\omega(\zeta)$) of the function $\lambda_0'(\zeta)$ ($\omega_0''(\zeta)$) within the class of inner ($\star$-inner) operator functions such that the pair $\{\omega(\zeta), \lambda(\zeta)\} \in \mathcal{J}_*^+(\theta) \times \mathcal{J}^+(\theta)$ is maximal in the subset of pairs from $\mathcal{J}_*^+(\theta) \times \mathcal{J}^+(\theta)$ satisfying condition (8.13).

Lemmas 8.13 and 8.14 lead us to

Definition 8.15 Let $\theta(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$, σ be a minimal unitary coupling of form (2.1) such that $\theta_\sigma(e^{it}) = \theta(e^{it})$, and $T := T_\sigma$. Let $\Xi(e^{it}) \in \mathcal{K}^{++}_{r,\max}(\theta)$, $\{\tilde{V}, V\} \in \tilde{\mathcal{V}}_T \times \mathcal{V}_T$ be the corresponding maximal pair in the subset of orthogonal pairs of the set $\tilde{\mathcal{V}}_T \times \mathcal{V}_T$ and $\{\omega(\zeta), \lambda(\zeta)\} \in \mathcal{J}_*^+(\theta) \times \mathcal{J}^+(\theta)$ be the corresponding maximal pair in the subset of all pairs from $\mathcal{J}_*^+(\theta) \times \mathcal{J}^+(\theta)$ satisfying condition (8.13). Then the coshift $\tilde{V}$ and the shift V, as well as the $\star$-inner function $\omega(\zeta)$ and the inner function $\lambda(\zeta)$, will be called mutually complementary (with respect to the set $\mathcal{K}^{++}_{r,\max}(\theta)$). The pairs

$$\{\tilde{V}_0', V_0'\} \text{and} \{\tilde{V}_0'', V_0''\}, \quad \{\omega_0'(\zeta), \lambda_0'(\zeta)\} \text{and} \{\omega_0''(\zeta), \lambda_0''(\zeta)\},$$

as well as the corresponding extensions

$$\Xi_0'(e^{it}) \in CM[\mathfrak{K}_*' \oplus \mathfrak{G}, \mathfrak{K} \oplus \mathfrak{F}] \text{ and } \Xi_0''(e^{it}) \in CM[\mathfrak{K}_* \oplus \mathfrak{G}, \mathfrak{K}'' \oplus \mathfrak{F}],$$

will be called extreme in the corresponding subsets of $\tilde{\mathcal{V}}_T \times \mathcal{V}_T$, $\mathcal{J}_*^+(\theta) \times \mathcal{J}^+(\theta)$ and in the set $\mathcal{K}_{r,\max}^{++}(\theta)$, respectively.

It is clear that for any $\omega(\zeta) \in \mathcal{J}_*^+(\theta)$ ($\lambda(\zeta) \in \mathcal{J}^+(\theta)$ such that $\omega(\zeta)$ ($\lambda(\zeta)$) is a right (left) divisor of the function $\omega_0''(\zeta)$ ($\lambda_0'(\zeta)$) the complementary function $\lambda(\zeta) \in \mathcal{J}^+(\theta)$ ($\omega(\zeta) \in \mathcal{J}_*^+(\theta)$) is the greatest left (right) divisor among all left (right) divisors $\tilde{\lambda}(\zeta)$ ($\tilde{\omega}(\zeta)$) of $\lambda_0'(\zeta)$ ($\omega_0''(\zeta)$) such that $\omega(e^{it})\chi(e^{it})\tilde{\lambda}(e^{it})$ ($\tilde{\omega}(e^{it})\chi(e^{it})\lambda(e^{it})$) is the boundary value function of some Schur operator function.
Note that

$$\Xi_0'(e^{it}) = [\psi_+(e^{it}; \Omega_0^+), \Omega_0^+(e^{it})], \quad \Xi_0''(e^{it}) = \mathrm{col}[\varphi_+(e^{it}; \Lambda_0^+), \Lambda_0^+(e^{it})],$$

where $\Omega_0^+(e^{it}) \in \mathcal{U}_r^+(\theta)$ and $\Lambda_0^+(e^{it}) \in \mathcal{L}_r^+(\theta)$ are the largest extensions of form (6.8) in the sets $\mathcal{U}_r^+(\theta)$ and $\mathcal{L}_r^+(\theta)$, respectively. Note also that in the case of $\chi(e^{it}) \in L_+^\infty[\mathfrak{K}_*, \mathfrak{K}]$ the extreme maximal extensions $\Xi_0'(e^{it})$ and $\Xi_0''(e^{it})$ coincide with the largest extension $\Xi_0^+(e^{it})$ of form (8.1) in the set $\mathcal{K}_r^+(\theta)$ ($= \mathcal{K}_r^{++}(\theta)$). Moreover, in this case, $\mathcal{K}_{r,\max}^{++}(\theta) = \{\Xi_0^+(e^{it})\}$.
From (8.19)) and (8.20) the next obvious assertion follows.

Lemma 8.16 *Let $\theta(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$, σ be a minimal unitary coupling of form (2.1) such that $\theta_\sigma(e^{it}) = \theta(e^{it})$, and $T := T_\sigma$. Let $\Xi_1(e^{it})$ and $\Xi_2(e^{it})$ be two extensions from $\mathcal{K}_{r,\max}^{++}(\theta)$, $\{\tilde{V}_1, V_1\}$ and $\{\tilde{V}_2, V_2\}$ be the two corresponding pairs of mutually complementary internal coshifts and shifts of σ, $\{\omega_1(\zeta), \lambda_1(\zeta)\}$ and $\{\omega_2(\zeta), \lambda_2(\zeta)\}$ be the two corresponding pairs of mutually complementary $\star$-inner and inner functions. Then the following conditions are equivalent:*

(1) $\tilde{V}_1 \subset \tilde{V}_2$; (2) $V_2 \subset V_1$; (3) $\omega_1(\zeta) \vdots \omega_2(\zeta)$ *(on the right)*; (4) $\lambda_2(\zeta) \vdots \lambda_1(\zeta)$ *(on the left)*.

Refining Remarks 4.6 and 8.6, we come to

Remark 8.17 The general form of $\Xi(e^{it}) \in \mathcal{K}_{r,\max}^{++}(\theta)$ for any function $\theta(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$ is given by the formula

$$\Xi(e^{it}) = A(e^{it})\Xi_0^+(e^{it})B(e^{it}),$$

where $\Xi_0^+(e^{it}) \in CM[\mathfrak{K}_* \oplus \mathfrak{G}, \mathfrak{K} \oplus \mathfrak{F}]$ is the extension of form (8.1),

$$A(e^{it}) \in L_+^\infty[\mathfrak{K} \oplus \mathfrak{F}, \mathfrak{F}^{(1)} \oplus \mathfrak{F}], \quad B(e^{it}) \in L_+^\infty[\mathfrak{G}^{(1)} \oplus \mathfrak{G}, \mathfrak{K}_* \oplus \mathfrak{G}]$$

are operator functions of form (4.18), $\omega(\zeta) \in S[\mathfrak{K}, \mathfrak{F}^{(1)}]$, $\lambda(\zeta) \in S[\mathfrak{G}^{(1)}, \mathfrak{K}_*]$ is an arbitrary pair of mutually complementary $\star$-inner and inner functions, respectively.

8.5 Description of the Set $\mathcal{K}_r^{++}(\theta)$

Let σ be a minimal unitary coupling and $T := T_\sigma$. For any orthogonal pair $\{\tilde{V}, V\} \in \tilde{\mathcal{V}}_T \times \mathcal{V}_T$ we define

$$V' := \sup\{V_1 \in \mathcal{V}_T : V_1 \perp \tilde{V}\}, \quad \tilde{V}'' := \sup\{\tilde{V}_1 \in \tilde{\mathcal{V}}_T : \tilde{V}_1 \perp V\}. \tag{8.23}$$

Denote by $\tilde{V}' \in \tilde{\mathcal{V}}_T$ and $V'' \in \mathcal{V}_T$ the internal coshift and shift of σ that are complementary to the shift $V' \in \mathcal{V}_T$ and the coshift $\tilde{V}'' \in \tilde{\mathcal{V}}_T$, respectively.

This definition is correct, because, obviously, the inclusions $V_0' \subset V'$, $\tilde{V}_0'' \subset \tilde{V}''$ are valid (see Lemma 8.13 (part (*b*))). By Definition 8.15 and Lemma 8.13 (part (*b*)), the pairs $\{\tilde{V}', V'\}$, $\{\tilde{V}'', V''\}$ are maximal in the subset of orthogonal pairs of the set $\tilde{\mathcal{V}}_T \times \mathcal{V}_T$ and they are preceded by the pair $\{\tilde{V}, V\}$.

Definition 8.18 Let σ be a minimal unitary coupling and $T := T_\sigma$. Let $\{\tilde{V}, V\} \in \tilde{\mathcal{V}}_T \times \mathcal{V}_T$ be an orthogonal pair of internal coshift and shift. The pairs $\{\tilde{V}', V'\}$ and $\{\tilde{V}'', V''\}$ defined above will be called the extreme maximal pairs in the subset of orthogonal pairs of the set $\tilde{\mathcal{V}}_T \times \mathcal{V}_T$ for the orthogonal pair $\{\tilde{V}, V\}$.

To justify the term "extreme"introduced above we prove

Lemma 8.19 *Let σ be a minimal unitary coupling and $T := T_\sigma$. Let $\{\tilde{V}, V\} \in \tilde{\mathcal{V}}_T \times \mathcal{V}_T$ be an orthogonal pair and $\{\tilde{V}', V'\}\{\tilde{V}'', V''\}$ be the extreme maximal pairs for $\{\tilde{V}, V\}$ in the subset of orthogonal pairs of the set $\tilde{\mathcal{V}}_T \times \mathcal{V}_T$. A maximal orthogonal pair $\{\tilde{V}''', V'''\} \in \tilde{\mathcal{V}}_T \times \mathcal{V}_T$ in the same subset is preceded by the pair $\{\tilde{V}, V\}$ iff any of the following equivalent conditions*

$$1)\ \tilde{V}' \subset \tilde{V}''' \subset \tilde{V}''; \quad 2)\ V'' \subset V''' \subset V'$$

is satisfied.

Proof The equivalence of these conditions follows from Lemma 8.16. The sufficiency of each of them is trivial.

Now let $\{\tilde{V}''', V'''\}$ is a maximal pair in the subset of subset of orthogonal pairs of the set $\tilde{\mathcal{V}}_T \times \mathcal{V}_T$ that is preceded by the pair $\{\tilde{V}, V\}$. Since $\tilde{V} \subset \tilde{V}'''$, $V \subset V'''$, and $\tilde{V}''' \perp V'''$, we infer that $\tilde{V} \perp V'''$, $\tilde{V}''' \perp V$. Hence, in view of (8.23), the inclusions $\tilde{V}''' \subset \tilde{V}''$, $V''' \subset V'$ are valid and, by Lemma 8.16, the inclusions $\tilde{V}' \subset \tilde{V}'''$, $V'' \subset V'''$ hold.

□

Definition 8.20 Let $\theta(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$, σ be a minimal unitary coupling of form (2.1) such that $\theta_\sigma(e^{it}) = \theta(e^{it})$, and $T := T_\sigma$. Let $\Xi(e^{it}) \in \mathcal{K}_r^{++}(\theta)$ and $\{\tilde{V}, V\} \in \tilde{\mathcal{V}}_T \times \mathcal{V}_T$ be the corresponding orthogonal pair. Let $\{\tilde{V}', V'\}$, $\{\tilde{V}'', V''\}$ be the extreme maximal pairs for $\{\tilde{V}, V\}$ in the subset of orthogonal pairs of the set $\tilde{\mathcal{V}}_T \times \mathcal{V}_T$. Then the corresponding, by Theorem 8.10 (part (b)), extensions $\Xi'(e^{it})$,

$\Xi''(e^{it})$ will be called extreme among all maximal extensions in the set $\mathcal{K}^{++}_{r,\max}(\theta)$ that are preceded by $\Xi(e^{it})$. If $\{\omega(\zeta), \lambda(\zeta)\}$, $\{\omega'(\zeta), \lambda'(\zeta)\}$, $\{\omega''(\zeta), \lambda''(\zeta)\}$ are the pairs that correspond, by Theorem 8.4 (part (b)), to the extensions $\Xi(e^{it})$, $\Xi'(e^{it})$, $\Xi''(e^{it})$, respectively, then the pairs $\{\omega'(\zeta), \lambda'(\zeta)\}$, $\{\omega''(\zeta), \lambda''(\zeta)\}$ will be called extreme for the pair $\{\omega(\zeta), \lambda(\zeta)\}$ among all maximal pairs $\{\tilde{\omega}(\zeta), \tilde{\lambda}(\zeta)\}$ in the subset of pairs satisfying condition (8.13) and the conditions $\omega(\zeta) \overset{\cdot}{:} \tilde{\omega}(\zeta)$ (on the right), $\lambda(\zeta) \overset{\cdot}{:} \tilde{\lambda}(\zeta)$ (on the left).

If $\Omega(e^{it}) \in \mathcal{U}^+_r(\theta)$, $\Lambda(e^{it}) \in \mathcal{L}^+_r(\theta)$ is the pair of unidirectional extensions corresponding to $\Xi(e^{it}) \in \mathcal{K}^{++}_r(\theta)$, then the extreme maximal extensions $\Xi'(e^{it}) \in \mathcal{K}^{++}_{r,\max}(\theta)$, $\Xi''(e^{it}) \in \mathcal{K}^{++}_{r,\max}(\theta)$ can be obtained in the following way. According to (8.23), first we construct the extensions

$$\Xi_1(e^{it}) := [\psi_+(e^{it}; \Omega), \Omega(e^{it})], \quad \Xi_2(e^{it}) := \mathrm{col}[\varphi_+(e^{it}; \Lambda), \Lambda(e^{it})].$$

Then, writing $\Xi_1(e^{it})$ and $\Xi_2(e^{it})$ in the form

$$\Xi_1(e^{it}) = \mathrm{col}[\Lambda_{12}(e^{it}), \Lambda'(e^{it})], \; \Xi_2(e^{it}) = [\Omega_{21}(e^{it}), \Omega''(e^{it})],$$

we come to

$$\Xi'(e^{it}) := \mathrm{col}[\varphi_+(e^{it}; \Lambda'), \Lambda'(e^{it})], \quad \Xi''(e^{it}) := [\psi_+(e^{it}; \Omega''), \Omega''(e^{it})].$$

Note that the extensions $\Xi'_0(e^{it})$ and $\Xi''_0(e^{it})$, introduced in Definition 8.15, obviously provide examples of the case of the unique extension in the set $\mathcal{K}^{++}_{r,\max}(\theta)$ which is preceded by a given, possibly non-maximal, extension $\Xi(e^{it})$ from $\mathcal{K}^{++}_r(\theta)$. (Here we mean that the extensions $\Omega^+_0(e^{it})$ and $\Lambda^+_0(e^{it})$ of form (6.8) are considered as $\Xi(e^{it})$ from $\mathcal{K}^{++}_r(\theta)$).

Reformulating Lemma 8.19, we obtain a description of the subset of extensions from $\mathcal{K}^{++}_{r,\max}(\theta)$ preceded by a given extensions $\Xi(e^{it}) \in \mathcal{K}^{++}_r(\theta)$.

Theorem 8.21 *Let $\theta(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$, $\Xi(e^{it}) \in \mathcal{K}^{++}_r(\theta)$ and let $\Xi'(e^{it}) \in \mathcal{K}^{++}_{r,\max}(\theta)$, $\Xi''(e^{it}) \in \mathcal{K}^{++}_{r,\max}(\theta)$ be the extrem maximal extensions for $\Xi(e^{it})$. Let $\{\omega(\zeta), \lambda(\zeta\}$, $\{\omega'(\zeta), \lambda'(\zeta\}$, $\{\omega''(\zeta), \lambda''(\zeta\}$ be the pairs from $\mathcal{J}^+_*(\theta) \times \mathcal{J}^+(\theta)$ that, by Theorem 8.4 (part (b)), correspond to $\Xi(e^{it})$, $\Xi'(e^{it})$, $\Xi''(e^{it})$, respectively.Then a maximal extension $\hat{\Xi}(e^{it}) \in \mathcal{K}^{++}_{r,\max}(\theta)$ is preceded by $\Xi(e^{it})$ iff the pair $\{\hat{\omega}(\zeta), \hat{\lambda}(\zeta\} \in \mathcal{J}^+_*(\theta) \times \mathcal{J}^+(\theta)$ corresponding to $\hat{\Xi}(e^{it})$ satisfies any of the following equivalent conditions:*

(1) *$\omega'(\zeta) \overset{\cdot}{:} \hat{\omega}(\zeta)$ (on the right) and $\hat{\omega}(\zeta) \overset{\cdot}{:} \omega''(\zeta)$ (on the right);*

(2) *$\lambda''(\zeta) \overset{\cdot}{:} \hat{\lambda}(\zeta)$ (on the left) and $\hat{\lambda}(\zeta) \overset{\cdot}{:} \lambda'(\zeta)$ (on the left).*

In conclusion of this section, taking into account Theorem 8.4, 8.10, and 8.21, we give a description of the set $\mathcal{K}^{++}_r(\theta)$ as their direct corollary.

Theorem 8.22 *Let* $\theta(e^{it}) \in CM[\mathfrak{G}, \mathfrak{F}]$ *and let* $\varphi_+(\zeta) \in S[\mathfrak{G}, \mathfrak{K}]$, $\psi_+(\zeta) \in S[\mathfrak{K}_*, \mathfrak{F}]$ *be its defect and* $\star$*-defect functions in the Schur class, respectively.*

(a) A non-trivial extension $\Xi(e^{it}) \in \mathcal{K}_r^{++}(\theta)$ *exists iff* $\varphi_+(\zeta) \not\equiv 0$ *or* $\psi_+(\zeta) \not\equiv 0$.
Let $\Xi(e^{it}) \in CM[\mathfrak{G}^{(1)} \oplus \mathfrak{G}, \mathfrak{F}^{(1)} \oplus \mathfrak{F}]$ *be a function of form* (3.32).

(b) $\Xi(e^{it}) \in \mathcal{K}_r^{++}(\theta)$ *iff there exist* $\star$*-inner and inner functions* $\omega(\zeta) \in S[\mathfrak{K}, \mathfrak{F}^{(1)}]$ *and* $\lambda(\zeta) \in S[\mathfrak{G}^{(1)}, \mathfrak{K}_*]$*, respectively, such that, in addition to the equalities* (8.5)*, the functions* $\omega(\zeta)$ *and* $\lambda(\zeta)$ *admit the representation*

$$\omega(\zeta) = \tilde{\omega}(\zeta)\hat{\omega}(\zeta), \quad \lambda(\zeta) = \hat{\lambda}(\zeta)\tilde{\lambda}(\zeta), \tag{8.24}$$

where $\hat{\omega}(\zeta) \in S[\mathfrak{K}, \hat{\mathfrak{F}}]$, $\hat{\lambda}(\zeta) \in S[\hat{\mathfrak{G}}, \mathfrak{K}_*]$ *is a maximal pair of mutually complementary* $\star$*-inner and inner functions and* $\tilde{\omega}(\zeta) \in S[\hat{\mathfrak{F}}, \mathfrak{F}^{(1)}]$, $\tilde{\lambda}(\zeta) \in S[\mathfrak{G}^{(1)}, \hat{\mathfrak{G}}]$ *is a pair of* $\star$*-inner and inner functions, respectively.*

(c) There exists an isometric (coisometric) extension $\Xi(e^{it}) \in \mathcal{K}_r^{++}(\theta)$ *iff the condition* (8.6) *is satisfied. If the condition* (8.6) *is valid, then all isometric (coisometric) extensions are given by the formulas* (3.32), (8.5)*, and* (8.24)*, where* $\tilde{\omega}(\zeta)$ *and* $\hat{\omega}(\zeta)$ ($\tilde{\lambda}(\zeta)$ *and* $\hat{\lambda}(\zeta)$)*, in addition to the properties listed in part (b), are two-sided inner.*

A unitary extension $\Xi(e^{it}) \in \mathcal{K}_r^{++}(\theta)$ *exists iff both dual equalities* (8.6) *are valid and there exists a pair of two-sided inner functions* $\omega(\zeta) \in S[\mathfrak{K}, \mathfrak{F}^{(1)}]$, $\lambda(\zeta) \in S[\mathfrak{G}^{(1)}, \mathfrak{K}_*]$ *such that condition* (8.13) *is satisfied. In this case, all such extensions are given by the same formulas* (3.32), (8.5)*, and* (8.24)*, where* $\hat{\omega}(\zeta) \in S[\mathfrak{K}, \hat{\mathfrak{F}}]$, $\hat{\lambda}(\zeta) \in S[\hat{\mathfrak{G}}, \mathfrak{K}_*]$ *is a maximal pair of mutually complementary two-sided inner functions and* $\tilde{\omega}(\zeta) \in S[\hat{\mathfrak{F}}, \mathfrak{F}^{(1)}]$, $\tilde{\lambda}(\zeta) \in S[\mathfrak{G}^{(1)}, \hat{\mathfrak{G}}]$ *are two-sided inner functions.*

Remark 8.23 Recall that a description of the subset of the set $\mathcal{K}_r^{++}(\theta)$ consisting of all completely regular extensions is obtained in Theorem 8.4 (*part* (*c*)).

Remark 8.24 Let $\theta(\zeta)$ belongs to $S[\mathfrak{G}, \mathfrak{F}]$. Define its holomorphic regular extension $\Xi(\zeta) \in S[\mathfrak{G}^{(1)} \oplus \mathfrak{G}, \mathfrak{F}^{(1)} \oplus \mathfrak{F}]$ as a Schur function whose boundary value function $\Xi(e^{it}) \in L_+^\infty[\mathfrak{G}^{(1)} \oplus \mathfrak{G}, \mathfrak{F}^{(1)} \oplus \mathfrak{F}]$ is a regular extension of $\theta(e^{it}) \in L_+^\infty[\mathfrak{G}, \mathfrak{F}]$. In this case, we can consider Theorem 8.22 as a description of all such extensions. Moreover, part (c) gives a description of all inner, $\star$-inner and two-sided inner regular up-leftward extensions for the function $\theta(\zeta)$.

References

1. V.M. Adamyan, D.Z. Arov: *On the unitary couplings of semiunitary operators* (Russian), Mat. Issled., Kishinev **I**(2) (1966), 3–64.
2. D.Z. Arov: *On the Darlington method in the theory of dissipative systems* (Russian), Dokl. Akad. Nauk SSSR, Vol. 201, No.3 (1971), 559–562.

3. D.Z. Arov: *Darlington realization of matrix-valued functions* (Russian), Izv. Akad. Nauk SSSR, Ser. Mat. **37**, No. 6 (1973), 1299–1331; English translation: Math. USSR, Izvestija **7** (1973), 1295–1326.
4. D.Z. Arov: *On unitary coupling with losses* (Russian), Funkts. Anal. i ego Prilozhen. **8(4)** (1974), 5–22.
5. D.Z. Arov: *Stable dissipative linear stationary dynamical scattering systems.* (Russian), J. Operator Theory 2 (1979), No. 1, 95–126; English translation with appendices by the author and J.Rovnjak in Operator Theory Adv. Appl., 134, Interpolation Theory, Systems theory and Related Topics (Eds.: D. Alpay, I. Gohberg and V. Vinnikov), (1999), 99–136.
6. D.Z. Arov, M.A. Nudelman: *A criterion of unitary similarity of minimal passive scattering systems with a given transfer function* (Russian), Ukrain. Mat. Zhurn. **52**, No. 6 (2000), 147–156.
7. D.Z. Arov, M.A. Nudelman: *Conditions of similarity of all minimal passive realizations for a given transfer function (scattering matrix or resistence matrix)* (Russian), Mat. Sborn. **193**, No. 6 (2002), 3–24.
8. S.S. Boiko, V.K. Dubovoy: *Unitary couplings and regular factorizations of operator functions in* L^{∞}, Dopovidi NAN Ukr. **1** (1997), 41–44.
9. S.S. Boiko, V.K. Dubovoy: *On some extremal problem connected with the suboperator of the scattering through inner channels of the system*, Dopovidi NAN Ukr. **4** (1997), 8–11.
10. S.S. Boiko, V.K. Dubovoy, A.Ya. Kheifets: *On some extremal problem for contractive harmonic operator functions on the unit disk*, Dopovidi NAN Ukr. **9** (1999), 37–41.
11. S.S. Boiko, V.K. Dubovoy: *Unitary couplings, scattering suboperator and regular factorizatons of bounded operator valued functions*, Math. Nachr. **236**, (2002), 47–89.
12. S.S. Boiko, V.K. Dubovoy: *Unitary couplings, scattering suboperator and regular factorizatons of bounded operator valued functions. Part II*, Math. Nachr. **278**, (2005), 1117–1136.
13. S.S. Boiko, V.K. Dubovoy: *Unitary couplings, scattering suboperator and regular factorizations of bounded operator valued functions. Part III*, Math. Nachr. **291**, (2018), 1941–1978.
14. S.S. Boiko, V.K. Dubovoy: *Defect functions of holomorphic contractive operator functions and the scattering suboperator through the internal channels of a system. Part I*, Complex Anal. Oper. Theory, **5**, (2011), 157–196.
15. S.S. Boiko, V.K. Dubovoy, B. Fritzsche and B. Kirstein: *Contractive operators, the defect functions and the scattering theory* (Russian), Ukrain. Mat. Zhurn. **49**, No. 4 (1997), 481–489.
16. S.S. Boiko, V.K. Dubovoy, B. Fritzsche and B. Kirstein: *Shift operators contained in contractions and pseudocontinuable Schur functions*, Math. Nachr. **278**, (2005), 784–807.
17. S.S. Boiko, V.K. Dubovoy and A.Ya. Kheifets: *Measure Schur complements and spectral functions of unitary operators with respect to different scales*, Operator Theory, System Theory and Related Topics - The Moshe Livšic Anniversary Volume (Eds.: D. Alpay, V. Vinnikov), Operator Theory: Advances and Applications, **123**, (2001), 89–138.
18. S.S. Boiko, V.K. Dubovoy and A.Ya. Kheifets: *Defect functions of holomorphic contractive operator functions and the scattering suboperator through the internal channels of a system. Part II*, Complex Anal. Oper. Theory, **8**, (2014), 991–1036.
19. M.S. Brodskii: *Unitary operator colligations and their characteristic functions* (Russian), Uspekhi Mat. Nauk **33**, No. 4 (202) (1978), 141–168; English translation: Russian Math. Surveys **33**, No. 4 (1987), 159–191.
20. S. Darlington: *Synthesis of reactance 4-poles which produce prescribed insertion loss characteristics including special applications to filter design*, J. Math. and Phys., **18** (1939), 257–353.
21. P. De Wilde: *Roomy scattering matrix synthesis*, Technical Report, Berkeley, (1971).
22. R.G. Douglas, J.W. Helton: *Inner dilations of analytic matrix functions and Darlington synthesis*, Acta Sci.Math. (Szeged) **34** (1973), 61–67.
23. V.K. Dubovoy, R.K. Mohammed: *Defect functions of holomorphic contractive matrix functions, regular extensions and open systems*, Math. Nachr. **160** (1993), 69–110.
24. V.K. Dubovoy: *The indefinite metric in the interpolation Schur problem for analytic matrix functions* (Russian), Teor. Funktsii, Funktsional. Anal. i Prilozhen., part VI: **47** (1987), 112–119; English translation: J. Sov. Math. **48**, No. 6 (1990), 701–706.

25. V.K. Dubovoy: *Shift operators contained in contractions, Schur parameters and pseudocontinuable Schur functions*, in: Interpolation, Schur Functions and Moment Problems (Eds.: D. Alpay, I. Gohberg), Oper. Theory Adv. Appl., Vol. 165, Birkhäuser, Basel, 2006, p. 175–250.
26. S.R. Garcia: *Inner matrices and Darlington synthesis*, Methods Funct. Anal. Topo-logy, **11**, No. 1 (2005), 37–47.
27. M.S. Livshits, A.A. Yantsevich: *Operator colligations in Hilbert spaces* (Russian), Har'kov. Gos. University Press, Har'kov, (1971); English translation: American Mathematical Society, Washington, D.C., (1979).
28. M.Rosenblum and J.Rovnyak: *Hardy classes and operator theory*, Oxford Mathematical Monographs, Oxford University Press, New-York, (1985).
29. B. Sz.-Nagy, C. Foias, H. Bercovici, L. Kerchy: *Harmonic analysis of operators on Hilbert space*, Springer Science + Business Media, LLC, (2010).

Free-Homomorphic Relations Induced by Certain Free Semicircular Families

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Dedicated to Prof. Victor Katsnelson

Abstract Starting from a given countable and mutually orthogonal system of projections in Hilbert space, we give a construction of an induced maximal free semicircular noncommutative $*$-probability space $\mathbb{L}_Q$. Harmonic analysis with the use of this new Banach $*$-probability space $\mathbb{L}_Q$ is studied. In particular, we concentrate on studying free-homomorphic relations of free-probabilistic sub-structures of $\mathbb{L}_Q$.

Keywords Free probability · Projections · Weighted-semicircular elements · Semicircular elements · Free-homomorphisms · Free-isomorphisms

Mathematics Subject Classification (2000) 46L40, 46L54, 47L15, 47L30, 47L55

1 Introduction

In this paper, certain *Banach-space operators* acting on *semicircular elements* induced by mutually orthogonal $|\mathbb{Z}|$-many *projections* are constructed-and-considered, where $\mathbb{Z}$ is the set of all *integers*. In particular, we are interested in the cases where such operators preserves (fully, or partially) the (*weighted-*) *semicircular law*(*s*) under $*$-*homomorphisms* induced by certain bijective maps on $\mathbb{Z}$. The main results show not only how such Banach-space operators affect the original free-distributional data on (*weighted-*)*semicircular elements*, but also

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how they induce free-homomorphic sub-structures of Banach $*$-probability space $\mathbb{L}_Q$ generated by the $|\mathbb{Z}|$-many semicircular elements. Roughly speaking, *free-homomorphic relations* are $*$-homomorphic relations preserving free probability. *Free-isomorphic relations* are naturally defined by bijective-free-homomorphic relations. The main results of this paper characterize some free-homomorphic, or free-isomorphic relations in $\mathbb{L}_Q$.

1.1 Motivations

There are various different approaches to construct semicircular elements (e.g., [5, 8, 10, 26, 27]). They fall naturally in two groups: (1) analysis on *measure spaces* (i.e., an approach from classical statistics), and approaches used in (2) *topological $*$-probability spaces* (including that of *C^*-probability spaces*, or *W^*-probability spaces*, or *Banach $*$-probability spaces*, etc. e.g., see [16–19, 25]).

Our present approach is different. We offer a new construction of semicircular elements. It is motivated by a systematic study of weighted-semicircular elements in certain Banach $*$-probability spaces from [9] and [12], and is based on our use of a new analysis on the *p-adic number fields* $\mathbb{Q}_p$, for *primes* p. In conclusion, our present construction is different from, and independent of, those in earlier papers. Also, see [10, 11] for additional discussion.

In [9, 12], we studied weighted-semicircular elements induced by measurable functions on p-adic number fields $\mathbb{Q}_p$, and those from a *free product Banach $*$-algebra* generated by the weighted-semicircular elements. The authors applied number-theoretic results (e.g., [14, 23]), and free-probabilistic techniques (e.g., [6–8, 17, 18, 20]) to consider free-probabilistic models of [12], and they realized there are well-defined semicircular-like elements, called *weighted-semicircular elements*. Interestingly, these operators automatically generate corresponding semicircular elements. In [9], the constructions and results of [12] are extended under *free product* over *primes*.

Motivated by Cho [9] and Cho and Jorgensen [12], the first-named author considered the similar constructions of (weighted-)semicircular elements from "arbitrary" C^*-probability spaces containing $|\mathbb{Z}|$-many mutually *orthogonal projections* in [10]. The main results of [10] show that whenever one can have mutually orthogonal $|\mathbb{Z}|$-many projections in a C^*-probability space, the corresponding weighted-semicircular elements whose weights are characterized by the free-distributional data of the projections; moreover, under suitable (additional) conditions, semicircular elements are well-defined (see short Sects. 3–5, below). As an application of [10], the authors consider free, semicircular elements induced by orthogonal projections acting on infinite-dimensional *separable Hilbert spaces* in [11].

In this paper, we are interested in certain *adjointable Banach-space operators* (in the sense of [13]) acting on weighted-semicircular elements of [10]. We focus on studying how such operators preserve free probability on inner sub-structures of the Banach $*$-algebra $\mathbb{L}_Q$ generated by our semicircular elements.

1.2 Overview

Section 2 begins with a brief review of background material on free probability as it will be needed inside the paper. In Sects. 3–5, we prove lemmas for families of weighted-semicircular elements. We discuss induction of semicircular elements, induction from prescribed mutually orthogonal $|\mathbb{Z}|$-many projections. For related results in detail, see [10, 11].

In Sect. 6, we construct a suitable free-probabilistic, operator-algebraic structure $\mathbb{L}_Q$ generated by our (weighted-)semicircular elements. And free-distributional data on $\mathbb{L}_Q$ are studied.

In Sect. 7, certain adjointable Banach-space operators acting on $\mathbb{L}_Q$ are constructed-and-studied. In particular, shifting processes on $\mathbb{Z}$ are defined in Sect. 7.1, and the corresponding $*$-isomorphisms on $\mathbb{L}_Q$ are introduced in Sect. 7.2. We realize that the collection of such $*$-isomorphisms forms a subgroup $\mathfrak{B}$ of the *automorphism group* $Aut(\mathbb{L}_Q)$ *of* $\mathbb{L}_Q$. The structure theorem of this group $\mathfrak{B}$ is provided in Sect. 7.2: $\mathfrak{B}$ is group-isomorphic to the infinite cyclic abelian group $(\mathbb{Z}, +)$. We then study how the group $\mathfrak{B}$ generate our Banach-space operators (in the sense of [13]) on $\mathbb{L}_Q$, and how they affects the free-probabilistic information on $\mathbb{L}_Q$ in Sect. 7.3.

In Sect. 8, by using the group $\mathfrak{B}$ of Sect. 7, we construct a *noncommutative monoid* $\mathfrak{B}(\mathbb{Z})$ consisting of certain $*$-homomorphisms induced by *restrictions* of the $*$-isomorphisms of $\mathfrak{B}$, contained in the *homomorphism semigroup* $Hom(\mathbb{L}_Q)$. The algebraic properties of $\mathfrak{B}(\mathbb{Z})$ is studied in details.

In Sect. 9, the free-homomorphic relations in $\mathbb{L}_Q$ are considered. We study how to construct suitable free-homomorphisms ($*$-homomorphisms preserving free-distributional data) among the subalgebras of $\mathbb{L}_Q$ from the monoid $\mathfrak{B}(\mathbb{Z})$, in Sect. 9.1. The free-homomorphic relations among the subalgebras of $\mathbb{L}_Q$, generated by "finitely" many, free, semicircular elements, are studied in Sect. 9.2 up to free-homomorphisms of Sect. 9.1. In Sect. 9.3, the free-isomorphic relations among the subalgebras of $\mathbb{L}_Q$, generated by "infinitely" many, free, semicircular elements, are characterized. Finally, we discuss interesting open problems in Sect. 9.4.

2 Preliminaries

For a review of relevant and fundamental tools from free probability theory, analytic-and-combinatorial, we refer to [21, 25] (and the cited papers therein). In rough outline, free probability serves as the noncommutative operator-algebraic version of classical *measure theory* and *statistics*. We get a new noncommutative, i.e., operator-algebraic, framework, which parallel notions, commutative vs. noncommutative. In the better known commutative world from statistics (or measure theory), we deal with measure spaces and function algebras. In passing to the free probability, the commutative concept of independence if replaced by what is called freeness. Similarly, measures on sample spaces are replaced with linear functionals defined on noncommutative $*$-algebras. Free probability has many applications, not only in pure mathematics (e.g., [6–8, 16, 18, 19, 22, 24]), but also in related fields, especially in quantum physics (e.g., [2–4, 9–12, 17, 20, 26, 27]).

In particular, we use combinatorial free probability of *Speicher* (e.g., [21]). *Free moments* and *free cumulants* of operators will be computed without detailed definitions. Also, *free product* (in the sense of [21, 25]) will be used without precise introduction.

3 Fundamental Settings

In this section, we establish basic settings of our works. Let (B, φ) be a topological $*$-probability space (a C^*-probability space, or a W^*-probability space, or a Banach $*$-probability space, etc), where B is a topological $*$-algebra (a C^*-algebra, resp., a W^*-algebra, resp., a Banach $*$-algebra, etc), and φ is a (bounded or unbounded) linear functional on B.

An operator a of B is said to be a *free random variable*, whenever it is regarded as an element of (B, φ). As usual in *operator theory*, an operator a is said to be *self-adjoint*, if $a^* = a$ in B, where a^* is the *adjoint of* a (e.g., [15]).

Definition 3.1 A self-adjoint free random variable a is said to be weighted-semicircular in (B, φ) with its weight $t_0 \in \mathbb{C}^{\times} = \mathbb{C} \setminus \{0\}$ (or, in short, t_0-semicircular), if a satisfies the free cumulant computations,

$$k_n(a, \ldots, a) = \begin{cases} k_2(a, a) = t_0 & \text{if } n = 2 \\ 0 & \text{otherwise,} \end{cases} \tag{3.1}$$

for all $n \in \mathbb{N}$, where $k_n(\ldots)$ is the free cumulant on B in terms of φ under the Möbius inversion of [21].

If $t_0 = 1$ in (3.1), the 1-semicircular element a is said to be semicircular in (B, φ), i.e., a is semicircular in (B, φ), if a satisfies

$$k_n(a, \ldots, a) = \begin{cases} 1 & \text{if } n = 2 \\ 0 & \text{otherwise,} \end{cases} \tag{3.2}$$

for all $n \in \mathbb{N}$.

Remark 3.1 There is a number of differences and intriguing similarities between properties of free random variables from operator algebra theory on the one hand, and on the other, the more familiar random variables in commutative case. For example, in the free case, the distribution is the semicircular law, while the "preferred" law in the commutative case is the Gaussian distribution. This distinction is stressed in [1]; in [1], a comparative study of the two theories with emphasis on correspondences between free probability on the one hand, vs. Gaussian processes on the other, is provided. Our present paper offers a new approach to the calculus of a class of free random variables, arising naturally in free probability theory (e.g., see [2, 3, 22]).

By the Möbius inversion of [21], one can characterize the weighted-semicircularity (3.1) as follows: a self-adjoint operator a is t_0-semicircular in $(B,\ \varphi)$, if and only if

$$\varphi(a^n) = \omega_n \left(t_0^{\frac{n}{2}} c_{\frac{n}{2}} \right), \tag{3.3}$$

where

$$\omega_n \overset{def}{=} \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

for all $n \in \mathbb{N}$, and c_k are the k-th Catalan numbers,

$$c_k \overset{def}{=} \frac{1}{k+1} \binom{2k}{k} = \frac{(2k)!}{k!(k+1)!},$$

for all $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Similarly, a self-adjoint free random variable a is semicircular in $(B,\ \varphi)$, if and only if a is 1-semicircular in $(B,\ \varphi)$, if and only if

$$\varphi(a^n) = \omega_n c_{\frac{n}{2}}, \tag{3.4}$$

by (3.2) and (3.3), for all $n \in \mathbb{N}$, where ω_n are in the sense of (3.3).

So, we use the t_0-semicircularity (3.1) (or the semicircularity (3.2)) and its characterization (3.3) (resp., (3.4)) alternatively from below.

If a is a self-adjoint free random variable in $(B,\ \varphi)$, then the sequences consisting of

$$\text{the } \textit{free moments } \left(\varphi(a^n)\right)_{n=1}^{\infty},$$

and

$$\text{the } \textit{free cumulants } \left(k_n(a, \ldots, a)\right)_{n=1}^{\infty}$$

provide equivalent free-distributional data of a in $(B,\ \varphi)$, characterizing the *free distribution of a* (e.g., [21]).

In the rest of this section, we fix a C^*-probability space (A, ψ), and assume that there are $|\mathbb{Z}|$-many projections $\{q_j\}_{j\in\mathbb{Z}}$ in the C^*-algebra A, i.e., the operators q_j satisfy

$$q_j^* = q_j = q_j^2 \text{ in } A,$$

for all $j \in \mathbb{Z}$. Assume further that these projections $\{q_j\}_{j\in\mathbb{Z}}$ are *mutually orthogonal* in A, in the sense that:

$$q_i q_j = \delta_{i,j} q_j \text{ in } A, \text{ for all } i, j \in \mathbb{Z}, \tag{3.5}$$

where δ is the *Kronecker delta*.

Now, we fix the family $\{q_j\}_{j\in\mathbb{Z}}$ of mutually orthogonal projections (3.5) of A, and we denote it by $\mathbf{Q}$, i.e.,

$$\mathbf{Q} = \{q_j : q_j \text{ satisfy (3.5)}\}_{j\in\mathbb{Z}} \text{ in } A. \tag{3.6}$$

Remark 3.2 One can have such a C^*-algebraic structure A containing a family $\mathbf{Q}$ of (3.6), naturally, or artificially. Clearly, in the settings of [9, 12], one can naturally take such structures.

Suppose there is a C^*-algebra A_0 containing a family $\mathbf{Q}_0 = \{q_1, \ldots, q_N\}$ of mutually orthogonal N-many projections $q_1, \ldots, q_N$, for $N \in \mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$. Then, under suitable direct product, or tensor product, or free product of copies of A_0 under product topology, one can construct a C^*-algebra A containing a family $\mathbf{Q}$ with $|\mathbb{Z}|$-many mutually orthogonal projections, where $\mathbf{Q}_0$ is contained in $\mathbf{Q}$, and every projection of $\mathbf{Q}$ is unitarily equivalent to a projection of $\mathbf{Q}_0$ in A (e.g., see [10, 11]).

Let Q be the C^*-subalgebra of A generated by the family $\mathbf{Q}$ of (3.6),

$$Q \overset{def}{=} C^*(\mathbf{Q}) \subseteq A. \tag{3.7}$$

Proposition 3.1 *Let Q be the C^*-subalgebra (3.7) of a fixed C^*-algebra A. Then*

$$Q \overset{*\text{-iso}}{=} \underset{j\in\mathbb{Z}}{\oplus} \left(\mathbb{C} \cdot q_j\right) \overset{*\text{-iso}}{=} \mathbb{C}^{\oplus|\mathbb{Z}|}, \tag{3.8}$$

in A.

Proof The proof of (3.8) is done by the orthogonality (3.5) of the generator set $\mathbf{Q}$ of (3.6) on Q. □

Define now linear functionals ψ_j on the C^*-algebra Q of (3.7) by

$$\psi_j(q_i) = \delta_{ij}\psi(q_j), \text{ for all } i \in \mathbb{Z}, \tag{3.9}$$

for all $j \in \mathbb{Z}$, where ψ is the linear functional of the fixed C^*-probability space (A, ψ). The linear functionals $\{\psi_j\}_{j \in \mathbb{Z}}$ of (3.9) are well-defined on Q by the structure theorem (3.8).

Assumption Let (A, ψ) be a fixed C^*-probability space, and let Q be the C^*-subalgebra (3.7) of A. In the rest of this paper, we further assume that

$$\psi(q_j) \in \mathbb{C}^{\times}, \text{ for all } j \in \mathbb{Z}.$$

□

Definition 3.2 The C^*-probability spaces $\left(Q, \psi_j\right)$ are called the j-th C^*-probability spaces of Q in a given C^*-probability space (A, ψ), where Q is in the sense of (3.7), and ψ_j are the linear functionals of (3.9), for all $j \in \mathbb{Z}$.

Now, let's define bounded linear transformations c and a acting on the C^*-algebra Q, by linear morphisms satisfying

$$c\left(q_j\right) = q_{j+1}, \text{ and } a\left(q_j\right) = q_{j-1}, \tag{3.10}$$

for all $j \in \mathbb{Z}$. Then c and a are well-defined bounded linear operators "on Q." One can understand they are *Banach-space operators* in the *operator space* $B(Q)$ consisting of all bounded linear transformations acting on Q, by regarding Q as a Banach space equipped with its C^*-norm (e.g., [13]).

Definition 3.3 We call these Banach-space operators c and a of (3.10), the *creation*, respectively, the *annihilation* on Q.

The creation c and the annihilation a on Q are indeed well-defined by the structure theorem (3.8) of Q. Define now a new Banach-space operator l on Q by

$$l = c + a \in B(Q). \tag{3.11}$$

Definition 3.4 We call the Banach-space operator $l \in B(Q)$ of (3.11), the *radial operator on* Q.

Now, define a closed subspace $\mathfrak{L}$ of the operator space $B(Q)$ by

$$\mathfrak{L} \overset{def}{=} \overline{\mathbb{C}[\{l\}]}^{\|.\|}, \tag{3.12}$$

generated by the radial operator l of (3.11), where the operator norm $\|.\|$ on the operator space $B(Q)$ is defined to be

$$\|T\| = \sup\{\|Tq\|_Q : \|q\|_Q = 1\},$$

for all $T \in B(Q)$, where $\|.\|_Q$ is the C^**-norm on* Q (inherited from the C^*-norm on A), and where $\overline{X}^{\|.\|}$ mean the *operator-norm closures* of subsets X of the *operator* *space* $B(Q)$ (e.g., [13]). It is not difficult to check that, by the definition (3.12), this subspace $\mathfrak{L}$ forms an algebra in the vector space $B(Q)$, i.e., it forms a Banach algebra embedded in the topological vector space $B(Q)$.

On this Banach algebra $\mathfrak{L}$ of (3.12), define a unary operation $(*)$ by

$$\left(\sum_{n=0}^{\infty} t_n l^n\right)^* = \sum_{n=0}^{\infty} \overline{t_n} l^n \text{ in } \mathfrak{L}, \tag{3.13}$$

where $\overline{z}$ mean the *conjugates of* $z \in \mathbb{C}$.

Then this operation (3.13) becomes a well-defined *adjoint on* the Banach algebra $\mathfrak{L}$ of (3.12) (e.g., [10, 15]), and hence, every element of $\mathfrak{L}$ is *adjointable* in $B(Q)$ in the sense of [13]. So, the algebra $\mathfrak{L}$ forms a *Banach* $*$*-algebra* in $B(Q)$ with the adjoint (3.13).

Definition 3.5 We call the Banach $*$-algebra $\mathfrak{L}$ of (3.12), the *radial* (*Banach* $*$-) *algebra on* Q.

Now, let $\mathfrak{L}$ be the radial algebra on Q. Define the *tensor product Banach* $*$*-algebra* $\mathfrak{L}_Q$ by

$$\mathfrak{L}_Q = \mathfrak{L} \otimes_{\mathbb{C}} Q, \tag{3.14}$$

where $\otimes_{\mathbb{C}}$ is the tensor product of Banach $*$-algebras. Since $\mathfrak{L}$ is a Banach $*$-algebra, and Q is a C^*-algebra, the tensor product $\mathfrak{L}_Q$ of (3.14) is a well-defined Banach $*$-algebra under product topology.

Definition 3.6 We call the tensor product Banach $*$-algebra $\mathfrak{L}_Q$ of (3.14), the radial projection (Banach $*$-)algebra on Q.

4 Weighted-Semicircular Elements Induced by Q

We here construct weighted-semicircular elements induced by the family **Q** of mutually orthogonal projections inducing the radial projection algebra $\mathfrak{L}_Q$ of (3.14). Let $(Q, \ \psi_j)$ be the j-th C^*-probability spaces of Q in $(A, \ \psi)$, where ψ_j are in the sense of (3.9), for all $j \in \mathbb{Z}$.

Remark that, if u_j are the generating operators of $\mathfrak{L}_Q$,

$$u_j \overset{def}{=} l \otimes q_j \in \mathfrak{L}_Q, \text{ for all } j \in \mathbb{Z}, \tag{4.1}$$

then

$$u_j^n = \left(l \otimes q_j\right)^n = l^n \otimes q_j, \text{ for all } n \in \mathbb{N},$$

since $q_j^k = q_j$, for all $k \in \mathbb{N}$, with axiomatization:

$$u_j^0 \overset{\text{axiom}}{=} l^0 \otimes q_j = 1_Q \otimes q_j,$$

where 1_Q is the identity operator of $B(Q)$, satisfying

$$1_Q(T) = T, \text{ for all } T \in Q,$$

for all $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, for $j \in \mathbb{Z}$.

Then one can construct a linear functional φ_j on $\mathfrak{L}_Q$ by a linear morphism satisfying that

$$\varphi_j\left((l \otimes q_i)^n\right) \overset{def}{=} \psi_j\left(l^n(q_i)\right), \tag{4.2}$$

for all $n \in \mathbb{N}_0$, for all $i, j \in \mathbb{Z}$.

These linear functionals φ_j of (4.2) are well-defined by (3.8), (3.12) and (3.14), for all $j \in \mathbb{Z}$.

Definition 4.1 We call the Banach $*$-probability spaces

$$\left(\mathfrak{L}_Q, \varphi_j\right), \text{ for all } j \in \mathbb{Z}, \tag{4.3}$$

the j-th (*Banach-*)$*$*-probability spaces on* Q.

Observe that, if c and a are the creation, respectively, the annihilation on Q in the sense of (3.10), then

$$ca = 1_Q = ac. \tag{4.4}$$

Indeed, for any generators $q_j \in \mathbf{Q}$ of Q,

$$ca\left(q_j\right) = c\left(a\left(q_j\right)\right) = c\left(q_{j-1}\right) = q_{j-1+1} = q_j,$$

and

$$ac\left(q_j\right) = a\left(c\left(q_j\right)\right) = a\left(q_{j+1}\right) = q_{j+1-1} = q_j,$$

for all $j \in \mathbb{Z}$. More generally, one has

$$c^n a^n = 1_Q = a^n c^n, \text{ for all } n \in \mathbb{N}, \tag{4.4'}$$

and

$$c^{n_1} a^{n_2} = a^{n_2} c^{n_1}, \text{ for all } n_1, n_2 \in \mathbb{N},$$

by (4.4).

Thus, one obtains that

$$l^n = (c + a)^n = \sum_{k=0}^{n} \binom{n}{k} c^k a^{n-k}, \tag{4.5}$$

for all $n \in \mathbb{N}$, by (4.4′), where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \forall k \leq n \in \mathbb{N}_0.$$

Note that, for any $n \in \mathbb{N}$,

$$l^{2n-1} = \sum_{k=0}^{2n-1} \binom{2n-1}{k} c^k a^{n-k}, \tag{4.6}$$

by (4.5). So, the formula (4.6) does not contain 1_Q-terms by (4.4) and (4.4′).

Note also that, for any $n \in \mathbb{N}$, one has

$$l^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} c^k a^{n-k} = \binom{2n}{n} c^n a^n + [\text{Rest terms}], \tag{4.7}$$

by (4.5). So, l^{2n} contains $\binom{2n}{n}$-many 1_Q-terms by (4.7).

Proposition 4.1 *Let l be the radial operator (3.11) on Q. Then, for any $n \in \mathbb{N}$,*

$$l^{2n-1} \text{ does not contain } 1_Q\text{-terms in } \mathfrak{L}, \tag{4.8}$$

$$l^{2n} \text{ contains } \binom{2n}{n} \cdot 1_Q \text{ in } \mathfrak{L}. \tag{4.9}$$

Proof The statements (4.8) and (4.9) are proven by (4.6), respectively, by (4.7). □

Remark that, since

$$u_j^n = (l \otimes q_j)^n = l^n \otimes q_j,$$

one has

$$\varphi_j\left(u_j^{2n-1}\right) = \psi_j\left(l^{2n-1}\left(q_j\right)\right) = 0, \tag{4.10}$$

for all $n \in \mathbb{N}$, by (3.9) and (4.8).

Similarly, we have

$$\varphi_j\left(u_j^{2n}\right) = \psi_j\left(l^{2n}\left(q_j\right)\right) = \psi_j\left(\binom{2n}{n} q_j + [\text{Rest terms}]\right)$$

by (4.7)

$$= \binom{2n}{n} \psi_j\left(q_j\right) = \binom{2n}{n} \psi\left(q_j\right),$$

by (3.9) and (4.9). i.e.,

$$\varphi_j\left(u_j^{2n}\right) = \binom{2n}{n}\psi\left(q_j\right), \tag{4.11}$$

for all $n \in \mathbb{N}$.

Thus, one obtains the following free-distributional data on the j-th probability space $\left(\mathfrak{L}_Q,\ \varphi_j\right)$, for $j \in \mathbb{Z}$.

Theorem 4.1 *Fix $j \in \mathbb{Z}$, and let $u_k = l \otimes q_k$ be the k-th generating operators of the j-th $*$-probability space $(\mathfrak{L}_Q, \varphi_j)$, for all $k \in \mathbb{Z}$. Then*

$$\varphi_j\left(u_k^n\right) = \delta_{j,k}\omega_n\left(\left(\frac{n}{2}+1\right)\psi\left(q_j\right)\right)c_{\frac{n}{2}}, \tag{4.12}$$

where ω_n are in the sense of (3.3) for all $n \in \mathbb{N}$, and c_k are the k-th Catalan numbers for all $k \in \mathbb{N}_0$.

Proof First, take the j-th generating operator u_j in the j-th $*$-probability space $\left(\mathfrak{L}_Q,\ \varphi_j\right)$. By (4.10) and (4.11), one can get that:

$$\varphi_j\left(u_j^{2n-1}\right) = 0,$$

and

$$\varphi_j\left(u_j^{2n}\right) = \binom{2n}{n}\psi\left(q_j\right) = \left(\frac{n+1}{n+1}\right)\binom{2n}{n}\psi\left(q_j\right)$$
$$= \left((n+1)\psi\left(q_j\right)\right)c_n,$$

for all $n \in \mathbb{N}$.

Assume now that $k \neq j$ in $\mathbb{Z}$. Then, by the definition (4.2) of φ_j (and by the definition (3.9) of ψ_j),

$$\varphi_j\left(u_k^n\right) = 0, \text{ for all } n \in \mathbb{N}.$$

Therefore, the formula (4.12) holds. □

Motivated by (4.12), we define a linear morphism,

$$E_{j,Q} : \mathfrak{L}_Q \to \mathfrak{L}_Q$$

by a bounded linear transformation satisfying

$$E_{j,Q}\left(u_i^n\right) \overset{def}{=} \begin{cases} \frac{\psi\left(q_j\right)^{n-1}}{\left(\left[\frac{n}{2}\right]+1\right)}\, u_j^n & \text{if } i = j \\ \\ 0_{\mathfrak{L}_Q}, \text{ the zero operator of } \mathfrak{L}_Q & \text{otherwise,} \end{cases} \tag{4.13}$$

for all $n \in \mathbb{N}$, i, $j \in \mathbb{Z}$, where $[\frac{n}{2}]$ mean the *minimal integers* greater than or equal to $\frac{n}{2}$, for example,

$$\left[\frac{3}{2}\right] = 2 = \left[\frac{4}{2}\right].$$

The linear transformations $E_{j,Q}$ of (4.13) are well-defined bounded linear transformations on $\mathfrak{L}_Q$, because of the cyclicity (3.12) of the tensor factor $\mathfrak{L}$ of $\mathfrak{L}_Q$, and the structure theorem (3.8) of the other tensor factor Q of $\mathfrak{L}_Q$, for all $j \in \mathbb{Z}$.

Define now new linear functionals τ_j on $\mathfrak{L}_Q$ by

$$\tau_j \overset{def}{=} \varphi_j \circ E_{j,Q} \text{ on } \mathfrak{L}_Q, \text{ for all } j \in \mathbb{Z}, \tag{4.14}$$

where φ_j are in the sense of (4.2), and $E_{j,Q}$ are in the sense of (4.13).

Definition 4.2 The well-defined Banach $*$-probability spaces

$$\mathfrak{L}_Q(j) \overset{\text{denote}}{=} \left(\mathfrak{L}_Q, \ \tau_j\right) \tag{4.15}$$

are called the j-th filtered (Banach-)$*$-probability spaces of $\mathfrak{L}_Q$, where τ_j are the linear functionals (4.14) on the radial projection algebra $\mathfrak{L}_Q$, for all $j \in \mathbb{Z}$.

On the j-th filtered $*$-probability space $\mathfrak{L}_Q(j)$ of (4.15), One can get that

$$\begin{aligned}
\tau_j\left(u_j^n\right) &= \varphi_j\left(E_{j,Q}\left(u_j^n\right)\right) \\
&= \varphi_j\left(\frac{\psi(q_j)^{n-1}}{([\frac{n}{2}]+1)}\left(u_j^n\right)\right) = \frac{\psi(q_j)^{n-1}}{([\frac{n}{2}]+1)}\varphi_j\left(u_j^n\right) \\
&= \frac{\psi(q_j)^{n-1}}{([\frac{n}{2}]+1)}\omega_n\left(\left(\frac{n}{2}+1\right)\psi\left(q_j\right)\right)c_{\frac{n}{2}},
\end{aligned}$$

by (4.12), i.e.,

$$\tau_j\left(u_j^n\right) = \omega_n \psi(q_j)^n c_{\frac{n}{2}}, \tag{4.16}$$

for all $n \in \mathbb{N}$, for $j \in \mathbb{Z}$, where ω_n are in the sense of (3.3).

Lemma 4.1 *Let $\mathfrak{L}_Q(j) = (\mathfrak{L}_Q, \ \tau_j)$ be the j-th filtered $*$-probability space of $\mathfrak{L}_Q$, for an arbitrarily fixed $j \in \mathbb{Z}$. Then*

$$\tau_j\left(u_i^n\right) = \delta_{j,i}\left(\omega_n \psi(q_j)^n c_{\frac{n}{2}}\right), \tag{4.17}$$

for all $n \in \mathbb{N}$, for all $i \in \mathbb{Z}$.

Proof If $i = j$ in $\mathbb{Z}$, then the free-momental data (4.17) holds true by (4.16), for all $n \in \mathbb{N}$.

If $i \neq j$ in $\mathbb{Z}$, then, by the very definition (4.13) of $E_{j,Q}$, and also by the definition (4.2) of φ_j,

$$\tau_j \left(u_i^n\right) = 0, \text{ for all } n \in \mathbb{N}.$$

Therefore, the free-distributional data (4.17) holds true, for all $i \in \mathbb{Z}$. □

The following theorem is proven by the above free-distributional data (4.17) in terms of the weighted-semicircularity characterization (3.3) of the weighted-semicircularity (3.1).

Theorem 4.2 *Let $\mathfrak{L}_Q(j)$ be the j-th filtered $*$-probability space $\left(\mathfrak{L}_Q,\ \tau_j\right)$ of $\mathfrak{L}_Q$, for $j \in \mathbb{Z}$, and let $u_j = l \otimes q_j$ be the "j-th" generating operator of $\mathfrak{L}_Q$. Then u_j is $\psi(q_j)^2$-semicircular in $\mathfrak{L}_Q(j)$.*

Proof First of all, the generating operators u_i are self-adjoint in $\mathfrak{L}_Q$, for all $i \in \mathbb{Z}$. Indeed,

$$u_i^* = (l \otimes q_i)^* = l \otimes q_i = u_i \text{ in } \mathfrak{L}_Q,$$

for all $i \in \mathbb{Z}$, by (3.13).

Let's fix $j \in \mathbb{Z}$, and let $u_j = l \otimes q_j$ be the j-th generating operator of the j-th filtered $*$-probability space $\mathfrak{L}_Q(j)$. Then, by (4.17), we have that

$$\tau_j \left(u_j^n\right) = \omega_n \left(\psi\left(q_j\right)^2\right)^{\frac{n}{2}} c_{\frac{n}{2}},$$

for all $n \in \mathbb{N}$, where c_k are the k-th Catalan numbers, for all $k \in \mathbb{N}_0$. Therefore, this self-adjoint element u_j is $\psi(q_j)^2$-semicircular in $\mathfrak{L}_Q(j)$, by (3.3). □

The above theorem shows that, for any $j \in \mathbb{Z}$, the j-th generating operator u_j is $\psi(q_j)^2$-semicircular in the j-th filtered $*$-probability space $\mathfrak{L}_Q(j)$. Meanwhile, also by (4.17), one can verify the following result, too.

Theorem 4.3 *Let $u_i = l \otimes u_i$ be the i-th generating operators of the j-th filtered $*$-probability space $\mathfrak{L}_Q(j)$, for all $j \neq i \in \mathbb{Z}$. Then u_i have the zero free distribution in $\mathfrak{L}_Q(j)$.*

Proof Let $\mathfrak{L}_Q(j)$ be the j-th filtered $*$-probability space for a fixed $j \in \mathbb{Z}$, and assume $i \neq j$ in $\mathbb{Z}$. Consider the i-th generating operators u_i of $\mathfrak{L}_Q(j)$. It is shown already that u_i are self-adjoint in $\mathfrak{L}_Q$, and hence, the free distributions of u_i are completely characterized by the free-momental sequences,

$$\left(\tau_j(u_i^n)\right)_{n=1}^{\infty} = (0,\ 0,\ 0,\ \dots),$$

the zero sequence, by (4.17). It guarantees that the free distributions of $u_i \in \mathfrak{L}_Q(j)$ are the zero free distribution, for all $j \neq i \in \mathbb{Z}$. □

The above two theorems characterize the free-probabilistic information of the generators $\{u_i\}_{i \in \mathbb{Z}}$ of our j-th filtered $*$-probability space $\mathfrak{L}_Q(j)$, for $j \in \mathbb{Z}$. From below, we focus on "non-zero" free-distributional data on $\mathfrak{L}_Q(j)$, for $j \in \mathbb{Z}$.

Note that, by the Möbius inversion of [21], if u_i are the i-th generating operators of the j-th filtered $*$-probability space $\mathfrak{L}_Q(j)$, then

$$k_n^j\,(u_i, \ldots, u_i) = \begin{cases} \delta_{j,i}\psi\left(q_j\right)^2 & \text{if } n = 2 \\ 0 & \text{otherwise,} \end{cases} \tag{4.18}$$

for all $n \in \mathbb{N}$, and $i \in \mathbb{Z}$, by (4.17), where $k_n^j(\ldots)$ is the free cumulant on $\mathfrak{L}_Q$ with respect to the linear functional τ_j, for $j \in \mathbb{Z}$.

5 Semicircular Elements Induced by Q

As in Sect. 4, let $\mathfrak{L}_Q(j)$ be the j-th filtered $*$-probability space of Q for $j \in \mathbb{Z}$. Then the j-th generating operator $u_j = l \otimes q_j$ of $\mathfrak{L}_Q$ is $\psi(q_j)^2$-semicircular in $\mathfrak{L}_Q(j)$, satisfying that

$$\tau_j\left(u_j^n\right) = \omega_n \psi(q_j)^n c_{\frac{n}{2}}, \tag{5.1}$$

equivalently,

$$k_n^j\left(u_j, \ldots, u_j\right) = \begin{cases} \psi(q_j)^2 & \text{if } n = 2 \\ 0 & \text{otherwise,} \end{cases}$$

for all $n \in \mathbb{N}$, by (4.17) and (4.18).

By the weighted-semicircularity (5.1), one may/can obtain the following semicircular element U_j of $\mathfrak{L}_Q(j)$ (under an additional condition),

$$U_j \overset{def}{=} \frac{1}{\psi(q_j)}\, u_j \in \mathfrak{L}_Q(j), \tag{5.2}$$

for $j \in \mathbb{Z}$. Recall that we assumed $\psi(q_k) \in \mathbb{C}^{\times}$, for all $k \in \mathbb{Z}$, and hence, the above operator U_j of (5.2) is well-defined in $\mathfrak{L}_Q(j)$.

Theorem 5.1 *Let $U_j = \frac{1}{\psi(q_j)}\, u_j$ be a free random variable (5.2) of $\mathfrak{L}_Q(j)$, for $j \in \mathbb{Z}$, where u_j is the j-th generating operator of $\mathfrak{L}_Q$. If*

$$\psi(q_j) \in \mathbb{R}^{\times} = \mathbb{R} \setminus \{0\} \textit{ in } \mathbb{C}^{\times},$$

then U_j is semicircular in $\mathfrak{L}_Q(j)$.

Proof Fix $j \in \mathbb{Z}$, and assume $\psi(q_j) \in \mathbb{R}^{\times}$ in $\mathbb{C}^{\times}$. Then

$$U_j^* = \left(\frac{1}{\psi(q_j)} u_j\right)^* = U_j,$$

by the self-adjointness of u_j in $\mathfrak{L}_Q$. Thus,

$$\begin{aligned} \tau_j\left(U_j^n\right) &= \left(\tfrac{1}{\psi(q_j)}\right)^n \tau_j\left(u_j^n\right) \\ &= \left(\tfrac{1}{\psi(q_j)^n}\right)\left(\omega_n \psi(q_j)^n c_{\frac{n}{2}}\right) = \omega_n c_{\frac{n}{2}}, \end{aligned} \tag{5.3}$$

for all $n \in \mathbb{N}$.

So, by (5.3), and by the free-momental characterization (3.4) of the semicircularity (3.2), the self-adjoint operator U_j is semicircular in $\mathfrak{L}_Q(j)$. □

The above theorem shows that, from our $\psi(q_j)^2$-semicircular elements $u_j = l \otimes q_j$, the corresponding semicircular elements $U_j = \frac{1}{\psi(q_j)} u_j$ are canonically obtained in $\mathfrak{L}_Q(j)$, whenever $\psi(q_j) \in \mathbb{R}^{\times}$ in $\mathbb{C}$, for $j \in \mathbb{Z}$.

Assumption 5.1 (In Short, A 5.1, from Below) For convenience, we will assume that

$$\psi(q_j) \in \mathbb{R}^{\times} \text{ in } \mathbb{C}, \text{ for } q_j \in \mathbf{Q},$$

for all $j \in \mathbb{Z}$. □

6 The Free Filterization $\underset{j \in \mathbb{Z}}{\star} \mathfrak{L}_Q(j)$ of Q

Let (A, ψ) be a fixed C^*-probability space containing a family $\mathbf{Q} = \{q_k\}_{k \in \mathbb{Z}}$ of mutually orthogonal projections q_k satisfying

$$\psi(q_k) \in \mathbb{R}^{\times}, \text{ for all } k \in \mathbb{Z},$$

and let $\mathfrak{L}_Q(j)$ be the corresponding j-th filtered $*$-probability space of Q, for all $j \in \mathbb{Z}$.

For the system

$$\{\mathfrak{L}_Q(j) : j \in \mathbb{Z}\}$$

of Banach $*$-probability spaces, define the *free product Banach $*$-probability space* $\mathfrak{L}_Q(\mathbb{Z})$ by

$$\begin{aligned}\mathfrak{L}_Q(\mathbb{Z}) &\overset{denote}{=} \left(\mathfrak{L}_Q(\mathbb{Z}),\ \tau\right) \\ &\overset{def}{=} \underset{j\in\mathbb{Z}}{\star}\mathfrak{L}_Q(j) = \left(\underset{j\in\mathbb{Z}}{\star}\mathfrak{L}_{Q,j},\quad \underset{j\in\mathbb{Z}}{\star}\tau_j\right),\end{aligned} \tag{6.1}$$

with

$$\mathfrak{L}_Q(\mathbb{Z}) = \underset{j\in\mathbb{Z}}{\star}\mathfrak{L}_{Q,j}, \text{ with } \mathfrak{L}_{Q,j} = \mathfrak{L}_Q, \forall j \in \mathbb{Z},$$

and

$$\tau = \underset{j\in\mathbb{Z}}{\star}\tau_j \text{ on } \mathfrak{L}_Q(\mathbb{Z}).$$

i.e., our j-th filtered $*$-probability spaces $\mathfrak{L}_Q(j)$ of (4.15) are the *free blocks of* $\mathfrak{L}_Q(\mathbb{Z})$, for all $j \in \mathbb{Z}$. For more about free product $*$-probability spaces, see [21, 25].

Definition 6.1 Let $\mathfrak{L}_Q(\mathbb{Z})$ be the free product Banach $*$-probability space (6.1) of the system $\{\mathfrak{L}_Q(j)\}_{j\in\mathbb{Z}}$ of all j-th filtered $*$-probability spaces of Q. Then it is said to be the free filterization of $Q \subset (A,\ \psi)$.

Now, construct two subsets $\mathcal{X}$ and $\mathcal{S}$ of $\mathfrak{L}_Q(\mathbb{Z})$,

$$\mathcal{X} = \{u_j \in \mathfrak{L}_Q(j) : j \in \mathbb{Z}\}, \tag{6.2}$$

and

$$\mathcal{S} = \{U_j \in \mathfrak{L}_Q(j) : j \in \mathbb{Z}\},$$

where u_j are the j-th generating operators (4.1) of the radial projection algebra $\mathfrak{L}_Q$, and $U_j = \frac{1}{\psi(q_j)}u_j$ are the operators (5.2) in $\mathfrak{L}_Q$, under **A 5.1**, for all $j \in \mathbb{Z}$.

Recall that a subset $\mathcal{Y}$ of an arbitrary topological $*$-probability space $(B,\ \varphi)$ is said to be a *free family*, if all elements of $\mathcal{Y}$ are free from each other in $(B,\ \varphi)$. Also, a free family $\mathcal{Y}$ is called a *free* (*weighted-*)*semicircular family* in $(B,\ \varphi)$, if this family $\mathcal{Y}$ is not only a free family in $(B,\ \varphi)$, but also a subset of B whose elements are (weighted-)semicircular in $(B,\ \varphi)$. (e.g., [11, 25]).

Theorem 6.1 *Let $\mathcal{X}$ and $\mathcal{S}$ be in the sense of* (6.2) *in the free filterization $\mathfrak{L}_Q(\mathbb{Z})$ of* (6.1).

$$\textit{The family } \mathcal{X} \textit{ is a free weighted-semicircular family in } \mathfrak{L}_Q(\mathbb{Z}). \tag{6.3}$$

$$\textit{The family } \mathcal{S} \textit{ is a free semicircular family in } \mathfrak{L}_Q(\mathbb{Z}). \tag{6.4}$$

Proof Let $\mathcal{X}$ be in the sense of (6.2) in $\mathfrak{L}_Q(\mathbb{Z})$. All elements u_j of $\mathcal{X}$ are taken from mutually distinct free blocks $\mathfrak{L}_Q(j)$ of $\mathfrak{L}_Q(\mathbb{Z})$, for all $j \in \mathbb{Z}$, and hence, they are free from each other in $\mathfrak{L}_Q(\mathbb{Z})$. Thus, this family $\mathcal{X}$ is a free family in $\mathfrak{L}_Q(\mathbb{Z})$. Moreover, every element u_j is $\psi(q_j)^2$-semicircular in $\mathfrak{L}_Q(j)$ by (5.1). Note now that the powers u_j^n of each self-adjoint operator $u_j \in \mathcal{X}$ are again contained in the free block $\mathfrak{L}_Q(j)$ as free reduced words of $\mathfrak{L}_Q(\mathbb{Z})$ with their lengths-1, for all $n \in \mathbb{N}$, for $j \in \mathbb{Z}$. Thus,

$$\tau\left(u_j^n\right) = \tau_j\left(u_j^n\right) = \omega_n \psi(q_j)^n c_{\frac{n}{2}},$$

for all $n \in \mathbb{N}$, for all $j \in \mathbb{Z}$. It shows that each element $u_j \in \mathcal{X}$ is $\psi(q_j)^2$-semicircular in $\mathfrak{L}_Q(\mathbb{Z})$, for all $j \in \mathbb{Z}$. Therefore, the family $\mathcal{X}$ of (6.2) is a free weighted-semicircular family in $\mathfrak{L}_Q(\mathbb{Z})$. Equivalently, the statement (6.3) holds.

Similarly, one can verify that the family $\mathcal{S}$ of (6.2) is a free family in $\mathfrak{L}_Q(\mathbb{Z})$, because U_j are the scalar-products $\frac{1}{\psi(q_j)}\, u_j$ of u_j in the free family $\mathcal{X}$ of $\mathfrak{L}_Q(\mathbb{Z})$, for all $j \in \mathbb{Z}$. So, the semicircularity (5.3) of U_j's (under **A 5.1**) guarantees that this free family $\mathcal{S}$ is a free semicircular family in the free filterization $\mathfrak{L}_Q(\mathbb{Z})$. i.e., the statement (6.4) holds. □

By (4.17) and (4.18), the only "j-th" generating operators u_j of the free blocks $\mathfrak{L}_Q(j)$ provide possible non-zero free distributions on $\mathfrak{L}_Q(\mathbb{Z})$ by (6.1). Thus, we now restrict our interests to the Banach $*$-subalgebra $\mathbb{L}_Q$ of the free filterization $\mathfrak{L}_Q(\mathbb{Z})$, whose elements have possible non-zero free distributions.

Definition 6.2 Let $\mathfrak{L}_Q(\mathbb{Z})$ be the free filterization of Q. Define a Banach $*$-subalgebra $\mathbb{L}_Q$ of $\mathfrak{L}_Q(\mathbb{Z})$ by

$$\mathbb{L}_Q \overset{\text{def}}{=} \overline{\mathbb{C}[\mathcal{X}]}, \tag{6.5}$$

where $\mathcal{X}$ is the free weighted-semicircular family (6.3) in $\mathfrak{L}_Q(\mathbb{Z})$, and $\overline{Y}$ are the Banach-topology closures of subsets Y of $\mathfrak{L}_Q(\mathbb{Z})$. Construct the Banach $*$-probability space,

$$\mathbb{L}_Q \overset{\text{denote}}{=} \left(\mathbb{L}_Q,\ \tau = \tau\mid_{\mathbb{L}_Q}\right), \tag{6.6}$$

as a free-probabilistic sub-structure of $\mathfrak{L}_Q(\mathbb{Z}) = \left(\mathfrak{L}_Q(\mathbb{Z}),\ \tau\right)$.

We call the Banach $*$-algebra $\mathbb{L}_Q$ of (6.5), or the Banach $*$-probability space $\mathbb{L}_Q$ of (6.6), the semicircular (free-sub-)filterization of $\mathfrak{L}_Q(\mathbb{Z})$.

By the definitions (6.5) and (6.6), the operators of the semicircular filterization $\mathbb{L}_Q$ are the free random variables in the free filterization $\mathfrak{L}_Q(\mathbb{Z})$, having "possible" non-zero free distributions. In particular, all free reduced words of $\mathfrak{L}_Q(\mathbb{Z})$ in the free weighted-semicircular family $\mathcal{X}$ of (6.3) (and hence, those of $\mathbb{L}_Q$) have non-zero free distributions in $\mathfrak{L}_Q(\mathbb{Z})$ by (4.17) and (4.18).

Theorem 6.2 *Let $\mathbb{L}_Q$ be the semicircular filterization (6.5) in the free filterization $\mathfrak{L}_Q(\mathbb{Z})$. Then*

$$\mathbb{L}_Q \overset{def}{=} \overline{\mathbb{C}\,[\mathcal{X}]} = \overline{\mathbb{C}[\mathcal{S}]} \overset{*\text{-}iso}{=} \underset{j\in\mathbb{Z}}{\star}\,\overline{\mathbb{C}[\{u_j\}]} \overset{*\text{-}iso}{=} \overline{\mathbb{C}\left[\underset{j\in\mathbb{Z}}{\star}\{u_j\}\right]}, \tag{6.7}$$

in $\mathfrak{L}_Q(\mathbb{Z})$, where "$\overset{*\text{-}iso}{=}$" means "being Banach-$*$-isomorphic," and where ($\star$) in the first $*$-isomorphic relation of (6.7) means the free-probabilistic free product of [21, 25], and ($\star$) in the second $*$-isomorphic relation of (6.7) is the pure-algebraic free product inducing noncommutative free words in $\mathcal{X}$.

Proof The free weighted-semicircular family $\mathcal{X}$ of (6.3) can be re-written by

$$\mathcal{X} = \{\psi(q_j)U_j \in \mathfrak{L}_Q(j) : j \in \mathbb{Z}\}$$

in the free filterization $\mathfrak{L}_Q(\mathbb{Z})$ of Q, where U_j are the semicircular elements $\frac{1}{\psi(q_j)}u_j$ of the free semicircular family $\mathcal{S}$ of (6.4). Therefore,

$$\overline{\mathbb{C}[\mathcal{X}]} = \overline{\mathbb{C}[\mathcal{S}]} \text{ in } \mathfrak{L}_Q(\mathbb{Z}).$$

It shows that the (set-)equality (=) of (6.7) holds.

By the definition (6.5) of $\mathbb{L}_Q$, it is generated by the free family $\mathcal{X}$ by (6.3), and hence, the first $*$-isomorphic relation of (6.7) holds in the free filterization $\mathfrak{L}_Q(\mathbb{Z})$ by (6.1), because

$$\overline{\mathbb{C}\left[\{u_j\}\right]} \subset \mathfrak{L}_Q(j) \text{ in } \mathfrak{L}_Q(\mathbb{Z}), \text{ for all } j \in \mathbb{Z}.$$

Since

$$\mathbb{L}_Q \overset{*\text{-iso}}{=} \underset{j\in\mathbb{Z}}{\star}\,\overline{\mathbb{C}[\{u_j\}]} \text{ in } \mathfrak{L}_Q(\mathbb{Z}),$$

every element T of $\mathbb{L}_Q$ is a limit of linear combinations of free reduced words (in the sense of [21, 25]). Also, all (pure-algebraic) free words in $\mathcal{X}$ have their unique free-reduced-word forms under operator-multiplication on $\mathfrak{L}_Q(\mathbb{Z})$. Furthermore, if we have a free (reduced) word

$$W = \prod_{l=1}^{N} u_{j_l} \text{ in } \mathcal{X},$$

then, as an operator, its adjoint W^* satisfies

$$W^* = \prod_{l=1}^{N} u_{j_{N-l+1}} \text{ in } \mathbb{L}_Q,$$

by the self-adjointness of $u_j \in \mathcal{X}$. Therefore, the second $*$-isomorphic relation of (6.7) holds, too. □

7 Shifts on $\mathbb{Z}$ and Integer-Shifts on $\mathbb{L}_Q$

Let (A, ψ) be the fixed C^*-probability space containing a family $\mathbf{Q} = \{q_j\}_{j \in \mathbb{Z}}$ of mutually-orthogonal projections q_j's having

$$\psi(q_j) \in \mathbb{R}^{\times}, \text{ for all } j \in \mathbb{Z},$$

and let $\mathbb{L}_Q$ be the semicircular filterization of the free filterization $\mathfrak{L}_Q(\mathbb{Z})$ of $Q = C^*(\mathbf{Q})$.

7.1 ($\pm$)-Shifts on $\mathbb{Z}$

Let $\mathbb{Z}$ be the set of all integers. Define bijective functions h_+ and h_- on $\mathbb{Z}$ by

$$h_+(j) = j + 1, \tag{7.1.1}$$

and

$$h_-(j) = j - 1,$$

for all $j \in \mathbb{Z}$.

Then, for these bijections $h_{\pm}$ of (7.1.1), one can construct the following bijections $h_{\pm}^{(n)}$ on $\mathbb{Z}$,

$$h_{\pm}^{(n)} = \underbrace{h_{\pm} \circ h_{\pm} \circ \cdots \circ h_{\pm}}_{n\text{-times}}, \tag{7.1.2}$$

for all $n \in \mathbb{N}_0$, with axiomatization:

$$h_{\pm}^{0} = id_{\mathbb{Z}}, \text{ the identity function on } \mathbb{Z},$$

satisfying, $h_{\pm}^{(1)} = h_{\pm}$, where $(\circ)$ is the usual functional composition. i.e.,

$$h_{\pm}^{(n)}(j) = j \pm n, \text{ for all } j \in \mathbb{Z},$$

for all $n \in \mathbb{N}_0$, by (7.1.2).

Definition 7.1 Let $h_{\pm}^{(n)}$ be in the sense of (7.1.2), for all $n \in \mathbb{N}_0$. Then we call $h_{\pm}^{(n)}$, the n-($\pm$)-shifts on $\mathbb{Z}$. If $n = 1$, then the 1-($\pm$)-shifts $h_{\pm}$ of (7.1.1) are simply said to be ($\pm$)-shifts on $\mathbb{Z}$.

From these shifting processes $h_{\pm}^{(n)}$ on $\mathbb{Z}$, we construct certain $*$-isomorphisms on the semicircular filterization $\mathbb{L}_Q$.

7.2 Integer-Shifts on $\mathbb{L}_Q$

Let $\mathbb{L}_Q$ be the semicircular filterization in the free filterization $\mathfrak{L}_Q(\mathbb{Z})$ of Q, and let $h_{\pm}^{(n)}$ be n-($\pm$)-shifts on $\mathbb{Z}$, for all $n \in \mathbb{N}_0$. In this section, by using $h_{\pm}^{(n)}$, the corresponding $*$-isomorphisms $\beta_{\pm}^{(n)}$ on $\mathbb{L}_Q$ are constructed, and we study how these $*$-isomorphisms act on $\mathbb{L}_Q$, for $n \in \mathbb{N}_0$.

Define a "multiplicative" bounded linear transformation $\beta_{\pm}$ on $\mathbb{L}_Q$ by morphisms satisfying that:

$$\beta_{\pm}\left(U_j\right) = U_{h_{\pm}(j)}, \tag{7.2.1}$$

for $U_j \in \mathcal{S}$, for all $j \in \mathbb{Z}$, where $\mathcal{S}$ is the free semicircular family (6.4) generating $\mathbb{L}_Q$.

Remark that, by (6.7), the free semicircular family $\mathcal{S}$ is the generator set of $\mathbb{L}_Q$. So, by (6.6), the above multiplicative linear transformation $\beta_{\pm}$ of (7.2.1) is well-defined on $\mathbb{L}_Q$. By (7.2.1), we obtain the following computations.

Lemma 7.1 *Let* $Y = \prod_{l=1}^{N} U_{j_l}^{n_l} \in \mathbb{L}_Q$, *for* $U_{j_1}, \ldots, U_{j_N} \in \mathcal{S}$, *and* $n_1, \ldots, n_N \in \mathbb{N}$, *for* $N \in \mathbb{N}$. *Then*

$$\beta_{\pm}(Y) = \prod_{l=1}^{N} U_{j_l \pm 1}^{n_l}. \tag{7.2.2}$$

Proof Let Y be given as above in $\mathbb{L}_Q$. Then, by the multiplicativity of the linear transformations $\beta_{\pm}$ of (7.2.1), one has that

$$\beta_{\pm}(Y) = \prod_{l=1}^{N} \beta_{\pm}\left(U_{j_l}^{n_l}\right) = \prod_{l=1}^{N} \left(\beta_{\pm}\left(U_{j_l}\right)\right)^{n_l} = \prod_{l=1}^{N} U_{h_{\pm}(j_l)}^{n_l}.$$

Therefore, the formula (7.2.2) holds. □

Now, let $u_{j_1}, \ldots, u_{j_N} \in \mathcal{X}$ be weighted-semicircular elements generating $\mathbb{L}_Q$, for $N \in \mathbb{N}$, where $\mathcal{X}$ is the free weighted-semicircular family (6.3), and let

$$X = \prod_{l=1}^{N} u_{j_l}^{n_l}, \text{ for } n_1, \ldots, n_N \in \mathbb{N}.$$

Then

$$\beta_{\pm}(X) = \beta_{\pm}\left(\left(\prod_{l=1}^{N}\psi(q_{j_l})^{n_l}\right)\left(\prod_{l=1}^{N}U_{j_l}^{n_l}\right)\right)$$

since

$$U_{j_l} = \frac{1}{\psi(q_{j_l})}u_{j_l} \Longleftrightarrow u_{j_l} = \psi(q_{j_l})U_{j_l} \text{ in } \mathbb{L}_Q,$$

and hence, the above equality goes to

$$= \left(\prod_{l=1}^{N}\psi(q_{j_l})^{n_l}\right)\ \beta_{\pm}\left(\prod_{l=1}^{N}U_{j_l}^{n_l}\right)$$
$$= \left(\prod_{l=1}^{N}\psi(q_{j_l})^{n_l}\right)\ \left(\prod_{l=1}^{N}U_{j_l\pm 1}^{n_l}\right),$$

by (7.2.2).

Corollary 7.1 *Let* $X = \prod_{l=1}^{N}u_{j_l}^{n_l} \in \mathbb{L}_Q$, *for* $u_{j_1}, \ldots, u_{j_N} \in \mathcal{X}$ *in* $\mathbb{L}_Q$, *for* $n_1, \ldots, n_N,\ N \in \mathbb{N}$. *Then*

$$\begin{aligned}\beta_{\pm}(X) &= \left(\prod_{l=1}^{N}\psi(q_{j_l})^{n_l}\right)\left(\prod_{l=1}^{N}U_{h_{\pm}(j_l)}^{n_l}\right)\\ &= \left(\prod_{l=1}^{N}\psi(q_{j_l})^{n_l}\right)\left(\beta_{\pm}\left(\prod_{l=1}^{N}U_{j_l}^{n_l}\right)\right),\end{aligned} \tag{7.2.2'}$$

in $\mathbb{L}_Q$, *where* $U_{j_l} = \frac{1}{\psi(q_{j_l})}u_{j_l} \in \mathcal{S}$ *in* $\mathbb{L}_Q$, *for all* $l = 1, \ldots, N$.

Proof The proof of (7.2.2′) is done by (7.2.2), and by the discussion in the very above paragraph. □

By (7.2.2) and (7.2.2′), one can realize that the freeness on $\mathbb{L}_Q$ is preserved by that on the set $\beta_{\pm}(\mathbb{L}_Q)$. Indeed, if an arbitrary N-tuple $(j_1, \ldots, j_N)$ is alternating in $\mathbb{Z}$, then the N-tuples $(h_{\pm}(j_1), \ldots, h_{\pm}(j_N))$ are alternating in $\mathbb{Z}$, too, for all $N \in \mathbb{N}$. It guarantees that $\beta_{\pm}$ preserves the freeness on the semicircular filterization $\mathbb{L}_Q$. So, if the operators Y and X are in the sense of the above lemma, respectively, of the above corollary, and if we further assume they are free reduced words with their lengths-N in $\mathbb{L}_Q$, then the images

$$\beta_{\pm}(Y), \text{ and } \beta_{\pm}(X)$$

are again free reduced words with their lengths-N in $\mathbb{L}_Q$.

Theorem 7.1 *Let $\beta_{\pm}$ be the multiplicative linear transformations* (7.2.1) *on $\mathbb{L}_Q$. Then they are $*$-isomorphisms on $\mathbb{L}_Q$.*

Proof By (6.5), (6.6) and (6.7), all elements of the semicircular filterization $\mathbb{L}_Q$ are the limits of linear combinations of free reduced words in the free semicircular family $\mathcal{S}$ of (6.4). So, let's focus on free reduced words of $\mathbb{L}_Q$ in $\mathcal{S}$.

Let $(j_1, \ldots, j_N)$ be an alternating N-tuple in $\mathbb{Z}$ for $N \in \mathbb{N}$, and

$$Y = \prod_{l=1}^{N} U_{j_l}^{n_l}, \text{ for } n_1, \ \ldots, \ n_N \in \mathbb{N}.$$

By the alternating-ness of $(j_1, \ \ldots, \ j_N)$, the above operator Y is a free reduced word with its length-N in $\mathbb{L}_Q$.

Then, by (7.2.2′),

$$\beta_{\pm}(Y) = \prod_{l=1}^{N} U_{h_{\pm}(j_l)}^{n_l}, \tag{7.2.3}$$

where $h_{\pm}$ are the $(\pm)$-shifts (7.1.1) on $\mathbb{Z}$.

By the bijectivity of $h_{\pm}$ on $\mathbb{Z}$, the relation (7.2.3) guarantees the bijectivity of $\beta_{\pm}$ on $\mathbb{L}_Q$. i.e., these multiplicative linear transformations $\beta_{\pm}$ of (7.2.1) are not only generator-preserving but also freeness-preserving. And hence, they are bounded and bijective on $\mathbb{L}_Q$ by (6.7). (Note that the restrictions $\beta_{\pm} \mid_{\mathcal{S}}$ are bijective functions on the generator set $\mathcal{S}$ of $\mathbb{L}_Q$.)

Consider now that if Y is as above, then

$$\beta_{\pm}(Y^*) = \beta_{\pm}\left(\prod_{l=1}^{N} U_{j_{N-j+1}}^{n_{N-l+1}}\right)$$

by the self-adjointness of $U_{j_1}, \ \ldots, \ U_{j_N}$

$$= \prod_{l=1}^{N} U_{h_{\pm}(j_{N-l+1})}^{n_{N-l+1}}$$

by (7.2.2′)

$$= \left(\prod_{l=1}^{N} U_{h_{\pm}(j_l)}^{n_l}\right)^* = (\beta_{\pm}(Y))^* . \tag{7.2.4}$$

So,

$$\beta_{\pm}(S^*) = (\beta_{\pm}(S))^* , \text{ for all } S \in \mathbb{L}_Q,$$

by (7.2.4).

Therefore, the bounded multiplicative linear transformations $\beta_{\pm}$ of (7.2.1) are both bijective, and adjoint-preserving on $\mathbb{L}_Q$, equivalently, they are well-defined $*$-isomorphisms on $\mathbb{L}_Q$. □

From the above theorem, one can realize that the $(\pm)$-shifts $h_{\pm}$ on $\mathbb{Z}$ induce the corresponding $*$-isomorphisms $\beta_{\pm}$ on $\mathbb{L}_Q$.

Definition 7.2 Let $\beta_{\pm}$ be the $*$-isomorphisms (7.2.1) on the semicircular filterization $\mathbb{L}_Q$, induced by the $(\pm)$-shifts $h_{\pm}$ of (7.1.1) on $\mathbb{Z}$. Then they are said to be $(\pm)$-integer-shift(-$*$-isomorphism)s on $\mathbb{L}_Q$.

These two $*$-isomorphisms $\beta_{\pm}$ satisfy the following identity relation on $\mathbb{L}_Q$.

Proposition 7.1 *Let $\beta_{\pm}$ be the $(\pm)$-integer-shifts (7.2.1) on $\mathbb{L}_Q$. Then*

$$\beta_+\beta_- = 1_{\mathbb{L}_Q} = \beta_-\beta_+ \text{ on } \mathbb{L}_Q, \tag{7.2.5}$$

where $1_{\mathbb{L}_Q}$ is the identity map on $\mathbb{L}_Q$, satisfying

$$1_{\mathbb{L}_Q}(T) = T, \text{ for all } T \in \mathbb{L}_Q.$$

Proof As we discussed above, it suffices to consider the cases where we have free reduced words

$$Y = \prod_{l=1}^{N} U_{j_l}^{n_l} \text{ of } \mathbb{L}_Q, \text{ for } n_1, \ldots, n_N \in \mathbb{N},$$

for $N \in \mathbb{N}$, where $U_{j_l} \in \mathcal{S}$, for $l = 1, \ldots, N$, and $(j_1, \ldots, j_N)$ is alternating in $\mathbb{Z}$, by (7.2.2), (7.2.2′), and (6.7).

Observe that

$$\beta_+\beta_-(Y) = \beta_+\left(\prod_{l=1}^{N} U_{h_-(j_l)}^{n_l}\right) = \beta_+\left(\prod_{l=1}^{N} U_{j_l-1}^{n_l}\right)$$
$$= \prod_{l=1}^{N} U_{h_+(j_l-1)}^{n_l} = \prod_{l=1}^{N} U_{j_l-1+1}^{n_l} = Y,$$

similarly,

$$\beta_-\beta_+(Y) = Y.$$

Therefore, for any arbitrary operators $S \in \mathbb{L}_Q$,

$$\beta_+\beta_-(S) = \beta_-\beta_+(S) \text{ in } \mathbb{L}_Q.$$

Therefore, the identity (7.2.5) holds. □

Let $\beta_{\pm}$ be the $(\pm)$-integer-shifts on $\mathbb{L}_Q$. Then one can construct $*$-isomorphisms $\beta_{\pm}^n$,

$$\beta_{\pm}^n = \underbrace{\beta_{\pm}\beta_{\pm} \cdot \cdots \cdot \beta_{\pm}}_{n\text{-times}} \text{ on } \mathbb{L}_Q, \tag{7.2.6}$$

for all $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, with axiomatization:

$$\beta_+^0 = 1_{\mathbb{L}_Q} = \beta_-^0.$$

Since $\beta_{\pm}$ and $1_{\mathbb{L}_Q}$ are $*$-isomorphisms, these morphisms $\beta_{\pm}^n$ of (7.2.6) are well-defined $*$-isomorphisms on $\mathbb{L}_Q$, too, for all $n \in \mathbb{N}_0$.

Definition 7.3 Let $\beta_{\pm}^n$ be the $*$-isomorphisms (7.2.6) on the semicircular filterization $\mathbb{L}_Q$, for all $n \in \mathbb{N}_0$, with axiomatization $\beta_{\pm}^0 = 1_{\mathbb{L}_Q}$. Then they are called the n-$(\pm)$-(integer-)shifts on $\mathbb{L}_Q$, for all $n \in \mathbb{N}_0$.

By (7.2.5) and (7.2.6), one obtains the following relations on the system $\{\beta_{\pm}^n : n \in \mathbb{N}_0\}$ of $*$-isomorphisms on the semicircular filterization $\mathbb{L}_Q$.

Lemma 7.2 *Let $\beta_{\pm}^n$ be the n-$(\pm)$-shifts on the semicircular filterization $\mathbb{L}_Q$, for $n \in \mathbb{N}_0$. Then they satisfy*

$$\beta_+^{n_1}\beta_-^{n_2} = \beta_-^{n_2}\beta_+^{n_1} = \begin{cases} 1_{\mathbb{L}_Q} & \text{if } n_1 = n_2 \\ \beta_+^{n_1-n_2} & \text{if } n_1 > n_2 \\ \beta_-^{n_2-n_1} & \text{if } n_1 < n_2, \end{cases} \tag{7.2.7}$$

on $\mathbb{L}_Q$, for all $n_1, \ n_2 \in \mathbb{N}_0$. Also,

$$\beta_+^{n_1}\beta_+^{n_2} = \beta_+^{n_1+n_2}, \text{ and } \beta_-^{n_1}\beta_-^{n_2} = \beta_-^{n_1+n_2}, \tag{7.2.8}$$

on $\mathbb{L}_Q$, for all $n_1, \ n_2 \in \mathbb{N}_0$.

Proof By the identity (7.2.5), two $*$-isomorphisms β_+ and β_- are not only commutative on $\mathbb{L}_Q$, but also their products $\beta_+\beta_-$ and $\beta_-\beta_+$ become the identity map $1_{\mathbb{L}_Q}$ on $\mathbb{L}_Q$. So, for any $n_1, \ n_2 \in \mathbb{N}_0$,

$$\beta_+^{n_1}\beta_-^{n_2} = \beta_-^{n_2}\beta_+^{n_1} \text{ on } \mathbb{L}_Q.$$

Thus, let's focus on the $*$-isomorphisms $\beta_+^{n_1}\beta_-^{n_2}$, for arbitrarily fixed $n_1, n_2 \in \mathbb{N}_0$.

Suppose first that $n_1 = n_2 = n$ in $\mathbb{N}_0$. Then, by (7.2.5),

$$\beta_+^{n_1}\beta_-^{n_2} = \beta_+^n\beta_-^n = (\beta_+\beta_-)^n = \left(1_{\mathbb{L}_Q}\right)^n = 1_{\mathbb{L}_Q}. \tag{7.2.9}$$

Assume now that $n_1 > n_2$ in $\mathbb{N}_0$. Then

$$\beta_+^{n_1}\beta_-^{n_2} = \beta_+^{n_1-n_2}\beta_+^{n_2}\beta_-^{n_2} = \beta_+^{n_1-n_2}, \tag{7.2.10}$$

on $\mathbb{L}_Q$, by (7.2.9).

Similar to (7.2.10), if $n_1 < n_2$ in $\mathbb{N}_0$, then

$$\beta_+^{n_1}\beta_-^{n_2} = \beta_+^{n_1}\beta_-^{n_1}\beta_-^{n_2-n_1} = \beta_-^{n_2-n_1}, \tag{7.2.11}$$

on $\mathbb{L}_Q$.

So, the formula (7.2.7) is proven by (7.2.9), (7.2.10) and (7.2.11).

For any free generators $U_j \in \mathcal{S}$ of $\mathbb{L}_Q$ (by (6.7)), one can get that

$$\begin{aligned}\beta_+^{n_1}\beta_+^{n_2}\left(U_j^n\right) &= \beta_+^{n_1}\left(U_{j+n_2}^n\right)\\ &= U_{j+n_1+n_2}^n = \beta_+^{n_1+n_2}\left(U_j^n\right),\end{aligned} \tag{7.2.12}$$

and

$$\begin{aligned}\beta_-^{n_1}\beta_-^{n_2}\left(U_j^n\right) = \beta_-^{n_1}\left(U_{j-n_2}^n\right) &= U_{j-n_2-n_1}^n\\ = U_{j-(n_1+n_2)}^n &= \beta_-^{n_1+n_2}\left(U_j^n\right),\end{aligned}$$

for all $j \in \mathbb{Z}$, for all $n \in \mathbb{N}$, for all $n_1,\ n_2 \in \mathbb{N}_0$.

Therefore, the formula (7.2.8) holds on $\mathbb{L}_Q$ by (7.2.2), (7.2.2′), and (7.2.12). □

The above relations (7.2.7) and (7.2.8) can be re-expressed as follows;

$$\beta_{e_1}^{n_1}\beta_{e_2}^{n_2} = \beta_{e_2}^{n_2}\beta_{e_1}^{n_1} = \beta_{sgn(e_1n_1+e_2n_2)}^{|e_1n_1+e_2n_2|} \text{ on } \mathbb{L}_Q, \tag{7.2.13}$$

with

$$sgn(e_1n_1+e_2n_2) = \begin{cases} + \text{ if } e_1n_1+e_2n_2 \geq 0 \\ - \text{ if } e_1n_1+e_2n_2 < 0,\end{cases}$$

for all $e_1,\ e_2 \in \{\pm\}$, and $n_1,\ n_2 \in \mathbb{N}_0$, where sgn in (7.2.13) is the *sign map on* $\mathbb{Z}$,

$$sgn(j) \overset{def}{=} \begin{cases} + & \text{if } j \geq 0 \\ - & \text{if } j < 0,\end{cases}$$

for all $j \in \mathbb{Z}$, and $|.|$ is the *absolute value on* $\mathbb{Z}$. From below, we use the re-expression (7.2.13) for the results (7.2.7) and (7.2.8) for convenience.

Now, consider the system $\mathfrak{B}$ of n-($\pm$)-shifts $\beta_\pm^n$ on $\mathbb{L}_Q$, i.e.,

$$\mathfrak{B} = \{\beta_\pm^n\}_{n\in\mathbb{N}_0} = \{\beta_\pm^n\}_{n\in\mathbb{N}} \cup \{1_{\mathbb{L}_Q}\}. \tag{7.2.14}$$

Let $Aut(\mathbb{L}_Q)$ be the group,

$$Aut\left(\mathbb{L}_Q\right) = \left(\left\{\alpha : \mathbb{L}_Q \to \mathbb{L}_Q \,\middle|\, \begin{array}{c}\alpha \text{ are} \\ *\text{-isomorphisms} \\ \text{on } \mathbb{L}_Q\end{array}\right\},\ \cdot\right) \tag{7.2.15}$$

consisting of all $*$-isomorphisms on $\mathbb{L}_Q$, where the operation $(\cdot)$ is the product (or composition) of $*$-isomorphisms. We call $Aut(\mathbb{L}_Q)$ of (7.2.15), the ($*$-) *automorphism group on* $\mathbb{L}_Q$. (Recall that $*$-isomorphisms from a $*$-algebra onto the same $*$-algebra are called $*$-*automorphisms*.)

By the construction (7.2.14), the system $\mathfrak{B}$ is definitely a "subset" of the automorphism group $Aut(\mathbb{L}_Q)$ of (7.2.15), by Theorem 7.1. Note that the operation $(\cdot)$ is closed on $\mathfrak{B}$, in the sense that:

$$\left(\beta_{e_1}^{n_1},\ \beta_{e_2}^{n_2}\right) \in \mathfrak{B} \times \mathfrak{B} \longmapsto \beta_{e_1}^{n_1}\beta_{e_2}^{n_2} \in \mathfrak{B}, \tag{7.2.16}$$

for all $e_1,\ e_2 \in \{\pm\}$, and $n_1,\ n_2 \in \mathbb{N}_0$, by (7.2.13).

Clearly, by (7.2.8), one can get that

$$\left(\beta_e^{n_1}\beta_e^{n_2}\right)\beta_e^{n_3} = \beta_e^{n_1+n_2+n_3} = \beta_e^{n_1}\left(\beta_e^{n_2}\beta_e^{n_3}\right), \tag{7.2.17}$$

for all $e \in \{\pm\}$, and $n_1,\ n_2,\ n_3 \in \mathbb{N}_0$.

Observe now that

$$\left(\beta_+^{n_1}\beta_-^{n_2}\right)\beta_+^{n_3} = \beta_{\sigma(n_1,n_2)}^{|n_1-n_2|}\beta_+^{n_3} = \beta_{\sigma(|n_1-n_2|,n_3)}^{||n_1-n_2|-n_3|}, \tag{7.2.18}$$

and

$$\beta_+^{n_1}\left(\beta_-^{n_2}\beta_+^{n_3}\right) = \beta_+^{n_1}\beta_{\sigma(n_2,n_3)}^{|n_2-n_3|} = \beta_{\sigma(n_1,|n_2-n_3|)}^{|n_1-|n_2-n_3||},$$

by (7.2.7) (and (7.2.13)), for all $n_1,\ n_2,\ n_3 \in \mathbb{N}_0$, where

$$\sigma(n,\ k) \overset{def}{=} sgn(n-k),\ \text{ for all } n,\ k \in \mathbb{N}_0,$$

in (7.2.18).

Consider two positive quantities a_1 and a_2,

$$a_1 = ||n_1 - n_2| - n_3|, \tag{7.2.19}$$

and

$$a_2 = |n_1 - |n_2 - n_3||,$$

for $n_1,\ n_2,\ n_3 \in \mathbb{N}_0$.

If either $n_1 \leq n_2 \leq n_3$, or $n_1 \geq n_2 \geq n_3$ in $\mathbb{N}_0$, then

$$a_1 = |n_2 - n_1 - n_3| = a_2; \tag{7.2.20}$$

and if either $n_1 \leq n_3 \leq n_2$, or $n_1 \geq n_3 \geq n_2$ in $\mathbb{N}_0$, then

$$a_1 = |n_2 - n_1 - n_3| = a_2; \tag{7.2.21}$$

and if either $n_2 \leq n_3 \leq n_1$, or $n_2 \geq n_3 \geq n_1$ in $\mathbb{N}_0$, then

$$a_1 = |n_1 - n_2 - n_3| = a_2, \tag{7.2.22}$$

where a_1 and a_2 are the quantities (7.2.19).

Lemma 7.3 *Let* $\mathfrak{B} = \{\beta_{\pm}^{n}\}_{n\in\mathbb{N}_0}$ *be the system* (7.2.14). *Then*

$$\left(\beta_{e_1}^{n_1}\beta_{e_2}^{n_2}\right)\beta_{e_3}^{n_3} = \beta_{e_1}^{n_1}\left(\beta_{e_2}^{n_2}\beta_{e_3}^{n_3}\right) \text{ on } \mathbb{L}_Q, \tag{7.2.23}$$

for all $e_1,\ e_2,\ e_3 \in \{\pm\}$, *and* $n_1,\ n_2,\ n_3 \in \mathbb{N}_0$.

Proof By (7.2.17), we have

$$\left(\beta_{e}^{n_1}\beta_{e}^{n_2}\right)\beta_{e}^{n_3} = \beta_{e}^{n_1}\left(\beta_{e}^{n_2}\beta_{e}^{n_3}\right) \text{ on } \mathbb{L}_Q,$$

for all $e \in \{\pm\}$, and $n_1,\ n_2,\ n_3 \in \mathbb{N}_0$.

By (7.2.18), (7.2.20), (7.2.21) and (7.2.22),

$$\begin{aligned}\left(\beta_{+}^{n_1}\beta_{-}^{n_2}\right)\beta_{+}^{n_3} &= \beta_{sgn(|n_1-n_2|-n_3)}^{||n_1-n_2|-n_3|} = \beta_{sgn(a_1')}^{a_1} \\ &= \beta_{sgn(a_2')}^{a_2} = \beta_{sgn(n_1-|n_2-n_3|)}^{|n_1-|n_2-n_3||} \\ &= \beta_{+}^{n_1}\left(\beta_{-}^{n_2}\beta_{+}^{n_3}\right),\end{aligned} \tag{7.2.24}$$

on $\mathbb{L}_Q$, for all $n_1,\ n_2,\ n_3 \in \mathbb{N}_0$, where $a_1 = \left|a_1'\right|$ and $a_2 = \left|a_2'\right|$ are in the sense of (7.2.19), and sgn is the sign map on $\mathbb{Z}$ in (7.2.13).

Similar to (7.2.24), one can obtain that

$$\begin{aligned}\left(\beta_{-}^{n_1}\beta_{+}^{n_2}\right)\beta_{-}^{n_3} &= \beta_{sgn(|n_1-n_2|-n_3)}^{||n_1-n_2|-n_3|} \\ &= \beta_{sgn(n_1-|n_2-n_3|)}^{|n_1-|n_2-n_3||} = \beta_{-}^{n_1}\left(\beta_{+}^{n_2}\beta_{-}^{n_3}\right),\end{aligned} \tag{7.2.25}$$

on $\mathbb{L}_Q$, for all $n_1,\ n_2,\ n_3 \in \mathbb{N}_0$.

Therefore, the formula (7.2.23) holds on $\mathfrak{B}$, by (7.2.17), (7.2.24) and (7.2.25). □

By the above lemma, we obtain the following structure theorem of the system $\mathfrak{B}$ of (7.2.14) in the automorphism group $Aut(\mathbb{L}_Q)$.

Theorem 7.2 *Let $\mathfrak{B}$ be the subset (7.2.14) of the automorphism group $Aut(\mathbb{L}_Q)$ of (7.2.15). Then $\mathfrak{B}$ is a subgroup of $Aut(\mathbb{L}_Q)$.*

Proof Let $\mathfrak{B}$ be in the sense of (7.2.14). Then, by (7.2.16), the operation $(\cdot)$ is closed on $\mathfrak{B}$. So, the algebraic pair $\mathfrak{B} = (\mathfrak{B}, \cdot)$ is well-constructed as an algebraic sub-structure of $Aut(\mathbb{L}_Q)$. By (7.2.23), this operation is associative on $\mathfrak{B}$, and hence, it forms a semigroup. Since

$$\beta_+^0 = 1_{\mathbb{L}_Q} = \beta_-^0 \text{ in } \mathfrak{B},$$

and since

$$\beta_e^n \cdot 1_{\mathbb{L}_Q} = \beta_e^n = 1_{\mathbb{L}_Q} \cdot \beta_e^n,$$

for all $e \in \{\pm\}$, and $n \in \mathbb{N}_0$, the semigroup $\mathfrak{B}$ contains its $(\cdot)$-identity $1_{\mathbb{L}_Q}$. Thus, it forms a monoid.

Finally, by (7.2.7), all elements $\beta_\pm^n \in \mathfrak{B}$ have their unique $(\cdot)$-inverses $\beta_\mp^n \in \mathfrak{B}$, such that

$$\beta_+^n \beta_-^n = 1_{\mathbb{L}_Q} = \beta_-^n \beta_+^n \text{ on } \mathbb{L}_Q,$$

for all $n \in \mathbb{N}_0$, i.e.,

$$\left(\beta_\pm^n\right)^{-1} = \beta_\mp^n on \mathbb{L}_Q, \text{ for all } n \in \mathbb{N}_0,$$

where x^{-1} mean the group-inverses of x. So, this monoid $\mathfrak{B}$ forms a group.

Therefore, the system $\mathfrak{B}$ is a subgroup of the automorphism group $Aut(\mathbb{L}_Q)$. □

By the above theorem, the system $\mathfrak{B}$ of (7.2.14) forms a group. As a group, $\mathfrak{B}$ satisfies the following group-property.

Theorem 7.3 *Let $\mathfrak{B}$ be the subgroup (7.2.14) of the automorphism group $Aut(\mathbb{L}_Q)$. Then $\mathfrak{B}$ is group-isomorphic to the infinite abelian cyclic group $\mathbb{Z} = (\mathbb{Z}, +)$. i.e.,*

$$\mathfrak{B} \overset{Group}{=} (\mathbb{Z}, +), \tag{7.2.26}$$

where "$\overset{Group}{=}$" means "being group-isomorphic."

Proof Define now a function $\Phi : \mathbb{Z} \to \mathfrak{B}$ by

$$\Phi : j \in \mathbb{Z} \longmapsto \beta_{sgn(j)}^{|j|} \in \mathfrak{B}, \tag{7.2.27}$$

with $\Phi(0) = 1_{\mathbb{L}_Q}$.

It is not hard to check that this function Φ of (7.2.27) is a well-defined bijection from $\mathbb{Z}$ onto $\mathfrak{B}$, by (7.2.14). Consider now that

$$\begin{aligned}\Phi(j_1+j_2) &= \beta_{sgn(j_1+j_2)}^{|j_1+j_2|} = \beta_{sgn(j_1)}^{|j_1|}\beta_{sgn(j_2)}^{|j_2|} \\ &= \Phi(j_1)\Phi(j_2),\end{aligned} \tag{7.2.28}$$

in $\mathfrak{B}$, by (7.2.13), for all $j_1, \ j_2 \in \mathbb{Z}$.

So, the bijection Φ of (7.2.27) is a group-homomorphism by (7.2.28), equivalently, it is a group-isomorphism from $\mathbb{Z}$ onto $\mathfrak{B}$. Therefore, the group-isomorphic relation (7.2.26) holds true. □

The above theorem characterizes the group-structure of the subgroup $\mathfrak{B} = \{\beta_{\pm}^{n}\}_{n\in\mathbb{N}_0}$ of the automorphism group $Aut(\mathbb{L}_Q)$. i.e., $\mathfrak{B}$ is an infinite cyclic abelian group.

Definition 7.4 Let $\mathfrak{B}$ be the subgroup (7.2.14) of the automorphism group $Aut(\mathbb{L}_Q)$. We call $\mathfrak{B}$, the integer-shift (sub)group (of $Aut(\mathbb{L}_Q)$ acting) on $\mathbb{L}_Q$.

7.3 Free Distributions on $\mathbb{L}_Q$ Under the Action of $\mathfrak{B}$

Let $\mathfrak{B}$ be the integer-shift group (7.2.14) acting on the semicircular filterization $\mathbb{L}_Q$ of Q. Let

$$u_j = l \otimes q_j \in \mathcal{X}, \text{ and } U_j = \frac{1}{\psi(q_j)}u_j \in \mathcal{S}, \tag{7.3.1}$$

in $\mathbb{L}_Q$, for $j \in \mathbb{Z}$, where $\mathcal{X}$ is the free weighted-semicircular family (6.3), and $\mathcal{S}$ is the free semicircular family (6.4) of $\mathbb{L}_Q$.

Theorem 7.4 *Let u_j and U_j be in the sense of* (7.3.1) *in $\mathbb{L}_Q$. Then, for $\beta_e^n \in \mathfrak{B}$, we have*

$$\tau\left(\left(\beta_e^n(u_j)\right)^k\right) = \omega_k \psi\left(q_j\right)^k c_{\frac{k}{2}} = \tau\left(u_j^k\right), \tag{7.3.2}$$

and

$$\tau\left(\left(\beta_e^n(U_j)\right)^k\right) = \omega_k c_{\frac{k}{2}} = \tau\left(U_j^k\right),$$

for all $k \in \mathbb{N}$, for all $e \in \{\pm 1\}$, and $n \in \mathbb{N}_0$.

Proof Let $e \in \{\pm 1\}$, and $n \in \mathbb{N}_0$ be arbitrarily taken, and let β_e^n be the corresponding $*$-isomorphism of the integer-shift group $\mathfrak{B}$. For the fixed $\psi(q_j)^2$

-semicircular element u_j of (7.3.1), one has that

$$\beta_e^n\left(u_j\right)^k = \left(\beta_e^n\left(\psi(q_j)U_j\right)\right)^k \\ = \psi(q_j)^k\beta_e^n\left(U_j\right)^k = \psi(q_j)^k U_{jen}^k, \tag{7.3.3}$$

where

$$jen = \begin{cases} j+n & \text{if } e = 1 \\ j-n & \text{if } e = -1, \end{cases}$$

for all $k \in \mathbb{N}$. Thus,

$$\tau\left(\left(\beta_e^n(u_j)\right)^k\right) = \psi(q_j)^k\tau\left(U_{jen}^k\right) = \psi(q_j)^k\left(\omega_k c_{\frac{k}{2}}\right), \tag{7.3.4}$$

since $U_{jen} \in \mathcal{S}$ in $\mathbb{L}_Q$, for all $k \in \mathbb{N}$. Since

$$\tau\left(U_m^k\right) = \omega_k c_{\frac{k}{2}}, \text{ for all } U_m \in \mathcal{S},$$

by the semicircularity (5.3) (under **A 5.1**),

$$\tau\left(\left(\beta_e^n(u_j)\right)^k\right) = \psi(q_j)^k\tau\left(U_j^k\right), \ \forall k \in \mathbb{N},$$

by (7.3.4). Therefore, the first free-distributional data of (7.3.2) holds.

By (7.3.3) and (7.3.4), one can get that

$$\tau\left(\left(\beta_e^n(U_j)\right)^k\right) = \tau\left(U_{jen}^k\right) = \omega_k c_{\frac{k}{2}} = \tau\left(U_j^k\right),$$

for all $k \in \mathbb{N}$. It shows that the second free-distributional data of (7.3.2) holds, too. ■

The above theorem shows how the free probability on the semicircular filterization $\mathbb{L}_Q$ is affected by (actions of) the n-(e)-shift $\beta_e^n \in \mathfrak{B}$ on $\mathbb{L}_Q$.

Corollary 7.2 *Let $\mathbb{L}_Q$ be the semicircular filterization.*

$$\textit{The semicircular law on } \mathbb{L}_Q \textit{ is preserved by the action of } \mathfrak{B}. \tag{7.3.5}$$

The $\psi(q_j)^2$-semicircular laws induced by $u_j \in \mathcal{X}$ on $\mathbb{L}_Q$ is preserved to be the $\psi\left(q_j\right)^2$-semicircular laws induced by $\psi(q_j)U_{jen}$ on $\mathbb{L}_Q$, for $U_j \in \mathcal{S}$, for all $\beta_e^n \in \mathfrak{B}$, where jen are in the sense of (7.3.3). (7.3.6)

Proof Now, let $U_j \in \mathcal{S}$ be a semicircular element (7.3.1) in the semicircular filterization $\mathbb{L}_Q$. Then

$$\tau\left(\left(\beta_e^n(U_j)\right)^k\right) = \omega_k c_{\frac{k}{2}} = \tau\left(U_j^k\right),$$

for all $k \in \mathbb{N}$, for all $e \in \{\pm\}$, $n \in \mathbb{N}_0$, by (7.3.2).

It shows that the semicircular law induced by $U_j \in \mathcal{S}$ on $\mathbb{L}_Q$ is preserved to be the semicircular law induced by $U_{jen} \in \mathcal{S}$ on $\mathbb{L}_Q$, for all $\beta_e^n \in \mathfrak{B}$. Therefore, the statement (7.3.5) holds.

Now, consider the $\psi(q_j)^2$-semicircular element $u_j = \psi(q_j)U_j \in \mathcal{X}$ of (7.3.1) in the semicircular filterization $\mathbb{L}_Q$. One has

$$\tau\left(\left(\beta_e^n(u_j)\right)^k\right) = \omega_n \psi\left(q_j\right)^n c_{\frac{n}{2}} = \tau\left(u_j^k\right),$$

for all $k \in \mathbb{N}$, for all $e \in \{\pm 1\}$, $n \in \mathbb{N}_0$, by (7.3.2).

Thus, the $\psi(q_j)^2$-semicircular laws on $\mathbb{L}_Q$ induced by $u_j \in \mathcal{X}$ are preserved to be the $\psi(q_j)^2$-semicircular laws on $\mathbb{L}_Q$ induced by

$$\beta_e^n\left(u_j\right) = \psi(q_j)U_{jen} \in \mathbb{L}_Q.$$

So, the statement (7.3.6) holds true. □

The (weighted-)semicircular law(s) induced by our free (weighted-)semicircular family $(\mathcal{X}\cup)\mathcal{S}$ on $\mathbb{L}_Q$ is (resp., are) preserved to be the (weighted-)semicircular law(s) induced by $(\mathcal{X}\cup)\mathcal{S}$, under the action of the integer-shift group $\mathfrak{B}$, by (7.3.5) and (7.3.6). So, one can verify that the free probability on the semicircular filterization $\mathbb{L}_Q$ is preserved under the action of $\mathfrak{B}$, by (6.7).

Definition 7.5 Let (B_1, φ_1) and (B_2, φ_2) be arbitrary topological $*$-probability spaces. We say that (B_1, φ_1) is free-($*$-)homomorphic to (B_2, φ_2), if (i) there is a $*$-homomorphism $\Omega : B_1 \to B_2$, and (ii)

$$\varphi_2\left(\Omega(a)\right) = \varphi_1(a), \text{ for all } a \in (B_1, \varphi_1),$$

where $\Omega(a) \in (B_2, \varphi_2)$. The $*$-homomorphism Ω is called a free-($*$-) homomorphism.

If Ω in (i) is a $*$-isomorphism satisfying (ii), then it is said to be a free-($*$-) isomorphism. In such a case, (B_1, φ_1) and (B_2, φ_2) are said to be free-($*$-) isomorphic.

By (6.5), (6.7), (7.3.2), (7.3.5) and (7.3.6), we obtain the following free-isomorphic relation on $\mathbb{L}_Q$.

Theorem 7.5 *Let* $\mathbb{L}_Q = (\mathbb{L}_Q, \tau)$ *be the semicircular filterization, and* $\mathfrak{B}$, *the integer-shift group. Then the Banach* $*$*-probability spaces*

$$\left\{\left(\beta\left(\mathbb{L}_Q\right), \ \tau \circ \beta\right) : \beta \in \mathfrak{B}\right\} \tag{7.3.7}$$

are free-isomorphic from each other, where $(\circ)$ *is the functional composition.*

Proof Let $\mathbb{L}_Q = (\mathbb{L}_Q, \tau)$ be our semicircular filterization on Q. Since the integer-shift group $\mathfrak{B}$ is a subgroup of the automorphism group $Aut(\mathbb{L}_Q)$, all elements $\beta \in \mathfrak{B}$ are $*$-isomorphisms from $\mathbb{L}_Q$ onto $\mathbb{L}_Q$, and hence,

$$\beta(\mathbb{L}_Q) = \mathbb{L}_Q, \text{ for all } \beta \in \mathfrak{B}.$$

So, the above family (7.3.7) is identified with

$$\left\{\left(\mathbb{L}_Q, \ \tau \circ \beta\right) : \beta \in \mathfrak{B}\right\}.$$

Thus, it suffices to show that

$$\tau(T) = \tau\left(\beta(T)\right), \text{ for all } T \in \mathbb{L}_Q, \tag{7.3.8}$$

for all $\beta \in \mathfrak{B}$.

By (7.3.2), the free distributions of the free generators of $\mathcal{S}$ (or, of $\mathcal{X}$) are preserved under the action of $\mathfrak{B}$ in $\mathbb{L}_Q$. It shows that the free distributions of all free reduced words of $\mathbb{L}_Q$ in $\mathcal{S}$ (or, in $\mathcal{X}$) are preserved by the action of $\mathfrak{B}$. Since all elements of $\mathbb{L}_Q$ are the limits of linear combinations of free reduced words, free distributions of $T \in \mathbb{L}_Q$ are preserved to be that of $\beta(T) \in \mathbb{L}_Q$, for all $\beta \in \mathfrak{B}$. i.e., $(\mathbb{L}_Q, \tau)$ and $(\mathbb{L}_Q, \tau \circ \beta)$ are free-isomorphic, for all $\beta \in \mathfrak{B}$. □

The above theorem shows that all elements β of the integer-shift group $\mathfrak{B}$ are not only $*$-isomorphisms on $\mathbb{L}_Q$, but also free-isomorphisms on $\mathbb{L}_Q$, in the automorphism group $Aut(\mathbb{L}_Q)$.

8 The Monoid $\mathfrak{B}(\mathbb{Z})$ Acting on $\mathbb{L}_Q$

In this section, we consider certain tools, and techniques for studying free-homomorphic relations of embedded free-probabilistic sub-structures of the semicircular filterization $\mathbb{L}_Q$.

Let $J = (j_1, \ldots, j_N)$ be the N-tuple of "mutually distinct" integers $j_1, \ldots, j_N \in \mathbb{Z}$, for

$$N \in \mathbb{N}_{>1} \overset{def}{=} \mathbb{N} \setminus \{1\},$$

and define the corresponding Banach $*$-subalgebra $\mathbb{L}[J]$ of $\mathbb{L}_Q$ by

$$\mathbb{L}[J] \overset{def}{=} \underset{l=1}{\overset{N}{\star}} \mathbb{L}_{j_l}, \text{ with } \mathbb{L}_{j_l} = \overline{\mathbb{C}\left[\{U_{j_l}\}\right]}, \tag{8.1}$$

where $\mathbb{L}_{j_l}$ are the free blocks of $\mathbb{L}_Q$, for $U_{j_l} \in \mathcal{S}$, for all $l = 1, \ldots, N$ (see (6.7)).

As a sub-structure of the semicircular filterization $\mathbb{L}_Q = (\mathbb{L}_Q, \tau)$, one can regard the Banach $*$-algebra $\mathbb{L}[J]$ of (8.1) as a Banach $*$-probability space,

$$\mathbb{L}[J] \overset{denote}{=} \left(\mathbb{L}[J], \ \tau = \tau \mid_{\mathbb{L}[J]}\right), \tag{8.2}$$

in $\mathbb{L}_Q$.

Extending the definitions of (8.1) and (8.2), if an integer-sequence $J = (j_1, j_2, j_3, \ldots)$ of mutually distinct infinitely many integers $j_1, j_2, \ldots$ is given, one can construct the Banach $*$-subalgebra $\mathbb{L}[J]$ of $\mathbb{L}_Q$, and the corresponding Banach $*$-probability space,

$$\mathbb{L}[J] = \left(\mathbb{L}[J], \ \tau = \tau \mid_{\mathbb{L}[J]}\right). \tag{8.3}$$

Including the extended case (8.3), for any $J = (j_1, \ldots, j_N)$ of mutually distinct integers $j_1, \ldots, j_N \in \mathbb{Z}$, for

$$N \in \mathbb{N}_{>1}^{\infty} \overset{def}{=} \mathbb{N}_{>1} \cup \{\infty\},$$

one can get the free-probabilistic sub-structure

$$\mathbb{L}[J] = (\mathbb{L}[J], \ \tau) \text{ in } \mathbb{L}_Q. \tag{8.4}$$

Now, let $\mathfrak{B}$ be the integer-shift group (7.2.14), an infinite cyclic abelian subgroup of the automorphism group $Aut(\mathbb{L}_Q)$ of $\mathbb{L}_Q$, and let $\beta_e^n \in \mathfrak{B}$, for

$$(e, n) \in \mathbb{N}_0^{\pm} \overset{def}{=} \{\pm\} \times \mathbb{N}_0.$$

For any arbitrarily fixed $k \in \mathbb{Z}$, and $\beta_e^n \in \mathfrak{B}$, define a new morphism $\beta_e^n(k)$ on $\mathbb{L}_Q$ by a bounded "multiplicative" linear transformation satisfying

$$\beta_e^n(k)\left(U_j\right) = \begin{cases} \beta_e^n\left(U_k\right) = U_{ken} \text{ if } k = j \\ 1_{\mathbb{L}_Q}\left(U_j\right) = U_j \text{ if } k \neq j \end{cases} \tag{8.5}$$

for all $U_j \in \mathcal{S}$, for $j \in \mathbb{Z}$.

By the multiplicativity of the morphism $\beta_e^n(k)$ of (8.5), we have

$$\beta_e^n(k)\left(\prod_{l=1}^{N} U_{j_l}^{n_{j_l}}\right) \overset{def}{=} \begin{cases} \left(\prod_{l=1}^{N-t} U_{j_l}^{n_{j_l}}\right)\left(U_{ken}^{n_k}\right)\left(\prod_{l=t+1}^{N} U_{j_l}^{n_{j_l}}\right) & \text{if } \exists t \in \{1, \ldots, N\} \\ & \text{s.t., } j_t = k \text{ in } \mathbb{Z} \\ \prod_{l=1}^{N} U_{j_l}^{n_{j_l}} & \text{otherwise,} \end{cases} \tag{8.6}$$

by (8.5), for all free reduced words $\prod_{l=1}^{N} U_{j_l}^{n_{j_l}}$ of $\mathbb{L}_Q$ in $\mathcal{S}$, for $n_{j_1}, \ \ldots, \ n_{j_N} \in \mathbb{N}$.

For example, one has

$$\begin{aligned}\beta_-^2(3)\left(U_2^2 U_3 U_{-1}^3 U_3^2 U_1^2\right) &= U_2^2 U_{3-2} U_{-1}^3 U_{3-2}^2 U_1^2 \\ &= U_2^2 U_1 U_{-1}^3 U_1^4,\end{aligned}$$

and

$$\begin{aligned}\beta_-^2(3)\left(U_2^2 U_3 U_{-1}^3 U_1^4\right) &= U_2^2 U_{3-2} U_{-1}^3 U_1^4 \\ &= U_2^2 U_1 U_{-1}^3 U_1^4,\end{aligned}$$

in $\mathbb{L}_Q$.

The above two examples demonstrate that the bounded multiplicative linear transformation $\beta_-^2(3)$ is not injective, and hence, the morphisms $\beta_e^n(k)$ of (8.5) are not injective (and hence, not bijective) on $\mathbb{L}_Q$, in general. However, we have the following result.

Lemma 8.1 *Let $(e, n) \in \mathbb{N}_0^{\pm}$, and $k \in \mathbb{Z}$, and let $\beta_e^n(k)$ be a bounded multiplicative linear transformation (8.5) satisfying (8.6) on $\mathbb{L}_Q$, where $\beta_e^n \in \mathfrak{B}$. Then*

$$\textit{the morphism } \beta_e^n(k) \textit{ of (8.5) is a } *\textit{-homomorphism on } \mathbb{L}_Q. \tag{8.7}$$

Proof Let $\beta_e^n(k)$ be a morphism (8.5) on $\mathbb{L}_Q$. By definition, this morphism $\beta_e^n(k)$ is a bounded multiplicative linear transformation on $\mathbb{L}_Q$. Indeed, $\beta_e^n(k)$ satisfies that

$$\beta_e^n(k)\,(T_1 T_2) = \left(\beta_e^n(k)\,(T_1)\right)\left(\beta_e^n(k)(T_2)\right), \tag{8.8}$$

by (8.6) and the linearity, for all $T_1, \ T_2 \in \mathbb{L}_Q$.

Moreover,

$$\beta_e^n(k)\left(W^*\right) = \left(\beta_e^n(k)(W)\right)^*,$$

for all free reduced words W of $\mathbb{L}_Q$ in the free semicircular family $\mathcal{S}$, by (8.6). It guarantees that

$$\beta_e^n(k)\left(T^*\right) = \left(\beta_e^n(k)(T)\right)^* \text{ in } \mathbb{L}_Q, \tag{8.9}$$

for all $T \in \mathbb{L}_Q$.

Therefore, the morphism $\beta_e^n(k)$ is a $*$-homomorphism on $\mathbb{L}_Q$ by (8.8) and (8.9), for any $(e, n) \in \mathbb{N}_0^{\pm}$, and $k \in \mathbb{Z}$. Equivalently, the statement (8.7) holds. □

Define now families $\mathfrak{B}(k)$ by

$$\mathfrak{B}(k) \overset{def}{=} \left\{ \beta_e^n(k) \,\middle|\, \begin{array}{c} \beta_e^n(k) \text{ are in the sense of (8.5),} \\ \forall \beta_e^n \in \mathfrak{B},\ \forall (e, n) \in \mathbb{N}_0^{\pm} \end{array} \right\}, \tag{8.10}$$

for all $k \in \mathbb{Z}$.

Let $Hom(\mathbb{L}_Q)$ be the $(*\text{-})$*homomorphism semigroup of* $\mathbb{L}_Q$. i.e.,

$$Hom\left(\mathbb{L}_Q\right) \overset{def}{=} \left(\left\{\theta : \mathbb{L}_Q \to \mathbb{L}_Q \,|\theta \text{ is a } * \text{-homomorphism}\right\},\ \cdot\right),$$

where the operation $(\cdot)$ is the product (or, the composition) of $*$-homomorphisms on $\mathbb{L}_Q$.

By definition, clearly, the automorphism group $Aut(\mathbb{L}_Q)$ is a subgroup embedded in the homomorphism semigroup $Hom(\mathbb{L}_Q)$.

Corollary 8.1 *Let $\mathfrak{B}(k)$ be the families (8.10) induced by the integer-shift group $\mathfrak{B}$, for all $k \in \mathbb{Z}$. Then*

$$\left(\underset{k \in \mathbb{Z}}{\sqcup} \mathfrak{B}(k)\right) \subset Hom(\mathbb{L}_Q). \tag{8.11}$$

Proof The proof of the set-inclusion (8.11) is clear by (8.7). □

Now, for a fixed $k \in \mathbb{Z}$, let $\mathfrak{B}(k)$ be the family (8.10) in $Hom(\mathbb{L}_Q)$ of $\mathbb{L}_Q$. Then it is not difficult to check that

$$\beta_{e_1}^{n_1}(k)\beta_{e_2}^{n_2}(k) = \begin{cases} \beta_{e_2}^{n_2}(k) \text{ if } n_2 \neq 0 \\ \beta_{e_1}^{n_1}(k) \text{ if } n_2 = 0, \end{cases} \tag{8.12}$$

for all $(e_l, n_l) \in \mathbb{N}_0^{\pm}$, for $l = 1,\ 2$.

Indeed, if $n_2 = 0 \in \mathbb{N}_0$, then

$$\beta_{e_2}^{n_2}(k) = \beta_{e_2}^{0}(k) = 1_{\mathbb{L}_Q}(k) = 1_{\mathbb{L}_Q},$$

by (8.5) and (8.6). So, if $n_2 = 0$, then

$$\begin{aligned}\left(\beta_{e_1}^{n_1}(k)\beta_{e_2}^{0}(k)\right)(T) &= \beta_{e_1}^{n_1}(k)\left(1_{\mathbb{L}_Q}(T)\right)\\ &= \beta_{e_1}^{n_1}(k)\,(T)\,,\end{aligned} \tag{8.13}$$

for all $T \in \mathbb{L}_Q$.

Meanwhile, if $n_2 \neq 0 \in \mathbb{N}_0$, then, for any free reduced word $W = \prod_{l=1}^{N} U_{j_l}^{n_{j_l}}$ of $\mathbb{L}_Q$ in the free semicircular family $\mathcal{S}$ of (6.4), we have that: (i) if there is no integers j_s such that $j_s = k$ in $\mathbb{Z}$, for $s \in \{1, \ldots, N\}$, then

$$\begin{aligned}\left(\beta_{e_1}^{n_1}(k)\beta_{e_2}^{n_2}(k)\right)(W) &= \beta_{e_1}^{n_1}(k)\left(\beta_{e_2}^{n_2}(k)(W)\right)\\ &= \beta_{e_1}^{n_1}(k)\,(W)\\ &= W = \beta_{e_2}^{n_2}(k)(W);\end{aligned} \tag{8.13$'$}$$

and (ii) if there exists j_s such that $j_s = k$ in $\mathbb{Z}$, for some $s \in \{1, \ldots, N\}$, then

$$\begin{aligned}\left(\beta_{e_1}^{n_1}(k)\beta_{e_2}^{n_2}(k)\right)(W) &= \beta_{e_1}^{n_1}(k)\left(U_{j_1}^{n_{j_1}} \ldots U_{ke_2n_2}^{n_k} \ldots U_{j_N}^{n_{j_N}}\right)\\ &= U_{j_1}^{n_{j_1}} \ldots U_{ke_2n_2}^{n_k} \ldots U_{j_N}^{n_{j_N}}\\ &= \beta_{e_2}^{n_2}(k)\,(W)\,,\end{aligned} \tag{8.13$''$}$$

by (8.6), for all $\beta_{e_1}^{n_1}(k),\ \beta_{e_2}^{n_2}(k) \in \mathfrak{B}(k)$.

Thus, by (8.13), (8.13)' and (8.13)", the formula (8.12) indeed holds true.

Lemma 8.2 *The inherited operation* $(\cdot)$ *on the homomorphism semigroup* $Hom(\mathbb{L}_Q)$ *is closed in the family* $\mathfrak{B}(k)$ *of* (8.10)*, for all* $k \in \mathbb{Z}$*. More precisely, if* $\beta_{e_1}^{n_1}(k),\ \beta_{e_2}^{n_2}(k) \in \mathfrak{B}(k)$*, then*

$$\beta_{e_1}^{n_1}(k)\beta_{e_2}^{n_2}(k) = \begin{cases}\beta_{e_2}^{n_2}(k) \in \mathfrak{B}(k) \text{ if } n_2 \neq 0\\ \beta_{e_1}^{n_1}(k) \in \mathfrak{B}(k) \text{ if } n_2 = 0,\end{cases}$$

for all $(e_1, n_1),\ (e_2, n_2) \in \mathbb{N}_0^{\pm}$*, for all* $k \in \mathbb{Z}$*.*

Proof The proof is done by (8.12). □

The above lemma shows that the pairs $(\mathfrak{B}(k),\ \cdot)$ form well-determined algebraic sub-structures of $Hom(\mathbb{L}_Q)$, for all $k \in \mathbb{Z}$. By Lemmas 8.1 and 8.2, we obtain the following structure theorem of $\mathfrak{B}(k)$, for $k \in \mathbb{Z}$. Recall that a semigroup having its binary-operation-identity is said to be a *monoid*. i.e., an algebraic structure $(Y, \cdot)$ of a set Y and a binary operation $(\cdot)$ is a monoid, if and only if $(\cdot)$ is associative, and there exists a unique $(\cdot)$-identity.

Theorem 8.1 *Let $k \in \mathbb{Z}$, and $\mathfrak{B}(k)$, the corresponding family* (8.10)*. Then $\mathfrak{B}(k)$ is a noncommutative sub-monoid of the homomorphism semigroup $Hom(\mathbb{L}_Q)$. i.e.,*

$$\mathfrak{B}(k) \overset{denote}{=} (\mathfrak{B}(k), \ \cdot) \overset{Monoid}{\subset} Hom(\mathbb{L}_Q), \tag{8.14}$$

where "$\overset{Monoid}{\subset}$" means "being a sub-monoid of."

Proof Note that the algebraic pair

$$\mathfrak{B}(k) \overset{denote}{=} (\mathfrak{B}(k), \ \cdot)$$

is a well-determined sub-structure of $Hom(\mathbb{L}_Q)$ by (8.11) and (8.12).

Also, if $n_3 \neq 0$, then

$$\left(\beta_{e_1}^{n_1}(k)\beta_{e_2}^{n_2}(k)\right)\beta_{e_3}^{n_3}(k) = \begin{cases} \beta_{e_2}^{n_2}(k)\beta_{e_3}^{n_3}(k) = \beta_{e_3}^{n_3}(k) \text{ if } n_2 \neq 0 \\ \beta_{e_1}^{n_1}(k)\beta_{e_3}^{n_3}(k) = \beta_{e_3}^{n_3}(k) \text{ if } n_2 = 0, \end{cases}$$

and

$$\beta_{e_1}^{n_1}(k)\left(\beta_{e_2}^{n_2}(k)\beta_{e_3}^{n_3}(k)\right) = \beta_{e_1}^{n_1}(k)\beta_{e_3}^{n_3}(k) = \beta_{e_3}^{n_3}(k),$$

in $\mathfrak{B}(k)$, by (8.12).

And, if $n_3 = 0$ in $\mathbb{N}_0$, then

$$\begin{aligned}\left(\beta_{e_1}^{n_1}(k)\beta_{e_2}^{n_2}(k)\right)\beta_{e_3}^{0}(k) &= \beta_{e_1}^{n_1}(k)\beta_{e_2}^{n_2}(k) \\ &= \beta_{e_1}^{n_1}(k)\left(\beta_{e_2}^{n_2}(k)\beta_{e_3}^{0}(k)\right),\end{aligned}$$

by (8.12), too.

So, the operation $(\cdot)$ is associative, i.e., the algebraic pair $\mathfrak{B}(k)$ forms a semigroup.

Note that, if $(e_1, \ n_1) \neq (e_2, \ n_2)$ in $\mathbb{N}_0^{\pm}$, and if $n_1 \neq 0$, and $n_2 \neq 0$ in $\mathbb{N}_0$, then

$$\beta_{e_1}^{n_1}(k)\beta_{e_2}^{n_2}(k) = \beta_{e_2}^{n_2}(k) \neq \beta_{e_1}^{n_1}(k) = \beta_{e_2}^{n_2}(k)\beta_{e_1}^{n_1}(k),$$

in $\mathfrak{B}(k)$, by (8.12). Thus, this semigroup $\mathfrak{B}(k)$ is not commutative in $Hom(\mathbb{L}_Q)$.

By the definition (8.10) of $\mathfrak{B}(k)$, there exists an element

$$\beta_e^0(k) = 1_{\mathbb{L}_Q}(k) = 1_{\mathbb{L}_Q} \in \mathfrak{B}(k), \text{ for } e \in \{\pm\},$$

such that

$$\beta_e^n(k) \cdot 1_{\mathbb{L}_Q} = \beta_e^n(k) = 1_{\mathbb{L}_Q} \cdot \beta_e^n(k),$$

for all $(e, \ n) \in \mathbb{N}_0^{\pm}$.

i.e., the noncommutative semigroup $\mathfrak{B}(k)$ of (8.10) is a monoid containing its $(\cdot)$-identity $1_{\mathbb{L}_Q}$. Therefore, $\mathfrak{B}(k)$ is a sub-monoid of $Hom(\mathbb{L}_Q)$. So, the relation (8.14) is satisfied.

Note that, as we discussed above in examples, if $n \neq 0$ in $\mathbb{N}_0$, then the elements $\beta_e^n(k)$ are not injective on $\mathbb{L}_Q$. It means that $\beta_e^n(k)$ are not invertible in $\mathfrak{B}(k)$ as $*$-homomorphisms in $Hom(\mathbb{L}_Q)$, in general. So, the monoid $\mathfrak{B}(k)$ is not a group in $Hom(\mathbb{L}_Q)$. Thus, the family $\mathfrak{B}(k)$ is a noncommutative sub-monoid of $Hom(\mathbb{L}_Q)$, for all $k \in \mathbb{Z}$, equivalently, the relation (8.14) holds. □

The above theorem illustrates how our new families $\mathfrak{B}(k)$ of (8.10) are different from our integer-shift group

$$\mathfrak{B} \subset Aut(\mathbb{L}_Q) \subset Hom(\mathbb{L}_Q)$$

as embedded sub-structures of the homomorphism semigroup $Hom(\mathbb{L}_Q)$, for $k \in \mathbb{Z}$. In fact,

$$\left(\mathfrak{B}(k) \setminus \{1_{\mathbb{L}_Q}\}\right) \subset \left(Hom(\mathbb{L}_Q) \setminus Aut(\mathbb{L}_Q)\right), \tag{8.14'}$$

set-theoretically, for all $k \in \mathbb{Z}$.

Definition 8.1 Let $\mathfrak{B}(k)$ be a noncommutative sub-monoid (8.10) of the homomorphism semigroup $Hom(\mathbb{L}_Q)$, for $k \in \mathbb{Z}$. Then we call $\mathfrak{B}(k)$, the k-(sub-)monoid of $Hom(\mathbb{L}_Q)$, for all $k \in \mathbb{Z}$.

Define now a sub-monoid $\mathfrak{B}(\mathbb{Z})$ of $Hom(\mathbb{L}_Q)$ generated by $\{\mathfrak{B}(k)\}_{k\in\mathbb{Z}}$, i.e.,

$$\mathfrak{B}(\mathbb{Z}) \overset{def}{=} \left\langle \underset{k\in\mathbb{Z}}{\cup} \mathfrak{B}(k) \right\rangle, \tag{8.15}$$

where $\mathfrak{B}(k)$ are the k-monoids (8.14) satisfying (8.14′), for all $k \in \mathbb{Z}$, and $\langle X \rangle$ mean the *sub-monoids generated by subsets* X of $Hom(\mathbb{L}_Q)$. i.e., the monoid $\mathfrak{B}(\mathbb{Z})$ of (8.15) is the minimal sub-monoid of $Hom(\mathbb{L}_Q)$ with its identity $1_{\mathbb{L}_Q}$, containing all k-monoids $\{\mathfrak{B}(k)\}_{k\in\mathbb{Z}}$, i.e.,

$$\mathfrak{B}(\mathbb{Z}) = \cap \left\{ M \subseteq Hom(\mathbb{L}_Q) \;\middle|\; \begin{array}{c} M \text{ is a sub-monoid of } \mathbb{L}_Q, \\ \text{and} \\ M \supseteq \left(\underset{k\in\mathbb{Z}}{\cup} \mathfrak{B}(k) \right) \end{array} \right\}.$$

Thus, for any $\beta_{e_1}^{n_1}(k_1) \in \mathfrak{B}(k_1)$, and $\beta_{e_2}^{n_2}(k_2) \in \mathfrak{B}(k_2)$,

$$\beta_{e_1}^{n_1}(k_1)\beta_{e_2}^{n_2}(k_2) \in \mathfrak{B}(\mathbb{Z}), \tag{8.16}$$

in $Hom(\mathbb{L}_Q)$, for all $(e_l, n_l) \in \mathbb{N}_0^{\pm}$, and $k_l \in \mathbb{Z}$, for all $l = 1,\ 2$.

If $k_1 = k = k_2$ in $\mathbb{Z}$, then

$$\beta_{e_1}^{n_1}(k_1)\beta_{e_2}^{n_2}(k_2) = \beta_{e_1}^{n_1}(k)\beta_{e_2}^{n_2}(k),$$

dictated by (8.12) in $\mathfrak{B}(k) \subset \mathfrak{B}(\mathbb{Z})$; while if $k_1 \neq k_2$ in $\mathbb{Z}$, then $\beta_{e_1}^{n_1}(k_1)\beta_{e_2}^{n_2}(k_2)$ is simply an element of $\mathfrak{B}(\mathbb{Z})$ contained in $Hom(\mathbb{L}_Q)$.

For example,

$$\begin{aligned}\left(\beta_+^2(-2)\beta_-^1(-2)\right)&\left(U_1^3U_{-2}^2U_4U_1^2\right)\\ &= \beta_+^2(-2)\left(U_1^3U_{-2-1}^2U_4U_1^2\right)\\ &= \beta_+^2(-2)\left(U_1^3U_{-3}^2U_4U_1^2\right)\\ &= U_1^3U_{-3}^2U_4U_1^2\\ &= \beta_-^1(-2)\left(U_1^3U_{-2}^2U_4U_1^2\right),\end{aligned}$$

and

$$\begin{aligned}\left(\beta_+^2(-2)\beta_-^3(1)\right)&\left(U_1^3U_{-2}^2U_4U_1^2\right)\\ &= \beta_+^2(-2)\left(U_{1-3}^3U_{-2}^2U_4U_{1-3}^2\right)\\ &= \beta_+^2(-2)\left(U_{-2}^3U_{-2}^2U_4U_{-2}^2\right)\\ &= \beta_+^2(-2)\left(U_{-2}^5U_4U_{-2}^2\right)\\ &= U_{-2+2}^5U_4U_{-2+2}^2 = U_0^5U_4U_0^2,\end{aligned}$$

etc.

Definition 8.2 Let $\mathfrak{B}(\mathbb{Z})$ be the sub-monoid (8.15) of the homomorphism semigroup $Hom(\mathbb{L}_Q)$ of the semicircular filterization $\mathbb{L}_Q$. Then we call it the integer-shift monoid (acting) on $\mathbb{L}_Q$.

As we have seen, even though $\mathfrak{B}(\mathbb{Z})$ is induced by $\mathfrak{B}$, the integer-shift monoid $\mathfrak{B}(\mathbb{Z})$ and the integer-shift group $\mathfrak{B}$ are totally different algebraic structures in $Hom(\mathbb{L}_Q)$ (e.g., see (7.2.26), (8.14) and (8.14)′).

9 Free-Homomorphic Relations on $\mathbb{L}_Q$

In this section, based on the results of Sects. 7 and 8, we study free-homomorphic relations in the semicircular filterization $\mathbb{L}_Q$. Recall first that, the main results of Sect. 7 show that all integer-shifts $\beta \in \mathfrak{B}$ are free-isomorphisms on $\mathbb{L}_Q$, and hence, the Banach $*$-probability spaces

$$\left\{\left(\beta\left(\mathbb{L}_Q\right),\ \tau\circ\beta\right):\beta\in\mathfrak{B}\right\}$$

are free-isomorphic from each other.

So, here, we are interested in sub-structures of $\mathbb{L}_Q$, and their free-homomorphic relations among them in $\mathbb{L}_Q$.

9.1 Free-Homomorphisms Among $\mathbb{L}[J]$'s in $\mathbb{L}_Q$

Let $J=(j_1,\ldots,j_N)$ be an N-tuple of "mutually-distinct" integers $j_1,\ \ldots,\ j_N\in\mathbb{Z}$, for

$$N\in\mathbb{N}_{>1}^{\infty}\overset{def}{=}(\mathbb{N}\ \setminus\ \{1\})\cup\{\infty\}.$$

For such an integer-sequence J, define the Banach $*$-subalgebra $\mathbb{L}[J]$ of the semicircular filterization $\mathbb{L}_Q$ by (8.1). i.e.,

$$\mathbb{L}[J]\overset{def}{=}\underset{n=1}{\overset{N}{\star}}\mathbb{L}_{j_n},\ \text{with}\ \mathbb{L}_{j_n}=\overline{\mathbb{C}\left[\{U_{j_n}\}\right]},\tag{9.1.1}$$

where $\mathbb{L}_{j_n}$ are the free blocks of $\mathbb{L}_Q$, for all $n=1,\ \ldots,\ N$.

As we discussed in Sect. 8, by the structure theorem (6.7) of $\mathbb{L}_Q$, the free product Banach $*$-algebra $\mathbb{L}[J]$ of (9.1.1) is indeed a well-defined Banach $*$-subalgebra of $\mathbb{L}_Q$, understood as a free-probabilistic sub-structure (8.2), i.e.,

$$\mathbb{L}[J]\overset{denote}{=}\left(\mathbb{L}[J],\ \tau=\tau\mid_{\mathbb{L}[J]}\right),\tag{9.1.2}$$

in $\mathbb{L}_Q=(\mathbb{L}_Q,\ \tau)$, as a Banach $*$-probability space.

Now, let

$$J_l=\left(j_{l1},\ j_{l2},\ldots,j_{lN_l}\right)\tag{9.1.3}$$

be N_l-tuples of mutually-distinct integers for $N_l\in\mathbb{N}_{>1}^{\infty}$, for all $l=1,\ 2$, and let $\mathbb{L}[J_l]$ be the corresponding Banach $*$-probability spaces (9.1.2) in $\mathbb{L}_Q$, for $l=1,\ 2$.

Theorem 9.1 *Let $\mathbb{L}[J_1]$ and $\mathbb{L}[J_2]$ be Banach $*$-subalgebras (9.1.1) of the semicircular filterization $\mathbb{L}_Q$, where J_1 and J_2 are in the sense of (9.1.3). Assume that $N_1\leq N_2$ in $\mathbb{N}_{>1}^{\infty}$. Then the Banach $*$-probability space $\mathbb{L}[J_1]$ of (9.1.2) is free-homomorphic to the Banach $*$-probability space $\mathbb{L}[J_2]$ of (9.1.2) in $\mathbb{L}_Q$. i.e.,*

$$\mathbb{L}[J_1]\overset{\text{free-homo}}{\subseteq}\mathbb{L}[J_2]\ \textit{in}\ \mathbb{L}_Q,\tag{9.1.4}$$

where "$\overset{\text{free-homo}}{\subseteq}$" means "being free-homomorphic to."

Proof Let J_1 and J_2 be in the sense of (9.1.3) under an additional assumption that $N_1 \leq N_2$ in $\mathbb{N}_{>1}^{\infty}$. Note that, since J_1 and J_2 are consisting of mutually distinct integers, the subsets

$$[J_l] = \{j_{l1},\ j_{l2}, \ldots, j_{lN_l}\}$$

of $\mathbb{Z}$ have their cardinalities N_l, too, where $[J_l]$ means the set consisting of all entries of J_l, for all $l = 1,\ 2$.

Suppose there exists

$$\{k_1,\ \ldots,\ k_{N_1}\} \subseteq \{1,\ \ldots,\ N_2\},$$

such that

$$j_{2k_1} = j_{11}e_1n_1,\ j_{2k_2} = j_{12}e_2n_2, \ldots, \tag{9.1.5}$$

and

$$j_{2k_{N_1}} = j_{1N_1}e_{N_1}n_{N_1},$$

for some $(e_i,\ n_i) \in \mathbb{N}_0^{\pm}$, for $i = 1,\ \ldots,\ N_1$.

Since $N_1 \leq N_2$ in $\mathbb{N}_{>1}^{\infty}$, one can naturally take the above relation (9.1.5). For convenience, let

$$j_{21} = j_{11}e_1n_1,\ j_{22} = j_{12}e_2n_2,\ \ldots, \tag{9.1.5$'$}$$

and

$$j_{2N_1} = j_{1N_1}e_{N_1}n_{N_1},$$

for some $(e_i,\ n_i) \in \mathbb{N}_0^{\pm}$, for $i = 1, \ldots,\ N_1$.

Take now an element $\beta(J_1 : J_2) \in \mathfrak{B}(\mathbb{Z})$,

$$\beta(J_1 : J_2) = \prod_{l=1}^{N_1} \beta_{e_l}^{n_l}(j_l) \in \mathfrak{B}(\mathbb{Z}), \tag{9.1.6}$$

where $\mathfrak{B}(\mathbb{Z})$ is the integer-shift monoid (8.15) contained in the homomorphism semigroup $Hom(\mathbb{L}_Q)$. And then, define the restriction $\beta(J_1 \to J_2)$ from $\mathbb{L}[J_1]$ into $\mathbb{L}_Q$, by

$$\beta(J_1 \to J_2) \overset{def}{=} \beta(J_1 : J_2) \mid_{\mathbb{L}[J_1]} : \mathbb{L}[J_1] \to \mathbb{L}_Q. \tag{9.1.6$'$}$$

Then, by (9.1.6), the restriction $\beta(J_1 \to J_2)$ of $\beta(J_1 : J_2)$, in the sense of (9.1.6′), is a well-defined $*$-homomorphism from $\mathbb{L}[J_1]$ into $\mathbb{L}_Q$. Furthermore, the range

$ran\,(\beta(J_1 \to J_2))$ satisfies

$$ran\,(\beta(J_1 \to J_2)) \subseteq \mathbb{L}[J_2] \text{ in } \mathbb{L}_Q, \tag{9.1.7}$$

by (9.1.5′) (or, by (9.1.5)).

So, by (9.1.6′) and (9.1.7), one can regard $\beta(J_1 \to J_2)$ as a $*$-homomorphism from $\mathbb{L}[J_1]$ to $\mathbb{L}[J_2]$, i.e.,

$$\beta(J_1 \to J_2) \;:\; \mathbb{L}[J_1] \overset{*\text{-homo}}{\longrightarrow} \mathbb{L}[J_2]. \tag{9.1.8}$$

Therefore

$$\mathbb{L}[J_1] \overset{*\text{-homo}}{\subseteq} \mathbb{L}[J_2] \text{ in } \mathbb{L}_Q,$$

by the existence of a $*$-homomorphism $\beta(J_1 \to J_2)$ of (9.1.8), where "$\overset{*\text{-homo}}{\subseteq}$" means "being $*$-homomorphic to."

Now, let X_1 and X_2 be free reduced words,

$$X_1 = \prod_{k=1}^{K_1} U_{j_k}^{n_k},\, X_2 = \prod_{k=2}^{K_2} U_{i_k}^{l_k} \in \mathbb{L}[J_1],$$

where $(j_1, \ldots, j_{K_1})$, $(i_1, \ldots, i_{K_2})$ are alternating tuples in $\{j_{11}, \ldots, j_{1N_1}\}$, for $K_1,\ K_2 \in \mathbb{N}$, and suppose

$$X_1 \neq X_2 \text{ in } \mathbb{L}[J_1].$$

Then

$$\begin{aligned}\beta(J_1 \to J_2)\,(X_1) &= \prod_{k=1}^{K_1} U_{j_k e_{j_k} n_{j_k}}^{n_k} \\ &\neq \prod_{k=1}^{K_2} U_{i_k e_{i_k} n_{i_k}}^{l_k} = \beta(J_1 \to J_2)(X_2),\end{aligned}$$

by (8.16), (9.1.5)′ and (9.1.6).

i.e., the $*$-homomorphism $\beta(J_1 \to J_2)$ of (9.1.8) is injective from $\mathbb{L}[J_1]$ into $\mathbb{L}[J_2]$. (Note that the injectivity happens because J_1 and J_2 have mutually distinct entries satisfying (9.1.5)′!)

Also, the injectivity of $\beta(J_1 \to J_2)$ guarantees that every free reduced word $W \in \mathbb{L}[J_1]$ with length-N becomes a free reduced word $\beta(J_1 \to J_2)(W)$ with its length-N in $\mathbb{L}[J_2]$, for all $N \in \mathbb{N}$.

Moreover, this injective $*$-homomorphism $\beta(J_1 \to J_2)$ of (9.1.8) satisfies that

$$\begin{aligned}\tau\left(\beta(J_1 \to J_2)\left(U_{j_{k_l}}^n\right)\right) &= \tau\left(U_{j_{k_l} e_{k_l} n_{k_l}}^n\right) \\ &= \omega_n c_{\frac{n}{2}} = \tau\left(U_{j_{k_l}}^n\right),\end{aligned} \tag{9.1.9}$$

in $\mathbb{L}[J_2]$, for all generators $U_{j_{k_l}} \in \mathbb{L}[J_1]$, for $l = 1, \ldots, N_1$, for all $n \in \mathbb{N}$, by (8.6) and (9.1.5)′.

Therefore, this injective $*$-homomorphism $\beta(J_1 \to J_2)$ is a free-homomorphism by (9.1.9), equivalently, the free-homomorphic relation (9.1.4) holds. □

The following corollary is a direct consequence of (9.1.4).

Corollary 9.1 *Let J be an integer-sequence with mutually distinct entries with its length-N, for $N \in \mathbb{N}_{>1}^{\infty}$, and let $\mathbb{L}[J]$ be the corresponding Banach $*$-probability space* (9.1.2) *in the semicircular filterization $\mathbb{L}_Q$.*

$$\mathbb{L}[J] \overset{\text{free-homo}}{\subseteq} \mathbb{L}[W], \text{ if } W \text{ is an integer-sequence of mutually distinct integers with its length-}n\text{, such that } n \geq N \text{ in } \mathbb{N}_{>1}^{\infty}. \tag{9.1.10}$$

$$\mathbb{L}[J] \overset{\text{free-homo}}{\subseteq} \mathbb{L}_Q. \tag{9.1.11}$$

Proof The proof of (9.1.10) is clear by (9.1.4). The proof of (9.1.11) is done by the canonical embedding map

$$\Phi_J : \mathbb{L}[J] \to \mathbb{L}_Q,$$

defined by

$$\Phi_J(T) = T \in \mathbb{L}_Q, \text{ for all } T \in \mathbb{L}[J].$$

Definitely, this embedding map Φ_J is an injective $*$-homomorphism from $\mathbb{L}[J]$ into $\mathbb{L}_Q$ by (9.1.1). And it satisfies that

$$\tau\left(\Phi_J(T)\right) = \tau(T) \text{ in } \mathbb{L}_Q,$$

for all $T \in \mathbb{L}[J]$. Therefore, $\mathbb{L}[J]$ is free-homomorphic to $\mathbb{L}_Q$. □

Now, let's consider the following special case of (9.1.4).

Theorem 9.2 *Let $J_l = (j_{l1}, \ldots, j_{lN})$ be an integer-sequence with N-many mutually distinct entries for $N \in \mathbb{N}_{>1}^{\infty}$, and let $\mathbb{L}[J_l]$ be the Banach $*$-probability spaces* (9.1.2) *in the semicircular filterization $\mathbb{L}_Q$, for all $l = 1, 2$. Then $\mathbb{L}[J_1]$ and $\mathbb{L}[J_2]$ are free-isomorphic in $\mathbb{L}_Q$. i.e.,*

$$\mathbb{L}[J_1] \overset{\text{free-iso}}{=} \mathbb{L}[J_2] \text{ in } \mathbb{L}_Q, \tag{9.1.12}$$

where "$\overset{\text{free-iso}}{=}$" means "being free-isomorphic to."

Proof Let J_1 and J_2 be given as above, and suppose

$$j_{2l} = j_{1l} e_l n_l \text{ in } \mathbb{Z}, \tag{9.1.13}$$

for some $(e_l, n_l) \in \mathbb{N}_0^{\pm}$, for all $l = 1, \ldots, N$, for the fixed $N \in \mathbb{N}_{>1}^{\infty}$.

Then, by (9.1.4),

$$\mathbb{L}[J_1] \overset{\text{free-homo}}{\subseteq} \mathbb{L}[J_2] \text{ in } \mathbb{L}_Q,$$

with an injective free-homomorphism (9.1.6)′,

$$\beta(J_1 \to J_2) = \prod_{k=1}^{N} \beta_{e_k}^{n_k}(j_{1k}),$$

and

$$\mathbb{L}[J_2] \overset{\text{free-homo}}{\subseteq} \mathbb{L}[J_1] \text{ in } \mathbb{L}_Q,$$

with an injective free-homomorphism in the sense of (9.1.6)′,

$$\beta(J_2 \to J_1) = \prod_{k=1}^{N} \beta_{-e_k}^{n_k}(j_{2k}).$$

Also, by (9.1.13), the morphisms $\beta(J_1 \to J_2)$ and $\beta(J_2 \to J_1)$ are not only injective but also bijective because they preserves the generators.

Note that, for any free reduced word $W = \prod_{l=1}^{n} U_{j_{1k_l}}^{n_l} \in \mathbb{L}[J_1]$, we have

$$\begin{aligned} &(\beta(J_2 \to J_1)\beta\,(J_1 \to J_2))\,(W) \\ &= \beta(J_2 \to J_1)\,(\beta(J_1 \to J_2)(W)) \\ &= \beta(J_2 \to J_1)\left(\prod_{l=1}^{n} U_{j_{2k_l}}^{n_l}\right) = \prod_{l=1}^{n} U_{j_{1k_l}}^{n_l} = W, \end{aligned}$$

and hence,

$$\beta(J_2 \to J_1)\beta(J_1 \to J_2) = 1_{\mathbb{L}[J_1]} \text{ on } \mathbb{L}[J_1], \tag{9.1.14}$$

by the injectivity of $\beta(J_1 \to J_2)$ and $\beta(J_2 \to J_1)$, where $1_{\mathbb{L}[J_l]}$ are the identity maps on $\mathbb{L}[J_l]$,

$$1_{\mathbb{L}[J_l]}(T) = T, \text{ for all } T \in \mathbb{L}[J_l],$$

for all $l = 1, 2$.

Similar to (9.1.14), one obtains that

$$\beta(J_1 \to J_2)\beta(J_2 \to J_1) = 1_{\mathbb{L}[J_2]} \text{ on } \mathbb{L}[J_2]. \tag{9.1.15}$$

By (9.1.13), (9.1.14) and (9.1.15), the injective $*$-homomorphism $\beta(J_1 \to J_2)$ is not only bijective but also

$$\beta(J_1 \to J_2)^{-1} = \beta(J_2 \to J_1) : \mathbb{L}[J_2] \to \mathbb{L}[J_1],$$

and

$$\beta(J_2 \to J_1)^{-1} = \beta(J_1 \to J_2) : \mathbb{L}[J_1] \to \mathbb{L}[J_2].$$

Since both $\beta(J_1 \to J_2)$ and $\beta(J_2 \to J_1)$ are free-homomorphisms, these bijective $*$-homomorphisms are free-isomorphisms. Therefore,

$$\mathbb{L}[J_1] \overset{\text{free-iso}}{=} \mathbb{L}[J_2] \text{ in } \mathbb{L}_Q,$$

i.e., the free-isomorphic relation (9.1.12) holds true. □

The above theorem shows that, for any same-length integer-sequences J_1 and J_2 of mutually distinct entries,

$$\mathbb{L}[J_1] \overset{\text{free-iso}}{=} \mathbb{L}[J_2] \text{ in } \mathbb{L}_Q,$$

by (9.1.12). In particular, one can take a free-isomorphism,

$$\beta(J_1 \to J_2) : \mathbb{L}[J_1] \to \mathbb{L}[J_2],$$

or

$$\beta(J_2 \to J_1) : \mathbb{L}[J_2] \to \mathbb{L}[J_1],$$

in the sense of (9.1.6)′.

9.2 Free-Homomorphic Relations of $\mathbb{L}[J]$'s Where $|J| < \infty$

In this section, we use same notations and concepts used in previous sections. In Sect. 9.1, we considered certain free-homomorphic relations in $\mathbb{L}_Q$. In particular, we showed that

$$\mathbb{L}[J_1] \overset{\text{free-homo}}{\subseteq} \mathbb{L}[J_2] \text{ in } \mathbb{L}_Q, \tag{9.2.1}$$

by (9.1.4), if integer-sequences J_1 and J_2 of mutually distinct entries satisfy $|J_1| \leq |J_2|$ in $\mathbb{N}_{>1}^{\infty}$, where $|J|$ mean the lengths of the sequences J; and

$$\mathbb{L}[J_1] \overset{\text{free-iso}}{=} \mathbb{L}[J_2] \text{ in } \mathbb{L}_Q, \tag{9.2.2}$$

by (9.1.12), whenever $|J_1| = |J_2|$ in $\mathbb{N}_{>1}^{\infty}$.

In this section, we restrict our interests to the cases where given integer-sequences have finite lengths.

Corollary 9.2 *Let* $J_1 = (j_{11}, \ldots, j_{1N_1})$ *and* $J_2 = (j_{21}, \ldots, j_{2N_2})$ *be integer-sequences of mutually distinct entries for* $N_1, N_2 \in \mathbb{N}_{>1}^{\infty}$, *and assume that* $j_{11}, \ldots, j_{1N_1}, j_{21}, \ldots, j_{2N_2}$ *are mutually distinct from each other in* $\mathbb{Z}$. *Let*

$$J = J_1 \vee J_2 = \left(j_{11}, \ldots, j_{1N_1}, j_{21}, \ldots, j_{2N_2}\right),$$

and let $\mathbb{L}[J_1]$, $\mathbb{L}[J_2]$ *and* $\mathbb{L}[J]$ *be in the sense of* (9.1.1), *or* (9.1.2) *in* $\mathbb{L}_Q$.

$$\mathbb{L}[J] \overset{*\text{-iso}}{=} \mathbb{L}[J_1] \star \mathbb{L}[J_2] \textit{ in } \mathbb{L}_Q. \tag{9.2.3}$$

$$\begin{gathered}\textit{If } W \textit{ is an integer-sequence of mutually distinct entries satisfying } |W| \\ \geq (N_1 + N_2) \textit{ in } \mathbb{N}_{>1}^{\infty}, \textit{ then}\end{gathered} \tag{9.2.4}$$

$$\mathbb{L}[J_l] \overset{\textit{free-homo}}{\subseteq} \mathbb{L}[J] \overset{\textit{free-homo}}{\subseteq} \mathbb{L}[W] \textit{ in } \mathbb{L}_Q,$$

for all $l = 1, 2$.

$$\begin{gathered}\textit{If } W \textit{ is an integer-sequence of mutually distinct entries with } |W| \\ = N_1 + N_2 \textit{ in } \mathbb{N}_{>1}^{\infty}, \textit{ then}\end{gathered} \tag{9.2.5}$$

$$\mathbb{L}[J] \overset{\textit{free-iso}}{=} \mathbb{L}[W] \textit{ in } \mathbb{L}_Q.$$

$$\mathbb{L}[J] \overset{\textit{free-homo}}{\subseteq} \mathbb{L}_Q. \tag{9.2.6}$$

Proof The statement (9.2.3) is shown by the very definition (9.1.1), and by the structure theorem (6.7) of $\mathbb{L}_Q$. Here, remark the additional condition that $j_{11}, \ldots, j_{1N_1}, j_{21}, \ldots, j_{2N_2}$ are mutually distinct in $\mathbb{Z}$.

In the statement (9.2.4), the free-homomorphic relations are proven by (9.2.1).
The free-isomorphic relation (9.2.5) is shown by (9.2.2).
Finally, the free-homomorphic relation (9.2.6) is proven by (9.1.11).

□

Now, let J be an N-tuple of mutually distinct "finitely" many integers for $N \in \mathbb{N}_{>1}$, and let $J^{(+n)}$ be the $(N+n)$-tuples of mutually distinct entries in $\mathbb{Z}$, for all $n \in \mathbb{N}$. Then we obtain the following free-homomorphic relation in $\mathbb{L}_Q$.

Corollary 9.3 *Let $J_1, \ldots, J_k$ be integer-sequences of mutually distinct entries satisfying*

$$|J_1| = \cdots = |J_k| \in \mathbb{N}_{>1}, \text{ for } k \in \mathbb{N}_{>1}^{\infty},$$

and let $J_1^{(+n)}, \ldots, J_k^{(+n)}$ be defined as in the above paragraph, for all $n \in \mathbb{N}$. Then

$$\begin{array}{ccccccccc} & \mathbb{L}[J_1] & \overset{\text{free-homo}}{\subseteq} & \mathbb{L}[J_1^{(+1)}] & \overset{\text{free-homo}}{\subseteq} & \mathbb{L}[J_1^{(+2)}] & \overset{\text{free-homo}}{\subseteq} & \cdots \\ \text{free-iso} & \shortparallel & & \shortparallel & & \shortparallel & & \\ & \mathbb{L}[J_2] & \overset{\text{free-homo}}{\subseteq} & \mathbb{L}[J_2^{(+1)}] & \overset{\text{free-homo}}{\subseteq} & \mathbb{L}[J_2^{(+2)}] & \overset{\text{free-homo}}{\subseteq} & \cdots \\ \text{free-iso} & \shortparallel & & \shortparallel & & \shortparallel & & \\ & \vdots & & \vdots & & \vdots & & \vdots \\ \text{free-iso} & \shortparallel & & \shortparallel & & \shortparallel & & \\ & \mathbb{L}[J_k] & \overset{\text{free-homo}}{\subseteq} & \mathbb{L}[J_k^{(+1)}] & \overset{\text{free-homo}}{\subseteq} & \mathbb{L}[J_k^{(+2)}] & \overset{\text{free-homo}}{\subseteq} & \cdots, \end{array} \tag{9.2.7}$$

in $\mathbb{L}_Q$.

Moreover, all Banach $*$-probability spaces in (9.2.7) are free-homomorphic to the semicircular filterization $\mathbb{L}_Q$.

Proof The free-homomorphic relation (9.2.7) is proven by (9.2.3), (9.2.4), (9.2.5) and (9.2.6). And, all Banach $*$-probability spaces in (9.2.7) are free-homomorphic to $\mathbb{L}_Q$, by (9.1.11). □

9.3 Free-Isomorphic Relation of $\mathbb{L}[J]$'s Where $|J| = \infty$

In Sects. 9.1 and 9.2, we studied free-homomorphic relations among

$$\left\{ \mathbb{L}[J] \subseteq \mathbb{L}_Q \;\middle|\; \begin{array}{c} \mathbb{L}[J] \text{ are Banach } *\text{-probability spaces} \\ \text{in } \mathbb{L}_Q, \text{ for integer-sequences } J \\ \text{of mutually distinct entries in } \mathbb{Z}, \text{ where} \\ |J| \in \mathbb{N}_{>1}^{\infty} \end{array} \right\} \cup \{\mathbb{L}_Q\}.$$

And the free-homomorphic relations, where $|J| < \infty$, are illustrated in (9.2.7).

In this section, we consider the "converse" of a homomorphic relation,

$$\mathbb{L}[J] \overset{\text{free-homo}}{\subseteq} \mathbb{L}_Q, \tag{9.3.1}$$

where $J = (j_1, \ j_2, \ j_3, \ \ldots)$ is an "infinite" sequence of mutually distinct integers in $\mathbb{Z}$. Note that, by (9.1.12) and (9.2.2), if J is such an infinite integer-sequence, then

$$\mathbb{L}[J] \overset{\text{free-iso}}{=} \mathbb{L}\left[(0, \ 1, \ 2, \ 3, \ 4, \ \ldots)\right], \tag{9.3.2}$$

in the semicircular filterization $\mathbb{L}_Q$.

By (9.3.2), without loss of generality, we regard $\mathbb{L}[J]$ for all infinite sequences J of mutually distinct integers, as their free-isomorphic Banach $*$-probability space, $\mathbb{L}\left[(0, 1, 2, \ldots)\right]$. i.e., from below,

$$\mathbb{L}[J] \overset{\text{let}}{=} \mathbb{L}[J_\infty] = \mathbb{L}\left[(0, \ 1, \ 2, \ 3, \ \ldots)\right], \tag{9.3.3}$$

equivalently,

$$J \overset{\text{let}}{=} J_\infty = (0, \ 1, \ 2, \ 3, \ \ldots).$$

Under (9.3.3), let's consider the converse of (9.3.1). i.e., is $\mathbb{L}_Q$ free-homomorphic to $\mathbb{L}[J_\infty]$? To answer this question, we study the existence of a certain "injective" $*$-homomorphism from $\mathbb{L}_Q$ "onto" $\mathbb{L}[J_\infty]$, preserving the free-distributional data, independent from our approaches in Sects. 8, 9.1, and 9.2.

First, define a bijection θ from $\mathbb{Z}$ onto $\mathbb{N}_0$, by

$$\theta(0) = 0, \ \theta(\mathbb{N}) = 2\mathbb{N} - 1, \tag{9.3.4}$$

and

$$\theta\,(-\mathbb{N}) = 2\mathbb{N},$$

where

$$\mathbb{Z} = (-\mathbb{N}) \sqcup \{0\} \sqcup (\mathbb{N}), \tag{9.3.4$'$}$$

and

$$\mathbb{N}_0 = \{0\} \sqcup (2\mathbb{N} - 1) \sqcup (2\mathbb{N}),$$

with

$$-\mathbb{N} = \{-n \in \ \mathbb{Z} \ : \ n \in \mathbb{N}\},$$

$$2\mathbb{N} - 1 = \{2n - 1 \in \mathbb{N} \ : \ n \in \mathbb{N}\},$$

and

$$2\mathbb{N} = \{2n \in \mathbb{N} \ : \ n \in \mathbb{N}\}.$$

In other words,

$$\theta(j) = \begin{cases} 0 & \text{if } j = 0 \\ 2j - 1 & \text{if } j > 0 \\ -2j & \text{if } j < 0, \end{cases} \tag{9.3.4''}$$

for all $j \in \mathbb{Z}$. For example,

$$\theta(0) = 0, \ \theta(1) = 2 \cdot 1 - 1 = 1, \ \theta(-1) = -2(-1) = 2,$$
$$\theta(2) = 2 \cdot 2 - 1 = 3, \ \theta(-2) = -2 \cdot (-2) = 4,$$
$$\theta(3) = 2 \cdot 3 - 1 = 5, \ \text{and } \theta(-3) = -2\left((-3)\right) = 6,$$

etc.

By the definition (9.3.4), this map θ of (9.3.4$''$) is a well-defined bijection from $\mathbb{Z}$ onto $\mathbb{N}_0$, by (9.3.4$'$).

Now, define a bounded "multiplicative" linear transformation

$$\Theta \ : \ \mathbb{L}_Q \to \mathbb{L}[J_\infty],$$

by a multiplicative linear morphism satisfying

$$\Theta\left(U_j\right) = U_{\theta(j)}, \tag{9.3.5}$$

for all $j \in \mathbb{Z}$, where $U_j \in \mathcal{S}$ in $\mathbb{L}_Q$, and θ is the bijection (9.3.4) from $\mathbb{Z}$ onto $\mathbb{N}_0$.

By the multiplicativity of the morphism Θ of (9.3.5), we have that

$$\Theta\left(\prod_{l=1}^{N} U_{j_l}^{n_l}\right) = \prod_{l=1}^{N} U_{\theta(j_l)}^{n_l}, \tag{9.3.5$'$}$$

for all free reduced words $\prod_{l=1}^{N} U_{j_l}^{n_l}$ of $\mathbb{L}_Q$ in $\mathcal{S}$, for any $n_1, \ldots, n_N, \ N \in \mathbb{N}$.

For example,

$$\Theta\left(U_{-3}^2 U_{-2} U_0^3 U_1^2 U_{-1}^4 U_2^2\right)$$
$$= U_{\theta(-3)}^2 U_{\theta(-2)} U_{\theta(0)}^3 U_{\theta(1)}^2 U_{\theta(-1)}^4 U_{\theta(2)}^2$$

by (9.3.5$'$)

$$= U_6^2 U_4 U_0^3 U_1^2 U_2^4 U_3^2,$$

by (9.3.4), (9.3.4$''$), and (9.3.5).

Lemma 9.1 *Let* $\Theta : \mathbb{L}_Q \to \mathbb{L}[J_\infty]$ *be the bounded multiplicative linear transformation* (9.3.5) *satisfying* (9.3.5′). *Then it is a well-defined* $*$*-homomorphism from* $\mathbb{L}_Q$ *onto* $\mathbb{L}[J_\infty]$. *i.e.,*

$$\text{the morphism } \Theta \text{ of (9.3.5) is a } * \text{-isomorphism.} \tag{9.3.6}$$

Proof Let Θ be in the sense of (9.3.5). Then, by the bijectivity of θ in the sense of (9.3.4) or (9.3.4′), this linear transformation Θ preserves the free generators $\{U_j\}_{j\in\mathbb{Z}}$ of $\mathbb{L}_Q$ onto the free generators $\{U_j\}_{j\in\mathbb{N}_0}$ of $\mathbb{L}[J_\infty]$. i.e., this morphism Θ is bounded and bijective from $\mathbb{L}_Q$ onto $\mathbb{L}[J_\infty]$. In particular, the injectivity of Θ is guaranteed by the generator-preserving property, and the bijectivity of θ.

Let $W = \prod_{l=1}^{N} U_{j_l}^{n_l}$ be an arbitrary free reduced word of the semicircular filterization $\mathbb{L}_Q$ in the free semicircular family $\mathcal{S}$ of (6.4). Then $\Theta(W)$ forms a free reduced word in $\mathbb{L}[J_\infty]$, too, by the generator-preserving property of Θ. So, if W_1 and W_2 are free reduced words forming a new free reduced word W_1W_2 in $\mathbb{L}_Q$, then

$$\Theta(W_1W_2) = \Theta(W_1)\Theta(W_2) \text{ in } \mathbb{L}[J_\infty].$$

It implies that

$$\Theta(T_1T_2) = \Theta(T_1)\ \Theta(T_2) \text{ in } \mathbb{L}[J_\infty], \tag{9.3.7}$$

for all $T_1,\ T_2 \in \mathbb{L}_Q$.

Moreover,

$$\Theta(W^*) = \Theta\left(\prod_{l=1}^{N} U_{j_{N-l+1}}^{n_{N-l+1}}\right) = \prod_{l=1}^{N} U_{\theta(j_{N-l+1})}^{n_{N-l+1}}$$

$$= \left(\prod_{l=1}^{N} U_{\theta(j_l)}^{n_l}\right)^* = (\Theta(W))^*,$$

implying that

$$\Theta(T^*) = (\Theta(T))^* \text{ in } \mathbb{L}[J_\infty], \tag{9.3.8}$$

for all $T \in \mathbb{L}_Q$.

Therefore, this bijective bounded multiplicative linear transformation Θ forms a $*$-isomorphism from $\mathbb{L}_Q$ onto $\mathbb{L}[J_\infty]$, by (9.3.7) and (9.3.8).

It is not difficult to check that

$$\Theta^{-1} : \mathbb{L}[J_\infty] \to \mathbb{L}_Q$$

is the morphism satisfying

$$\Theta^{-1}(U_k) = U_{\theta^{-1}(k)},$$

for all $U_k \in \mathcal{S}$, where

$$\theta^{-1}(k) = \begin{cases} 0 & \text{if } k = 0 \\ \frac{k+1}{2} & \text{if } k \in 2\mathbb{N} - 1 \\ -\frac{k}{2} & \text{if } k \in 2\mathbb{N}, \end{cases}$$

for all $k \in \mathbb{Z}$.

Note that

$$\Theta\Theta^{-1} = 1_{\mathbb{L}[J_\infty]}, \text{ and } \Theta^{-1}\Theta = 1_{\mathbb{L}_Q}.$$

Therefore, the relation (9.3.6) holds. □

By (9.3.6), one can take a $*$-isomorphism Θ of (9.3.5) from the semicircular filterization $\mathbb{L}_Q$ onto the Banach $*$-probability space $\mathbb{L}[J_\infty]$ of (9.3.3).

Theorem 9.3 *Let J be an integer sequence of mutually distinct entries in $\mathbb{Z}$, with $|J| = \infty$ in $\mathbb{N}^{\infty}_{>1}$. Then*

$$\mathbb{L}[J] \overset{\text{free-iso}}{=} \mathbb{L}_Q. \tag{9.3.9}$$

Proof In (9.3.6), there does exist a $*$-isomorphism Θ of (9.3.5) from $\mathbb{L}_Q$ onto $\mathbb{L}[J_\infty]$, where

$$J_\infty = (0,\ 1,\ 2,\ 3, \ldots).$$

Therefore,

$$\mathbb{L}_Q \overset{*\text{-iso}}{=} \mathbb{L}[J_\infty].$$

Moreover, if $\prod_{l=1}^{N} U_{j_l}^{n_l}$ is a free reduced word of $\mathbb{L}_Q$ in $\mathcal{S}$ with its length-N, then

$$\Theta\left(\prod_{l=1}^{N} U_{j_l}^{n_l}\right) = \prod_{l=1}^{N} U_{\theta(j_l)}^{n_l} \in \mathbb{L}[J_\infty]$$

is a free reduced word of $\mathbb{L}[J_\infty]$ in the generator set $\{U_j\}_{j\in\mathbb{N}_0}$ with the same length-N, by (9.3.4″), and (9.3.5). And, one has

$$\tau\left(U_j^n\right) = \omega_n c_{\frac{n}{2}} = \tau\left(U_{\theta(j)}^n\right) = \tau\left(\Theta(U_j)^n\right), \ \forall n \in \mathbb{N},$$

for all $U_j \in \mathcal{S}$, implying that

$$\tau\left(\Theta(T)\right) = \tau\left(T\right) \text{ in } \mathbb{L}[J_\infty], \text{ for all } T \in \mathbb{L}_Q.$$

Thus, the $*$-isomorphism Θ is a free-isomorphism, i.e.,

$$\mathbb{L}_Q \overset{\text{free-iso}}{=} \mathbb{L}[J_\infty]. \tag{9.3.10}$$

By (9.1.12), if $|J| = \infty = |J_\infty|$ in $\mathbb{N}^\infty_{>1}$, then

$$\mathbb{L}[J] \overset{\text{free-iso}}{=} \mathbb{L}[J_\infty] \text{ in } \mathbb{L}_Q.$$

Therefore,

$$\mathbb{L}_Q \overset{\text{free-iso}}{=} \mathbb{L}[J_\infty] \overset{\text{free-iso}}{=} \mathbb{L}[J],$$

whenever $|J| = \infty$ in $\mathbb{N}^\infty_{>1}$, by (9.1.12) and (9.3.10).

It proves the statement (9.3.9). □

The proof of the free-isomorphic relation (9.3.9) is summarized by that, for any integer sequences J of mutually distinct "infinitely" many entries,

$$\mathbb{L}[J] \underset{(9.1.12)}{\overset{\text{free-iso}}{=}} \mathbb{L}\left[(0,\ 1,\ 2,\ 3,\ \ldots)\right] \underset{(9.3.10)}{\overset{\text{free-iso}}{=}} \mathbb{L}_Q,$$

by the free-isomorphism Θ of (9.3.5). By (9.3.9), one can obtain the following corollary.

Corollary 9.4 *Let J be an integer sequence of mutually distinct infinitely many entries, i.e., $|J| = \infty \in \mathbb{N}^\infty_{>1}$. Suppose $J^{(+n)}$ is an integer sequence obtained by adding mutually distinct n-many integers to the sequence J, where all such n-many integers are mutually distinct from the entries of J, for $n \in \mathbb{N}$. Then*

$$\mathbb{L}[J] \overset{\textit{free-iso}}{=} \mathbb{L}_Q \overset{\textit{free-iso}}{=} \mathbb{L}[J^{(+n)}], \tag{9.3.11}$$

in the semicircular filterization $\mathbb{L}_Q$.

Proof Since $\left|J^{(+n)}\right| = \infty + n = \infty$ in $\mathbb{N}^\infty_{>1}$, for all $n \in \mathbb{N}$, we have

$$\mathbb{L}[J^{(+n)}] \overset{\text{free-iso}}{=} \mathbb{L}_Q,$$

by (9.3.9). Therefore, free-isomorphic relation (9.3.11) holds. □

By (9.3.9) and (9.3.11), we obtain the following corollary similar to (9.2.7).

Corollary 9.5 *Let $J_1, \ldots, J_k$ be integer sequences of mutually distinct "infinitely" many entries in $\mathbb{Z}$, for any $k \in \mathbb{N}^{\infty}_{>1}$, and let $J_l^{(+n)}$ be integer sequences obtained by adding mutually distinct n-more integers to the integer sequences J_l, which are distinct from the entries of J_l, for all $l = 1, \ldots, k$, for all $n \in \mathbb{N}$, then*

$$\begin{array}{ccccccc} & \mathbb{L}[J_1] & \overset{\textit{free-iso}}{=} & \mathbb{L}[J_1^{(+1)}] & \overset{\textit{free-iso}}{=} & \mathbb{L}[J_1^{(+2)}] & \overset{\textit{free-iso}}{=} \cdots \\ \textit{free-iso} & \shortparallel & & \shortparallel & & \shortparallel & \\ & \mathbb{L}[J_2] & \overset{\textit{free-iso}}{=} & \mathbb{L}[J_2^{(+1)}] & \overset{\textit{free-iso}}{=} & \mathbb{L}[J_2^{(+2)}] & \overset{\textit{free-iso}}{=} \cdots \\ \textit{free-iso} & \shortparallel & & \shortparallel & & \shortparallel & \\ & \vdots & & \vdots & & \vdots & \vdots \\ \textit{free-iso} & \shortparallel & & \shortparallel & & \shortparallel & \\ & \mathbb{L}[J_k] & \overset{\textit{free-iso}}{=} & \mathbb{L}[J_k^{(+1)}] & \overset{\textit{free-iso}}{=} & \mathbb{L}[J_k^{(+2)}] & \overset{\textit{free-iso}}{=} \cdots . \end{array} \tag{9.3.12}$$

Moreover, all Banach $*$-probability spaces in (9.3.12) are free-isomorphic to the semicircular filterization $\mathbb{L}_Q$.

Proof The proof of the free-isomorphic relations in (9.3.12) are done by (9.3.9) and (9.3.11). □

9.4 Summary and Discussion

The main results of Sects. 9.1–9.3 show certain free-homomorphic relations in the semicircular filterization $\mathbb{L}_Q$ up to semicircular free generators of $\mathcal{S}$, as follows; (i) if $J_1, \ldots, J_k$ are "finite" integer sequences with $|J_1| = \ldots = |J_k| \in \mathbb{N}_{>1}$, for $k \in \mathbb{N}^{\infty}_{>1}$, then

$$\begin{array}{ccccccc} & \mathbb{L}[J_1] & \overset{\text{free-homo}}{\subseteq} & \mathbb{L}[J_1^{(+1)}] & \overset{\text{free-homo}}{\subseteq} & \mathbb{L}[J_1^{(+2)}] & \overset{\text{free-homo}}{\subseteq} \cdots \\ \text{free-iso} & \shortparallel & & \shortparallel & & \shortparallel & \\ & \mathbb{L}[J_2] & \overset{\text{free-homo}}{\subseteq} & \mathbb{L}[J_2^{(+1)}] & \overset{\text{free-homo}}{\subseteq} & \mathbb{L}[J_2^{(+2)}] & \overset{\text{free-homo}}{\subseteq} \cdots \\ \text{free-iso} & \shortparallel & & \shortparallel & & \shortparallel & \\ & \vdots & & \vdots & & \vdots & \vdots \\ \text{free-iso} & \shortparallel & & \shortparallel & & \shortparallel & \\ & \mathbb{L}[J_k] & \overset{\text{free-homo}}{\subseteq} & \mathbb{L}[J_k^{(+1)}] & \overset{\text{free-homo}}{\subseteq} & \mathbb{L}[J_k^{(+2)}] & \overset{\text{free-homo}}{\subseteq} \cdots , \end{array} \tag{9.4.1}$$

and all Banach $*$-probability spaces in (9.4.1) are free-homomorphic to $\mathbb{L}_Q$, by (9.2.7), and (ii) if J are the integer sequences of mutually distinct "infinitely" many entries in $\mathbb{Z}$, then

$$\mathbb{L}[J] \overset{\text{free-iso}}{=} \mathbb{L}_Q, \tag{9.4.2}$$

by (9.3.9) and (9.3.11), and hence, the equivalence diagram (9.3.12) is obtained.

Let

$$J = (-j,\ j)\,, \text{ with } j \in \mathbb{N} \subset \mathbb{Z}, \tag{9.4.3}$$

and

$$J_0 = (-j,\ 0,\ j)\,.$$

By the condition that j is taken from $\mathbb{N}$ in $\mathbb{Z}$, the pair J consists of distinct integers $-j$ and j, and the triple J_0 consists of mutually distinct integers $-j,\ 0$ and j in $\mathbb{Z}$. For such integer sequences J and J_0 of (9.4.3), one has the corresponding Banach $*$-probability spaces,

$$\mathbb{L}[J] \text{ and } \mathbb{L}[J_0],$$

as free-probabilistic sub-structures of $\mathbb{L}_Q$.

By (9.4.1), we directly obtain that

$$\mathbb{L}[J] \overset{\text{free-homo}}{\subseteq} \mathbb{L}[J_0] \text{ in } \mathbb{L}_Q. \tag{9.4.4}$$

How about the converse of (9.4.4)? In other words, is $\mathbb{L}[J_0]$ free-homomorphic to $\mathbb{L}[J]$ in $\mathbb{L}_Q$?

Even though $|J| = 2 \neq 3 = |J_0|$, can we find at least one injective $*$-homomorphism from $\mathbb{L}[J_0]$ to $\mathbb{L}[J]$? If we can, can it be a free-homomorphism? As expected, it is not easy to answer this question.

Remark 9.1 Let F_N be the free group with N-generators for $N \in \mathbb{N}_{>1}^{\infty}$, and let $L(F_N)$ be the free group factor, which is the group von Neumann algebra generated by F_N equipped with its canonical trace under the unitary regular representation of F_N. The famous main result of [19] shows that either the statement (9.4.5), or the statement (9.4.6) holds, where

$$L(F_n) \cong L(F_\infty), \text{ for all } n \in \mathbb{N}_{>1}^{\infty}, \tag{9.4.5}$$

$$L(F_{n_1}) \ncong L(F_{n_2}), \text{ if and only if } n_1 \neq n_2 \text{ in } \mathbb{N}_{>1}^{\infty}, \tag{9.4.6}$$

where "$\cong$" means "being W^*-isomorphic." Under the authors' knowledge, no proof, showing which one holds true, is known yet.

Here, we have the similar difficulties. i.e., we are not sure

$$\mathbb{L}\,[J] \overset{*\text{-iso}}{=} \mathbb{L}\,[J_0]\, in \mathbb{L}_Q,$$

or

$$\mathbb{L}\left[J\right] \overset{*\text{-iso}}{\neq} \mathbb{L}\left[J_0\right] in \mathbb{L}_Q,$$

where J and J_0 are in the sense of (9.4.3).

The only relations clear now are that

$$\mathbb{L}\left[J\right] \overset{*\text{-homo}}{\subseteq} \mathbb{L}\left[J_0\right] \text{ in } \mathbb{L}_Q,$$

because

$$\mathbb{L}[J] \overset{\text{free-homo}}{\subseteq} \mathbb{L}[J_0] \text{ in } \mathbb{L}_Q,$$

by (9.4.1) and (9.4.4). i.e., one can find a free-homomorphism $\beta(J \to J_0)$ in the integer-shift monoid $\mathfrak{B}(\mathbb{Z})$.

Thus it seems natural to check the existence of well-defined (injective) $*$-homomorphisms from $\mathbb{L}[J_0]$ to $\mathbb{L}[J]$, to answer our question, like in Sect. 9.3.

Note that

$$\mathbb{L}[J] \overset{\text{Banach-sp}}{=} \mathbb{C} \oplus \left(\overset{\infty}{\underset{n=1}{\oplus}} \left(\underset{(j_{i_1}, \ldots, j_{i_n}) \in Alt(\{-j, j\}^n)}{\oplus} \left(\overset{n}{\underset{l=1}{\otimes}} \mathbb{L}^o_{j_{i_l}} \right) \right) \right), \tag{9.4.7}$$

and

$$\mathbb{L}[J_0] \overset{\text{Banach-sp}}{=} \mathbb{C} \oplus \left(\overset{\infty}{\underset{n=1}{\oplus}} \left(\underset{(k_{i_1}, \ldots, k_{i_n}) \in Alt(\{-j, 0, j\}^n)}{\oplus} \left(\overset{n}{\underset{l=1}{\otimes}} \mathbb{L}^o_{k_{i_l}} \right) \right) \right),$$

where

$$\mathbb{L}^o_j \overset{def}{=} \mathbb{L}_j \ominus \left(\mathbb{C} \cdot U^0_j \right) = \mathbb{L}_j \ominus \mathbb{C},$$

for all $j \in \mathbb{Z}$, as Banach spaces, where "$\overset{\text{Banach-sp}}{=}$" means "being Banach-space-isomorphic to," and where "$\ominus$" means the "Banach-space orthogonal complement" in terms of the Banach-space direct product $\oplus$, and where $\otimes$ is the Banach-space tensor product (e.g., [21, 25]).

One can understand the vectors of

$$\overset{n}{\underset{l=1}{\otimes}} \mathbb{L}^o_{j_{i_l}} \text{ of } \mathbb{L}[J],$$

and those of

$$\overset{n}{\underset{l=1}{\otimes}} \mathbb{L}^o_{k_{i_l}} \text{ of } \mathbb{L}[J_0]$$

in (9.4.7) are the scalar products of free reduced words with their lengths-n in the free semicircular family $\{U_{-j}, U_j\}$, respectively, those in the free semicircular family $\{U_{-j}, U_0, U_j\}$.

The Banach-space expression (9.4.7) can be re-expressed by

$$\mathbb{L}[J] \overset{\text{Banach-sp}}{=} \mathbb{C} \oplus \left(\mathbb{L}_{-j}^o \oplus \mathbb{L}_j^o\right) \oplus \left(\left(\mathbb{L}_{-j}^o \otimes \mathbb{L}_j^o\right) \oplus \left(\mathbb{L}_j^o \otimes \mathbb{L}_{-j}^o\right)\right) \oplus \left((\mathbb{L}_j^o \otimes \mathbb{L}_{-j}^o \otimes \mathbb{L}_j^o) \oplus \left(\mathbb{L}_{-j}^o \otimes \mathbb{L}_j^o \otimes \mathbb{L}_{-j}^o\right)\right) \oplus \cdots, \tag{9.4.8}$$

and

$$\mathbb{L}[J_0] \overset{\text{Banach-sp}}{=} \mathbb{C} \oplus \left(\mathbb{L}_{-j}^o \oplus \mathbb{L}_0^o \oplus \mathbb{L}_j^o\right) \oplus \left(\left(\mathbb{L}_{-j}^o \otimes \mathbb{L}_0^o\right) \oplus \left(\mathbb{L}_0^o \otimes \mathbb{L}_{-j}^o\right) \oplus \left(\mathbb{L}_0^o \otimes \mathbb{L}_j^o\right) \oplus \left(\mathbb{L}_j^o \otimes \mathbb{L}_0^o\right) \oplus \left(\mathbb{L}_{-j}^o \otimes \mathbb{L}_j^o\right) \oplus \left(\mathbb{L}_j^o \otimes \mathbb{L}_{-j}^o\right)\right) \oplus \cdots.$$

The above Banach-space expression (9.4.7) illustrates that, to obtain a suitable (injective) $*$-homomorphism (and hence, a possible free-homomorphism) from $\mathbb{L}[J_0]$ to $\mathbb{L}[J]$, one has to assign the elements of $\mathbb{L}_0^o$, and the vectors of the direct summands of $\mathbb{L}[J_0]$, containing their tensor factor $\mathbb{L}_0^o$, wisely. See (9.4.8).

Problem 9.1 $\mathbb{L}[J_0] \overset{*\text{-homo}}{\subseteq} \mathbb{L}[J]$, injectively?

Problem 9.2 $\mathbb{L}[J_0] \overset{\text{free-homo}}{\subseteq} \mathbb{L}[J]$?

In the long run, can we answer the following question?

Problem 9.3 $\mathbb{L}[J] \overset{\text{free-iso}}{=} \mathbb{L}[J_0]$ in $\mathbb{L}_Q$?

Conjecture $\mathbb{L}[J_0]$ is not free-homomorphic to $\mathbb{L}[J]$.

Even though it is possible that $\mathbb{L}[J_0]$ is $*$-isomorphic to $\mathbb{L}[J]$, as Banach $*$-algebras, the Banach $*$-probability spaces $\mathbb{L}[J_0]$ and $\mathbb{L}[J]$ may not be free-isomorphic.

References

1. D. Alpay, P. E. T. Jorgensen, and G. Salomon, On Free Stochastic Processes and Their Derivatives, Stochastic Process. Appl., 124, no. 10, (2014) 3392–3411.
2. D. Alpay, P. E. T. Jorgensen, and D. Levanony, On the Equivalence of Probability Spaces, J. Theoret. Probab., 30, no. 3, (2017) 813–841.

3. D. Alpay, and P. E. T. Jorgensen, Spectral Theory for Gaussian Processes: Reproducing Kernels, Boundaries, and L^2-Wavelet Generators with Fractional Scales, Numer. Funct. Anal. Optim., 36, no. 10, (2015) 1239–1285.
4. D. Alpay, P. E. T. Jorgensen, and D. Levanony, A Class of Gaussian Processes with Fractional Spectral Measures, J. Funct. Anal., 261, no. 2, (2011) 507–541.
5. M. Ahsanullah, Some Inferences on Semicircular Distribution, J. Stat. Theo. Appl., 15, no. 3, (2016) 207–213.
6. H. Bercovici, and D. Voiculescu, Superconvergence to the Central Limit and Failure of the Cramer Theorem for Free Random Variables, Probab. Theo. Related Fields, 103, no. 2, (1995) 215–222.
7. M. Bozejko, W. Ejsmont, and T. Hasebe, Noncommutative Probability of Type D, Internat. J. Math., 28, no. 2, (2017) 1750010, 30.
8. M. Bozheuiko, E. V. Litvinov, and I. V. Rodionova, An Extended Anyon Fock Space and Non-commutative Meixner-Type Orthogonal Polynomials in the Infinite-Dimensional Case, Uspekhi Math. Nauk., 70, no. 5, (2015) 75–120.
9. I. Cho, Free Semicircular Families in Free Product Banach $*$-Algebras Induced by p-Adic Number Fields over Primes p, Compl. Anal. Oper. Theo., 11, no. 3, (2017) 507–565.
10. I. Cho, Semicircular-Like Laws and the Semicircular Law Induced by Orthogonal Projections, Compl. Anal. Oper. Theo., 12, (2018) 1657–1695.
11. I. Cho, and P. E. T. Jorgensen, Semicircular Elements Induced by Projections on Separable Hilbert Spaces, a chapter of Operator Theory: Advances and Applications, 275, ISBN: 978-030-18483-4, (2019) Published by Birkhauser, Cham, 167–209.
12. I. Cho, and P. E. T. Jorgensen, Semicircular Elements Induced by p-Adic Number Fields, Opuscula Math., 35, no. 5, (2017) 665–703.
13. A. Connes, Noncommutative Geometry, ISBN: 0-12-185860-X, (1994) Academic Press (San Diego, CA).
14. T. Gillespie, Prime Number Theorems for Rankin-Selberg L-Functions over Number Fields, Sci. China Math., 54, no. 1, (2011) 35–46.
15. P. R. Halmos, Hilbert Space Problem Books, Grad. Texts in Math., 19, ISBN: 978-0387906850, (1982) Published by Springer.
16. B. Meng, and M. Guo, Operator-Valued Semicircular Distribution and its Asymptotically Free Matrix Models, J. Math. Res. Exposition, 28, no. 4, (2008) 759–768.
17. I. Nourdin, G. Peccati, and R. Speicher, Multi-Dimensional Semicircular Limits on the Free Wigner Chaos, Progr. Probab., 67, (2013) 211–221.
18. V. Pata, The Central Limit Theorem for Free Additive Convolution, J. Funct. Anal., 140, no. 2, (1996) 359–380.
19. F. Radulescu, Random Matrices, Amalgamated Free Products and Subfactors of the C^*-Algebra of a Free Group of Nonsingular Index, Invent. Math., 115, (1994) 347–389.
20. P. Shor, Quantum Information Theory: Results and Open Problems, Geom. Funct. Anal (GAFA), Special Volume: GAFA2000, (2000) 816–838.
21. R. Speicher, Combinatorial Theory of the Free Product with Amalgamation and Operator-Valued Free Probability Theory, Amer. Math. Soc. Mem., vol 132, no. 627, (1998).
22. R. Speicher, Free Probability and Random Matrices, Proceedings of the International Congress of Mathematicians, Seoul 2014, vol. III, Published by Kyung Moon Sa, (2014) 477–501.
23. V. S. Vladimirov, I. V. Volovich, and E. I. Zelenov, p-Adic Analysis and Mathematical Physics, Ser. Soviet & East European Math., vol 1, ISBN: 978-981-02-0880-6, (1994) World Scientific.
24. D. Voiculescu, Free Probability for Pairs of Faces I, Comm. Math. Phy., 332, no. 3, (2014) 955–980.
25. D. Voiculescu, K. J. Dykema, and A. Nica, Free Random Variables, CRM Monograph Series, vol 1., ISBN: 978-0-8218-1140-5, (1992) Published by Ame. Math. Soc..
26. Y. Yin, Z. Bai, and J. Hu, On the Semicircular Law of Large-Dimensional Random Quaternion Matrices, J. Theo. Probab., 29, no. 3, (2016) 1100–1120.
27. Y. Yin, and J. Hu, On the Limit of the Spectral Distribution of Large-Dimensional Random Quaternion Covariance Matrices, Random Mat. Theo. Appl., 6, no. 2, (2017) 1750004, 20.

Self-Adjoint Extensions of a Symmetric Linear Relation with Finite Defect: Compressions and Straus Subspaces

Aad Dijksma and Heinz Langer

Dedicated to our colleague and friend Victor Emanuelovich Katsnelson on the occasion of his 75-th birthday

Abstract Let S be a symmetric relation with finite and equal defect numbers in the Hilbert space $\mathfrak{H}$. If $\widetilde{A}$ is a self-adjoint extension of S in some larger Hilbert space $\widetilde{\mathfrak{H}}$, the compression of $\widetilde{A}$ to $\mathfrak{H}$ is a symmetric extension of S. We study this compression in dependence of the parameter $\mathcal{T}$, which parametrizes the extensions $\widetilde{A}$ according to M.G. Krein's resolvent formula. By means of a fractional transformation, analogous results are proved for the Straus extensions of S at a real point.

Keywords Hilbert space · Symmetric and self-adjoint operators · Linear relations · Self-adjoint extensions · Compressions · Straus extensions · Generalized resolvents · Krein's resolvent formula · Q-functions · Nevanlinna functions

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1 Introduction

This note is a continuation of [4, 5]. Given are a symmetric operator or linear relation S with finite and equal defect numbers $d > 0$ in some Hilbert space $\mathfrak{H}$, and a self-adjoint extension $\widetilde{A}_{\mathcal{T}}$ of S with exit. The latter means that $\widetilde{A}_{\mathcal{T}}$ acts in some larger Hilbert space $\widetilde{\mathfrak{H}} \supset \mathfrak{H}$. Here $\mathcal{T}$ denotes a Nevanlinna $d \times d$ matrix or relation function (see Sect. 2.1), which is the parameter for the extension $\widetilde{A}_{\mathcal{T}}$ in Krein's resolvent formula (2.15). We study the compression $C_{\mathfrak{H}}(\widetilde{A}_{\mathcal{T}})$ of $\widetilde{A}_{\mathcal{T}}$ to $\mathfrak{H}$ (see (3.1)) and also the Straus subspaces $S_{\widetilde{A}_{\mathcal{T}}}(\lambda)$ related to $\widetilde{A}_{\mathcal{T}}$ (see (4.1)); the latter are also called the Straus extensions of S, see [16, 6].

The compression $C_{\mathfrak{H}}(\widetilde{A}_{\mathcal{T}})$ is a symmetric or self-adjoint extension of S. Among others, we are interested in its defect, in the dimension of its intersection with A_0 (modulo S), and, roughly speaking, in the dimension of the subspace where the compression does not coincide with the linear span of this intersection and of $\widetilde{A}_{\mathcal{T}}$. These numbers are determined by the parameter $\mathcal{T}$. In fact, if $\mathcal{T}$ is matrix valued and admits the integral representation (2.1):

$$\mathcal{T}(z) = \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \mathrm{d}\Sigma(t) + \mathcal{A} + z\mathcal{B}, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

we define subspaces $\mathbb{L}_c, \mathbb{L}_0, \mathbb{L}_f$ and $\mathbb{L}^\infty$ of $\mathbb{C}^d$ in terms of $\mathcal{A}, \mathcal{B}$ and Σ (see (3.4), (3.6), and (3.7)): their dimensions determine the numbers we are interested in. We call this result, which takes a central place in the paper, *Dimension theorem*.

If the parameter $\mathcal{T}$ is rational the compression of $\widetilde{A}_{\mathcal{T}}$ is self-adjoint: this follows immediately from Stenger's lemma [15]. It was shown in [5] that the parameter corresponding to this self-adjoint extension in Krein's formula is $\mathcal{T}(\infty)$. Here we show that in general, that means also for a non-rational parameter $\mathcal{T}$, the existence of $\mathcal{T}(\infty)$ implies that the corresponding extension $A_{\mathcal{T}(\infty)}$ in $\mathfrak{H}$ is a self-adjoint extension of the compression $C_{\mathfrak{H}}(\widetilde{A}_{\mathcal{T}})$.

V.I. Mogilevskii showed in [14] by the method of boundary triplets, that some implications for compressions in [5] are in fact equivalences, even if the defect d is infinite. For the case of finite defect as considered in the present paper these results follow also from the Dimension theorem below. After a first version of this paper was submitted we realized that some of our results are close to those in [2, Subsection 7.4].

The compression of a self-adjoint extension $\widetilde{A}_{\mathcal{T}}$ of a symmetry S can be considered as a Straus extension $S_{\widetilde{A}_{\mathcal{T}}}(\lambda)$ for $\lambda = \infty$. By means of a fractional linear transformation we show that analogues of the above results do also hold for the Straus extension $S_{\widetilde{A}_{\mathcal{T}}}(\lambda_0)$ at a real point λ_0.

A short synopsis is as follows. In Sect. 2 we introduce Nevanlinna $d \times d$ matrix and relation functions and formulate Krein's resolvent formula. In Sect. 3 we introduce the compressions $C_{\mathfrak{H}}(\widetilde{A}_{\mathcal{T}})$ and describe them as well as $\widetilde{A}_{\mathcal{T}}$ explicitly in terms of a model. The subspaces $\mathbb{L}_c, \mathbb{L}_0, \mathbb{L}_f, \mathbb{L}^\infty$ are introduced in Sect. 3.3 and the Dimension theorem is proved in Sect. 3.4. There we also formulate in

Corollaries 3.8–3.13 results of V.I. Mogilevskii [14] for the special case of finite defect d.

Straus subspaces are introduced in Sect. 4.1. In Sect. 4.2 we show that the Straus subspace $S_{\mathcal{T}}(\lambda)$ corresponding to the self-adjoint extension $\widetilde{A}_{\mathcal{T}}$ at some nonreal point λ arises from Krein's formula for the (constant) parameter $\mathcal{T}(\lambda)$, see Proposition 4.3. Using a limiting procedure (see Proposition 4.2) we show in Theorem 4.6 that $A_{\mathcal{T}(\infty)}$ is a self-adjoint extension of the compression $C_{\mathfrak{H}}(\widetilde{A}_{\mathcal{T}})$. Finally, by means of a fractional linear transformation, in Sect. 5 we prove analogues of the results of Sects. 3 and 4 for Straus subspaces at a real point.

We use subspace or closed linear relation notation as in [1, 3]. For Nevanlinna matrix functions we refer to [9], and for Nevanlinna relation functions to [13]. Finally, we thank Professor Annemarie Luger for useful discussions.

2 Preliminaries

2.1 Nevanlinna Functions

Let $d \in \mathbb{N}$. A function $\mathcal{T}$, defined on $\mathbb{C} \setminus \mathbb{R}$, is a *Nevanlinna $d \times d$ matrix function* if $\mathcal{T}$ is a $d \times d$ matrix function and has one of the following equivalent properties:

(a) $\mathcal{T}$ is holomorphic and satisfies

$$\mathcal{T}(z^*) = \mathcal{T}(z)^* \text{ and } \frac{\mathcal{T}(z) - \mathcal{T}(z)^*}{z - z^*} \geq 0, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

(b) $\mathcal{T}$ admits the *integral representation*

$$\mathcal{T}(z) = \int_{\mathbb{R}} \left(\frac{1}{t - z} - \frac{t}{1 + t^2} \right) \mathrm{d}\Sigma(t) + \mathcal{A} + z\mathcal{B}, \quad z \in \mathbb{C} \setminus \mathbb{R}, \tag{2.1}$$

where $\mathcal{A}$ and $\mathcal{B}$ are symmetric $d \times d$ matrices, $\mathcal{B} \geq 0$, and Σ is a $d \times d$ matrix valued measure such that

$$\int_{\mathbb{R}} \frac{\mathrm{d}\Sigma(t)}{t^2 + 1} < \infty.$$

(c) $\mathcal{T}$ admits a *relation representation*, that is, there exist a Hilbert space $\mathfrak{H}_{\mathcal{T}}$, a self-adjoint linear relation $B_{\mathcal{T}}$ in $\mathfrak{H}_{\mathcal{T}}$ and, after fixing a point $z_0 \in \mathbb{C} \setminus \mathbb{R}$, a linear mapping $\delta : \mathbb{C}^d \to \mathfrak{H}_{\mathcal{T}}$, such that

$$\mathcal{T}(z) = \mathcal{T}(z_0)^* + (z - z_0^*)\, \delta^* \big(I + (z - z_0)(B_{\mathcal{T}} - z)^{-1} \big) \delta, \quad z \in \mathbb{C} \setminus \mathbb{R}. \tag{2.2}$$

Ad (b): Clearly, for each bounded interval Δ, $\Sigma(\Delta)$ is a non-negative symmetric $d \times d$ matrix. For later reference we set

$$\ker \Sigma := \cap\{\ker \Sigma(\Delta) \;:\; \Delta \text{ bounded real interval}\}, \quad \operatorname{ran} \Sigma := \mathbb{C}^d \ominus \ker \Sigma. \tag{2.3}$$

If the Nevanlinna $d \times d$ matrix function $\mathcal{T}$ is rational its representation (2.1) becomes

$$\mathcal{T}(z) = \sum_{j=1}^{\ell} \frac{\mathcal{B}_j}{\alpha_j - z} + \mathcal{A} + z\mathcal{B}, \quad z \in \mathbb{C} \setminus \mathbb{R}, \tag{2.4}$$

with $\ell \in \mathbb{N} \cup \{0\}$, real points $\alpha_1 < \alpha_2 < \cdots < \alpha_\ell$, and nonzero $d \times d$ matrices $\mathcal{B}_j \geq 0,\ j = 1, 2, \ldots, \ell$, a symmetric $d \times d$ matrix $\mathcal{A}$, and a $d \times d$ matrix $\mathcal{B} \geq 0$. Later we use the following result:

$$\lim_{y \to \pm\infty} y \operatorname{Im} \big\langle \mathcal{T}(\mathrm{i}y)\mathbf{x}, \mathbf{x} \big\rangle = \int_{\mathbb{R}} \mathrm{d}\langle \Sigma(t)\mathbf{x}, \mathbf{x}\rangle \ (\leq \infty), \quad \mathbf{x} \in \ker \mathcal{B}, \tag{2.5}$$

which implies the following equivalence:

$$\begin{aligned} &\lim_{y \to +\infty} y \operatorname{Im} \big\langle \mathcal{T}(\mathrm{i}y)\mathbf{x}, \mathbf{x} \big\rangle = \infty \text{ for all } \mathbf{x} \in \mathbb{C}^d \setminus \{0\} \\ &\iff \begin{cases} \mathcal{B} > 0 & \text{if } \mathcal{T} \text{ is rational,} \\ \displaystyle\int_{\mathbb{R}} \mathrm{d}\langle \Sigma(t)\mathbf{x}, \mathbf{x}\rangle = \infty \text{ for all } \mathbf{x} \in \ker \mathcal{B} \setminus \{0\} & \text{otherwise.} \end{cases} \end{aligned} \tag{2.6}$$

Ad (c): For z in the resolvent set $\rho(B_{\mathcal{T}})$, we denote by $R_{\mathcal{T}}(z) := (B_{\mathcal{T}} - z)^{-1}$ the resolvent operator of $B_{\mathcal{T}}$, and set

$$\delta_z := \big(I + (z - z_0) R_{\mathcal{T}}(z)\big)\delta, \quad z \in \mathbb{C} \setminus \mathbb{R}. \tag{2.7}$$

It follows that

$$\delta_z = (I + (z - w) R_{\mathcal{T}}(z))\delta_w, \quad z, w \in \mathbb{C} \setminus \mathbb{R},$$

and

$$\frac{\mathcal{T}(z) - \mathcal{T}(w)^*}{z - w^*} = \delta_w^* \delta_z, \quad z, w \in \mathbb{C} \setminus \mathbb{R},\ w \neq z^*. \tag{2.8}$$

The relation representation (2.2) will always be chosen *minimal*, which means that

$$\mathfrak{H}_{\mathcal{T}} = \overline{\operatorname{span}}\, \{\delta_z \mathbf{x} \;:\; \mathbf{x} \in \mathbb{C}^d,\ z \in \mathbb{C} \setminus \mathbb{R}\}.$$

The triplet $(\mathfrak{H}_{\mathcal{T}}, B_{\mathcal{T}}, \delta_{z_0})$ is sometimes called a *model* of the function $\mathcal{T}$. The above relations extend to points $z \in \mathbb{R}$ into which $\mathcal{T}$ can be continued analytically or, equivalently, which belong to $\rho(B_{\mathcal{T}})$.

Finally, $\mathcal{T}$ will be called a *Nevanlinna $d \times d$ relation function* on $\mathbb{C} \setminus \mathbb{R}$, if there exists an orthogonal projection P_m in $\mathbb{C}^d$ such that $\mathcal{T}(z)$ is the linear relation

$$\mathcal{T}(z) = \big\{\{P_m\mathbf{x}, \mathcal{T}_m(z)P_m\mathbf{x} + (I - P_m)\mathbf{x}\} \,:\, \mathbf{x} \in \mathbb{C}^d\big\}, \; z \in \mathbb{C} \setminus \mathbb{R}, \tag{2.9}$$

where, with $d_m := \dim \operatorname{ran} P_m$ and $\operatorname{ran} P_m$ identified with $\mathbb{C}^{d_m}$ (as we shall do throughout this note), $\mathcal{T}_m$ is a Nevanlinna $d_m \times d_m$ matrix function, called the *matrix part* of $\mathcal{T}$. We also identify $\ker P_m$ with $\mathbb{C}^{d_\infty}$, where $d_\infty := d - d_m$, and $\operatorname{ran} P_m \oplus \ker P_m$ with $\mathbb{C}^d$. Relative to these identifications we can write $\mathcal{T}(z)$ as the orthogonal direct sum in $\mathbb{C}^d \times \mathbb{C}^d$

$$\mathcal{T}(z) = \mathcal{T}_m(z) \oplus \mathcal{T}_\infty,$$

where $\mathcal{T}_\infty := \{\{0, \mathbf{y}\} \,:\, \mathbf{y} \in \mathbb{C}^{d_\infty}\}$ is called the *multi-valued part* of $\mathcal{T}$.

When $\mathcal{T}$ is a Nevanlinna $d \times d$ relation function and when we refer to the equivalent definitions (a), (b) and (c) above and the formulas following these definitions we mean that there d and $\mathcal{T}$ are replaced everywhere by d_m and $\mathcal{T}_m$.

2.2 Krein's Formula

Let S be a closed symmetric linear relation in a Hilbert space $\mathfrak{H}$ with finite and equal defect numbers $d > 0$. Krein's formula for the generalized resolvents of S, which we describe below, depends on the choice of a self-adjoint extension A_0 of S in $\mathfrak{H}$, a point $z_0 \in \mathbb{C} \setminus \mathbb{R}$, and a bijective mapping $\gamma : \mathbb{C}^d \to \ker(S^* - z_0)$. Having chosen A_0, z_0 and γ, we define a so-called *γ-field*

$$\gamma_z : \mathbb{C}^d \to \ker(S^* - z), \quad \gamma_z := \big(I + (z - z_0)R_0(z)\big)\gamma, \quad z \in \rho(A_0),$$

where $R_0(z) := (A_0 - z)^{-1}$ is the resolvent of A_0. Evidently, γ_z is a bijection, and $\gamma_{z_0} = \gamma$. Note that for each $z \in \mathbb{C} \setminus \mathbb{R}$

$$\begin{aligned} S &= \big\{\{f, g\} \in A_0 \,:\, \gamma_{z^*}^*(g - zf) = 0\big\} \\ &= \big\{\{R_0(z)f, f + zR_0(z)f\} \,:\, f \in \mathfrak{H}, \; \gamma_{z^*}^* f = 0\big\}. \end{aligned} \tag{2.10}$$

For the given relation S, a self-adjoint extension A_0 of S in $\mathfrak{H}$ and a subspace $\mathbb{L} \subset \mathbb{C}^d$ we introduce the linear relation

$$S_{\mathbb{L}} := \big\{\{R_0(z)f, f + zR_0(z)f\} : f \in \mathfrak{H}, \; \gamma_{z^*}^* f \in \mathbb{L}\big\}. \tag{2.11}$$

Here the expression on the right-hand side is independent of $z \in \mathbb{C} \setminus \mathbb{R}$, and the mapping $\mathbb{L} \to S_{\mathbb{L}}$ defines a one-to-one correspondence between all subspaces $\mathbb{L} \subset \mathbb{C}^d$ and all closed symmetric extensions $S_{\mathbb{L}}$ of S contained in A_0. In particular we have $S_{\{0\}} = S$ (see (2.10)) and $S_{\mathbb{C}^d} = A_0$. Since $\gamma_{z^*}^* : \ker(S^* - z^*) \to \mathbb{C}^d$ is a bijection and $\ker \gamma_{z^*}^* = \operatorname{ran}(S - z)$,

$$\dim S_{\mathbb{L}}/S = \dim \mathbb{L}. \tag{2.12}$$

With the γ-field γ_z there is defined a corresponding *Q-function* $\mathcal{Q}_0$ by the relation

$$\frac{\mathcal{Q}_0(z) - \mathcal{Q}_0(w)^*}{z - w^*} = \gamma_w^* \gamma_z, \quad z, w \in \rho(A_0), \tag{2.13}$$

see [12]. It is a $d \times d$ matrix function, which is determined by (2.13) up to a constant symmetric $d \times d$ matrix summand. Evidently,

$$\operatorname{Im} \mathcal{Q}_0(z)/\operatorname{Im} z = \gamma_z^* \gamma_z > 0, \quad z \in \mathbb{C} \setminus \mathbb{R}, \tag{2.14}$$

and hence $\mathcal{Q}_0$ is a Nevanlinna $d \times d$ matrix function.

Now let $\widetilde{A}$ be a self-adjoint extension of S, a relation or an operator, acting in some Hilbert space $\widetilde{\mathfrak{H}} \supset \mathfrak{H}$. If $\widetilde{\mathfrak{H}} = \mathfrak{H}$ then the extension $\widetilde{A}$ is called *canonical*. The compressed resolvent $P_{\mathfrak{H}}(\widetilde{A} - z)^{-1}\big|_{\mathfrak{H}}$ of $\widetilde{A}$ is called the *generalized resolvent of* S, corresponding to the extension $\widetilde{A}$. The set of all generalized resolvents of S can be described as follows (see [11, Theorem 5.1], [13, Theorem 3.2] or [2, Theorem 6.2]):

There is a bijective correspondence between all generalized resolvents of S and all Nevanlinna $d \times d$ relation functions $\mathcal{T}$ given by the formula

$$P_{\mathfrak{H}}(\widetilde{A} - z)^{-1}\big|_{\mathfrak{H}} = (A_0 - z)^{-1} - \gamma_z(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1}\gamma_{z^*}^*, \quad z \in \rho(A_0) \cap \rho(\widetilde{A}). \tag{2.15}$$

We call (2.15) *Krein's resolvent formula based on* A_0. If the relation $\widetilde{A}$ on the left-hand side of (2.15) is a minimal extension of S and corresponds to $\mathcal{T}$, then it is uniquely determined by $\mathcal{T}$ up to unitary equivalence with a unitary mapping which is the identity on $\mathfrak{H}$. We then denote $\widetilde{A}$ by $\widetilde{A}_{\mathcal{T}}$ and recall the following two facts:

(A) The parameter $\mathcal{T}$ is independent of z if and only if $\widetilde{A}_{\mathcal{T}}$ is a canonical extension of S.

(B) The parameter $\mathcal{T}$ is a Nevanlinna $d \times d$ matrix function if and only if $\widetilde{A}_{\mathcal{T}} \cap A_0 = S$.

For item (A) see [13, Theorem 3.2]; item (B) has been generalized as follows (see [10, Section 3] and [5, Proposition 3.4]):

(C) The parameter $\mathcal{T}$ in Krein's formula (2.15) is a Nevanlinna $d \times d$ relation function of the form (2.9) if and only if $\widetilde{A}_{\mathcal{T}} \cap A_0 = S_{\ker P_m}$.

By (2.12),

$$\dim S_{\ker P_m}/S = d_\infty \tag{2.16}$$

and $S_{\ker P_m}$ has equal defect numbers $d_m = d - d_\infty$. If $\mathcal{T}$ is a relation function of the form (2.9) the last summand on the right-hand side of Krein's formula (2.15) is equal to

$$\gamma_z(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1}\gamma_{z^*}^* = \gamma_z P_m(P_m\mathcal{Q}_0(z)P_m + \mathcal{T}_m(z))^{-1}\gamma_{z^*}^* P_m. \tag{2.17}$$

With the above mentioned identifications, the inverse on the right-hand side of (2.17) is the inverse of an invertible $d_m \times d_m$ matrix, see [13]. The mappings $\gamma_z P_m$, which are bijections from $\mathbb{C}^{d_m}$ onto $\ker(S_{\ker P_m}^* - z)$, form a γ-field, and the $d_m \times d_m$ matrix $P_m\mathcal{Q}_0(z)P_m$ defines a corresponding Q-function associated with the symmetry $S_{\ker P_m}$ and its canonical self-adjoint extension A_0.

3 The Dimension Theorem for Compressions

3.1 Compressions

Let S be a closed symmetric linear relation in a Hilbert space $\mathfrak{H}$ with equal defect numbers $d \in \mathbb{N}$. Denote by $\widetilde{A}$ any self-adjoint extension of S in a Hilbert space $\widetilde{\mathfrak{H}}$ which contains $\mathfrak{H}$ as a subspace. We shall assume that $\widetilde{A}$ is *minimal* which means that for some $w \in \mathbb{C} \setminus \mathbb{R}$

$$\overline{\operatorname{span}}\left\{(I + (z-w)(\widetilde{A} - z)^{-1})h \,:\, h \in \mathfrak{H},\ z \in \mathbb{C} \setminus \mathbb{R}\right\} = \widetilde{\mathfrak{H}}.$$

Let $\widetilde{P}_{\mathfrak{H}}$ be the orthogonal projection in $\widetilde{\mathfrak{H}}$ onto $\mathfrak{H}$. The *compression* $C_{\mathfrak{H}}(\widetilde{A})$ of $\widetilde{A}$ to $\mathfrak{H}$ is defined by the equation

$$C_{\mathfrak{H}}(\widetilde{A}) := \widetilde{P}_{\mathfrak{H}}\widetilde{A}|_{\mathfrak{H}} = \left\{\{\widetilde{f}, \widetilde{P}_{\mathfrak{H}}\widetilde{g}\} \,:\, \{\widetilde{f}, \widetilde{g}\} \in \widetilde{A},\ \widetilde{f} \in \mathfrak{H}\right\}. \tag{3.1}$$

We denote by $\widetilde{P}_\infty$ the orthogonal projection in $\widetilde{\mathfrak{H}}$ onto $\widetilde{A}(0)$.

Proposition 3.1 *Let S be a closed symmetric linear relation in $\mathfrak{H}$ with equal defect numbers $d \in \mathbb{N}$. Let $\widetilde{A}$ be a self-adjoint extension of S in some Hilbert space $\widetilde{\mathfrak{H}} \supset \mathfrak{H}$. Then:*

(i) *$C_{\mathfrak{H}}(\widetilde{A})$ is a closed symmetric extension of S in $\mathfrak{H}$ with equal defect numbers $d - \dim(C_{\mathfrak{H}}(\widetilde{A})/S)$.*
(ii) *$\widetilde{P}_{\mathfrak{H}}\big(\widetilde{A}(0)\big) = C_{\mathfrak{H}}(\widetilde{A})(0)$, and if $\widetilde{A}$ is minimal, then $C_{\mathfrak{H}}(\widetilde{A})$ is an operator if and only if $\widetilde{A}$ is an operator.*
(iii) *If $\widetilde{A}$ is minimal, then $\widetilde{P}_\infty\mathfrak{H} = \widetilde{A}(0)$.*

Proof (i) From $S \subset \widetilde{A} \cap \mathfrak{H}^2$ it follows that $S \subset \widetilde{P}_{\mathfrak{H}} \widetilde{A}|_{\mathfrak{H}} = C_{\mathfrak{H}}(\widetilde{A})$. If $\{f, g\} \in C_{\mathfrak{H}}(\widetilde{A})$, then $\{f, \widetilde{g}\} \in \widetilde{A} = \widetilde{A}^*$ for some $\widetilde{g} \in \widetilde{\mathfrak{H}}$ with $\widetilde{P}_{\mathfrak{H}} \widetilde{g} = g$ and

$$\operatorname{Im}(g, f)_{\mathfrak{H}} = \operatorname{Im}(\widetilde{g}, f)_{\widetilde{\mathfrak{H}}} = 0.$$

Hence $C_{\mathfrak{H}}(\widetilde{A})$ is a symmetric extension of S and $S \subset C_{\mathfrak{H}}(\widetilde{A}) \subset C_{\mathfrak{H}}(\widetilde{A})^* \subset S^*$. Since $\dim S^*/S = 2d < \infty$, $C_{\mathfrak{H}}(\widetilde{A})$ is closed.

(ii) The equality follows directly from the definition of a compression. It implies that if $\widetilde{A}$ is an operator, then $C_{\mathfrak{H}}(\widetilde{A})$ is an operator. Conversely, assume that $C_{\mathfrak{H}}(\widetilde{A})$ is an operator. Then $\widetilde{P}_{\mathfrak{H}}(\widetilde{A}(0)) = \{0\}$. Using that $(\widetilde{A} - z)^{-1}\widetilde{A}(0) = \{0\}$ for all $z \in \rho(\widetilde{A})$ we find that for all $h \in \mathfrak{H}$ and $z, w \in \rho(\widetilde{A})$

$$(\widetilde{A}(0), (I + (z - w)(\widetilde{A} - z)^{-1})h)_{\widetilde{\mathfrak{H}}} = \{0\}.$$

This and the minimality of $\widetilde{A}$ imply that $\widetilde{A}(0) = \{0\}$, that is $\widetilde{A}$ is an operator.

(iii) Assume $\widetilde{A}$ is minimal. We first show that $\dim \widetilde{A}(0)/S(0) \leq 2d$ by proving that the inequality $\dim \widetilde{A}(0)/S(0) > 2d$ leads to a contradiction. This inequality implies that for some $k > 2d$ there exist elements $\widetilde{g}_1, \ldots, \widetilde{g}_k$ in $\widetilde{A}(0)$ which are linearly independent modulo $S(0)$. Set $g_j = \widetilde{P}_{\mathfrak{H}} \widetilde{g}_j$, $j = 1, \ldots, k$. Then

$$\{0, g_1\}, \ldots, \{0, g_k\} \in S^*$$

and hence, since $\dim S^*/S = 2d$, there exist complex numbers τ_j, not all zero, such that $\sum_{j=1}^k \tau_j \{0, g_j\} \in S$, that is $\sum_{j=1}^k \tau_j g_j \in S(0)$. Hence

$$\sum_{j=1}^k \tau_j \widetilde{g}_j - \sum_{j=1}^k \tau_j g_j \in \widetilde{A}(0) \cap (\widetilde{\mathfrak{H}} \ominus \mathfrak{H})$$

and consequently for all $h \in \mathfrak{H}$ and $z, w \in \rho(\widetilde{A})$

$$\left(\sum_{j=1}^k \tau_j \widetilde{g}_j - \sum_{j=1}^k \tau_j g_j, (I + (z - w)(\widetilde{A} - z)^{-1})h \right)_{\widetilde{\mathfrak{H}}} = 0.$$

The minimality of $\widetilde{A}$ implies that

$$\sum_{j=1}^k \tau_j \widetilde{g}_j = \sum_{j=1}^k \tau_j g_j \in S(0),$$

and hence all τ_j's are zero. The contradiction with not all τ_j's are zero proves that $\dim \widetilde{A}(0)/S(0) \leq 2d$. From this and the inclusion $S(0) \subset \widetilde{P}_\infty \widetilde{\mathfrak{H}} \subset \widetilde{A}(0)$ we infer that the linear set $\widetilde{P}_\infty \mathfrak{H}$ is closed. We show that it is dense in $\widetilde{A}(0)$ and hence it is equal to $\widetilde{A}(0)$. Assume $\widetilde{f} \in \widetilde{A}(0) \ominus \widetilde{P}_\infty \mathfrak{H}$. Then $\widetilde{f} = \widetilde{P}_\infty \widetilde{f}$ and for all $h \in \mathfrak{H}$ and $z, w \in \rho(\widetilde{A})$

$$\big(\widetilde{f}, (I + (z - w)(\widetilde{A} - z)^{-1})h\big)_{\widetilde{\mathfrak{H}}} = 0.$$

The minimality of $\widetilde{A}$ implies that $\widetilde{f} = 0$, that is $\widetilde{P}_\infty \mathfrak{H}$ is dense in $\widetilde{A}(0)$. □

3.2 The Self-Adjoint Extensions and Their Compressions

In the following we shall use explicit representations of the self-adjoint extensions $\widetilde{A}_\mathcal{T}$ of the symmetry S. The starting point is a generalization of [5, Theorem 2.4] to the situation that S is a linear relation. We formulate this result in the following proposition.

Proposition 3.2 *Let S be a closed symmetric relation in the Hilbert space $\mathfrak{H}$ with equal defect numbers $d \in \mathbb{N}$. Let A_0 be a canonical self-adjoint extension of S and denote by γ_z and $\mathcal{Q}_0(z)$ a γ-field and a corresponding Q-function associated with S and A_0. Let $\mathcal{T}$ be a Nevanlinna $d \times d$ matrix function with model $(\mathfrak{H}_\mathcal{T}, B_\mathcal{T}, \delta_{z_0})$, see (2.2). Then the operator function $\widetilde{R}_\mathcal{T}$:*

$$\begin{aligned}\widetilde{R}_\mathcal{T}(z) &:= \begin{pmatrix} R_0(z) - \gamma_z\big(\mathcal{Q}_0(z) + \mathcal{T}(z)\big)^{-1}\gamma_{z^*}^* & -\gamma_z\big(\mathcal{Q}_0(z) + \mathcal{T}(z)\big)^{-1}\delta_{z^*}^* \\ -\delta_z\big(\mathcal{Q}_0(z) + \mathcal{T}(z)\big)^{-1}\gamma_{z^*}^* & R_\mathcal{T}(z) - \delta_z\big(\mathcal{Q}_0(z) + \mathcal{T}(z)\big)^{-1}\delta_{z^*}^* \end{pmatrix} \\ &= \begin{pmatrix} R_0(z) & 0 \\ 0 & R_\mathcal{T}(z) \end{pmatrix} - \begin{pmatrix} \gamma_z \\ \delta_z \end{pmatrix} \big(\mathcal{Q}_0(z) + \mathcal{T}(z)\big)^{-1} \begin{pmatrix} \gamma_{z^*}^* & \delta_{z^*}^* \end{pmatrix}, \ z \in \mathbb{C} \setminus \mathbb{R},\end{aligned}$$

whose values are bounded operators in $\mathfrak{H} \oplus \mathfrak{H}_\mathcal{T}$, is the resolvent $(\widetilde{A} - z)^{-1}$ of a self-adjoint relation $\widetilde{A}$ in the Hilbert space $\mathfrak{H} \oplus \mathfrak{H}_\mathcal{T}$; $\widetilde{A}$ is a minimal self-adjoint extension of S.

The proof is the same as for [5, Theorem 2.4] where it is assumed that S is densely defined. But here S need not be densely defined, and hence $\widetilde{A}$ need not be an operator, but nevertheless it is a minimal extension of S. Since $\widetilde{P}_\mathfrak{H} \widetilde{R}_\mathcal{T}(z)\big|_\mathfrak{H} = \widetilde{P}_\mathfrak{H}(\widetilde{A} - z)^{-1}\big|_\mathfrak{H}$ is equal to the right-hand side of (2.15), $\widetilde{A}$ corresponds to the parameter $\mathcal{T}$, that is $\widetilde{A} = \widetilde{A}_\mathcal{T}$. We write the formula for $\widetilde{A}_\mathcal{T}$ with resolvent $\widetilde{R}_\mathcal{T}$ in full detail:

$$\widetilde{A}_{\mathcal{T}} = \left\{ \left\{ \begin{pmatrix} R_0(z)f - \gamma_z(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1}(\gamma_{z^*}^* f + \delta_{z^*}^* g) \\ R_{\mathcal{T}}(z)g - \delta_z(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1}(\gamma_{z^*}^* f + \delta_{z^*}^* g) \end{pmatrix}, \right.\right.$$
$$\left.\left. \begin{pmatrix} f \\ g \end{pmatrix} + z \begin{pmatrix} R_0(z)f - \gamma_z(\mathcal{Q}_0(z)+\mathcal{T}(z))^{-1}(\gamma_{z^*}^* f+\delta_{z^*}^* g) \\ R_{\mathcal{T}}(z)g - \delta_z(\mathcal{Q}_0(z)+\mathcal{T}(z))^{-1}(\gamma_{z^*}^* f+\delta_{z^*}^* g) \end{pmatrix} \right\} : f \in \mathfrak{H},\ g \in \mathfrak{H}_{\mathcal{T}} \right\}.$$

Here the right-hand side is independent of $z \in \mathbb{C} \setminus \mathbb{R}$; in the following we often fix a point $z \in \mathbb{C} \setminus \mathbb{R}$. The formula for $\widetilde{A}_{\mathcal{T}}$ implies that the restriction of $\widetilde{A}_{\mathcal{T}}$ to $\mathfrak{H}$ and the graph restriction of $\widetilde{A}_{\mathcal{T}}$ to $\mathfrak{H}^2$ are given by

$$\widetilde{A}_{\mathcal{T}}\big|_{\mathfrak{H}} = \left\{ \left\{ \begin{pmatrix} R_0(z)f - \gamma_z(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1}(\gamma_{z^*}^* f + \delta_{z^*}^* g) \\ 0 \end{pmatrix}, \right.\right.$$
$$\left. \begin{pmatrix} f \\ g \end{pmatrix} + z \begin{pmatrix} R_0(z)f - \gamma_z(\mathcal{Q}_0(z)+\mathcal{T}(z))^{-1}(\gamma_{z^*}^* f+\delta_{z^*}^* g) \\ 0 \end{pmatrix} \right\} :$$
$$\left. f \in \mathfrak{H},\ g \in \mathfrak{H}_{\mathcal{T}},\ R_{\mathcal{T}}(z)g = \delta_z(\mathcal{Q}_0(z)+\mathcal{T}(z))^{-1}\left(\gamma_{z^*}^* f+\delta_{z^*}^* g\right) \right\},$$

and

$$\widetilde{A}_{\mathcal{T}} \cap \mathfrak{H}^2 = \left\{ \left\{ R_0(z)f - \gamma_z(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1}\gamma_{z^*}^* f, \right.\right.$$
$$\left. f + z(R_0(z)f - \gamma_z(\mathcal{Q}_0(z)+\mathcal{T}(z))^{-1}\gamma_{z^*}^* f) \right\} : \tag{3.2}$$
$$\left. f \in \mathfrak{H},\ \delta_z(\mathcal{Q}_0(z)+\mathcal{T}(z))^{-1}\gamma_{z^*}^* f = 0 \right\}.$$

The compression is given by

$$C_{\mathfrak{H}}(\widetilde{A}_{\mathcal{T}}) = \left\{ \left\{ R_0(z)f - \gamma_z(\mathcal{Q}_0(z)+\mathcal{T}(z))^{-1}\left(\gamma_{z^*}^* f+\delta_{z^*}^* g\right), \right.\right.$$
$$\left. f + z\big(R_0(z)f - \gamma_z(\mathcal{Q}_0(z)+\mathcal{T}(z))^{-1}(\gamma_{z^*}^* f+\delta_{z^*}^* g)\big) \right\} :$$
$$\left. f \in \mathfrak{H},\ g \in \mathfrak{H}_{\mathcal{T}},\ R_{\mathcal{T}}(z)g = \delta_z(\mathcal{Q}_0(z)+\mathcal{T}(z))^{-1}(\gamma_{z^*}^* f+\delta_{z^*}^* g) \right\},$$

and hence

$$\begin{aligned} C_{\mathfrak{H}}(\widetilde{A}_{\mathcal{T}}) \cap A_0 &= \left\{ \left\{ R_0(z)f,\ f + zR_0(z)f \right\} \ :\ f \in \mathfrak{H},\ \gamma_{z^*}^* f \in \delta_{z^*}^* B_{\mathcal{T}}(0) \right\} \\ &= S_{\delta_{z^*}^* B_{\mathcal{T}}(0)}. \end{aligned} \tag{3.3}$$

For the first equality in (3.3) we used that $g \in B_{\mathcal{T}}(0)$ if and only if $R_{\mathcal{T}}(z)g = 0$ for any (and every) $z \in \mathbb{C} \setminus \mathbb{R}$; for the second equality see (2.11).

3.3 Decomposition of $\mathbb{C}^d$

For a Nevanlinna $d \times d$ matrix function $\mathcal{T}$, we introduce a decomposition of the space $\mathbb{C}^d$, which will play an essential role in what follows. In the next three lemmas we assume that $\mathcal{T}$ has the integral representation (2.1) and the relation representation (2.2).

First we define the subspace $\mathbb{L}_c \subset \mathbb{C}^d$ with $\ker \Sigma$ as in (2.3) by

$$\mathbb{L}_c := (\ker \mathcal{B}) \cap (\ker \Sigma). \tag{3.4}$$

Lemma 3.3 *For every $z \in \mathbb{C} \setminus \mathbb{R}$*

$$\mathbb{L}_c = \ker \operatorname{Im} \mathcal{T}(z) = \ker \delta_z = \{\mathbf{x} \in \mathbb{C}^d \;:\; \mathcal{T}(z)\mathbf{x} = \mathcal{A}\mathbf{x}\}. \tag{3.5}$$

Since $\operatorname{Im} \mathcal{T}(z)/\operatorname{Im} z \geq 0$, the lemma implies that

$$\mathbb{L}_c = \{0\} \Longleftrightarrow \operatorname{Im} \mathcal{T}(z)/\operatorname{Im} z > 0.$$

Proof Since Σ is non-negative and $\mathcal{B} \geq 0$ the first equality in (3.5) follows from the relation

$$\operatorname{Im} \mathcal{T}(z)/\operatorname{Im} z = \int_{\mathbb{R}} \frac{\mathrm{d}\Sigma(t)}{|t-z|^2} + \mathcal{B}, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

The second equality is a consequence of the following implications for $\mathbf{x} \in \mathbb{C}^d$:

$$\begin{aligned} &\operatorname{Im} \mathcal{T}(z)\mathbf{x} = 0 \overset{(2.8)}{\Longrightarrow} \delta_z^* \delta_z \mathbf{x} = 0 \Longrightarrow (\delta_z \mathbf{x}, \delta_z \mathbf{x})_{\mathfrak{H}_{\mathcal{T}}} = 0 \Longrightarrow \delta_z \mathbf{x} = 0 \overset{(2.2)}{\Longrightarrow} \\ &\mathcal{T}(z)\mathbf{x} = \mathcal{T}(z_0)^* \mathbf{x} \text{ is independent of } z \Longrightarrow \mathcal{T}(z)\mathbf{x} = \mathcal{T}(z^*)\mathbf{x} = \mathcal{T}(z)^* \mathbf{x} \\ &\Longrightarrow \operatorname{Im} \mathcal{T}(z)\mathbf{x} = 0. \end{aligned}$$

The definition of $\mathbb{L}_c$ and the integral representation of $\mathcal{T}$ imply the last equality in (3.5). □

The orthogonal complement of $\mathbb{L}_c$ in $\mathbb{C}^d$ is given by

$$\mathbb{L}_c^{\perp} = \operatorname{span} \{\operatorname{ran} \mathcal{B}, \operatorname{ran} \Sigma\},$$

where $\operatorname{ran} \Sigma$ is defined in (2.3). In $\mathbb{L}_c^{\perp}$ we consider two subspaces which are orthogonal to each other:

$$\mathbb{L}_r := \operatorname{ran} \mathcal{B}, \quad \mathbb{L}_f := \left\{\mathbf{x} \in \mathbb{C}^d : \mathbf{x} \in (\ker \mathcal{B}) \cap \mathbb{L}_c^{\perp}, \int_{\mathbb{R}} \mathrm{d}\langle \Sigma(t)\mathbf{x}, \mathbf{x}\rangle < \infty \right\}. \tag{3.6}$$

Lemma 3.4 *The mapping $\delta_z^* : B_{\mathcal{T}}(0) \to \mathbb{L}_r$ is a bijection for every $z \in \mathbb{C} \setminus \mathbb{R}$.*

Proof This follows from [5, Lemma 2.5(ii)], which states that δ^* is a bijection, and from the relation (2.7) and the equality $R_{\mathcal{T}}(z)B_{\mathcal{T}}(0) = \{0\}$, $z \in \mathbb{C} \setminus \mathbb{R}$, which imply that $\delta_z^* = \delta^*$ on $B_{\mathcal{T}}(0)$. □

Finally, we define the remaining subspace $\mathbb{L}^\infty$ in $\mathbb{L}_c^\perp$ by the relation

$$\mathbb{L}_c^\perp = \mathbb{L}_r \oplus \mathbb{L}_f \oplus \mathbb{L}^\infty. \tag{3.7}$$

It follows that

$$\mathbf{x} \in \mathbb{L}^\infty,\ \mathbf{x} \neq 0 \Longrightarrow \mathbf{x} \in \ker \mathcal{B},\ \int_{\mathbb{R}} \mathrm{d}\langle \Sigma(t)\mathbf{x}, \mathbf{x}\rangle = \infty. \tag{3.8}$$

Lemma 3.5 *For* $\mathbf{x} \in \ker \mathcal{B}$

$$\int_{\mathbb{R}} \mathrm{d}\langle \Sigma(t)\mathbf{x}, \mathbf{x}\rangle = \infty \Longleftrightarrow \delta_z \mathbf{x} \notin \operatorname{dom} B_{\mathcal{T}}.$$

Proof Let $\mathbf{x} \in \ker \mathcal{B}$. Then the lemma follows from the equivalence (see (2.1))

$$\int_{\mathbb{R}} \mathrm{d}\langle \Sigma(t)\mathbf{x}, \mathbf{x}\rangle = \infty \Longleftrightarrow \lim_{y \uparrow \infty} y \operatorname{Im} \big\langle \mathcal{T}(\mathrm{i}y)\mathbf{x}, \mathbf{x}\big\rangle = \infty$$

and [13, Theorem 2.4 (2)], according to which

$$\lim_{y \uparrow \infty} y \operatorname{Im} \big\langle \mathcal{T}(\mathrm{i}y)\mathbf{x}, \mathbf{x}\big\rangle = \infty \Longleftrightarrow \delta_z \mathbf{x} \notin \operatorname{dom} B_{\mathcal{T}}.$$

□

Thus we obtain the following orthogonal decomposition of $\mathbb{C}^d$:

$$\mathbb{C}^d = \mathbb{L}_c \oplus \mathbb{L}_r \oplus \mathbb{L}_f \oplus \mathbb{L}^\infty. \tag{3.9}$$

We denote the dimensions of the subspaces on the right-hand side of (3.9) by d_c, d_r, d_f, and d^∞, respectively. Clearly,

$$d = d_c + d_r + d_f + d^\infty. \tag{3.10}$$

Remark 3.6 If $\mathcal{T}(z)$ is a *rational* Nevanlinna $d \times d$ matrix function with representation (2.4), then

$$\mathbb{L}_c = (\ker \mathcal{B}) \cap \big(\cap_{j=1}^{\ell} \ker \mathcal{B}_j \big), \quad \mathbb{L}_c^\perp = \operatorname{span}\{\operatorname{ran} \mathcal{B}, \operatorname{ran} \mathcal{B}_1, \ldots, \operatorname{ran} \mathcal{B}_\ell\}$$

and

$$\mathbb{L}_r = \operatorname{ran}\mathcal{B}, \quad \mathbb{L}_f = (\ker\mathcal{B}) \cap \mathbb{L}_c^{\perp}, \quad \mathbb{L}^{\infty} = \{0\}.$$

The last equality follows from the two preceding equalities and (3.7) as they imply

$$\mathbb{L}_f = \mathbb{L}_r^{\perp} \cap \mathbb{L}_c^{\perp} = \mathbb{L}_f \oplus \mathbb{L}^{\infty}.$$

3.4 The Dimension Theorem

In the following theorem we suppose that the parameter $\mathcal{T}$ is a Nevanlinna $d \times d$ matrix function. The first equality is well-known and is listed for completeness. The extension of the theorem to a relation function $\mathcal{T}$ is described in Remark 3.15.

Theorem 3.7 *Suppose that the parameter $\mathcal{T}$ is a Nevanlinna $d \times d$ matrix function. Then the following relations hold*:

(i) $\dim\Big(\big(\widetilde{A}_{\mathcal{T}} \cap A_0\big)/S\Big) = 0,$

(ii) $\dim\Big(\big(\widetilde{A}_{\mathcal{T}} \cap \mathfrak{H}^2\big)/S\Big) = d_c,$

(iii) $\dim\Big(\big(C_{\mathfrak{H}}(\widetilde{A}_{\mathcal{T}}) \cap A_0\big)/S\Big) = d_r,$

(iv) $\dim\Big(C_{\mathfrak{H}}(\widetilde{A}_{\mathcal{T}})/\big(\widetilde{A}_{\mathcal{T}} \cap \mathfrak{H}^2 \,+\, C_{\mathfrak{H}}(\widetilde{A}_{\mathcal{T}}) \cap A_0\big)\Big) = d_f,$

(v) $\dim\Big(C_{\mathfrak{H}}(\widetilde{A}_{\mathcal{T}})/S\Big) = d_c + d_r + d_f.$

Proof Item (i) or, equivalently, $S = \widetilde{A}_{\mathcal{T}} \cap A_0$, holds because of the assumption that the parameter $\mathcal{T}$ is a *matrix* function, see (B) in Sect. 2.2.

To prove the remaining equalities we introduce three (possibly empty) sets $\{\mathsf{u}_i : i = 1, \dots, d_c\}$, $\{\mathsf{v}_j : j = 1, \dots, d_r\}$ and $\{\mathsf{w}_k : k = 1, \dots, d_f\}$ of elements in $\mathfrak{H}^2$. We use without recalling that the mappings $\gamma_z : \mathbb{C}^d \to \ker(S^* - z)$ and $\gamma_{z^*}^*|_{\ker(S^*-z^*)} : \ker(S^* - z^*) \to \mathbb{C}^d$ are bijections and $\ker\gamma_{z^*}^* = \operatorname{ran}(S - z)$. Further, we fix a point $z \in \mathbb{C} \setminus \mathbb{R}$, and denote by $\dot{+}$ the direct sum in $\mathfrak{H}^2$.

If $d_c > 0$, for $i = 1, \dots, d_c$ set

$$\mathsf{u}_i := \big\{R_0(z) f_i' - \gamma_z(\cdots)^{-1}\gamma_{z^*}^* f_i',\, f_i' + z(R_0(z) f_i' - \gamma_z(\cdots)^{-1}\gamma_{z^*}^* f_i')\big\}$$

where $(\cdots)^{-1}$ stands for $(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1}$ and the elements f_i', $i = 1, \dots, d_c$, span a d_c-dimensional subspace of $\ker(S^* - z^*)$ such that (see Lemma 3.3)

$$(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1}\gamma_{z^*}^* f_i' \in \ker\delta_z = \mathbb{L}_c.$$

According to (3.2), the u_i's belong to $\widetilde{A}_{\mathcal{T}} \cap \mathfrak{H}^2$. Since the f_i''s are linearly independent, so are the u_i's. We prove (ii) by showing that

$$\widetilde{A}_{\mathcal{T}} \cap \mathfrak{H}^2 = S \dot{+} \operatorname{span}\{\mathsf{u}_1, \ldots, \mathsf{u}_{d_c}\}. \tag{3.11}$$

To show that it is a direct sum let u belong to the intersection of the sets on the right-hand side. Then for some $f' \in \ker(S^* - z^*)$

$$\mathsf{u} = \left\{R_0(z)f' - \gamma_z(\cdots)^{-1}\gamma_{z^*}^* f',\, f' + z(R_0(z)f' - \gamma_z(\cdots)^{-1}\gamma_{z^*}^* f')\right\} \in S.$$

This implies that

$$\left\{f',\, R_0(z)f' - \gamma_z(\cdots)^{-1}\gamma_{z^*}^* f'\right\} \in (S - z)^{-1} \subset R_0(z).$$

Since also $\{f', R_0(z)f'\} \in R_0(z)$ and $R_0(z)$ is an operator, $\gamma_z(\cdots)^{-1}\gamma_{z^*}^* f' = 0$. Hence $f' = 0$ and therefore $\mathsf{u} = \{0, 0\}$. This shows that the sum on the right-hand side of (3.11) is direct. Clearly this sum is contained in $\widetilde{A}_{\mathcal{T}} \cap \mathfrak{H}^2$. We prove the reverse inclusion. Let $\mathsf{u} \in \widetilde{A}_{\mathcal{T}} \cap \mathfrak{H}^2$. By (3.2), there is an $f \in \mathfrak{H}$ with

$$(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1}\gamma_{z^*}^* f \in \ker \delta_z = \mathbb{L}_c \tag{3.12}$$

such that

$$\mathsf{u} = \left\{R_0(z)f - \gamma_z(\cdots)^{-1}\gamma_{z^*}^* f,\, f + z(R_0(z)f - \gamma_z(\cdots)^{-1}\gamma_{z^*}^* f)\right\}.$$

Decompose f into the orthogonal sum $f = f_0 + f'$ with $f_0 \in \operatorname{ran}(S - z)$ and $f' \in \ker(S^* - z^*)$. Then $\gamma_{z^*}^* f_0 = 0$ and, by (3.12), $f' = \sum \alpha_i f_i'$ for some $\alpha_i \in \mathbb{C}$. By (2.10), $\{R_0(z)f_0, f_0 + zR_0(z)f_0\} \in S$. Hence

$$\mathsf{u} = \{R_0(z)f_0,\, f_0 + zR_0(z)f_0\} + \sum \alpha_i \mathsf{u}_i \in S + \operatorname{span}\{\mathsf{u}_1, \ldots, \mathsf{u}_{d_c}\},$$

and the reverse inclusion holds. This completes the proof of (3.11).

As to (iii) note that, by (3.3) and Lemma 3.4, $C_{\mathfrak{H}}(\widetilde{A}_{\mathcal{T}}) \cap A_0 = S_{\mathbb{L}_r}$. Hence (iii) follows from (2.12). If $d_r > 0$ set

$$\mathsf{v}_j := \left\{R_0(z)f_j,\, f_j + zR_0(z)f_j\right\}, \quad j = 1, \ldots, d_r,$$

where the f_j's span a d_r-dimensional subspace of $\ker(S^* - z^*)$ such that (see Lemma 3.4)

$$\gamma_{z^*}^* f_j \in \delta_{z^*}^* B_{\mathcal{T}}(0) = \operatorname{ran} \mathcal{B} = \mathbb{L}_r,$$

or, equivalently, there exist (unique) g_j's in $B_{\mathcal{T}}(0)$ such that

$$\delta_{z^*}^* g_j + \gamma_{z^*}^* f_j = 0.$$

By (3.3), the v_j's belong to $C_{\mathfrak{H}}(\widetilde{A}_{\mathcal{T}}) \cap A_0$ and are linearly independent, and the decomposition

$$C_{\mathfrak{H}}(\widetilde{A}_{\mathcal{T}}) \cap A_0 = S \dot{+} \operatorname{span}\{\mathsf{v}_1, \ldots, \mathsf{v}_{d_r}\} \tag{3.13}$$

can be proved in a way similar to the proof of (3.11).

If $d_f > 0$, let $\mathbf{x}_1, \ldots, \mathbf{x}_{d_f}$ be a basis for $\mathbb{L}_f$. Since $\int_{\mathbb{R}} \mathrm{d}\langle \Sigma(t)\mathbf{x}_k, \mathbf{x}_k\rangle < \infty$, according to Lemma 3.5 it holds that $\delta_z \mathbf{x}_k \in \operatorname{dom} B_{\mathcal{T}}$, and hence there exist elements $\widehat{g}_k \in \mathfrak{H}_{\mathcal{T}}$ such that

$$R_{\mathcal{T}}(z)\widehat{g}_k = \delta_z \mathbf{x}_k.$$

Choose elements $\widehat{f}_1, \ldots, \widehat{f}_{d_f}$ in $\ker(S^* - z^*)$ such that

$$(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1}(\delta_{z^*}^* \widehat{g}_k + \gamma_{z^*}^* \widehat{f}_k) = \mathbf{x}_k, \ k = 1, \ldots, d_f.$$

For $k = 1, \ldots, d_f$ define

$$\mathsf{w}_k := \left\{R_0(z)\widehat{f}_k - \gamma_z \mathbf{x}_k, \widehat{f}_k + z(R_0(z)\widehat{f}_k - \gamma_z \mathbf{x}_k)\right\}. \tag{3.14}$$

By (3.1), the w_k's belong to $C_{\mathfrak{H}}(\widetilde{A}_{\mathcal{T}})$. They are linearly independent. For, if for some τ_k's in $\mathbb{C}$ we have $\sum \tau_k \mathsf{w}_k = 0$, then $\sum \tau_k(R_0(z)\widehat{f}_k - \gamma_z \mathbf{x}_k) = 0$ and $\sum \tau_k \widehat{f}_k = 0$, which implies $\gamma_z \sum \tau_k \mathbf{x}_k = 0$, whence $\sum \tau_k \mathbf{x}_k = 0$, and thus $\tau_k = 0$ for all k.

We claim:

$$C_{\mathfrak{H}}(\widetilde{A}_{\mathcal{T}}) = S \dot{+} \operatorname{span}\{\mathsf{u}_1, \ldots, \mathsf{u}_{d_c}\} \dot{+} \operatorname{span}\{\mathsf{v}_1, \ldots, \mathsf{v}_{d_r}\} \dot{+} \operatorname{span}\{\mathsf{w}_1, \ldots, \mathsf{w}_{d_f}\}. \tag{3.15}$$

The claim implies (v) and also (iv), because of (3.11) and (3.13).

To prove that the sum on the right-hand side of (3.15) is direct, consider an element $\{f, g\} \in S$ and complex numbers α_i, $i = 1, \ldots, d_c$, β_j, $j = 1, \ldots, d_r$, τ_k, $k = 1, \ldots, d_f$, such that

$$\{f, g\} + \sum_{i=1}^{d_c} \alpha_i \mathsf{u}_i + \sum_{j=1}^{d_r} \beta_j \mathsf{v}_j + \sum_{k=1}^{d_f} \tau_k \mathsf{w}_k = \{0, 0\}.$$

Then we have two equalities:

$$\sum_{i=1}^{d_c} \alpha_i (R_0(z) f_i' - \gamma_z (\cdots)^{-1} \gamma_{z^*}^* f_i') + \sum_{j=1}^{d_r} \beta_j R_0(z) f_j + \sum_{k=1}^{d_f} \tau_k (R_0(z)\widehat{f}_k - \gamma_z \mathbf{x}_k) = -f$$

and

$$\sum_{i=1}^{d_c} \alpha_i f_i' + \sum_{j=1}^{d_r} \beta_j f_j + \sum_{k=1}^{d_f} \tau_k \widehat{f_k} = -(g - zf).$$

From $\{g - zf, f\} \in (S - z)^{-1} \subset R_0(z)$ and by applying $R_0(z)$ to both sides of the second equality we obtain

$$\sum_{i=1}^{d_c} \alpha_i R_0(z) f_i' + \sum_{j=1}^{d_r} \beta_j R_0(z) f_j + \sum_{k=1}^{d_f} \tau_k R_0(z) \widehat{f_k} = -f.$$

Combining this with the first equality we get

$$(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1} \gamma_{z^*}^* \sum_{i=1}^{d_c} \alpha_i f_i' = -\sum_{k=1}^{d_f} \tau_k \mathbf{x}_k \in \mathbb{L}_c \cap \mathbb{L}_f = \{0\}.$$

It follows that the α_i's and the τ_k's are zero. Hence

$$\sum_{j=1}^{d_r} \beta_j f_j = -(g - zf) \in \operatorname{ran}(S - z) \cap \ker(S^* - z^*) = \{0\},$$

which implies that the β_j's are zero. Finally we see that $\{f, g\} = \{0, 0\}$. This proves that the sum on the right-hand side of (3.15) is direct. Clearly this sum is contained in $C_{\mathfrak{H}}(\widetilde{A}_{\mathcal{T}})$, and it remains to prove the reverse inclusion.

Consider an element $\{x, y\} \in C_{\mathfrak{H}}(\widetilde{A}_{\mathcal{T}})$. Then there exist elements $h \in \mathfrak{H}$ and $k \in \mathfrak{H}_{\mathcal{T}}$ satisfying

$$R_{\mathcal{T}}(z)k = \delta_z(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1}(\delta_{z^*}^* k + \gamma_{z^*}^* h)$$

such that

$$x = R_0(z)h - \gamma_z(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1}(\delta_{z^*}^* k + \gamma_{z^*}^* h) \quad \text{and} \quad y = h + zx.$$

We decompose k as

$$k = g + \widehat{g}, \quad g \in B_{\mathcal{T}}(0), \ \widehat{g} \in B_{\mathcal{T}}(0)^{\perp},$$

and h as $h = h_0 + h_1$ with $h_0 \in \operatorname{ran}(S - z)$ and $h_1 \in \ker(S^* - z^*)$ decomposed as

$$h_1 = f + \widehat{f} + f',$$

where $f \in \ker(S^* - z^*)$ is chosen such that $\gamma_{z^*}^* f + \delta_{z^*}^* g = 0$, $f' := P_z(h_1 - f)$ and $\widehat{f} := (I - P_z)(h_1 - f)$; here P_z denotes the projection in $\ker(S^* - z^*)$ onto $\ker(S^* - z^*) \cap \ker\big(\delta_z(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1}\gamma_{z^*}^*\big)$. Hence $(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1}\gamma_{z^*}^* f' \in \mathbb{L}_c$,

$$R_{\mathcal{T}}(z)\widehat{g} = \delta_z \mathbf{x} \quad \text{with} \quad \mathbf{x} = (\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1}(\gamma_{z^*}^* \widehat{f} + \delta_{z^*}^* \widehat{g}) \in \mathbb{L}_f$$

and

$$\{x, y\} = \{R_0(z)h_0, h_0 + zR_0(z)h_0\} + \mathsf{u} + \mathsf{v} + \mathsf{w}$$

with $\mathsf{u}, \mathsf{v}, \mathsf{w} \in C_{\mathfrak{H}}(\widetilde{A}_{\mathcal{T}})$ given by

$$\begin{aligned}
\mathsf{u} &= \{R_0(z)f' - \gamma_z(\cdots)^{-1}\gamma_{z^*}^* f', f' + z(R_0(z)f' - \gamma_z(\cdots)^{-1}\gamma_{z^*}^* f')\},\\
\mathsf{v} &= \{R_0(z)f, f + R_0(z)f\},\\
\mathsf{w} &= \{R_0(z)\widehat{f} - \gamma_z \mathbf{x}, \widehat{f} + z(R_0(z)\widehat{f} - \gamma_z \mathbf{x})\}.
\end{aligned}$$

By (2.10) and since $\gamma_{z^*}^* h_0 = 0$, we have $\{R_0(z)h_0, h_0 + zR_0(z)h_0\} \in S$. Clearly, $\mathsf{u} \in \operatorname{span}\{\mathsf{u}_i\}$, $\mathsf{v} \in \operatorname{span}\{\mathsf{v}_j\}$ and $\mathsf{w} \in \operatorname{span}\{\mathsf{w}_k\}$. Hence $\{x, y\}$ belongs to the right-hand side of (3.15). This completes the proof of (3.15). □

In the following Corollaries 3.8–3.13 it is always assumed that the parameter $\mathcal{T}$ is a Nevanlinna $d \times d$ *matrix* function.

Corollary 3.8 *The compression $C_{\mathfrak{H}}(\widetilde{A}_{\mathcal{T}})$ has equal defect numbers d^∞. If $\mathcal{T}$ is rational, then $C_{\mathfrak{H}}(\widetilde{A}_{\mathcal{T}})$ is self-adjoint.*

Proof The first statement follows from Proposition 3.1(i), (3.10) and Theorem 3.7(v). The last statement follows from Stenger's lemma (see [15]), but also from the fact that if $\mathcal{T}$ is rational, then, by Remark 3.6, $\mathbb{L}^\infty = \{0\}$, that is $d^\infty = 0$. □

Corollary 3.9 ([14, Theorem 3.9]) *The following statements are equivalent*:

(a) $\lim\limits_{y \to +\infty} y \operatorname{Im}\big\langle \mathcal{T}(\mathrm{i}y)\mathbf{x}, \mathbf{x}\big\rangle = \infty$ *for all* $\mathbf{x} \in \mathbb{C}^d \setminus \{0\}$.

(b) $C_{\mathfrak{H}}(\widetilde{A}_{\mathcal{T}}) \subset A_0$.

Proof We show that (a) and (b) are equivalent to (c): $d_r + d^\infty = d$. First we prove that (a) and (c) are equivalent for the case that $\mathcal{T}$ is rational. Then, by Remark 3.6, $d^\infty = 0$. By (2.1), (a) holds if and only if $\mathcal{B} > 0$. Since $\mathcal{B} \geq 0$, this holds if and only if $d_r = \dim \operatorname{ran} \mathcal{B} = d$. This implies that (a) and (c) are equivalent. Now we consider the case that $\mathcal{T}$ is non-rational. By (3.10), (c) is equivalent to $d_c = 0$ and $d_f = 0$. Assume (c). Then $\mathbb{L}_c^\perp = \mathbb{C}^d$ and $\mathbb{L}_f = \{0\}$ imply

$$\int_{\mathbb{R}} \mathrm{d}\langle \Sigma(t)\mathbf{x}, \mathbf{x}\rangle = \infty \quad \text{for all } \mathbf{x} \in \ker \mathcal{B} \setminus \{0\}.$$

Hence (a) follows from (2.1). Assume (a). By Lemma 3.3, the kernel $\ker \operatorname{Im} \mathcal{T}(z)$ is independent of $z \in \mathbb{C} \setminus \mathbb{R}$. Hence (a) implies that $\ker \operatorname{Im} \mathcal{T}(z) = \{0\}$, and so, again by Lemma 3.3, $d_c = 0$. From $\mathbb{L}_c^{\perp} = \mathbb{C}^d$ and (2.1) we now obtain

$$\mathbb{L}_f = \left\{ \mathbf{x} \in \mathbb{C}^d \, : \, \mathbf{x} \in \ker \mathcal{B}, \int_{\mathbb{R}} \mathrm{d}\langle \Sigma \mathbf{x}, \mathbf{x} \rangle < \infty \right\} = \{0\}.$$

Hence (a) implies (c), and (a) and (c) are equivalent.

We show that (c) is equivalent to (b). Assume (c), that is the equalities $d_c = d_f = 0$. By Theorem 3.7(ii) and (iv), $d_c = 0$ implies $\widetilde{A}_{\mathcal{T}} \cap \mathfrak{H}^2 = S$ and $d_f = 0$ implies

$$C_{\mathfrak{H}}(\widetilde{A}_{\mathcal{T}}) = \widetilde{A}_{\mathcal{T}} \cap \mathfrak{H}^2 + C_{\mathfrak{H}}(\widetilde{A}_{\mathcal{T}}) \cap A_0.$$

From this and $S \subset A_0$ it follows that $C_{\mathfrak{H}}(\widetilde{A}_{\mathcal{T}}) \subset A_0$, that is (b) holds. Before proving that (b) implies $d_c = d_f = 0$ we first show that

$$\operatorname{span}\{\mathsf{w}_1, \ldots, \mathsf{w}_{d_f}\} \cap A_0 = \{0\}, \tag{3.16}$$

where w_k is given by (3.14). Assume w belongs to the intersection. Then there exist τ_k's in $\mathbb{C}$ and $u \in \mathfrak{H}$ such that

$$\mathsf{w} = \sum \tau_k \mathsf{w}_k = \{R_0(z)u, u + zR_0(z)u\},$$

that is

$$\begin{cases} R_0(z)u = R_0(z) \sum \tau_k \widehat{f_k} - \gamma_z \sum \tau_k \mathbf{x}_k, \\ u + zR_0(z)u = \sum \tau_k \widehat{f_k} + z(R_0(z) \sum \tau_k \widehat{f_k} - \gamma_z \sum \tau_k \mathbf{x}_k). \end{cases}$$

It successively follows that $u = \sum \tau_k \widehat{f_k}$, $\gamma_z \sum \tau_k \mathbf{x}_k = 0$ and $\sum \tau_k \mathbf{x}_k = 0$. Since the $\mathbf{x}_k$'s are linearly independent, the τ_k's are zero, hence $\mathsf{w} = 0$. This proves (3.16). Now assume (b). Then

$$\widetilde{A}_{\mathcal{T}} \cap \mathfrak{H}^2 \subset C_{\mathfrak{H}}(\widetilde{A}_{\mathcal{T}}) \subset A_0$$

and hence

$$S \subset \widetilde{A}_{\mathcal{T}} \cap \mathfrak{H}^2 \subset \widetilde{A}_{\mathcal{T}} \cap A_0 = S.$$

This proves that $\widetilde{A}_{\mathcal{T}} \cap \mathfrak{H}^2 = S$, that is $d_c = 0$. The inclusion in (b) and (3.15) imply that $\operatorname{span}\{\mathsf{w}_1, \ldots, \mathsf{w}_{d_f}\} \subset A_0$, whence, by (3.16), $d_f = 0$. □

Corollary 3.10 ([14, Corollary 3.10]) *The following statements are equivalent*:

(a) $\mathcal{B} > 0$.
(b) $C_{\mathfrak{H}}(\widetilde{A}_{\mathcal{T}}) = A_0$.

Proof We show that (a) and (b) are equivalent to (c): $d_r = d$. The equality in (c) holds if and only if $d = \dim \mathbb{L}_r = \dim \operatorname{ran} \mathcal{B}$. Since $\mathcal{B} \geq 0$, (c) is equivalent to (a). That (c) is equivalent to (b) follows from (3.10), Corollary 3.8, Corollary 3.9, $A_0 = A_0^*$ and the following equivalences

$$\begin{aligned} d_r = d &\iff d^\infty = 0,\ d_c + d_f = 0 \\ &\iff C_{\mathfrak{H}}(\widetilde{A}_{\mathcal{T}}) \text{ is self-adjoint, } C_{\mathfrak{H}}(\widetilde{A}_{\mathcal{T}}) \subset A_0 \\ &\iff C_{\mathfrak{H}}(\widetilde{A}_{\mathcal{T}}) = A_0. \end{aligned}$$

□

Corollary 3.11 ([2, Theorem 7.13], [14, Corollary 3.11]) *The following statements are equivalent*:

(a) $B = 0$ *and* $\lim_{y\to+\infty} y \operatorname{Im}\big\langle \mathcal{T}(\mathrm{i}y)\mathbf{x}, \mathbf{x}\big\rangle = \infty$ *for all* $\mathbf{x} \in \mathbb{C}^d \setminus \{0\}$.

(b) $C_{\mathfrak{H}}(\widetilde{A}_{\mathcal{T}}) = S$.

Proof We show that (a) and (b) are equivalent to (c): $d^\infty = d$. Assume (c). By (3.10), $d_r = d_c = d_f = 0$. From $d_r = 0$ it follows that $\mathcal{B} = 0$, the first equality in (a). Hence $\ker B = \mathbb{C}^d$, and since $d_c = 0$ implies that also $\mathbb{L}_c^\perp = \mathbb{C}^d$ we obtain

$$\mathbb{L}_f = \{\mathbf{x} \in \mathbb{C}^d \ :\ \int_{\mathbb{R}} \mathrm{d}\langle \Sigma(t)\mathbf{x}, \mathbf{x}\rangle < \infty\}. \tag{3.17}$$

The second equality in (a) now follows from $d_f = 0$ and (2.5). Thus (c) implies (a). Assume (a). Then $d_r = 0$. As shown in the proof of Corollary 3.9 the second equality in (a) implies $d_c = 0$. Thus $\mathbb{L}_f$ is given by (3.17). By (2.5), $d_f = 0$. Thus (c) holds, and (a) and (c) are equivalent. It follows from Theorem 3.7(v) that (c) and (b) are equivalent. □

Corollary 3.12 ([14, Proposition 3.14]) *If* $\lim_{y\to+\infty} y \operatorname{Im}\big\langle \mathcal{T}(\mathrm{i}y)\mathbf{x}, \mathbf{x}\big\rangle < \infty$ *for all* $\mathbf{x} \in \ker \mathcal{B}$*, then* $C_{\mathfrak{H}}(\widetilde{A}_{\mathcal{T}})$ *is self-adjoint.*

Proof The assumption and (2.5) imply

$$\int_{\mathbb{R}} \mathrm{d}\langle \Sigma(t)\mathbf{x}, \mathbf{x}\rangle < \infty \ \text{ for all } x \in \ker \mathcal{B},$$

and so, by (3.8), $\mathbb{L}^\infty = \{0\}$, that is $d^\infty = 0$ and the compression is self-adjoint. □

Corollary 3.13 ([14, Proposition 3.16]) *The following statements are equivalent*:

(a) $\lim_{y\to+\infty} y\,\mathrm{Im}\left\langle \mathcal{T}(\mathrm{i}y)\mathbf{x},\mathbf{x}\right\rangle < \infty$ *for all* $\mathbf{x}\in\mathbb{C}^d$.
(b) $C_{\mathfrak{H}}(\widetilde{A}_{\mathcal{T}})$ *is self-adjoint and* $C_{\mathfrak{H}}(\widetilde{A}_{\mathcal{T}})\cap A_0 = S$.

Proof We show that (a) and (b) are equivalent to (c): $d_c + d_f = d$. By (3.10), (c) holds if and only if $d^\infty = 0$ and $d_r = 0$. By Corollary 3.8 and Theorem 3.7(iii), these equalities are equivalent to (b). These equalities also imply $\mathcal{B} = 0$, because of the definition of d_r, and $\mathbb{L}_f = \mathbb{L}_c^\perp$, because of (3.7). Hence the following inequality holds:

$$\int_{\mathbb{R}} \mathrm{d}\langle \Sigma(t)\mathbf{x},\mathbf{x}\rangle < \infty \text{ for all } \mathbf{x}\in\mathbb{L}_c^\perp.$$

Since $\mathbb{L}_c \subset \ker\Sigma$, this inequality holds for all $x\in\mathbb{L}_c^\perp\oplus\mathbb{L}_c = \mathbb{C}^d$. Thus, by (2.5), (c) implies (a). Assume (a). Then $\mathcal{B} = 0$, and, by (2.5) and the definition of the space $\mathbb{L}_f$, $\mathbb{L}_f = \mathbb{L}_c^\perp$. Hence $\mathbb{C}^d = \mathbb{L}_c\oplus\mathbb{L}_f$, that is (c) holds. □

Remark 3.14 V.I. Mogilevskii (correspondence) proposed to introduce with the parameter $\mathcal{T}$ also the subspaces

$$\mathbb{T}_1 := \left\{\mathbf{x}\in\mathbb{C}^d \;:\; \lim_{y\to\infty} y\,\mathrm{Im}\langle\mathcal{T}(\mathrm{i}y)\mathbf{x},\mathbf{x}\rangle < \infty\right\}, \tag{3.18}$$

$$\mathbb{T}_2 := \left\{\mathbf{x}\in\mathbb{C}^d \;:\; \lim_{y\to\infty} \frac{1}{y}\mathcal{T}(\mathrm{i}y)\mathbf{x} = 0\right\}. \tag{3.19}$$

of $\mathbb{C}^d$. Then the following relations hold:

$$\mathbb{T}_1 = \mathbb{L}_c\oplus\mathbb{L}_f,\quad \mathbb{T}_2 = \mathbb{L}_r^\perp,$$

and, by (3.9), $\mathbb{T}_2 = \mathbb{T}_1\oplus\mathbb{L}^\infty$. According to Corollary 3.8, the defect numbers of the compression $C_{\mathfrak{H}}(\widetilde{A}_{\mathcal{T}})$ are now equal to $\dim\mathbb{T}_2 - \dim\mathbb{T}_1$, whereas, by Theorem 3.7 (iii), the defect numbers of the symmetric extension $C_{\mathfrak{H}}(\widetilde{A})\cap A_0$ of S equal $\dim\mathbb{T}_2$.

Remark 3.15 Now suppose that $\mathcal{T}$ is multi-valued, that is $\mathcal{T}$ is a Nevanlinna $d\times d$ *relation* function of the form (2.9) with integral and relation representations (2.1) and (2.2). Recall that this means that in these representations we replace d and $\mathcal{T}$ by d_m and $\mathcal{T}_m$. Similarly, the decomposition (3.9) of the space $\mathbb{C}^d$ should now be interpreted as the decomposition of ran P_m identified with $\mathbb{C}^{d_m}$ and so (3.10) becomes $d_m = d_c + d_r + d_f + d^\infty$, that is

$$d = d_\infty + d_c + d_r + d_f + d^\infty.$$

With this understanding, the formulas (i), (ii), (iii) and (v) of Theorem 3.7 hold if only on the left-hand sides S is replaced by its symmetric extension $S_{\ker P_m}$ or, equivalently (see (2.16)), if only the numbers on right-hand sides are raised by d_∞.

The equality (iv) in Theorem 3.7 remains correct as it stands. The conclusions in Corollary 3.8 also remain unaltered if $\mathcal{T}$ is a Nevanlinna $d \times d$ relation function. The equivalent statements in Corollaries 3.9–3.13 remain correct if d, $\mathcal{T}$ and S are replaced by d_m, $\mathcal{T}_m$ and $S_{\ker P_m}$. Of course, if the equalities in the last items of these corollaries remain as they are (that is with d on the right-hand side) then $d_\infty = 0$ and the relation function $\mathcal{T}$ is in fact a matrix function.

4 A Special Self-Adjoint Extension of the Compression

4.1 *Straus Extensions*

Let S be a closed symmetric linear relation in the Hilbert space $\mathfrak{H}$ with finite and equal defect numbers $d > 0$, and let $\widetilde{A}$ be a self-adjoint extension of S in some Hilbert space $\widetilde{\mathfrak{H}} \supset \mathfrak{H}$. If $\lambda \in \mathbb{C}$, the *Straus extension* $S_{\widetilde{A}}(\lambda)$ of S or *Straus subspace* associated with $\widetilde{A}$ is the linear relation

$$S_{\widetilde{A}}(\lambda) := \left\{ \{\widetilde{P}_{\mathfrak{H}} \widetilde{f}, \widetilde{P}_{\mathfrak{H}} \widetilde{g}\} : \{\widetilde{f}, \widetilde{g}\} \in \widetilde{A},\ \widetilde{g} - \lambda \widetilde{f} \in \mathfrak{H} \right\}. \tag{4.1}$$

Here $\widetilde{P}_{\mathfrak{H}}$ denotes the orthogonal projection in $\widetilde{\mathfrak{H}}$ onto $\mathfrak{H}$.

In the following proposition we collect some well-known properties of the linear relation $S_{\widetilde{A}}(\lambda)$.

Proposition 4.1 *If* $\lambda \in \mathbb{C}$*, then:*

(i) $S \subset \widetilde{A} \cap \mathfrak{H}^2 \subset S_{\widetilde{A}}(\lambda) \subset \left\{ \{\widetilde{P}_{\mathfrak{H}} \widetilde{f}, \widetilde{P}_{\mathfrak{H}} \widetilde{g}\} : \{\widetilde{f}, \widetilde{g}\} \in \widetilde{A} \right\}$,
(ii) $S_{\widetilde{A}}(\lambda^*) \subset S_{\widetilde{A}}(\lambda)^*$ *with equality if* $\lambda \in \rho(\widetilde{A})$,
(iii) *if* $\lambda \in \mathbb{C}^{\mp}$ *then* $\pm S_{\widetilde{A}}(\lambda)$ *is maximal dissipative,*
(iv) *if* $\lambda \notin \sigma_p(\widetilde{A})$*, the point spectrum of* $\widetilde{A}$*, then*

$$S_{\widetilde{A}}(\lambda) = \left\{ \{\widetilde{P}_{\mathfrak{H}}(\widetilde{A} - \lambda)^{-1} h, \widetilde{P}_{\mathfrak{H}}(I + \lambda(\widetilde{A} - \lambda)^{-1})h\} : h \in \operatorname{ran}(\widetilde{A} - \lambda) \cap \mathfrak{H} \right\}. \tag{4.2}$$

Proof The claims (i) and (ii) follow easily from the definition of $S_{\widetilde{A}}(\lambda)$. To prove (iii) consider $\{\widetilde{f}, \widetilde{g}\} \in \widetilde{A}$ with $\widetilde{g} - \lambda \widetilde{f} \in \mathfrak{H}$. Then

$$\begin{aligned}
(\widetilde{P}_{\mathfrak{H}} \widetilde{g}, \widetilde{P}_{\mathfrak{H}} \widetilde{f})_{\mathfrak{H}} &= (\widetilde{P}_{\mathfrak{H}}(\widetilde{g} - \lambda \widetilde{f}), \widetilde{P}_{\mathfrak{H}} \widetilde{f})_{\mathfrak{H}} + \lambda \|\widetilde{P}_{\mathfrak{H}} \widetilde{f}\|^2_{\mathfrak{H}} \\
&= (\widetilde{g} - \lambda \widetilde{f}, \widetilde{f})_{\widetilde{\mathfrak{H}}} + \lambda \|\widetilde{P}_{\mathfrak{H}} \widetilde{f}\|^2_{\mathfrak{H}} \\
&= (\widetilde{g}, \widetilde{f})_{\widetilde{\mathfrak{H}}} - \lambda \|(I - \widetilde{P}_{\mathfrak{H}}) \widetilde{f}\|^2_{\widetilde{\mathfrak{H}}}.
\end{aligned}$$

Since $\widetilde{A}$ is self-adjoint, $\operatorname{Im}(\widetilde{g}, \widetilde{f})_{\widetilde{\mathfrak{H}}} = 0$. Hence

$$\pm \operatorname{Im}(\widetilde{P}_{\mathfrak{H}} \widetilde{g}, \widetilde{P}_{\mathfrak{H}} \widetilde{f})_{\mathfrak{H}} = \mp(\operatorname{Im}\lambda) \|(I - \widetilde{P}_{\mathfrak{H}}) \widetilde{f}\|^2_{\widetilde{\mathfrak{H}}} \geq 0,\ \lambda \in \mathbb{C}^{\mp}.$$

This proves that $\pm S_{\widetilde{A}}(\lambda)$ is dissipative for all $\lambda \in \mathbb{C}^{\mp}$. It is maximal dissipative for these λ's because for all $z \in \mathbb{C} \setminus \mathbb{R}$ the operator $(S_{\widetilde{A}}(z) - z)^{-1} = \widetilde{P}_{\mathfrak{H}}(\widetilde{A} - z)^{-1}|_{\mathcal{H}}$ is bounded on all of $\mathfrak{H}$, that is $\mathbb{C}^- \cap \rho(\pm S_{\widetilde{A}}(\lambda)) \neq \emptyset$ and consequently $\mathbb{C}^- \subset \rho(\pm S_{\widetilde{A}}(\lambda))$ for all $\lambda \in \mathbb{C}^{\mp}$.

To prove (iv), let $\lambda \notin \sigma_p(\widetilde{A})$. Then $(\widetilde{A} - \lambda)^{-1}$ is an operator on $\operatorname{ran}(\widetilde{A} - \lambda)$. Consider $h \in \operatorname{ran}(\widetilde{A} - \lambda) \cap \mathfrak{H}$ and set

$$\widetilde{f} = (\widetilde{A} - \lambda)^{-1} h, \qquad \widetilde{g} = (I + \lambda(\widetilde{A} - \lambda)^{-1})h \tag{4.3}$$

Then $\{\widetilde{f}, \widetilde{g}\} \in \widetilde{A}$ and $\widetilde{g} - \lambda \widetilde{f} = h \in \mathfrak{H}$. Hence

$$\{\widetilde{P}_{\mathfrak{H}}(\widetilde{A} - \lambda)^{-1} h, \widetilde{P}_{\mathfrak{H}}(I + \lambda(\widetilde{A} - \lambda)^{-1})h\} = \{\widetilde{P}_{\mathfrak{H}} \widetilde{f}, \widetilde{P}_{\mathfrak{H}} \widetilde{g}\} \in S_{\widetilde{A}}(\lambda).$$

Conversely, consider $\{f, g\} \in S_{\widetilde{A}}(\lambda)$. Then there exists $\{\widetilde{f}, \widetilde{g}\} \in \widetilde{A}$ with $\widetilde{g} - \lambda \widetilde{f} \in \mathfrak{H}$ such that

$$\{f, g\} = \{\widetilde{P}_{\mathfrak{H}} \widetilde{f}, \widetilde{P}_{\mathfrak{H}} \widetilde{g}\}.$$

Set $h = \widetilde{g} - \lambda \widetilde{f}$. Then $h \in \operatorname{ran}(\widetilde{A} - \lambda) \cap \mathfrak{H}$ and (4.3) holds. Hence

$$\{f, g\} = \{\widetilde{P}_{\mathfrak{H}}(\widetilde{A} - \lambda)^{-1} h, \widetilde{P}_{\mathfrak{H}}(I + \lambda(\widetilde{A} - \lambda)^{-1})h\}$$

belongs to the set on the right-hand side of (4.2). □

Proposition 4.2 *Let $z \in \mathbb{C}^+$ and let A be a canonical self-adjoint extension and $\widetilde{A}$ be a minimal self-adjoint extension of S with exit. If the resolvent operators $(S_{\widetilde{A}}(\mathrm{i}\lambda) - z)^{-1}$ converge in norm to the resolvent operator $(A - z)^{-1}$ as $\lambda \to +\infty$, then $C_{\mathfrak{H}}(\widetilde{A}) \subset A$.*

Proof Choose a sequence of numbers $\lambda_n > 0$ converging to ∞ as $n \to \infty$. The assumption implies that if $\{u_n, v_n\} \in \mathfrak{H}^2$ converges to $\{u, v\} \in \mathfrak{H}^2$ as $n \to \infty$ and $v_n = (S_{\widetilde{A}}(\mathrm{i}\lambda_n) - z)^{-1} u_n$, then $v = (A - z)^{-1} u$. We apply this implication twice.

(a) Consider $f \in \mathfrak{H}$ and set

$$u_n := \widetilde{P}_{\mathfrak{H}} \left(I + (\mathrm{i}\lambda_n - z)(\widetilde{A} - \mathrm{i}\lambda_n)^{-1} \right) f, \quad v_n := \widetilde{P}_{\mathfrak{H}}(\widetilde{A} - \mathrm{i}\lambda_n)^{-1} f.$$

Then, by (4.2), $v_n = (S_{\widetilde{A}}(\mathrm{i}\lambda_n) - z)^{-1} u_n$ and, as $n \to \infty$, $\{u_n, v_n\}$ converges to $\{\widetilde{P}_{\mathfrak{H}} \widetilde{P}_\infty f, 0\}$ in $\mathfrak{H}^2$, where $\widetilde{P}_\infty$ stands for the orthogonal projection in $\widetilde{\mathfrak{H}}$ onto $\widetilde{A}(0)$. It follows that

$$(A - z)^{-1} \widetilde{P}_{\mathfrak{H}} \widetilde{P}_\infty f = 0,$$

that is $\widetilde{P}_{\mathfrak{H}}\widetilde{P}_\infty f \in A(0)$. Since $f \in \mathfrak{H}$ is arbitrary, observing Proposition 3.1(iii), we have shown that

$$\widetilde{P}_{\mathfrak{H}}\widetilde{A}(0) = \widetilde{P}_{\mathfrak{H}}\widetilde{P}_\infty\mathfrak{H} \subset A(0). \tag{4.4}$$

(b) Choose $\{g, h\} \in (C_{\mathfrak{H}}(\widetilde{A}) - z)^{-1}$, that is $\{h, g + zh\} \in C_{\mathfrak{H}}(\widetilde{A})$. Then there exists an element $\widetilde{k} \in \widetilde{\mathfrak{H}}$ such that $\widetilde{P}_{\mathfrak{H}}\widetilde{k} = g + zh$ and $\{h, \widetilde{k}\} \in \widetilde{A}$, and hence $(\widetilde{A} - z)^{-1}(\widetilde{k} - zh) = h$. Set

$$\begin{aligned}
u_n &:= -\mathrm{i}\lambda_n\widetilde{P}_{\mathfrak{H}}\big(I + (\mathrm{i}\lambda_n - z)(\widetilde{A} - \mathrm{i}\lambda_n)^{-1}\big)h \\
&= -\mathrm{i}\lambda_n\widetilde{P}_{\mathfrak{H}}\big(h + (\mathrm{i}\lambda_n - z)(\widetilde{A} - \mathrm{i}\lambda_n)^{-1}(\widetilde{A} - z)^{-1}(\widetilde{k} - zh)\big) \\
&= -\mathrm{i}\lambda_n\widetilde{P}_{\mathfrak{H}}\big(h + (\widetilde{A} - \mathrm{i}\lambda_n)^{-1}(\widetilde{k} - zh) - (\widetilde{A} - z)^{-1}(\widetilde{k} - zh)\big) \\
&= -\mathrm{i}\lambda_n\widetilde{P}_{\mathfrak{H}}\big(\widetilde{A} - \mathrm{i}\lambda_n\big)^{-1}(\widetilde{k} - zh\big)
\end{aligned}$$

and

$$\begin{aligned}
v_n &:= -\mathrm{i}\lambda_n\widetilde{P}_{\mathfrak{H}}(\widetilde{A} - \mathrm{i}\lambda_n)^{-1}h \\
&= -\mathrm{i}\lambda_n\widetilde{P}_{\mathfrak{H}}\frac{(\widetilde{A} - \mathrm{i}\lambda_n)^{-1} - (\widetilde{A} - z)^{-1}}{\mathrm{i}\lambda_n - z}(\widetilde{k} - zh).
\end{aligned}$$

Then, again by (4.2), $v_n = (S_{\mathcal{T}}(\mathrm{i}\lambda_n) - z)^{-1}u_n$. Observing that, as $n \to \infty$,

$$u_n \to u := \widetilde{P}_{\mathfrak{H}}(I - \widetilde{P}_\infty)(\widetilde{k} - zh) = g - \widetilde{P}_{\mathfrak{H}}\widetilde{P}_\infty(\widetilde{k} - zh)$$

and

$$v_n \to v := \widetilde{P}_{\mathfrak{H}}(\widetilde{A} - z)^{-1}(\widetilde{k} - zh) = h$$

we have

$$(A - z)^{-1}g = (A - z)^{-1}\big(g - \widetilde{P}_{\mathfrak{H}}\widetilde{P}_\infty(\widetilde{k} - zh)\big) = h.$$

Here we used that, by (4.4), $\widetilde{P}_{\mathfrak{H}}\widetilde{P}_\infty(\widetilde{k} - zh) \in \widetilde{P}_{\mathfrak{H}}\widetilde{A}(0) \subset A(0)$. Thus $\{g, h\} \in (A-z)^{-1}$. Hence $(C_{\mathfrak{H}}(\widetilde{A})-z)^{-1} \subset (A-z)^{-1}$ and this implies the asserted inclusion in the proposition. □

4.2 Straus Extensions and Krein's Resolvent Formula

In the following, the self-adjoint extension $\widetilde{A}$ of S is often the minimal self-adjoint extension which corresponds to the parameter $\mathcal{T}$ in Krein's formula and which we denote by $\widetilde{A}_{\mathcal{T}}$. In this situation, instead of $S_{\widetilde{A}_{\mathcal{T}}}(\lambda)$ we write $S_{\mathcal{T}}(\lambda)$.

Proposition 4.3 *If* $\mathcal{T}$ *is a Nevanlinna* $d \times d$ *relation function, then for* $\lambda, z \in \mathbb{C} \setminus \mathbb{R}$ *such that* $\operatorname{Im} \lambda \cdot \operatorname{Im} z > 0$ *we have*

$$(S_{\mathcal{T}}(\lambda) - z)^{-1} = (A_0 - z)^{-1} - \gamma_z(\mathcal{Q}_0(z) + \mathcal{T}(\lambda))^{-1}\gamma_{z^*}^*. \tag{4.5}$$

Proof Assume $\lambda, z \in \mathbb{C} \setminus \mathbb{R}$ and $\operatorname{Im} \lambda \cdot \operatorname{Im} z > 0$. Then $\lambda \in \rho(\widetilde{A}_{\mathcal{T}}) \cap \rho(A_0)$ and $z \in \rho(S_{\mathcal{T}}(\lambda)) \cap \rho(A_0)$ and, by Proposition 4.1 (iv),

$$\begin{aligned}(S_{\mathcal{T}}(\lambda) - z)^{-1} = \big\{\{\widetilde{P}_{\mathfrak{H}}(I + (\lambda - z)(\widetilde{A}_{\mathcal{T}} - \lambda)^{-1})h,\ \widetilde{P}_{\mathfrak{H}}(\widetilde{A}_{\mathcal{T}} - \lambda)^{-1}h\} :\\ h \in \operatorname{ran}(\widetilde{A}_{\mathcal{T}} - \lambda) \cap \mathfrak{H}\big\}.\end{aligned} \tag{4.6}$$

For $h \in \operatorname{ran}(\widetilde{A}_{\mathcal{T}} - \lambda) \cap \mathfrak{H}$, which equals $\mathfrak{H}$ because $\lambda \in \rho(\widetilde{A}_{\mathcal{T}})$, we set

$$k = \widetilde{P}_{\mathfrak{H}}\big(I + (\lambda - z)(\widetilde{A}_{\mathcal{T}} - \lambda)^{-1}\big)h.$$

Then, if h varies over all of $\mathfrak{H}$, k varies over all of $\operatorname{ran}(S_{\mathcal{T}}(\lambda) - z)$, which equals $\mathfrak{H}$ because $z \in \rho(S_{\mathcal{T}}(\lambda))$. Indeed, if $\{\widetilde{f}, \widetilde{g}\} \in \widetilde{A}_{\mathcal{T}}$ and $h = \widetilde{g} - \lambda \widetilde{f} \in \mathfrak{H}$, then $\{\widetilde{P}_{\mathfrak{H}}\widetilde{f}, \widetilde{P}_{\mathfrak{H}}\widetilde{g}\} \in S_{\mathcal{T}}(\lambda)$, and

$$k = \widetilde{P}_{\mathfrak{H}}(I + (\lambda - z)(\widetilde{A}_{\mathcal{T}} - \lambda)^{-1})h = \widetilde{P}_{\mathfrak{H}}\widetilde{g} - z\widetilde{P}_{\mathfrak{H}}\widetilde{f} \in \operatorname{ran}(S_{\mathcal{T}}(\lambda) - z) = \mathfrak{H}.$$

Conversely, if $k \in \mathfrak{H} = \operatorname{ran}(S_{\mathcal{T}}(\lambda) - z)$, then $k = g - zf$ for some $\{f, g\} \in S_{\mathcal{T}}(\lambda)$. There exists $\{\widetilde{f}, \widetilde{g}\} \in \widetilde{A}_{\mathcal{T}}$ such that $\widetilde{P}_{\mathfrak{H}}\widetilde{f} = f$, $\widetilde{P}_{\mathfrak{H}}\widetilde{g} = g$ and $\widetilde{g} - \lambda\widetilde{f} \in \mathfrak{H}$. If we set $h = \widetilde{g} - \lambda\widetilde{f}$, then $h \in \mathfrak{H}$ and $(\widetilde{A}_{\mathcal{T}} - \lambda)^{-1}h = \widetilde{f}$. It follows that

$$k = \widetilde{P}_{\mathfrak{H}}(\widetilde{g} - \lambda\widetilde{f}) + (\lambda - z)\widetilde{P}_{\mathfrak{H}}\widetilde{f} = \widetilde{P}_{\mathfrak{H}}(I + (\lambda - z)(\widetilde{A}_{\mathcal{T}} - \lambda)^{-1})h.$$

We determine $((A_0 - z)^{-1} - \gamma_z(\mathcal{Q}_0(z) + \mathcal{T}(\lambda))^{-1}\gamma_{z^*}^*)k$. Using Krein's resolvent formula for $\widetilde{P}_{\mathfrak{H}}(\widetilde{A}_{\mathcal{T}} - \lambda)^{-1}h$, we find

$$\begin{aligned}
&\big((A_0 - z)^{-1} - \gamma_z(\mathcal{Q}_0(z) + \mathcal{T}(\lambda))^{-1}\gamma_{z^*}^*\big)k \\
&\quad = \big((A_0 - z)^{-1} - \gamma_z(\mathcal{Q}_0(z) + \mathcal{T}(\lambda))^{-1}\gamma_{z^*}^*\big)\big(h + (\lambda - z)\widetilde{P}_{\mathfrak{H}}(\widetilde{A}_{\mathcal{T}} - \lambda)^{-1}h\big) \\
&\quad = (A_0 - z)^{-1}h - \gamma_z(\mathcal{Q}_0(z) + \mathcal{T}(\lambda))^{-1}\gamma_{z^*}^* h \\
&\qquad + (\lambda - z)\big((A_0 - z)^{-1} - \gamma_z(\mathcal{Q}_0(z) + \mathcal{T}(\lambda))^{-1}\gamma_{z^*}^*\big)\times \\
&\qquad\qquad \times \big((A_0 - \lambda)^{-1} - \gamma_\lambda(\mathcal{Q}_0(\lambda) + \mathcal{T}(\lambda))^{-1}\gamma_{\lambda^*}^*\big)h \\
&\quad = (A_0 - z)^{-1}h - \gamma_z(\mathcal{Q}_0(z) + \mathcal{T}(\lambda))^{-1}\gamma_{z^*}^* h + \mathrm{I} - \mathrm{II} - \mathrm{III} + \mathrm{IV}, \qquad (4.7)
\end{aligned}$$

where

I $:= (\lambda - z)(A_0 - z)^{-1}(A_0 - \lambda)^{-1}h = (A_0 - \lambda)^{-1}h - (A_0 - z)^{-1}h$ by the resolvent formula,

II $:= (\lambda - z)(A_0 - z)^{-1}\gamma_\lambda(\mathcal{Q}_0(\lambda) + \mathcal{T}(\lambda))^{-1}\gamma_{\lambda^*}^* h = (\gamma_\lambda - \gamma_z)(\mathcal{Q}_0(\lambda) + \mathcal{T}(\lambda))^{-1}\gamma_{\lambda^*}^* h$ by the relation $(z - \lambda)(A_0 - z)^{-1}\gamma_\lambda = \gamma_z - \gamma_\lambda$,

III $:= (\lambda - z)\gamma_z(\mathcal{Q}_0(z) + \mathcal{T}(\lambda))^{-1}\gamma_{z^*}^*(A_0 - \lambda)^{-1}h = \gamma_z(\mathcal{Q}_0(z) + \mathcal{T}(\lambda))^{-1}(\gamma_{\lambda^*}^* - \gamma_{z^*}^*)h$ by taking adjoints of both sides of the equality $(\lambda - z)(A_0 - \lambda)^{-1}\gamma_z = \gamma_\lambda - \gamma_z$,

IV $:= (\lambda - z)\gamma_z(\mathcal{Q}_0(z) + \mathcal{T}(\lambda))^{-1}\gamma_{z^*}^*\gamma_\lambda(\mathcal{Q}_0(\lambda) + \mathcal{T}(\lambda))^{-1}\gamma_{\lambda^*}^* h$
$\qquad = \gamma_z(\mathcal{Q}_0(z) + \mathcal{T}(\lambda))^{-1}\gamma_{\lambda^*}^* h - \gamma_z(\mathcal{Q}_0(\lambda) + \mathcal{T}(\lambda))^{-1}\gamma_{\lambda^*}^* h$
by the relation $(\lambda - z)\gamma_{z^*}^*\gamma_\lambda = \mathcal{Q}_0(\lambda) - \mathcal{Q}_0(z) = (\mathcal{Q}_0(\lambda) + \mathcal{T}(\lambda)) - (\mathcal{Q}_0(z) + \mathcal{T}(\lambda))$.

Inserting these expressions into (4.7) and using Krein's resolvent formula we obtain

$$\begin{aligned}
&\big((A_0 - z)^{-1} - \gamma_z(\mathcal{Q}_0(z) + \mathcal{T}(\lambda))^{-1}\gamma_{z^*}^*\big)k \\
&\qquad = \big((A_0 - \lambda)^{-1} - \gamma_\lambda(\mathcal{Q}_0(\lambda) + \mathcal{T}(\lambda))^{-1}\gamma_{\lambda^*}^*\big)h \\
&\qquad = \widetilde{P}_{\mathfrak{H}}(\widetilde{A}_{\mathcal{T}} - \lambda)^{-1}h.
\end{aligned}$$

Hence, by (4.6),

$$(S_{\mathcal{T}}(\lambda) - z)^{-1} = \big\{\{k, ((A_0 - z)^{-1} - \gamma_z(\mathcal{Q}_0(z) + \mathcal{T}(\lambda))^{-1}\gamma_{z^*}^*)k\} : k \in \mathfrak{H}\big\}. \qquad \square$$

The following corollary of Proposition 4.3 can be proved as [5, Proposition 3.4 (ii)]. For the definition of $S_{\mathbb{L}}$ see (2.11).

Corollary 4.4 *If $\mathcal{T}$ in Proposition 4.3 is given by (2.9), then $S_{\mathcal{T}}(\lambda) \cap A_0 = S_{\ker P_m}$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$.*

4.3 Assumption: $\mathcal{T}(\infty)$ Exists

In this subsection we consider the compression to $\mathfrak{H}$ of a self-adjoint extension $\widetilde{A}_{\mathcal{T}}$ under the assumption that for the Nevanlinna parameter $\mathcal{T}$ the following limits exist in resolvent sense and are equal:

$$\lim_{y\to+\infty} \mathcal{T}(\mathrm{i}y) = \lim_{y\to-\infty} \mathcal{T}(\mathrm{i}y) =: \mathcal{T}(\infty). \tag{4.8}$$

By this we mean that the limit of the $d \times d$ matrix $(\mathcal{T}(\mathrm{i}y) + \mathrm{i})^{-1}$ for $y \to +\infty$ and the limit of the $d \times d$ matrix $(\mathcal{T}(\mathrm{i}y) - \mathrm{i})^{-1}$ as $y \to -\infty$ exist and are equal to

$$\lim_{y\to+\infty} (\mathcal{T}(\mathrm{i}y)+\mathrm{i})^{-1} = (\mathcal{T}(\infty)+\mathrm{i})^{-1} \quad \text{and} \quad \lim_{y\to-\infty} (\mathcal{T}(\mathrm{i}y)-\mathrm{i})^{-1} = (\mathcal{T}(\infty)-\mathrm{i})^{-1}.$$

Since $\mathcal{T}(\mathrm{i}y)^* = \mathcal{T}(-\mathrm{i}y)$, $\mathcal{T}(\infty)$ in (4.8) is a self-adjoint relation in $\mathbb{C}^d$.

If $\mathcal{T}$ is of the form (2.9) then relative to the decomposition $\mathbb{C}^d = \operatorname{ran} P_m \oplus \ker P_m$, identified with $\mathbb{C}^d = \mathbb{C}^{d_m} \oplus \mathbb{C}^{d_\infty}$ as in Remark 3.15,

$$(\mathcal{T}(z) \pm \mathrm{i})^{-1} = \begin{bmatrix} (\mathcal{T}_m(z) \pm \mathrm{i})^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathbb{C}^{d_m} \\ \mathbb{C}^{d_\infty} \end{bmatrix} \to \begin{bmatrix} \mathbb{C}^{d_m} \\ \mathbb{C}^{d_\infty} \end{bmatrix}, \quad z \in C^{\pm}, \tag{4.9}$$

and hence (4.8) holds if and only if it holds with $\mathcal{T}$ replaced by $\mathcal{T}_m$, and then (4.9) holds with z replaced by ∞.

A sufficient condition which implies (4.8) is given in the following proposition. In view of the preceding observation we can suppose that $\mathcal{T}$ is a matrix function. In the following P_0 is the orthogonal projection in $\mathbb{C}^d$ onto $\ker \mathcal{B}$.

Proposition 4.5 *Let $\mathcal{T}$ be a Nevanlinna $d \times d$ matrix function with integral representation* (2.1)*. If*

$$\lim_{y\to+\infty} y \operatorname{Im}\langle \mathcal{T}(\mathrm{i}y)\mathbf{x}, \mathbf{x}\rangle < \infty \quad \text{for all } \mathbf{x} \in \ker \mathcal{B},$$

then (4.8) *holds where $\mathcal{T}(\infty)$ is the self-adjoint relation:*

$$\mathcal{T}(\infty) = \left\{\{P_0\mathbf{x}, \mathcal{T}_0 P_0\mathbf{x} + (I - P_0)\mathbf{x} \;:\; \mathbf{x} \in \mathbb{C}^d\right\} \tag{4.10}$$

in which $\mathcal{T}_0$ is the symmetric $(d - d_r) \times (d - d_r)$ matrix

$$\mathcal{T}_0 = P_0 \left(-\int_{\mathbb{R}} \frac{t}{t^2+1} \mathrm{d}\Sigma(t) + \mathcal{A} \right) P_0.$$

Proof Note that by (2.5)

$$\int_{\mathbb{R}} \mathrm{d}\langle \Sigma(t)\mathbf{x}, \mathbf{x}\rangle < \infty \quad \text{for all } \mathbf{x} \in \ker \mathcal{B},$$

hence

$$\lim_{y\to\pm\infty} \mathcal{T}(\mathrm{i}y)\mathbf{x} = \mathcal{M}\mathbf{x}, \quad \mathbf{x} \in \ker \mathcal{B}; \quad \mathcal{M} := -\int_{\mathbb{R}} \frac{t}{t^2+1} \mathrm{d}\Sigma(t) + \mathcal{A}$$

and $\mathcal{T}_0$ is well defined. Decompose $\mathcal{T}(z)$ relative to the decomposition $\mathbb{C}^d = \ker \mathcal{B} \oplus \operatorname{ran} \mathcal{B}$:

$$\mathcal{T}(z) = \begin{bmatrix} \mathcal{T}_{11}(z) & \mathcal{T}_{12}(z) \\ \mathcal{T}_{12}(z^*)^* & \mathcal{T}_{22}(z) \end{bmatrix} : \begin{bmatrix} \ker \mathcal{B} \\ \operatorname{ran} \mathcal{B} \end{bmatrix} \to \begin{bmatrix} \ker \mathcal{B} \\ \operatorname{ran} \mathcal{B} \end{bmatrix},$$

where $\mathcal{T}_{11}$ and $\mathcal{T}_{22}$ are Nevanlinna matrix functions of size $(d-d_r) \times (d-d_r)$ and $d_r \times d_r$ and $\mathcal{T}_{12}$ is a $(d-d_r) \times d_r$ matrix function. Then

$$\mathcal{T}_{12}(-\mathrm{i}y)^* = (I-P_0)\mathcal{T}(\mathrm{i}y)P_0 \to (I-P_0)\mathcal{M}P_0 \quad \text{as} \quad y \to \pm\infty,$$

and hence $\mathcal{T}_{12}(\mathrm{i}y)$ and $\mathcal{T}_{12}(-\mathrm{i}y)^*$ are bounded in norm for large values of $|y|$. From

$$\frac{1}{\mathrm{i}y}\mathcal{T}_{22}(\mathrm{i}y) = \frac{1}{\mathrm{i}y}(I-P_0)\mathcal{T}(\mathrm{i}y)(I-P_0) \to \mathcal{B}(I-P_0) \quad \text{as} \quad y \to \pm\infty,$$

we obtain that

$$\lim_{y\to\pm\infty} (\mathcal{T}_{22}(\mathrm{i}y) \pm \mathrm{i})^{-1} = 0.$$

Since for $z \in \mathbb{C}^{\pm}$ the square matrices $(\mathcal{T}_{11}(z) \pm \mathrm{i})$ and $(\mathcal{T}_{22}(z) \pm \mathrm{i})$ are invertible, the Frobenius-Schur factorization (see for example [17, Proposition 1.6.2]) implies that

$$\begin{aligned}&(\mathcal{T}(z) \pm \mathrm{i})^{-1} \\ &= \begin{bmatrix} I & 0 \\ -(\mathcal{T}_{22} \pm \mathrm{i})^{-1}\mathcal{T}_{12}(z^*)^* & I \end{bmatrix} \begin{bmatrix} \mathcal{R}(z)^{-1} & 0 \\ 0 & (\mathcal{T}_{22}(z) \pm \mathrm{i})^{-1} \end{bmatrix} \begin{bmatrix} I & -\mathcal{T}_{12}(z)(\mathcal{T}_{22} \pm \mathrm{i})^{-1} \\ 0 & I \end{bmatrix},\end{aligned}$$

where

$$\mathcal{R}(z) := (\mathcal{T}_{11}(z) \pm \mathrm{i}) - \mathcal{T}_{12}(z)(\mathcal{T}_{22}(z) \pm \mathrm{i})^{-1}\mathcal{T}_{12}(z^*)^*$$

is an invertible $(d-d_r) \times (d-d_r)$ matrix function and

$$\lim_{y\to\pm\infty} \mathcal{R}(\mathrm{i}y) = \lim_{y\to\pm\infty} (P_0\mathcal{T}(\mathrm{i}y)P_0 \pm \mathrm{i}) = \mathcal{T}_0 \pm \mathrm{i}.$$

Now if we set $z = \mathrm{i}y$ and let $y \to \pm\infty$, then the right-hand side of the factorization formula converges to

$$\begin{bmatrix} (\mathcal{T}_0 \pm \mathrm{i})^{-1} & 0 \\ 0 & 0 \end{bmatrix} = (\mathcal{T}(\infty) \pm \mathrm{i})^{-1},$$

which implies (4.10). □

Theorem 4.6 *If $\mathcal{T}$ is a Nevanlinna $d \times d$ relation function satisfying (4.8), then the canonical self-adjoint extension $A_{\mathcal{T}(\infty)}$ of S is a self-adjoint extension of the compression $C_{\mathfrak{H}}(\widetilde{A}_{\mathcal{T}})$:*

$$C_{\mathfrak{H}}(\widetilde{A}_{\mathcal{T}}) \subset A_{\mathcal{T}(\infty)}.$$

Proof Choose $z \in \mathbb{C}^+$ and a sequence of numbers $\lambda_n > 0$ converging to ∞ as $n \to \infty$. Set $\mathcal{U}(\mathrm{i}\lambda_n) := I - 2\mathrm{i}(\mathcal{T}(\mathrm{i}\lambda_n) + \mathrm{i})^{-1}$. Then $\mathcal{U}(\mathrm{i}\lambda_n)$ is a contractive $d \times d$ matrix which converges to the unitary matrix $\mathcal{U}(\infty) := I - 2\mathrm{i}(\mathcal{T}(\infty) + \mathrm{i})^{-1}$. With $\mathcal{Q}_0(z)$ as in Krein's formula and using that for square matrices $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ of the same size, which need not be invertible, we have the linear relation equality

$$\big(\mathcal{X} + \mathcal{Z}\mathcal{Y}^{-1}\big)^{-1} = \mathcal{Y}(\mathcal{X}\mathcal{Y} + \mathcal{Z})^{-1}$$

we obtain that

$$\begin{aligned}
(\mathcal{Q}_0(z)+\mathcal{T}(\mathrm{i}\lambda_n))^{-1} &= \big(\mathcal{Q}_0(z) - \mathrm{i} + 2\mathrm{i}(I - \mathcal{U}(\mathrm{i}\lambda_n))^{-1}\big)^{-1} \\
&= (I - \mathcal{U}(\mathrm{i}\lambda_n))\big((\mathcal{Q}_0(z) - \mathrm{i})(I - \mathcal{U}(\mathrm{i}\lambda_n)) + 2\mathrm{i}\big)^{-1} \\
&= (I - \mathcal{U}(\mathrm{i}\lambda_n))\big((\mathcal{Q}_0(z) + \mathrm{i}) - (\mathcal{Q}_0(z) - \mathrm{i})\mathcal{U}(\mathrm{i}\lambda_n)\big)^{-1} \\
&= (I - \mathcal{U}(\mathrm{i}\lambda_n))\big(I - (\mathcal{Q}_0(z) + \mathrm{i})^{-1}(\mathcal{Q}_0(z) - \mathrm{i})\mathcal{U}(\mathrm{i}\lambda_n)\big)^{-1}(\mathcal{Q}_0(z) + \mathrm{i})^{-1}.
\end{aligned}$$

Note that the inverses on the left-hand side and the right-hand side of this chain of equalities are all matrices. By (2.14) the matrix $(\mathcal{Q}_0(z) + \mathrm{i})^{-1}(\mathcal{Q}_0(z) - \mathrm{i})$ is a strict contraction and hence as $n \to \infty$

$$\begin{aligned}
&(\mathcal{Q}_0(z)+\mathcal{T}(\mathrm{i}\lambda_n))^{-1} \\
&\quad\longrightarrow (I - \mathcal{U}(\infty)\big(I - (\mathcal{Q}_0(z) + \mathrm{i})^{-1}(\mathcal{Q}_0(z) - \mathrm{i})\mathcal{U}(\infty)\big)^{-1}(\mathcal{Q}_0(z) + \mathrm{i})^{-1} \\
&\qquad = (\mathcal{Q}_0(z) + \mathcal{T}(\infty))^{-1}.
\end{aligned}$$

By (4.5) and Krein's formula for $A_{\mathcal{T}(\infty)}$, the resolvent operators $(S_{\mathcal{T}}(\mathrm{i}\lambda_n) - z)^{-1}$ converge in norm to the resolvent operator $(A_{\mathcal{T}(\infty)} - z)^{-1}$. It remains to apply Proposition 4.2. □

Corollary 4.7 *In Corollary 3.8 if $\mathcal{T}$ is rational and in Corollary 3.12 and Corollary 3.13 we have $C_{\mathfrak{H}}(\widetilde{A}_{\mathcal{T}}) = A_{\mathcal{T}(\infty)}$.*

By Remark 3.15 this corollary remains true if $\mathcal{T}$ is multi-valued.

Example 4.8 Let S be a symmetry with defect $d = 1$. An example of a parameter $\mathcal{T}$ such that the compression $C_{\mathfrak{H}}(\widetilde{A}_{\mathcal{T}})$ has defect one (and hence coincides with S) is given by the function

$$\mathcal{T}(z) = \int_{\mathbb{R}} \frac{1}{t-z} \frac{|t|}{1+t^2} \mathrm{d}t, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

Then $\mathcal{T}(\mathrm{i}y) = \mathrm{i}\dfrac{2y \ln |y|}{y^2 - 1}$, $y \neq 0, \pm 1$, hence $\mathcal{T}(\infty) = 0$, and $\displaystyle\int_{\mathbb{R}} \frac{|t|}{1+t^2} \mathrm{d}t = \infty$, which yields $\mathbb{L}^{\infty} = \mathbb{C}$.

5 Straus Extensions at a Real Point

In this section we consider a closed symmetric relation S with finite and equal defect numbers $d > 0$ in a Hilbert space $\mathfrak{H}$, a self-adjoint extension $\widetilde{A}_{\mathcal{T}}$ of S in some Hilbert space $\widetilde{\mathfrak{H}}$, where $\mathcal{T}$ is again the parameter in Krein's formula (2.15) for $\widetilde{A}_{\mathcal{T}}$ based on the canonical self-adjoint extension A_0 of S. We are interested in the Straus subspace $S_{\mathcal{T}}(\lambda_0)$ for a real point λ_0:

$$S_{\mathcal{T}}(\lambda_0) = \left\{ \{\widetilde{P}_{\mathfrak{H}} \widetilde{f}, \widetilde{P}_{\mathfrak{H}} \widetilde{g}\} : \{\widetilde{f}, \widetilde{g}\} \in \widetilde{A}_{\mathcal{T}},\ \widetilde{g} - \lambda_0 \widetilde{f} \in \mathfrak{H} \right\}.$$

This Straus subspace is a symmetric or self-adjoint extension of S in $\mathfrak{H}$. At least formally, for $\lambda_0 = \infty$ the Straus extension becomes the compression:

$$S_{\mathcal{T}}(\infty) = \left\{ \{\widetilde{P}_{\mathfrak{H}} \widetilde{f}, \widetilde{P}_{\mathfrak{H}} \widetilde{g}\} : \{\widetilde{f}, \widetilde{g}\} \in \widetilde{A}_{\mathcal{T}},\ \widetilde{f} \in \mathfrak{H} \right\} = C_{\mathfrak{H}}(\widetilde{A}_{\mathcal{T}}).$$

We prove for $S_{\mathcal{T}}(\lambda_0)$ analogous results as for compressions in Sects. 3 and 4 by means of a fractional linear transformation.

5.1 A Fractional Linear Transformation of Linear Relations

For fixed $\lambda_0 \in \mathbb{R}$ consider the following transformation φ of the complex plane:

$$\widehat{z} := \varphi(z) := \frac{1}{\lambda_0 - z}, \quad \text{or, equivalently, } z = \lambda_0 - \frac{1}{\widehat{z}}. \tag{5.1}$$

Evidently, $\widehat{\lambda}_0 = \infty$. For a linear relation G in $\mathfrak{H}^2$ we define

$$\widehat{G} := (G)\widehat{\ } := (\lambda_0 - G)^{-1}.$$

In particular, the corresponding transformations of S, A_0 and $\widetilde{A}_{\mathcal{T}}$ are:

$$\widehat{S} = (\lambda_0 - S)^{-1}, \quad \widehat{A_0} = (\lambda_0 - A_0)^{-1}, \quad (\widetilde{A}_{\mathcal{T}})\widehat{\ } = (\lambda_0 - \widetilde{A}_{\mathcal{T}})^{-1}.$$

Then with S also $\widehat{S}$ is a closed relation in $\mathfrak{H}$ which is symmetric, because

$$\widehat{S} = (\lambda_0 - S)^{-1} \subset (\lambda_0 - S^*)^{-1} = (\lambda_0 - S)^{-*} = \widehat{S}^*.$$

and $\widehat{S}(0) = \ker(S - \lambda_0)$. The operator or relation $\widehat{A_0}$ is a canonical self-adjoint extension of $\widehat{S}$, and $(\widetilde{A}_{\mathcal{T}})\widehat{\ }$ is a self-adjoint extension of $\widehat{S}$ in the Hilbert space $\widetilde{\mathfrak{H}}$. Then in Krein's formula for $\widehat{S}$, based on $\widehat{A_0}$, to the extension $(\widetilde{A}_{\mathcal{T}})\widehat{\ }$ there corresponds a parameter, which we denote by $\breve{\mathcal{T}}$: $(\widetilde{A}_{\mathcal{T}})\widehat{\ } = (\widetilde{A}_{\mathcal{T}})\widehat{\ }_{\breve{\mathcal{T}}}$; for simplicity we write $(\widetilde{A}_{\mathcal{T}})\widehat{\ }_{\breve{\mathcal{T}}} =: \breve{A}_{\breve{\mathcal{T}}}$ and, now without reference to the parameters, $(\widetilde{A}\,)\widehat{\ } = \breve{A}$.

Proposition 5.1 *$\widehat{S}$ is a closed symmetric relation with equal defect numbers d. If in Krein's resolvent formulas for S and $\widehat{S}$, based on A_0 and $\widehat{A}_0$, related γ-fields and corresponding Q-functions are denoted by γ_z and $\breve{\gamma}_{\widehat{z}}$ and by Q_0 and $\breve{Q}_0$, respectively, they can be chosen such that*

$$\breve{\gamma}_{\widehat{z}} := \frac{1}{\widehat{z}}\gamma_z : \mathbb{C}^d \to \ker(S^* - z) = \ker(\widehat{S}^* - \widehat{z}), \quad \breve{Q}_0(\widehat{z}) = Q_0(z), \tag{5.2}$$

and then the parameters $\breve{\mathcal{T}}$ and $\mathcal{T}$ are connected by the formula

$$\breve{\mathcal{T}}(\widehat{z}) = \mathcal{T}(z), \quad z \in \mathbb{C} \setminus \mathbb{R}. \tag{5.3}$$

For the proof of this proposition, in the following lemma we collect some simple formulas.

Lemma 5.2 *For $z, w \in \mathbb{C} \setminus \mathbb{R}$ we have the following equalities*:

(i) $\operatorname{ran}(\widehat{S} - \widehat{z}) = \operatorname{ran}(S - z)$,
(ii) $\ker(\widehat{S}^* - \widehat{z}) = \ker(S^* - z)$,
(iii) $(\widehat{A_0} - \widehat{z})^{-1} = \frac{1}{\widehat{z}^2}(A_0 - z)^{-1} - \frac{1}{\widehat{z}}$,
(iv) $I + (\widehat{z} - \widehat{w})(\widehat{A_0} - \widehat{z})^{-1} = \frac{\widehat{w}}{\widehat{z}}\big(I + (z - w)(A_0 - z)^{-1}\big)$,
(v) $\widetilde{P}_{\mathfrak{H}}(\breve{A}_{\breve{\mathcal{T}}} - \widehat{z})^{-1}|_{\mathfrak{H}} = \frac{1}{\widehat{z}^2}\widetilde{P}_{\mathfrak{H}}(\widetilde{A}_{\mathcal{T}} - z)^{-1}|_{\mathfrak{H}} - \frac{1}{\widehat{z}}$.

Proof To prove these items we observe that for an arbitrary linear relation G in a Hilbert space $\mathfrak{H}$ the resolvents of G and $\widehat{G}$ are related by the formula

$$(\widehat{G} - \widehat{z})^{-1} = \frac{1}{\widehat{z}^2}(G - z)^{-1} - \frac{1}{\widehat{z}}. \tag{5.4}$$

This readily follows from the equivalences

$$\begin{aligned}\{f,g\} \in (\widehat{G}-\widehat{z})^{-1} &\Longleftrightarrow \{g,f\} \in \widehat{G}-\widehat{z} \\ &\Longleftrightarrow \{f+\widehat{z}g, \frac{1}{\widehat{z}}f\} \in G-z \Longleftrightarrow \{f,g\} \in \frac{1}{\widehat{z}^2}(G-z)^{-1}-\frac{1}{\widehat{z}}.\end{aligned}$$

Now the equality in (i) follows from the second equivalence applied to $G=S$. The equality in (ii) follows from (i) by taking orthogonal complements. Items (iii)-(v) follow directly from (5.4). □

Proof of Proposition 5.1 By Lemma 5.2 (ii), $\widehat{S}$ has equal defect numbers d. With a γ-field γ_z and a corresponding Q-function $\mathcal{Q}_0$ associated with S and A_0 we define a γ-field $\check{\gamma}_{\widehat{z}}$ and a corresponding Q-function $\check{\mathcal{Q}}_0$ associated with $\widehat{S}$ and $\widehat{A_0}$. To this end we fix $z_1 \in \mathbb{C}\setminus\mathbb{R}$ and set

$$\check{\gamma} := \frac{1}{\widehat{z}_1}\gamma_{z_1} : \mathbb{C}^d \to \ker(S^*-z_1) = \ker(\widehat{S}^*-\widehat{z}_1).$$

Then (observe Lemma 5.2 (iv))

$$\check{\gamma}_{\widehat{z}} := (I+(\widehat{z}-\widehat{z}_1)(\widehat{A_0}-\widehat{z})^{-1})\check{\gamma} = \frac{\widehat{z}_1}{\widehat{z}}(I+(z-z_1)(A_0-z)^{-1})\frac{1}{\widehat{z}_1}\gamma_{z_1} = \frac{1}{\widehat{z}}\gamma_z$$

is a γ-field associated with $\widehat{S}$ and $\widehat{A_0}$. The corresponding Q-function $\check{\mathcal{Q}}_0$ will be defined such that for $\widehat{z}, \widehat{w} \in \mathbb{C}\setminus\mathbb{R}$

$$\frac{\check{\mathcal{Q}}_0(\widehat{z})-\check{\mathcal{Q}}_0(\widehat{w})^*}{\widehat{z}-\widehat{w}^*} = \check{\gamma}_{\widehat{w}}^*\check{\gamma}_{\widehat{z}} = \frac{1}{\widehat{z}\widehat{w}^*}\gamma_w^*\gamma_z = \frac{1}{\widehat{z}\widehat{w}^*}\frac{\mathcal{Q}_0(z)-\mathcal{Q}_0(w)^*}{z-w^*} = \frac{\mathcal{Q}_0(z)-\mathcal{Q}_0(w)^*}{\widehat{z}-\widehat{w}^*}.$$

We choose $\check{\mathcal{Q}}_0$ such that $\check{\mathcal{Q}}_0(\widehat{z}_1)=\mathcal{Q}_0(z_1)$ and thus $\check{\mathcal{Q}}_0(\widehat{z})=\mathcal{Q}_0(z)$ as in (5.2).

Recall that $\check{A}_{\check{\mathcal{T}}}$ in Krein's formula based on $\widehat{S}$ and $\widehat{A_0}$ corresponds to the parameter $\check{\mathcal{T}}$:

$$\widetilde{P}_{\mathfrak{H}}(\check{A}_{\check{\mathcal{T}}}-\widehat{z})^{-1}\big|_{\mathfrak{H}} = (\widehat{A_0}-\widehat{z})^{-1} - \check{\gamma}_{\widehat{z}}\left(\check{\mathcal{Q}}_0(\widehat{z})+\check{\mathcal{T}}(\widehat{z})\right)^{-1}\check{\gamma}_{\widehat{z}^*}^*.$$

Then, by Lemma 5.2 (iii) and (v),

$$\widetilde{P}_{\mathfrak{H}}\left(\tfrac{1}{\widehat{z}^2}(\widetilde{A}_{\mathcal{T}}-z)^{-1}-\tfrac{1}{\widehat{z}}\right)\big|_{\mathfrak{H}} = \tfrac{1}{\widehat{z}^2}(A_0-z)^{-1}-\tfrac{1}{\widehat{z}}-\tfrac{1}{\widehat{z}}\gamma_z\left(\mathcal{Q}_0(z)+\check{\mathcal{T}}(\widehat{z})\right)^{-1}\tfrac{1}{\widehat{z}}\gamma_{z^*}^*$$

and this simplifies to

$$\widetilde{P}_{\mathfrak{H}}(\widetilde{A}_{\mathcal{T}}-z)^{-1}\big|_{\mathfrak{H}} = (A_0-z)^{-1}-\gamma_z\left(\mathcal{Q}_0(z)+\check{\mathcal{T}}(\widehat{z})\right)^{-1}\gamma_{z^*}^*.$$

Comparing this formula with (2.15), we obtain (5.3).

We denote the Straus extension of $\widehat{S}$ related to $\breve{A}_{\breve{\mathcal{T}}}$ by $\breve{S}_{\breve{\mathcal{T}}}(\lambda)$:

$$\breve{S}_{\breve{\mathcal{T}}}(\lambda) = \left\{ \{\widetilde{P}_{\mathfrak{H}}\widetilde{f}, \widetilde{P}_{\mathfrak{H}}\widetilde{g}\} : \{\widetilde{f}, \widetilde{g}\} \in \breve{A}_{\breve{\mathcal{T}}},\ \widetilde{g} - \lambda\widetilde{f} \in \mathfrak{H} \right\}, \quad \lambda \in \mathbb{C},$$

and then, formally, $\breve{S}_{\breve{\mathcal{T}}}(\infty) = C_{\mathfrak{H}}(\breve{A}_{\mathcal{T}})$.

Proposition 5.3 *If* $\lambda_0 \in \mathbb{R}$, $\breve{A}_{\breve{\mathcal{T}}} = (\lambda_0 - \widetilde{A}_{\mathcal{T}})^{-1}$ *and* λ *and* $\widehat{\lambda}$ *are related as in* (5.1)*, then*

(i) $\breve{S}_{\breve{\mathcal{T}}}(\widehat{\lambda}) = (S_{\mathcal{T}}(\lambda))\widehat{\ } \quad \big(= (\lambda_0 - S_{\mathcal{T}}(\lambda))^{-1}\big), \quad \lambda \neq \lambda_0,$

(ii) $C_{\mathfrak{H}}(\breve{A}_{\breve{\mathcal{T}}}) = (S_{\mathcal{T}}(\lambda_0))\widehat{\ } \quad \big(= (\lambda_0 - S_{\mathcal{T}}(\lambda_0))^{-1}\big).$

Proof To prove (i) we observe that for $\widetilde{f}, \widetilde{g}, \widetilde{h} \in \widetilde{\mathfrak{H}}$ and with $\lambda, \widehat{\lambda}$ related as in (5.1) we have

$$\widetilde{h} = \lambda_0\widetilde{f} - \widetilde{g},\ \lambda \neq \lambda_0 \Longrightarrow \widetilde{f} - \widehat{\lambda}\widetilde{h} = \widehat{\lambda}(\widetilde{g} - \lambda\widetilde{f}).$$

It follows that for $\lambda \in \mathbb{C} \setminus \{\lambda_0\}$

$$\begin{aligned}(\lambda_0 - S_{\mathcal{T}}(\lambda))^{-1} &= \left\{ \{\widetilde{P}_{\mathfrak{H}}\widetilde{f}, \lambda_0\widetilde{P}_{\mathfrak{H}}\widetilde{f} - \widetilde{P}_{\mathfrak{H}}\widetilde{g}\} : \{\widetilde{f}, \widetilde{g}\} \in \widetilde{A}_{\mathcal{T}},\ \widetilde{g} - \lambda\widetilde{f} \in \mathfrak{H} \right\}^{-1} \\ &= \left\{ \{\widetilde{P}_{\mathfrak{H}}(\lambda_0\widetilde{f} - \widetilde{g}), \widetilde{P}_{\mathfrak{H}}\widetilde{f}\} : \{\widetilde{f}, \lambda_0\widetilde{f} - \widetilde{g}\} \in \lambda_0 - \widetilde{A}_{\mathcal{T}},\ \widetilde{g} - \lambda\widetilde{f} \in \mathfrak{H} \right\} \\ &= \left\{ \{\widetilde{P}_{\mathfrak{H}}\widetilde{h}, \widetilde{P}_{\mathfrak{H}}\widetilde{f}\} : \{\widetilde{h}, \widetilde{f}\} \in \breve{A}_{\breve{\mathcal{T}}},\ \widetilde{f} - \widehat{\lambda}\widetilde{h} \in \mathfrak{H} \right\} \\ &= \breve{S}_{\breve{\mathcal{T}}}(\widehat{\lambda}).\end{aligned}$$

Item (ii) can be proved similarly. □

5.2 A Transformation of the Parameter

In this subsection we fix $\lambda_0 \in \mathbb{R}$, and assume that $\mathcal{T}$ and $\breve{\mathcal{T}}$ are Nevanlinna $d \times d$ matrix functions with integral representations (2.1):

$$\begin{aligned}\mathcal{T}(z) &= \int_{\mathbb{R}} \left(\frac{1}{t - z} - \frac{t}{1 + t^2} \right) \mathrm{d}\Sigma(t) + \mathcal{A} + z\mathcal{B}, \\ \breve{\mathcal{T}}(\widehat{z}) &= \int_{\mathbb{R}} \left(\frac{1}{\widehat{t} - \widehat{z}} - \frac{\widehat{t}}{1 + \widehat{t}^2} \right) \mathrm{d}\breve{\Sigma}(\widehat{t}) + \breve{\mathcal{A}} + \widehat{z}\breve{\mathcal{B}},\end{aligned} \tag{5.5}$$

and such that (5.3) holds: $\breve{\mathcal{T}}(\widehat{z}) = \mathcal{T}(z)$. Recall that φ is the function $\varphi(t) = \dfrac{1}{\lambda_0 - t}$; for a real interval Δ we set $\widehat{\Delta} := \varphi(\Delta)$.

Lemma 5.4 *The following relations hold*:

(i) $\breve{\Sigma}(\widehat{\Delta}) = \int_{\Delta} \frac{\mathrm{d}\Sigma(t)}{(t-\lambda_0)^2}$, Δ *any bounded closed interval such that* $\lambda_0 \notin \Delta$.

(ii) $\breve{\Sigma}(\{0\}) = \mathcal{B}$.

(iii) $\breve{\mathcal{B}} = \Sigma(\{\lambda_0\})$.

Proof The relation $\breve{\mathcal{T}}(\widehat{z}) = \mathcal{T}(\lambda_0 - \frac{1}{\widehat{z}})$ reads as

$$\begin{aligned} &\int_{\mathbb{R}} \left(\frac{1}{\widehat{t} - \widehat{z}} - \frac{\widehat{t}}{1+\widehat{t}^2} \right) \mathrm{d}\breve{\Sigma}(\widehat{t}) + \breve{\mathcal{A}} + \widehat{z}\breve{\mathcal{B}} \\ &\qquad = \int_{\mathbb{R}} \left(\frac{1}{t - \lambda_0 + \frac{1}{\widehat{z}}} - \frac{t}{1+t^2} \right) \mathrm{d}\Sigma(t) + \mathcal{A} + \left(\lambda_0 - \frac{1}{\widehat{z}}\right)\mathcal{B}. \end{aligned} \tag{5.6}$$

Consider a closed interval $\Delta \subset \mathbb{R} \setminus \{\lambda_0\}$. The right-hand side of (5.6) can be written as

$$\int_{\Delta} \left(\frac{1}{t - \lambda_0 + \frac{1}{\widehat{z}}} - \frac{1}{t - \lambda_0} \right) \mathrm{d}\Sigma(t) + \cdots,$$

where $\cdots$ is an expression which is analytic with respect to $\widehat{z}$ in the interior of $\widehat{\Delta}$. Observing that

$$\frac{1}{t - \lambda_0 + \frac{1}{\widehat{z}}} - \frac{1}{t - \lambda_0} = \frac{1}{(t-\lambda_0)^2 (\frac{1}{\lambda_0 - t} - \widehat{z})},$$

(5.6) yields

$$\int_{\mathbb{R}} \left(\frac{1}{\widehat{t} - \widehat{z}} - \frac{\widehat{t}}{1+\widehat{t}^2} \right) \mathrm{d}\breve{\Sigma}(\widehat{t}) + \breve{\mathcal{A}} + \widehat{z}\breve{\mathcal{B}} = \int_{\Delta} \frac{\mathrm{d}\Sigma(t)}{(t-\lambda_0)^2} \frac{1}{\frac{1}{\lambda_0 - t} - \widehat{z}} + \cdots,$$

where again $\cdots$ is an expression analytic in the interior of $\widehat{\Delta}$ with respect to $\widehat{z}$. Under the substitution

$$\mathrm{d}\breve{\Sigma}_1(\widehat{t}) := \frac{\mathrm{d}\Sigma(t)}{(t-\lambda_0)^2}, \quad \widehat{t} = \frac{1}{\lambda_0 - t},$$

this relation becomes

$$\int_{\mathbb{R}} \left(\frac{1}{\widehat{t} - \widehat{z}} - \frac{\widehat{t}}{1+\widehat{t}^2} \right) \mathrm{d}\breve{\Sigma}(\widehat{t}) + \breve{\mathcal{A}} + \widehat{z}\breve{\mathcal{B}} = \int_{\widehat{\Delta}} \frac{\mathrm{d}\breve{\Sigma}_1(\widehat{t})}{\widehat{t} - \widehat{z}} + \cdots.$$

Now the Stieltjes inversion formula implies

$$\mathrm{d}\breve{\Sigma}(\widehat{t}) = \mathrm{d}\breve{\Sigma}_1(\widehat{t}) = \frac{\mathrm{d}\Sigma(t)}{(t-\lambda_0)^2}, \quad \widehat{t} = \frac{1}{\lambda_0 - t} \in \widehat{\Delta},$$

and (i) follows.

To prove (ii), we multiply both sides of $\mathcal{T}(z) = \breve{\mathcal{T}}(\widehat{z})$ by $\frac{1}{z}$, set $z = iy$ and obtain, by for example [9, (5.14) and (5.30)], that

$$\mathcal{B} = \lim_{y\to+\infty} \frac{1}{\mathrm{i}y}\mathcal{T}(\mathrm{i}y) = \lim_{y\to+\infty} -\frac{\mathrm{i}}{y}\breve{\mathcal{T}}\left(\lambda_0 + \frac{\mathrm{i}}{y}\right) = \breve{\Sigma}(\{\lambda_0\}).$$

The equality in (iii) can be proved in the same way. □

Corollary 5.5 *With the notation of Lemma 5.4, for* $\mathbf{x} \in \mathbb{C}^d$, $\mathbf{x} \neq 0$, *we have*

$$\int_{\mathbb{R}} \frac{\mathrm{d}\langle\Sigma(t)\mathbf{x},\mathbf{x}\rangle}{(t-\lambda_0)^2} = \infty,\ \mathbf{x} \perp \operatorname{ran}\Sigma(\{\lambda_0\}) \iff \int_{\mathbb{R}} \mathrm{d}\langle\breve{\Sigma}(s)\mathbf{x},\mathbf{x}\rangle = \infty,\ \mathbf{x} \perp \operatorname{ran}\breve{\mathcal{B}}.$$

5.3 A Dimension Theorem

In this subsection we prove a Dimension theorem for $S_{\mathcal{T}}(\lambda_0)$ using the fractional transformation of the previous subsection and Theorem 3.7 for the compression of $\widetilde{A}_{\mathcal{T}}$. To this end we decompose the space $\mathbb{C}^d$ with respect to $\mathcal{T}$ and λ_0 as follows (comp. (3.10)):

$$\mathbb{C}^d = \mathbb{L}_c \oplus \mathbb{L}_r(\lambda_0) \oplus \mathbb{L}_f(\lambda_0) \oplus \mathbb{L}^\infty(\lambda_0), \tag{5.7}$$

where $\mathbb{L}_c$ is the subset of $\mathbb{C}^d$ on which $\mathcal{T}(z)$ is constant, and

$$\mathbb{L}_r(\lambda_0) \oplus \mathbb{L}_f(\lambda_0) \oplus \mathbb{L}^\infty(\lambda_0) = \mathbb{L}_c^\perp,$$

with

$$\begin{aligned}
\mathbb{L}_r(\lambda_0) &:= \operatorname{ran}\Sigma(\{\lambda_0\}),\\
\mathbb{L}_f(\lambda_0) &:= \left\{\mathbf{x}\in\mathbb{C}^d : \mathbf{x}\in\mathbb{L}_r(\lambda_0)^\perp \cap \mathbb{L}_c^\perp,\ \int_{\mathbb{R}} \frac{\mathrm{d}\langle\Sigma(t)\mathbf{x},\mathbf{x}\rangle}{(t-\lambda_0)^2} < \infty\right\},\\
\mathbb{L}^\infty(\lambda_0) &:= \left\{\mathbf{x}\in\mathbb{C}^d : \mathbf{x}\in\mathbb{L}_c^\perp \cap \mathbb{L}_0(\lambda_0)^\perp \cap \mathbb{L}_f(\lambda_0)^\perp, \int_{\mathbb{R}} \frac{\mathrm{d}\langle\Sigma(t)\mathbf{x},\mathbf{x}\rangle}{(t-\lambda_0)^2} = \infty\right\}.
\end{aligned}$$

That is, $\mathbb{L}^\infty(\lambda_0)$ is the maximal subspace in $(\operatorname{ran}\Sigma(\{\lambda_0\})^\perp$ such that for its nonzero elements $\mathbf{x}$ we have $\int_{\mathbb{R}} \frac{d\langle\Sigma(t)\mathbf{x},\mathbf{x}\rangle}{(t-\lambda_0)^2} = \infty$. We denote the dimensions of the subspaces on the right-hand side of (5.7) by d_c, $d_r(\lambda_0)$, $d_f(\lambda_0)$ and $d^\infty(\lambda_0)$.

Remark 5.6 Define the Nevanlinna $d \times d$ matrix function $\breve{\mathcal{T}}$ by $\breve{\mathcal{T}}(\widehat{z}) = \mathcal{T}(z)$. Then the spaces in the decomposition (5.7) are related to the integral representation of the parameter $\breve{\mathcal{T}}$ in the same way as the spaces in (3.10) are related to the integral representation of $\mathcal{T}$. Indeed, by Lemma 3.3 and Lemma 5.4,

$$\mathbb{L}_c = \ker\mathcal{T}(z) = \ker\breve{\mathcal{T}}(\widehat{z}) = (\ker\breve{\mathcal{B}}) \cap (\ker\breve{\Sigma}),$$

$$\mathbb{L}_c^\perp = \operatorname{span}\left\{\operatorname{ran}\breve{\mathcal{B}}, \quad \operatorname{ran}\breve{\Sigma}\right\},$$

$$\mathbb{L}_r(\lambda_0) = \operatorname{ran}\breve{\mathcal{B}},$$

$$\mathbb{L}_f(\lambda_0) = \left\{\mathbf{x} \in \mathbb{C}^d \ : \ \mathbf{x} \in (\ker\breve{\mathcal{B}}) \cap \mathbb{L}_c^\perp, \quad \int_{\mathbb{R}} d\langle\breve{\Sigma}(t)\mathbf{x},\mathbf{x}\rangle < \infty\right\},$$

$$\mathbb{L}^\infty(\lambda_0) = \left\{\mathbf{x} \in \mathbb{C}^d \ : \ \mathbf{x} \in \mathbb{L}_c^\perp \cap (\ker\breve{\mathcal{B}}) \cap \mathbb{L}_f(\lambda_0)^\perp, \quad \int_{\mathbb{R}} d\langle\breve{\Sigma}(t)\mathbf{x},\mathbf{x}\rangle = \infty\right\}.$$

Theorem 5.7 *If the parameter $\mathcal{T}$ is a $d \times d$ matrix function, then the following relations hold*:

(i) $\dim\big((S_{\mathcal{T}}(\lambda_0) \cap A_0)/S\big) = d_r(\lambda_0)$,
(ii) $\dim\big(S_{\mathcal{T}}(\lambda_0)/(\widetilde{A}_{\mathcal{T}} \cap \mathfrak{H}^2 + S_{\mathcal{T}}(\lambda_0) \cap A_0)\big) = d_f(\lambda_0)$,
(iii) $\dim\big(S_{\mathcal{T}}(\lambda_0)/S\big) = d_c + d_r(\lambda_0) + d_f(\lambda_0)$.

Proof By Remark 5.6, the equalities in the theorem are a direct consequence of Theorem 3.7 and Proposition 5.3 (ii). As an example we prove (ii):

$$\begin{aligned}
&\dim\Big(S_{\mathcal{T}}(\lambda_0)/(\widetilde{A}_{\mathcal{T}} \cap \mathfrak{H}^2 + S_{\mathcal{T}}(\lambda_0) \cap A_0)\Big) \\
&\qquad = \dim\Big((S_{\mathcal{T}}(\lambda_0))\widehat{\ }/\big((\widetilde{A}_{\mathcal{T}})\widehat{\ } \cap \mathfrak{H}^2 + (S_{\mathcal{T}}(\lambda_0))\widehat{\ } \cap \widehat{A_0}\big)\Big) \\
&\qquad = \dim\Big(C_{\mathfrak{H}}(\breve{A}_{\breve{\mathcal{T}}})/(\breve{A}_{\breve{\mathcal{T}}} \cap \mathfrak{H}^2 + C_{\mathfrak{H}}(\breve{A}_{\breve{\mathcal{T}}}) \cap \widehat{A_0})\Big) \\
&\qquad = d_f(\lambda_0),
\end{aligned}$$

where the last equality follows from Theorem 3.7 (iv). □

Now also analogues of Corollaries 3.8–3.13 hold. They follow from Remark 5.6, their corresponding counterparts in Sect. 3.3 and Proposition 5.3 (ii). In the following six corollaries it is assumed that the parameter $\mathcal{T}$ is a Nevanlinna $d \times d$ matrix function with integral representation (2.1) and relation representation (2.2) and that $\lambda_0 \in \mathbb{R}$.

Corollary 5.8 *The Straus extension $S_{\mathcal{T}}(\lambda_0)$ of S has equal defect numbers $d^{\infty}(\lambda_0)$. If $\mathcal{T}$ is rational, then $S_{\mathcal{T}}(\lambda_0)$ is self-adjoint.*

The last statement is a special case of [7, Theorem 3.3].

Corollary 5.9 *The following statements are equivalent*:

(a) $\lim_{s\to 0+} \frac{1}{s}\,\mathrm{Im}\langle \mathcal{T}(\lambda_0+\mathrm{i}s)\mathbf{x},\mathbf{x}\rangle = \infty$ *for all* $\mathbf{x}\in\mathbb{C}^d\setminus\{0\}$.
(b) $S_{\mathcal{T}}(\lambda_0)\subset A_0$.
(c) $d_r(\lambda_0)+d^{\infty}(\lambda_0)=d$.

Corollary 5.10 *The following statements are equivalent*:

(a) $\Sigma(\{\lambda_0\})>0$.
(b) $S_{\mathcal{T}}(\lambda_0)=A_0$.
(c) $d_r(\lambda_0)=d$.

Corollary 5.11 *The following statements are equivalent*:

(a) $\Sigma(\{\lambda_0\})=0$ *and* $\lim_{s\to 0+} \frac{1}{s}\,\mathrm{Im}\langle \mathcal{T}(\lambda_0+\mathrm{i}s)\mathbf{x},\mathbf{x}\rangle = \infty$ *for all* $\mathbf{x}\in\mathbb{C}^d\setminus\{0\}$.
(b) $S_{\mathcal{T}}(\lambda_0)=S$.
(c) $d^{\infty}(\lambda_0)=d$.

Corollary 5.12 *If* $\lim_{s\to 0+} \frac{1}{s}\,\mathrm{Im}\langle \mathcal{T}(\lambda_0+\mathrm{i}s)\mathbf{x},\mathbf{x}\rangle < \infty$ *for all* $\mathbf{x}\in\ker\Sigma(\{\lambda_0\})$, *then* $S_{\mathcal{T}}(\lambda_0)$ *is self-adjoint.*

Corollary 5.13 *The following statements are equivalent*:

(a) $\lim_{s\to 0+} \frac{1}{s}\,\mathrm{Im}\langle \mathcal{T}(\lambda_0+\mathrm{i}s)\mathbf{x},\mathbf{x}\rangle < \infty$ *for all* $\mathbf{x}\in\mathbb{C}^d$.
(b) $S_{\mathcal{T}}(\lambda_0)$ *is self-adjoint and* $S_{\mathcal{T}}(\lambda_0)\cap A_0=S$.
(c) $d_c+d_f(\lambda_0)=d$.

Example 5.14 As examples for the case $d=1$ we mention the scalar Nevanlinna functions $\mathcal{T}_1(z)=\sqrt{z}$, positive on $(0,\infty)$, and $\mathcal{T}_2(z)=\log z$, real on $(0,\infty)$, considered in [8, p. 27]. It follows from the formulas given there that the measures in the integral representations of these functions have no point masses. We find that

$$\lim_{s\to 0+}\frac{1}{s}\,\mathrm{Im}\,\mathcal{T}_1(\lambda_0+\mathrm{i}s)=\begin{cases}\frac{1}{2} & \text{if } \lambda_0>0,\\ \frac{1}{2}\sqrt{2} & \text{if } \lambda_0=0,\\ \infty & \text{if } \lambda_0<0.\end{cases}$$

Hence, by Corollary 5.11, $d^{\infty}(\lambda_0)=1$ for $\lambda_0<0$ and, by Corollary 5.13, $S_{\mathcal{T}_1}(\lambda_0)$ is self-adjoint for $\lambda_0\geq 0$. Similarly, from

$$\lim_{s\to 0+}\frac{1}{s}\,\mathrm{Im}\,\mathcal{T}_2(\lambda_0+\mathrm{i}s)=\begin{cases}\frac{1}{\lambda_0} & \text{if } \lambda_0>0,\\ \infty & \text{if } \lambda_0\leq 0\end{cases}$$

we infer that $d^{\infty}(\lambda_0) = 1$ for $\lambda_0 \le 0$ and $S_{\mathcal{T}_2}(\lambda_0)$ is self-adjoint for $\lambda_0 > 0$.

Remark 5.15 If $\mathcal{T}$ is a Nevanlinna $d \times d$ *relation* function then the corollaries need to be adapted in a similar way as in Remark 3.15.

5.4 Assumption: $\mathcal{T}(\lambda_0)$ Exists

In analogy to Sect. 4.3, we finally consider the Straus extension of $S_{\mathcal{T}}$ at a real point λ_0 determined by a self-adjoint extension $\widetilde{A}_{\mathcal{T}}$ under the assumption that for the parameter $\mathcal{T}$ the following limits exist in resolvent sense and are equal:

$$\lim_{s \to 0+} \mathcal{T}(\lambda_0 + \mathrm{i}s) = \lim_{s \to 0-} \mathcal{T}(\lambda_0 + \mathrm{i}s) =: \mathcal{T}(\lambda_0), \tag{5.8}$$

that is

$$\lim_{s \to 0+} \Big(\mathcal{T}(\lambda_0{+}\mathrm{i}s){+}\mathrm{i}\Big)^{-1} = \Big(\mathcal{T}(\lambda_0){+}\mathrm{i}\Big)^{-1}, \quad \lim_{s \to 0-} \Big(\mathcal{T}(\lambda_0{+}\mathrm{i}s){-}\mathrm{i}\Big)^{-1} = \Big(\mathcal{T}(\lambda_0){-}\mathrm{i}\Big)^{-1}.$$

Then $\mathcal{T}(\lambda_0)$ in (5.8) is a self-adjoint relation in $\mathbb{C}^d$.

A sufficient condition for the limits in (5.8) to exist and to be equal is given by the following analogue of Proposition 4.5. We denote by P_{λ_0} the orthogonal projection in $\mathbb{C}^d$ onto $\ker \Sigma(\{\lambda_0\})$.

Proposition 5.16 *Let $\mathcal{T}$ be a Nevanlinna $d \times d$ matrix function with integral representation* (2.1). *If*

$$\lim_{s \to 0+} \frac{1}{s} \,\mathrm{Im}\big\langle \mathcal{T}(\lambda_0 + \mathrm{i}s)\mathbf{x}, \mathbf{x}\big\rangle < \infty \textit{ for all } \mathbf{x} \in \ker \Sigma(\{\lambda_0\}),$$

then (5.8) *holds where $\mathcal{T}(\lambda_0)$ is the self-adjoint relation :*

$$\mathcal{T}(\lambda_0) = \Big\{ \{P_{\lambda_0}\mathbf{x}, \mathcal{T}_{\lambda_0} P_{\lambda_0}\mathbf{x} + (I - P_{\lambda_0})\mathbf{x}\} \,:\, \mathbf{x} \in \mathbb{C}^d \Big\} \tag{5.9}$$

in which $\mathcal{T}_{\lambda_0}$ is the symmetric $(d - d_r(\lambda_0)) \times (d - d_r(\lambda_0))$ matrix

$$\mathcal{T}_{\lambda_0} = P_{\lambda_0} \left(\int_{\mathbb{R}} \left(\frac{1}{t - \lambda_0} - \frac{t}{t^2 + 1} \right) \mathrm{d}\Sigma(t) + \mathcal{A} + \lambda_0 \mathcal{B} \right) P_{\lambda_0}.$$

Proof Define the Nevanlinna $d \times d$ matrix function $\check{\mathcal{T}}$ by the relation $\check{\mathcal{T}}(\widehat{z}) = \mathcal{T}(z)$. Assume it has the integral representation (5.5). Then $\check{\mathcal{T}}$ satisfies the assumption of Proposition 4.5 and hence (4.8) holds. This implies (5.8) with $\mathcal{T}(\lambda_0) = \check{\mathcal{T}}(\infty)$. The equality (5.9) follows from (4.10), and

$$\mathrm{ran}(I - P_{\lambda_0}) = \mathcal{T}(\lambda_0)(0) = \check{\mathcal{T}}(\infty)(0) = \mathrm{ran}\,\check{\mathcal{B}} = \mathrm{ran}\,\Sigma(\{\lambda_0\}),$$

that is ran $P_{\lambda_0} = \ker \Sigma(\{\lambda_0\})$. The last equality follows from Lemma 5.4 (ii). The assumption in the proposition implies that

$$\int_{\mathbb{R}} \frac{\mathrm{d}\langle \Sigma(t)\mathbf{x}, \mathbf{x}\rangle}{(t-\lambda_0)^2} < \infty \text{ for all } \mathbf{x} \in \ker \Sigma(\{\lambda_0\}),$$

and hence the matrix part of $\mathcal{T}(\lambda_0)$ is given by

$$\begin{aligned}\mathcal{T}_{\lambda_0} &= \lim_{s\to 0+} P_{\lambda_0}\mathcal{T}(\lambda_0 + \mathrm{i}s)P_{\lambda_0} \\ &= P_{\lambda_0}\left(\int_{\mathbb{R}}\left(\frac{1}{t-\lambda_0} - \frac{t}{t^2+1}\right)\mathrm{d}\Sigma(t) + \mathcal{A} + \lambda_0\mathcal{B}\right)P_{\lambda_0}.\end{aligned}$$

□

Theorem 5.17 *If the Nevanlinna $d \times d$ relation function $\mathcal{T}$ satisfies* (5.8)*, then the canonical self-adjoint extension $A_{\mathcal{T}(\lambda_0)}$ of S corresponding to $\mathcal{T}(\lambda_0)$ in Krein's formula* (2.15) *is a self-adjoint extension of the Straus extension $S_{\mathcal{T}}(\lambda_0)$ of S :*

$$S_{\mathcal{T}}(\lambda_0) \subset A_{\mathcal{T}(\lambda_0)}.$$

Proof Choose $z \in \mathbb{C}^+$ and let $\mathcal{Q}_0(z)$ be the Q-function in Krein's formula. By the same reasoning as in the proof of Theorem 4.6

$$\lim_{s\to 0+}\Big(\mathcal{Q}_0(z) + \mathcal{T}(\lambda_0 + \mathrm{i}s)\Big)^{-1} = \Big(\mathcal{Q}_0(z) + \mathcal{T}(\lambda_0)\Big)^{-1}.$$

From (4.5) and Krein's formula for $A_{\mathcal{T}(\lambda_0)}$ we obtain that as $s \to 0+$

$$\Big(S_{\mathcal{T}}(\lambda_0 + \mathrm{i}s) - z\Big)^{-1} \to \Big(A_{\mathcal{T}(\lambda_0)} - z\Big)^{-1},$$

and hence, by Proposition 5.3 (i) and (5.4),

$$\left(\breve{S}_{\mathcal{T}}\left(\frac{\mathrm{i}}{s}\right) - \widehat{z}\right)^{-1} = \Big((S_{\mathcal{T}}(\lambda_0 + \mathrm{i}s))\widehat{\ } - \widehat{z}\Big)^{-1} \to \Big((A_{\mathcal{T}(\lambda_0)})\widehat{\ } - \widehat{z}\Big)^{-1},$$

both limits are in operator norm. By Proposition 5.3 (ii) and Proposition 4.2

$$(S_{\mathcal{T}}(\lambda_0))\widehat{\ } = \breve{S}_{\mathcal{T}}(\infty) = C_{\mathfrak{H}}(\breve{A}_{\mathcal{T}}) \subset (A_{\mathcal{T}(\lambda_0)})\widehat{\ },$$

whence $S_{\mathcal{T}}(\lambda_0) \subset A_{\mathcal{T}(\lambda_0)}$. □

Corollary 5.18 *In Corollary 5.8 if $\mathcal{T}$ is rational and in Corollary 5.12 and Corollary 5.13 we have $S_{\mathcal{T}}(\lambda_0) = A_{\mathcal{T}(\lambda_0)}$.*

By Remark 5.15 this corollary remains true if $\mathcal{T}$ is multi-valued.

References

1. E.A. Coddington, Extension theory of formally normal and symmetric subspaces, Mem. Amer. Math. Soc. **134** (1973)
2. V. Derkach, S. Hassi, M. Malamud, H. de Snoo, Boundary Relations and Generalized Resolvents of Symmetric Operators, Russian Journal of Mathematical Physics, Vol. **16**, 17–60 (2009)
3. A. Dijksma, H.S.V. de Snoo, Symmetric and selfadjoint relations in Krein spaces I, Oper. Theory Adv. Appl. **24**, 145–166 (1987)
4. A. Dijksma, H. Langer, Finite-dimensional self-adjoint extensions of a symmetric operator with finite defect and their compressions, Advances in Complex Analysis and Operator Theory, Festschrift in honor of Daniel Alpay, Birkhäuser, Basel, 135–163 (2017)
5. A. Dijksma, H. Langer, Compressions of self-adjoint extensions of a symmetric operator and M.G. Krein's resolvent formula, Integr. Equ. Oper. Theory 90:41 (2018)
6. A. Dijksma, H. Langer, H.S.V. de Snoo, Selfadjoint Π_κ-extensions of symmetric subspaces: an abstract approach to boundary problems with spectral parameter in the boundary conditions, Integr. Equ. Oper. Theory **7**, 459–515 (1984); addendum in Integr. Equ. Oper. Theory **7**, 905 (1984)
7. A. Dijksma, H. Langer, H.S.V. de Snoo, Unitary colligations in Π_κ-spaces, characteristic functions and Straus extensions, Pacific J. Math. **125**, 347–362 (1986)
8. W.F. Donoghue, Monotone matrix functions and analytic continuation, Springer-Verlag, Berlin-Heidelberg-New York, 1974
9. F. Gesztesy, E.R. Tsekanovskii, On matrix-valued Herglotz functions, Math. Nachr. **218**, 61–138 (2000)
10. S. Hassi, M. Kaltenbaeck, H.S.V. de Snoo, Selfadjoint extensions of the orthogonal sum of symmetric relations. I., in: Operator theory, operator algebras and related topics (Timisoara, 1996), Theta Found., Bucharest, 163–178 (1997)
11. M.G. Krein, G.K. Langer, Defect subspaces and generalized resolvents of an hermitian operator in the space Π_κ, Funkcional. Anal. i Priložen **5**(2), 59–71 (1971); **5**(3), 54–69 (1971) (Russian); English translation: Functional Anal. Appl. **5**(2), 136–146 (1971); **5**(3), 217–228 (1971)
12. M.G. Krein, H. Langer, Über die Q-Funktion eines π-hermiteschen Operators in Raume Π_κ, Acta Sci. Math. (Szeged) **34**, 191–230 (1973)
13. H. Langer, B. Textorius, On generalized resolvents and Q-functions of symmetric linear relations (subspaces) in Hilbert space, Pacific J. Math. **72**, 135–165 (1977)
14. V.I. Mogilevskii, On compressions of self-adjoint extensions of a symmetric linear relation, Integr. Equ. Oper. Theory 91:9 (2019)
15. W. Stenger, On the projection of a selfadjoint operator. Bull. Amer. Math. Soc. **74**, 369–372 (1968)
16. A.V. Straus, Extensions and generalized resolvents of a non-densely defined symmetric operator, Math. USSR-Izv. **4** 179–208 (1970)
17. C. Tretter, Spectral theory of block operator matrices and applications, Imperial College Press, London (2008)

On Conditions of Complete Indeterminacy for the Matricial Hamburger Moment Problem

Yu. M. Dyukarev

Dedicated to my teacher Viktor Emmanuilovich Katsnelson on the occasion of his 75th birthday

Abstract Conditions for the complete indeterminacy of the matricial Hamburger moment problem will be studied. These conditions are formulated in terms of series built on the basis of matrix-valued polynomials of first and second kind.

Keywords Matricial Hamburger moment problem · Matrix-valued polynomials of first and second kind · Completely indeterminate moment problem · Symmetric operators · Deficiency numbers · Deficiency vectors

Mathematics Subject Classification (2000) 44A60, 47A57

1 Introduction

Let integers $m, n \geq 1$ be given. The symbol $\mathbb{C}^{m\times n}$ stands for the set of all complex matrices with m rows and n columns. For each matrix $A \in \mathbb{C}^{m\times n}$, we denote by $A^* \in \mathbb{C}^{n\times m}$ its adjoint matrix. The unit matrix of order m we denote by I_m, whereas the zero matrix with m rows and n columns is denoted by $O_{m\times n}$. For simplicity, sometimes we omit the indices associated with the matrices I_m and $O_{m\times n}$ if these indices are clear from the context. The linear matrix space $\mathbb{C}^{m\times 1}$ will be denoted by $\mathbb{C}^m$. The elements of $\mathbb{C}^m$ are called m-dimensional vectors and will be written as $x = \mathrm{col}(x_1, \dots, x_m)$. The zero vector belonging to $\mathbb{C}^m$ will be

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denoted by O_m. For an arbitrary square matrix $A \in \mathbb{C}^{m\times m}$, we denote its kernel $\{x \in \mathbb{C}^m : Ax = O_m\}$ by ker A. The scalar product in $\mathbb{C}^m$ is defined by the formula $(x, y) = \sum_{j=1}^{m} \bar{x}_j y_j$. A matrix $A \in \mathbb{C}^{m\times m}$ is called Hermitian if $(x, Ay) = (Ax, y)$ for all $x, y \in \mathbb{C}^m$. The set of all Hermitian matrices is denoted by $\mathbb{C}_H^{m\times m}$. A Hermitian matrix $A \in \mathbb{C}_H^{m\times m}$ is called non-negative Hermitian if $(x, Ax) \geq 0$ for all $x \in \mathbb{C}^m$. The set of all non-negative Hermitian matrices will be denoted by $\mathbb{C}_{\geq}^{m\times m}$. A matrix $A \in \mathbb{C}_{\geq}^{m\times m}$ is called positive Hermitian if $(x, Ax) > 0$ for all non-zero vectors $x \in \mathbb{C}^m$. The set of all positive Hermitian matrices will be denoted by $\mathbb{C}_{>}^{m\times m}$. If $A, B \in \mathbb{C}_H^{m\times m}$, then we write $A > B$ (resp. $A \geq B$) if $A - B \in \mathbb{C}_{>}^{m\times m}$ (resp. $A - B \in \mathbb{C}_{>}^{m\times m}$). If some matrix A is invertible, then for the matrix $(A^{-1})^*$ we also write A^{-*}. If $f(z)$ is some matrix-valued function (MF), then the function $\big(f(z)\big)^*$ will shortly be written as $f^*(z)$. If the MF $f(z)$ is invertible, then we write $\big(f(z)\big)^{-1}$ and $\Big(\big(f(z)\big)^{-1}\Big)^*$ for short in the form $f^{-1}(z)$ and $f^{-*}(z)$, respectively. The open half-planes will be denoted by

$$\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}, \quad \mathbb{C}_- = \{z \in \mathbb{C} : \operatorname{Im} z < 0\}.$$

The symbol $\mathfrak{B}$ stands for the σ-algebra of the Borelian subsets of the real axis $\mathbb{R}$. A mapping $\sigma : \mathfrak{B} \longrightarrow \mathbb{C}^{m\times m}$ is called non-negative Hermitian measure if

$$\sigma\Big(\bigcup_{j=1}^{\infty} A_j\Big) = \sum_{j=1}^{\infty} \sigma(A_j)$$

is satisfied for each sequence of pairwise disjoint Borelian subsets of $\mathbb{R}$. In this paper we will use integrals with respect to non-negative Hermitian measure and the basic properties of these integrals (see, e.g. [1, 2]).

Let an infinite sequence $(s_l)_{l=0}^{\infty}$ of matrices from $\mathbb{C}^{m\times m}$ be given. Now we introduce several block matrices:

$$H_j = \big(s_{l+k}\big)_{l,\,k=0}^{j}, \quad K_j = \big(s_{l+k+1}\big)_{l,\,k=0}^{j}, \quad j \geq 0, \tag{1.1}$$

$$y_{j,k} = \begin{pmatrix} s_j \\ \vdots \\ s_k \end{pmatrix}, \quad z_{j,k} = \begin{pmatrix} s_j & \dots & s_k \end{pmatrix}, \quad 0 \leq j \leq k,$$

$$u_0 = O_{m\times m}, \quad u_j = \begin{pmatrix} O_{m\times m} \\ -y_{0,\,j-1} \end{pmatrix}, \quad j > 0,$$

$$v_j = \begin{pmatrix} I_m \\ O_{mj\times m} \end{pmatrix}, \quad V_j = \begin{pmatrix} O_{mj\times m} \\ I_m \end{pmatrix}, \quad j \geq 0, \tag{1.2}$$

$$T_0 = O_{m\times m}, \quad T_j = \begin{pmatrix} O_{m\times mj} & O_{m\times m} \\ I_{mj} & O_{mj\times m} \end{pmatrix}, \quad j > 0,$$

$$R_j(z) = \left(I_{(j+1)m} - zT_j\right)^{-1} = \begin{pmatrix} I_m & O_{m\times m} & \dots & O_{m\times m} \\ zI_m & I_m & \ddots & O_{m\times m} \\ \vdots & \ddots & \ddots & O_{m\times m} \\ z^j I_m & \dots & zI_m & I_m \end{pmatrix}, \quad j \geq 0.$$

The last identity is obvious for $j = 0$. For $j > 0$ we have

$$\begin{pmatrix} I_m & O_{m\times m} & \dots & O_{m\times m} \\ -zI_m & I_m & \ddots & O_{m\times m} \\ \vdots & \ddots & \ddots & O_{m\times m} \\ O_m & \dots & -zI_m & I_m \end{pmatrix} \begin{pmatrix} I_m & O_{m\times m} & \dots & O_{m\times m} \\ zI_m & I_m & \ddots & O_{m\times m} \\ \vdots & \ddots & \ddots & O_{m\times m} \\ z^j I_m & \dots & zI_m & I_m \end{pmatrix} = I$$

The blocks (of size $m \times m$) of the matrices (1.1) depend only on the sum of the numbers of its rows and the number of its columns. Such matrices are called block Hankel matrices.

A sequence

$$(s_l)_{l=0}^{\infty} \subset \mathbb{C}^{m\times m} \tag{1.3}$$

is called $\mathbb{R}$-positive (see, e.g. [3, Ch. I, Sect. 1.1]) if for all $j \geq 0$ the block Hankel matrices H_j are positive Hermitian. Let for an $\mathbb{R}$-positive sequence (1.3) the block Hankel matrices $H_j > O$ be constructed. For $j > 0$ we have

$$\begin{aligned} H_j &= \begin{pmatrix} H_{j-1} & y_{j,2j-1} \\ z_{j,2j-1} & s_{2j} \end{pmatrix} \\ &= \begin{pmatrix} I & O \\ z_{j,2j-1}H_{j-1}^{-1} & I \end{pmatrix} \begin{pmatrix} H_{j-1} & O \\ O & \widehat{H}_j \end{pmatrix} \begin{pmatrix} I & H_{j-1}^{-1}y_{j,2j-1} \\ O & I \end{pmatrix}. \end{aligned} \tag{1.4}$$

Here

$$\widehat{H}_j = \begin{cases} H_0 & j = 0 \\ s_{2j} - z_{j,2j-1}H_{j-1}^{-1}y_{j,2j-1} & j > 0. \end{cases} \tag{1.5}$$

The inequality $H_j > O$ implies that the product of the three matrices in (1.4) is non-singular. For this reason, each of the three matricial factors on the right side of (1.4) is non-singular, too, and in particular,

$$\widehat{H}_j > O_{m\times m}, \qquad j \geq 0. \tag{1.6}$$

Forming the inverse matrices in (1.4), we obtain

$$H_j^{-1} = \begin{pmatrix} I & -H_{j-1}^{-1} y_{j,2j-1} \\ O & I \end{pmatrix} \begin{pmatrix} H_{j-1}^{-1} & O \\ O & \widehat{H}_j^{-1} \end{pmatrix} \begin{pmatrix} I & O \\ -z_{j,2j-1} H_{j-1}^{-1} & I \end{pmatrix}. \tag{1.7}$$

From this and the second identity in (1.2) it follows

$$V_j^* H_j^{-1} = \widehat{H}_j^{-1} \left(-z_{j,2j-1} H_{j-1}^{-1} \; I \right), \quad V_j^* H_j^{-1} V_j = \widehat{H}_j^{-1}. \tag{1.8}$$

We associate the matricial Hamburger moment problem with the $\mathbb{R}$-positive matrix sequence (1.3). This means, we want to describe all non-negative Hermitian $m \times m$ measures σ such that

$$s_l = \int_{\mathbb{R}} t^l \sigma(dt), \quad l \geq 0. \tag{1.9}$$

The set of all solutions of problem (1.9) is denoted by $\mathcal{M}$. Under the required assumptions it is ensured that $\mathcal{M} \neq \emptyset$. In the case $m = 1$ the Hamburger moment problem (1.9) is called the classical moment problem.

Remark 1.1 Note that the sequence $(s_l)_{l=0}^{\infty}$ from $\mathbb{C}^{m\times m}$ admits a representation (1.9) with some non-negative Hermitian measure σ if and only if all block Hankel matrices $H_j = \left(s_{l+k}\right)_{l,\,k=0}^{j} \geq O, \quad j \geq 0$ are non-negative Hermitian (see, e.g. [4])

We consider matrix-valued polynomials of the type

$$P_j(t) = \sum_{l=0}^{j} F_l t^l, \quad t \in \mathbb{C}, \quad F_l \in \mathbb{C}^{m\times m}.$$

The matrix $F_j \neq O_{m\times m}$ is called leading coefficient of the matrix polynomial P and the number j is called the degree of the matrix polynomial.

We summarize several known facts about the matricial Hamburger moment problem (see [4–9]). Outgoing from the $\mathbb{R}$-positive matrix sequence (1.3), we form two sequences of matrix-valued polynomials

$$P_j(z) = \widehat{H}_j^{\frac{1}{2}} V_j{}^* H_j^{-1} R_j(z) v_j, \quad j \geq 0, \tag{1.10}$$

$$Q_j(z) = -\widehat{H}_j^{\frac{1}{2}} V_j^* H_j^{-1} R_j(z) u_j, \quad j \geq 0. \tag{1.11}$$

From this and the first formula in (1.8), it follows for $j \geq 1$ then

$$P_0 = H_0^{-\frac{1}{2}}, \quad P_j(z) = \widehat{H}_j^{-\frac{1}{2}} \left(-z_{j,2j-1} H_{j-1}^{-1} \; I \right) R_j(z) v_j,$$

$$Q_0 = O_{m\times m}, \quad Q_j(z) = -\widehat{H}_j^{-\frac{1}{2}} \left(-z_{j,2j-1} H_{j-1}^{-1} \; I \right) R_j(z) u_j.$$

It is obvious that the coefficients of the considered matrix polynomials are square matrices of order m. The matrix polynomials P_j are orthonormal with respect to each measure $\sigma \in \mathcal{M}$ what means that

$$\int_{\mathbb{R}} P_j(t)\sigma(dt)P_k{}^*(t) = \delta_{jk} I_m, \qquad \delta_{jk} = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}, \tag{1.12}$$

whereas the matrix polynomials Q_j satisfy with respect to each measure $\sigma \in \mathcal{M}$ the integral representation

$$Q_j(z) = \int_{\mathbb{R}} \frac{P_j(t) - P_j(z)}{t - z}\sigma(dt) \tag{1.13}$$

and, thus are matrix polynomials of second kind (see, e.g., [10, 11]).

From orthonormality conditions (1.12) it follows that the matrix polynomials $(P_j)_{j=0}^{\infty}$ satisfy the following initial conditions and recurrence formulas

$$P_0(z) \equiv H_0^{-\frac{1}{2}}, \quad zP_0(z) = B_0P_0(z) + A_0P_1(z), \tag{1.14}$$

$$zP_j(z) = A_{j-1}^* P_{j-1}(z) + B_j P_j(z) + A_j P_{j+1}(z), \quad j \geq 1, \tag{1.15}$$

where the matrices A_j and B_j are given by the formulas

$$A_j = \widehat{H}_j^{-\frac{1}{2}} \widehat{H}_{j+1}^{\frac{1}{2}}, \quad B_j = \widehat{H}_j^{\frac{1}{2}} V_j^* H_j^{-1} K_j H_j^{-1} V_j \widehat{H}_j^{\frac{1}{2}} \tag{1.16}$$

and fulfil the conditions

$$\det A_j \neq 0, \quad B_j \in \mathbb{C}_H^{m\times m}.$$

From (1.11), (1.13), and (1.15) it follows that the matrix polynomials $(Q_j)_{j=0}^{\infty}$ satisfy the following initial conditions and recurrence formulas

$$Q_0(z) \equiv O, \quad Q_1(z) \equiv \widehat{H}_1^{-1/2} s_0, \tag{1.17}$$

$$zQ_j(z) = A_{j-1}^* Q_{j-1}(z) + B_j Q_j(z) + A_j Q_{j+1}(z), \quad j \geq 1. \tag{1.18}$$

Taking into account that $P_0(z) \equiv H_0^{-1/2}$, the relations (1.14) and (1.15) can be formally written in the form

$$z \begin{pmatrix} P_0(z) \\ P_1(z) \\ P_2(z) \\ \vdots \end{pmatrix} = \begin{pmatrix} B_0 & A_0 & O & O & \dots \\ A_0^* & B_1 & A_1 & O & \dots \\ O & A_1^* & B_2 & A_2 & \dots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} P_0(z) \\ P_1(z) \\ P_2(z) \\ \vdots \end{pmatrix}, \tag{1.19}$$

whereas using $Q_0(z) \equiv O$ the relations (1.17) and (1.18) can be formally written in the form

$$z\begin{pmatrix} Q_0(z) \\ Q_1(z) \\ Q_2(z) \\ \vdots \end{pmatrix} + \begin{pmatrix} I \\ O \\ O \\ \vdots \end{pmatrix} s_0^{1/2} = \begin{pmatrix} B_0 & A_0 & O & O & \dots \\ A_0^* & B_1 & A_1 & O & \dots \\ O & A_1^* & B_2 & A_2 & \dots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} Q_0(z) \\ Q_1(z) \\ Q_2(z) \\ \vdots \end{pmatrix}. \tag{1.20}$$

We consider the infinite block Jacobi matrix

$$\mathbf{J} = \begin{pmatrix} B_0 & A_0 & O & O & \dots \\ A_0^* & B_1 & A_1 & O & \dots \\ O & A_1^* & B_2 & A_2 & \dots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix} \tag{1.21}$$

the blocks of which are matrices from $\mathbb{C}^{m\times m}$. We denote by $\ell^2(\mathbb{C}^m)$ the Hilbert space of all infinite column vectors

$$u = \mathrm{col}\,(u_0, u_1, u_2, \dots), \quad \sum_{j=0}^{\infty} u_j^* u_j < +\infty, \quad u_j \in \mathbb{C}^m,$$

with scalar product $(u, v) = \sum_{j=0}^{\infty} u_j^* v_j$. The symbol $\ell_0^2(\mathbb{C}^m)$ stands for the closed linear subspace of $\ell^2(\mathbb{C}^m)$ consisting of all finite vectors. With the aid of the block Jacobi matrix $\mathbf{J}$ we define a symmetric linear operator $\mathbf{L}_0 : \ell_0^2(\mathbb{C}^m) \longrightarrow \ell_0^2(\mathbb{C}^m)$ via the formula

$$\mathbf{L}_0 u = \mathbf{J}u, \ \ \forall u \in \ell_0^2(\mathbb{C}^m).$$

The closure of this symmetric operator is denoted by $\mathbf{L}$.

The deficiency space of the operator $\mathbf{L}$ in the non-real point z is given by

$$\mathcal{K}(z) = \left\{u \in \ell^2(\mathbb{C}^m) \mid \mathbf{L}^* u = zu\right\}, \quad z \in \mathbb{C} \setminus \mathbb{R}. \tag{1.22}$$

The numbers

$$m_+ = \dim \mathcal{K}(z), \ z \in \mathbb{C}_+, \quad m_- = \dim \mathcal{K}(z), \ z \in \mathbb{C}_- \tag{1.23}$$

do not depend on the concrete choice of the point z belonging to the upper or lower half plane, respectively. They are called the deficiency numbers of the operator $\mathbf{L}$. It is well-known that a closed symmetric operator is Hermitian (i.e., $\mathbf{L} = \mathbf{L}^*$) if and only if $m_+ = m_- = 0$. The deficiency numbers of the operator $\mathbf{L}$ satisfy the inequalities

$$0 \le m_+ \le m, \qquad 0 \le m_- \le m.$$

The deficiency number m_+ is maximal if and only if the deficiency number m_- is maximal (see [12]).

If the deficiency numbers are not maximal, then they can independently of each other obtain arbitrary values ranging from 0 to $m-1$ (see [13, 14]).

Definition 1.2 The matricial Hamburger moment problem (1.9) is called

- *completely indeterminate* if the deficiency numbers of the operator $\mathbf{L}$ are maximal, i.e., $m_+ = m_- = m$.
- *completely determinate* if the deficiency numbers of the operator $\mathbf{L}$ are minimal, i.e., $m_+ = m_- = 0$, and the operator $\mathbf{L}$ is Hermitian.
- *semi-determinate* if the deficiency numbers of the operator $\mathbf{L}$ satisfy the conditions $0 \le m_+ \le m-1,\ 0 \le m_- \le m-1$ and $m_+^2 + m_-^2 \ne 0$.

We consider infinite matricial column vectors of the type

$$V = \operatorname{col}(V_0, V_1, V_2, \dots), \quad V_j \in \mathbb{C}^{m\times m}.$$

We denote by $\ell^2(\mathbb{C}^{m\times m})$ the set of all infinite matricial columns V for which the matricial series $\sum_{j=0}^{\infty} V_j^* V_j$ converges.

Using the polynomials (1.10) and (1.11), we construct the infinite matricial columns

$$\pi(z) = \operatorname{col}(P_0(z), P_1(z), P_2(z), \dots), \tag{1.24}$$

$$\xi(z) = \operatorname{col}(Q_0(z), Q_1(z), Q_2(z), \dots), \tag{1.25}$$

$$\frac{d\pi}{dz}(z) = \operatorname{col}\Big(\frac{dP_0}{dz}(z), \frac{dP_1}{dz}(z), \frac{dP_2}{dz}(z), \dots\Big),$$

$$\frac{d\xi}{dz}(z) = \operatorname{col}\Big(\frac{dQ_0}{dz}(z), \frac{dQ_1}{dz}(z), \frac{dQ_2}{dz}(z), \dots\Big).$$

Further, for each vector $\phi \in \mathbb{C}^m$ we consider the infinite column vector

$$\pi(z)\phi = \operatorname{col}(P_0(z)\phi, P_1(z)\phi, P_2(z)\phi, \dots), \quad P_j(z)\phi \in \mathbb{C}^m, \quad j \ge 0.$$

Analogously, the infinite column vectors

$$\xi(z)\phi, \quad \frac{d\pi}{dz}(z)\phi, \quad \frac{d\xi}{dz}(z)\phi$$

are defined.

The following theorem is the main result of this paper

Theorem 1.3 *Let an $\mathbb{R}$-positive sequence (1.3) and the associated matricial Hamburger moment problem (1.9) be given. Further, let the matrix polynomials P_j and Q_j be defined by the formulas (1.10) and (1.11), respectively. Then the following statements (1)–(5) are equivalent:*

(1) *The matricial Hamburger moment problem (1.9) is completely indeterminate.*
(2) *For some point $z_0 \in \mathbb{C} \setminus \mathbb{R}$ the infinite matricial column $\pi(z_0)$ belongs to $\ell^2(\mathbb{C}^{m\times m})$.*
(3) *For some point $z_0 \in \mathbb{C} \setminus \mathbb{R}$ the infinite matricial column $\xi(z_0)$ belongs to $\ell^2(\mathbb{C}^{m\times m})$.*
(4) *For some $x_0 \in \mathbb{R}$ both infinite matricial columns $\pi(x_0)$ and $\xi(x_0)$ belong to $\ell^2(\mathbb{C}^{m\times m})$.*
(5) *For all $z \in \mathbb{C}$ both infinite matricial columns $\pi(z)$ and $\xi(z)$ belong to $\ell^2(\mathbb{C}^{m\times m})$.*

Furthermore, the following statements hold true:

(6) *If for some $x_0 \in \mathbb{R}$ and some non-null vector $\phi \in \mathbb{C}^m$ both infinite column vectors*

$$\pi(x_0)\phi, \quad \frac{d\pi}{dx}(x_0)\phi \tag{1.26}$$

belong to the Hilbert space $\ell^2(\mathbb{C}^m)$, then the matricial Hamburger moment problem (1.9) is not completely determinate.

(7) *If for some $x_0 \in \mathbb{R}$ and some non-null vector $\phi \in \mathbb{C}^m$ both infinite column vectors*

$$\xi(x_0)\phi, \quad \frac{d\xi}{dx}(x_0)\phi \tag{1.27}$$

belong to the Hilbert space $\ell^2(\mathbb{C}^m)$, then the matricial Hamburger moment problem (1.9) is not completely determinate.

Remark 1.4 In the case $m = 1$ of the classical Hamburger moment problem all statements of Theorem 1.3 were proved in [15, Theorem 3]. In this paper, corresponding results are proved for the matricial case $m > 1$. Our strategy is based on the application of methods originating in the theory of J-contractive analytic matrix functions by V. P. Potapov (see, e.g., [16–23]). What concerns a modern presentation of these methods we refer to the monograph [24].

It should be mentioned that in the classical scalar case it was proved in [15, Theorem 3] that even all conditions (1)–(7) listed in Theorem 1.3 are equivalent. However, in the matricial case the situation is different. What concerns the scalar case as well as the matricial case we are able to derive in the situation of statements (6) and (7) of Theorem 1.3 that the operator $\mathbf{L}$ is not Hermitian. From this it follows in the scalar case the equivalence of all statements (1)–(7) in Theorem 1.3.

In the case $m > 1$ the matricial Hamburger moment problem can turn out to be completely indeterminate or semi-determinate. Thus, the statements (6) and (7) are not equivalent to the statements (1)–(5) in Theorem 1.3.

Remark 1.5 For $m = 1$ and the choice $x_0 = 0$ the statement (4) in Theorem 1.3 is just the Hamburger criterion for indeterminacy of the classical moment problem (see, e.g., [3]).

2 Proofs of Auxiliary Results for the Main Theorem 1.3

Let $\mathbf{L}$ be the closed symmetric operator associated with the block Jacobi matrix (1.21) and let $\mathbf{L}^*$ be its adjoint operator. The domain of the adjoint operator $\mathbf{L}^*$ consists of those vectors from $\ell^2(\mathbb{C}^m)$, which after multiplication by the block Jacobi matrix $\mathbf{J}$ are in $\ell^2(\mathbb{C}^m)$. In this way, the action of the operator $\mathbf{L}^*$ applied to some vector u belonging to its domain is given by left multiplication $\mathbf{J}u$ of u by the block Jacobi matrix $\mathbf{J}$ (see, e.g., [3, 5, 7]). These observations enable us to write the definition (1.22) of the deficiency subspaces in the form

$$\mathcal{K}(z) = \left\{u \in \ell^2(\mathbb{C}^m) \mid \mathbf{J}u = zu\right\}, \quad z \in \mathbb{C} \setminus \mathbb{R}. \tag{2.1}$$

The infinite column vector $u = \operatorname{col}(u_0, u_1, u_2, \ldots)$, $u_j \in \mathbb{C}^m$, (not necessarily belonging to $\ell^2(\mathbb{C}^m)$) is called formal deficiency vector of the block Jacobi matrix $\mathbf{J}$ in the point $z \in \mathbb{C} \setminus \mathbb{R}$ if $\mathbf{J}u = zu$. The set

$$\mathcal{F}(z) = \left\{u \mid \mathbf{J}u = zu\right\}, \quad z \in \mathbb{C} \setminus \mathbb{R} \tag{2.2}$$

of all formal deficiency vectors of the block Jacobi matrix $\mathbf{J}$ in the point $z \in \mathbb{C} \setminus \mathbb{R}$ forms a linear space.

A formal deficiency vector u of the block Jacobi matrix $\mathbf{J}$ is a deficiency vector of the corresponding operator $\mathbf{L}$ if it belongs to $\ell^2(\mathbb{C}^m)$. For this reason, it holds the inclusion $\mathcal{K}(z) \subset \mathcal{F}(z)$. The space $\mathcal{F}(z)$ is called the formal deficiency space of the block Jacobi matrix $\mathbf{J}$ in the non-real point z, whereas the space $\mathcal{K}(z)$ is called the deficiency space of the corresponding operator $\mathbf{L}$ in the non-real point z.

From (1.19) it follows that for each $\phi \in \mathbb{C}^m$ and each $z \in \mathbb{C} \setminus \mathbb{R}$ the infinite column vector

$$\pi(z)\phi = \operatorname{col}(P_0(z)\phi, P_1(z)\phi, P_2(z)\phi, \ldots)$$

belongs to the formal deficiency space $\mathcal{F}(z)$ of the block Jacobi matrix $\mathbf{J}$. It can be easily seen that the mapping

$$\phi \in \mathbb{C}^m \leftrightarrow \pi(z)\phi = \operatorname{col}(P_0(z)\phi, P_1(z)\phi, P_2(z)\phi, \ldots) \in \mathcal{F}(z) \tag{2.3}$$

is an isomorphism between the spaces $\mathbb{C}^m$ and $\mathcal{F}(z)$.

A formal deficiency vector $\pi(z)\phi = \operatorname{col}(P_0(z)\phi, P_1(z)\phi, P_2(z)\phi, \ldots)$ of the block Jacobi matrix **J** in a non-real point z is a deficiency vector of the corresponding operator **L** if it belongs to the space $\ell^2(\mathbb{C}^m)$, i.e., if the series

$$\sum_{j=0}^{\infty} \phi^* P_j^*(z) P_j(z)\phi < +\infty. \tag{2.4}$$

converges. We denote by $\mathcal{L}(z)$ the set of all vectors $\phi \in \mathbb{C}^m$, which satisfy the condition (2.4):

$$\mathcal{L}(z) = \left\{\phi \in \mathbb{C}^m \;\middle|\; \pi(z)\phi \in \ell^2(\mathbb{C}^m)\right\}. \tag{2.5}$$

Clearly, $\mathcal{L}(z)$ is a subspace of $\mathbb{C}^m$ and the formula

$$\phi \in \mathcal{L}(z) \leftrightarrow \pi(z)\phi = \operatorname{col}(P_0(z)\phi, P_1(z)\phi, P_2(z)\phi, \ldots) \in \mathcal{D}(z) \tag{2.6}$$

is an isomorphism between the linear spaces $\mathcal{L}(z)$ and $\mathcal{D}(z)$. From this it follows that the deficiency numbers can be computed via the formulas

$$m_+ = \dim \mathcal{L}(z),\ z \in \mathbb{C}_+, \qquad m_- = \dim \mathcal{L}(z),\ z \in \mathbb{C}_-.$$

Lemma 2.1 *If the deficiency number m_+ and m_- of the operator* **L** *associated with the block Jacobi matrix* **J** *are equal to m, then the series*

$$\sum_{j=0}^{\infty} P_j^*(z) P_j(z) \tag{2.7}$$

converges for all $z \in \mathbb{C} \setminus \mathbb{R}$. If the series (2.7) converges for some $z \in \mathbb{C} \setminus \mathbb{R}$, then both deficiency numbers are equal to m.

Proof Assume that $m_+ = m_- = m$. Then all formal deficiency vectors of the block Jacobi matrix **J** in all non-real points z are deficiency vectors of the corresponding operator **L**. Hence, for each $\phi \in \mathbb{C}^m$ and each non-real z the series $\sum_{j=0}^{\infty} \phi^* P_j^*(z) \cdot P_j(z)\phi$ converges. From this it follows the convergence of the series (2.7) for all $z \in \mathbb{C} \setminus \mathbb{R}$.

Let the series (2.7) converge for some $z_0 \in \mathbb{C} \setminus \mathbb{R}$. Then for all $\phi \in \mathbb{C}^m$ the series $\sum_{j=0}^{\infty} \phi^* P_j^*(z_0) \cdot P_j(z_0)\phi$ converges. For this reason, $\mathcal{L}(z_0) = \mathbb{C}^m$ and one of the deficiency numbers is equal to m. Hence, both deficiency numbers are equal to m and $\mathcal{L}(z) = \mathbb{C}^m$ for all non-real z. Consequently, the series (2.7) converges for all $z \in \mathbb{C} \setminus \mathbb{R}$. Lemma 2.1 is proved. □

Lemma 2.2 *Let the sequences* $(A_j)_{j=0}^{\infty}$ *and* $(B_j)_{j=0}^{\infty}$ *of matrices from* $\mathbb{C}^{m\times m}$ *be such that the series*

$$\sum_{j=0}^{\infty} A_j^* A_j, \qquad \sum_{j=0}^{\infty} B_j^* B_j \tag{2.8}$$

converge. Then for all matrices $C, D, E, F \in \mathbb{C}^{m\times m}$ *the series*

$$\sum_{j=0}^{\infty} \big(CA_jD + EB_jF\big)^* \big(CA_jD + EB_jF\big). \tag{2.9}$$

converges.

The proof of this Lemma is obvious.

To each measure σ, which is a solution of the matricial Hamburger moment problem (1.9), we associate the MF

$$w(z) = \int_{\mathbb{R}} \frac{\sigma(dt)}{t-z}, \quad \sigma \in \mathcal{M}. \tag{2.10}$$

MF w is defined and holomorphic in the upper half plane $\mathbb{C}_+$ and in the lower half plane $\mathbb{C}_-$. The set of all MF of the form (2.10) will be denoted by the symbol $\mathcal{F}$ and it is called the set of MF's associated with the matricial Hamburger moment problem (1.9). The Stieltjes inversion formula establishes a bijective correspondence between $\mathcal{F}$ and $\mathcal{M}$.

It is known (see [9]) that for all $z \in \mathbb{C}\setminus\mathbb{R}$ and for all integers $l \geq 0$ the associated MF $w(z)$ satisfies V. P. Potapov's Fundamental Matrix Inequality (FMI)

$$\begin{pmatrix} H_l & R_l(z)(v_l w(z) - u_l) \\ \big(R_l(z)(v_l w(z) - u_l)\big)^* & (w(z) - w^*(z))/(z-\bar{z}) \end{pmatrix} \geq O. \tag{2.11}$$

Lemma 2.3 *Let an* $\mathbb{R}$*-positive matrix sequence* (1.3) *with corresponding matricial Hamburger moment problem* (1.9) *be given. Further, let MF* w *be associated with the matricial Hamburger moment problem* (1.9) *and let the sequences* $(P_j)_{j=0}^{\infty}$ *and* $(Q_j)_{j=0}^{\infty}$ *of matrix polynomials be constructed via formulas* (1.10) *and* (1.11)*, respectively. Then:*

1. *For all* $z \in \mathbb{C}\setminus\mathbb{R}$*, the series*

$$\sum_{j=0}^{\infty} \big(P_j(z)w(z) - Q_j(z)\big)^* (P_j(z)w(z) - Q_j(z)) \tag{2.12}$$

converges.

2. *If for some fixed* $z_0 \in \mathbb{C} \setminus \mathbb{R}$ *one of the series*

$$\sum_{j=0}^{\infty} P_j^*(z_0) P_j(z_0), \qquad \sum_{j=0}^{\infty} Q_j^*(z_0) Q_j(z_0). \tag{2.13}$$

converges (resp. diverges), then for all $z \in \mathbb{C} \setminus \mathbb{R}$ *the two series* (2.13) *converge (resp. diverge).*

Proof

1. We multiply FMI (2.11) from right by the matrix

$$\begin{pmatrix} I & -H_l^{-1} R_l(z)(v_l w(z) - u_l) \\ O & I \end{pmatrix},$$

and from left with its adjoint matrix. In this way, we obtain

$$\frac{w(z) - w^*(z)}{z - \bar{z}} - \big(R_l(z)(v_l w(z) - u_l)\big)^* H_l^{-1} R_l(z)(v_l w(z) - u_l) \geq O. \tag{2.14}$$

From (1.7), (1.10) and (1.11) it follows (note that $l > 0$)

$$\begin{aligned}
&\big(R_l(z)(v_l w(z) - u_l)\big)^* H_l^{-1} R_l(z)(v_l w(z) - u_l) \\
&\quad = \big(R_l(z)(v_l w(z) - u_l)\big)^* \begin{pmatrix} H_{l-1}^{-1} & O \\ O & O \end{pmatrix} R_l(z)(v_l w(z) - u_l) \\
&\qquad + \big(R_l(z)(v_l w(z) - u_l)\big)^* H_l^{-1} V_l \widehat{H}_l^{-1/2} \widehat{H}_l^{-1/2} V_l^* H_l^{-1} R_l(z)(v_l w(z) - u_l) \\
&\quad = \big(R_l(z)(v_l w(z) - u_l)\big)^* \begin{pmatrix} H_{l-1}^{-1} & O \\ O & O \end{pmatrix} R_l(z)(v_l w(z) - u_l) \\
&\qquad + \Big(\widehat{H}_l^{-1/2} V_l^* H_l^{-1} R_l(z) v_l w(z) - \widehat{H}_l^{-1/2} V_l^* H_l^{-1} R_l(z) u_l\Big)^* \\
&\qquad \times \Big(\widehat{H}_l^{-1/2} V_l^* H_l^{-1} R_l(z) v_l w(z) - \widehat{H}_l^{-1/2} V_l^* H_l^{-1} R_l(z) u_l\Big) \\
&\quad = \big(R_{l-1}(z)(v_{l-1} w(z) - u_{l-1})\big)^* H_{l-1}^{-1} R_{l-1}(z)(v_{l-1} w(z) - u_{l-1}) \\
&\qquad + \big(P_l(z) w(z) - Q_l(z)\big)^* (P_l(z) w(z) - Q_l(z)).
\end{aligned}$$

An obvious induction over l leads to the identity

$$\begin{aligned}
&\big(R_l(z)(v_l w(z) - u_l)\big)^* H_l^{-1} R_l(z)(v_l w(z) - u_l) \\
&\qquad = \sum_{j=0}^{l} \big(P_j(z) w(z) - Q_j(z)\big)^* (P_j(z) w(z) - Q_j(z)), \quad l \geq 0.
\end{aligned}$$

Now (2.14) can be written in the form

$$\sum_{j=0}^{l} \big(P_j(z)w(z) - Q_j(z)\big)^* (P_j(z)w(z) - Q_j(z)) \le \frac{w(z) - w^*(z)}{z - \bar{z}}.$$

This inequality holds for all non-real $z \in \mathbb{C}$ and all integers $l \ge 0$. This implies the first assertion of the Lemma.

2. Let for some $z_0 \in \mathbb{C} \setminus \mathbb{R}$ the first of the series in (2.13) converge. In view of Lemma 2.1, this series converges for all $z \in \mathbb{C} \setminus \mathbb{R}$. Hence, for all $z \in \mathbb{C} \setminus \mathbb{R}$ the series

$$\sum_{j=0}^{\infty} P_j^*(z)P_j(z), \qquad \sum_{j=0}^{\infty} \big(P_j(z)w(z) + Q_j(z)\big)^* \big(P_j(z)w(z) + Q_j(z)\big).$$

converge. Combining this with Lemma 2.2, it follows that it converges for all $z \in \mathbb{C} \setminus \mathbb{R}$ the series

$$\sum_{j=0}^{\infty} \big(P_j(z)w(z) + Q_j(z) - P_j(z)w(z)\big)^* (P_j(z)w(z) + Q_j(z) - P_j(z)w(z)),$$

which coincides with the series $\sum_{j=0}^{\infty} Q_j^*(z)Q_j(z)$. Analogously, it can be shown that the convergence for some $z_0 \in \mathbb{C} \setminus \mathbb{R}$ of the second of the two series in (2.13) implies the convergence of both series (2.13) in all non-real points z.

Lemma 2.3 is proved. □

Let the orthonormal matrix polynomials P_l of the first kind be defined by formulas (1.10), let the matrix polynomials Q_l of the second kind be defined by (1.11), and let A_l be the matrix coefficients occurring in the recurrence formulas (1.14)–(1.15). It is known (see, e.g., [2, 7]), for all integers $l \ge 0$, all $z \in \mathbb{C}$ and all $x \in \mathbb{R}$ it holds the Ostrogradskii-Liouville formulas

$$P_l(z)Q_l^*(\bar{z}) - Q_l(z)P_l^*(\bar{z}) = O, \tag{2.15}$$

$$P_l(z)Q_{l+1}^*(\bar{z}) - Q_l(z)P_{l+1}^*(\bar{z}) = A_l^{-*}, \tag{2.16}$$

the Christoffel-Darboux formulas

$$(z - x)\sum_{j=0}^{l} P_j^*(\bar{z})P_j(x) = P_{l+1}^*(\bar{z})A_l^* P_l(x) - P_l^*(\bar{z})A_l P_{l+1}(x), \tag{2.17}$$

$$(z - x)\sum_{j=0}^{l} Q_j^*(\bar{z})Q_j(x) = Q_{l+1}^*(\bar{z})A_l^* Q_l(x) - Q_l^*(\bar{z})A_l Q_{l+1}(x) \tag{2.18}$$

and the Green formulas

$$-I+(z-x)\sum_{j=0}^{l}P_j^*(\bar z)Q_j(x)=P_{l+1}^*(\bar z)A_l^*Q_l(x)-P_l^*(\bar z)A_lQ_{l+1}(x), \tag{2.19}$$

$$I+(z-x)\sum_{j=0}^{l}Q_j^*(\bar z)P_j(x)=Q_{l+1}^*(\bar z)A_l^*P_l(x)-Q_l^*(\bar z)A_lP_{l+1}(x). \tag{2.20}$$

Let

$$\mathcal{J}=\begin{pmatrix} O_m & -iI_m \\ iI_m & O_m \end{pmatrix}.$$

Then obviously

$$\mathcal{J}^*=\mathcal{J},\ \mathcal{J}^2=I_{2m}.$$

If $x_0\in\mathbb{R}$, then we denote by $\mathcal{P}_{x_0}$ the class of all entire MF $U:\mathbb{C}\to\mathbb{C}^{2m\times 2m}$, which satisfy the conditions

$$\mathcal{J}-U(z)\mathcal{J}U^*(z)=\begin{cases}\le O_{2m}, & z\in\mathbb{C}_+\\ = O_{2m}, & z\in\mathbb{R}\\ \ge O_{2m}, & z\in\mathbb{C}_-\end{cases}. \tag{2.21}$$

and

$$U(x_0)=I_{2m}. \tag{2.22}$$

The basic property of the class $\mathcal{P}_{x_0}$ is its multiplicativity. More precisely, if $U_1(z),\ U_2(z)\in\mathcal{P}_{x_0}$, then $U_1(z)\cdot U_2(z)\in\mathcal{P}_{x_0}$, too. Let the MF U belong to the class $\mathcal{P}_{x_0}$. Then from (2.21) it follows that for all $x\in\mathbb{R}$ it holds $\mathcal{J}-U(x)\mathcal{J}U^*(x)=O_{2m\times 2m}$. Hence, in view of the holomorphicity of U it follows that for all $z\in\mathbb{C}$ the formula $\mathcal{J}-U(z)\mathcal{J}U^*(\bar z)=O_{2m\times 2m}$ holds. From this it follows the invertibility of MF U for all $z\in\mathbb{C}$ and the formula ("principle of symmetry")

$$U^{-1}(z)=\mathcal{J}U^*(\bar z)\mathcal{J},\quad U\in\mathcal{P}_{x_0}. \tag{2.23}$$

We consider the matrix-valued Blaschke-Potapov factors

$$b_j(z)=I_{2m}-i(z-x_0)E_j(x_0),\quad z\in\mathbb{C},\quad j\ge 0, \tag{2.24}$$

where $x_0 \in \mathbb{R}$ and

$$E_j(x_0) = \begin{pmatrix} P_j^*(x_0) \\ -Q_j^*(x_0) \end{pmatrix} \begin{pmatrix} P_j(x_0) & -Q_j(x_0) \end{pmatrix} \mathcal{J}. \tag{2.25}$$

Lemma 2.4 *Let the matrix-valued Blaschke-Potapov factor $b_j(z)$ be given by formulas (2.24) and (2.25). Then:*

1. *The matrices $E_j(x_0)$ satisfy*

$$E_j(x_0)\mathcal{J} \geq O, \qquad \big(E_j(x_0)\big)^2 = O; \tag{2.26}$$

2. $b_j(z) \in \mathcal{P}_{x_0}$.

Proof

1. We have

$$E_j(x_0)\mathcal{J} = \begin{pmatrix} P_j^*(x_0) \\ -Q_j^*(x_0) \end{pmatrix} \big(P_j(x_0)\ , -Q_j(x_0) \big) \geq O.$$

   Furthermore,

$$\begin{aligned} &E_j(x_0)E_j(x_0) \\ &= \begin{pmatrix} P_j^*(x_0) \\ -Q_j^*(x_0) \end{pmatrix} \begin{pmatrix} P_j(x_0) & -Q_j(x_0) \end{pmatrix} \mathcal{J} \begin{pmatrix} P_j^*(x_0) \\ -Q_j^*(x_0) \end{pmatrix} \big(P_j(x_0)\ , -Q_j(x_0) \big) \mathcal{J} \\ &= i \begin{pmatrix} P_j^*(x_0) \\ -Q_j^*(x_0) \end{pmatrix} \Big(P_j(x_0)Q_j^*(x_0), -Q_j(x_0)P_j^*(x_0) \Big) (P_j(x_0)\ , -Q_j(x_0))\mathcal{J} = O. \end{aligned}$$

   Here, the last equality follows from the Ostrograskii-Liouville formula (2.15).
2. Using formulas (2.26), we get

$$\begin{aligned} \mathcal{J} - b_j(z)\mathcal{J}b_j^*(z) &= \mathcal{J} - \Big(I_{2m} - i(z-x_0)E_j(x_0)\Big)\mathcal{J}\Big(I_{2m} + i(\bar{z}-x_0)E_j(x_0)^*\Big) \\ &= i(z-x_0)E_j(x_0)\mathcal{J} - i(\bar{z}-x_0)\mathcal{J}E_j(x_0)^* + |z-x_0|^2 E_j(x_0)\mathcal{J}E_j(x_0)^* \\ &= i(z-x_0)E_j(x_0)\mathcal{J} - i(\bar{z}-x_0)E_j(x_0)\mathcal{J} + |z-x_0|^2 E_j(x_0)E_j(x_0)\mathcal{J} \\ &= i(z-\bar{z})E_j(x_0)\mathcal{J}. \end{aligned}$$

Hence,

$$\mathcal{J} - b_j(z)\mathcal{J}b_j^*(z) = i(z-\bar{z})E_j(x_0)\mathcal{J} = \begin{cases} \leq O_{2m}, & z \in \mathbb{C}_+ \\ = O_{2m}, & z \in \mathbb{R} \\ \geq O_{2m}, & z \in \mathbb{C}_- \end{cases}. \tag{2.27}$$

Combining this with the obvious relation $b_j(x_0) = I_{2m}$, we get $b_j(z) \in \mathcal{P}_{x_0}$.
Lemma 2.4 is proved. □

We consider a sequence of products of matrix-valued Blaschke-Potapov factors

$$U_l(z) = b_0(z)\cdot b_1(z)\cdot\ldots\cdot b_l(z) = \overrightarrow{\prod_{j=0}^{l}}\Big(I_{2m} - i(z-x_0)E_j(x_0)\Big), \quad l \geq 0. \tag{2.28}$$

The MF U_l belongs to the class $\mathcal{P}_{x_0}$ because all matrix-valued Blaschke-Potapov factors $b_j(z)$ belong to the class $\mathcal{P}_{x_0}$.

In the sequel we use the concept of $\mathcal{J}$-modulus, which goes back to V. P. Potapov [16, 26] (see also [19, Section 1.4], [24, Section 2.13]). Let U be a non-singular $\mathcal{J}$-contractive matrix. Then (see, e.g., [19, Proposition 1.4.1]) the eigenvalues of the matrix $G := \mathcal{J}U^*\mathcal{J}$ are positive. Hence, there exists a unique matrix R with positive eigenvalues such that $R^2 = G$. This matrix G is called the $\mathcal{J}$-modulus of U. The $\mathcal{J}$-modulus of a $\mathcal{J}$-Hermitian and $\mathcal{J}$-contractive matrix with positive eigenvalues coincides with itself (see [19, Theorem 1.4.2]).

A matrix R is the $\mathcal{J}$-modulus of some non-singular and J-contractive matrix U if and only if R is a $\mathcal{J}$-Hermitian and $\mathcal{J}$-contractive matrix with positive eigenvalues (see [19, Theorem 1.4.3]).

Theorem 2.5 ([26, Theorem 5.2], [24, Theorem 2.65 (a)]) *Let $E \in \mathbb{C}^{2m\times 2m}$ be such that $E\mathcal{J} \geq O_{2m\times 2m}$. Then the matrix $R := \exp E$ is the $\mathcal{J}$-modulus of itself.*

This result leads to the following notion.

Definition 2.6 Let the matrix $E \in \mathbb{C}^{2m\times 2m}$ be such that $E\mathcal{J} \geq O_{2m\times 2m}$. Then the matrix $R = \exp(E)$ is called a $\mathcal{J}$-modulus.

Lemma 2.7 *Let the matrix-valued Blaschke-Potapov factor b_j be given by the formulas (2.24) and (2.25). Then:*

1. *It holds*

$$b_j(z) = \exp\Big(-i(z-x_0)E_j(x_0)\Big). \tag{2.29}$$

2. *For each* $y > 0$ *it holds*

$$b_j(x_0 + iy) = \exp\Big(yE_j(x_0)\Big), \quad yE_j(x_0)\mathcal{J} \geq O_{2m}, \tag{2.30}$$

i.e., $b_j(x_0 + iy)$ *is a* $\mathcal{J}$*-modulus.*

The proof of this Lemma follows quickly from (2.26) and Definition 2.6.

Now the product (2.28) of matrix-valued Blaschke-Potapov factors can be written in the form

$$U_l(z) = \exp\Big(-i(z - x_0)E_0(x_0))\Big) \times \exp\Big(-i(z - x_0)E_1(x_0))\Big) \times \dots \times \exp\Big(-i(z - x_0)E_l(x_0)\Big) = \overrightarrow{\prod_{j=0}^{l}} \exp\Big(-i(z - x_0)E_j(x_0)\Big), \quad l \geq 0. \tag{2.31}$$

Infinite products of matrices will play an important role in our further considerations. The following two theorems on the convergence of infinite products of matrices are proved in [16] (see also [18]).

Theorem 2.8 *Let the infinite sequence of matrices* $(A_j)_{j=0}^{\infty}$ *from* $\mathbb{C}^{m\times m}$ *be such that the series* $\sum_{j=0}^{\infty} A_j$ *converges. Then the infinite matrix product* $\overrightarrow{\prod}_{j=0}^{\infty} \exp\big(A_j\big)$ *converges.*

The converse statement is not true. The following result on the convergence of an infinite product of $\mathcal{J}$-modulus matrices, which is due to V. P. Potapov plays a key role.

Theorem 2.9 *Let* $(E_j)_{j=0}^{\infty}$ *be an infinite sequence of matrices from* $\mathbb{C}^{2m\times 2m}$ *such that* $E_j\mathcal{J} \geq O_{2m\times 2m}$, $j \geq 0$*, and such that there exists a constant* $C > 0$ *such that for all* $l \geq 0$ *with the setting*

$$U_l = \overrightarrow{\prod_{j=0}^{l}} \exp\Big(E_j\Big), \quad l \geq 0 \tag{2.32}$$

the inequalities

$$\|\mathcal{J} - U_l\mathcal{J}U_l^*\| \leq C, \quad \|\mathcal{J} - U_l^{-*}\mathcal{J}U_l^{-1}\| \leq C \tag{2.33}$$

are satisfied. Then the series $\sum_{j=0}^{\infty} E_j$ *and consequently also the infinite matrix product* $\overrightarrow{\prod}_{j=0}^{\infty} \exp\Big(E_j\Big)$ *converges.*

Lemma 2.10 *Let the matrices $U_l,\ l \geq 0$ be defined by the formulas (2.31). For all $z \in \mathbb{C}$ then*

$$U_l(z) = I_{2m} - i(z - x_0) \sum_{j=0}^{l} \begin{pmatrix} P_j^*(\bar{z}) P_j(x_0) & -P_j^*(\bar{z}) Q_j(x_0) \\ -Q_j^*(\bar{z}) P_j(x_0) & Q_j^*(\bar{z}) Q_j(x_0) \end{pmatrix} \mathcal{J} \tag{2.34}$$

and the formulas

$$\mathcal{J} - U_l(z) \mathcal{J} U_l^*(z) = i(z - \bar{z}) \sum_{j=0}^{l} \begin{pmatrix} P_j^*(\bar{z}) P_j(\bar{z}) & -P_j^*(\bar{z}) Q_j(\bar{z}) \\ -Q_j^*(\bar{z}) P_j(\bar{z}) & Q_j^*(\bar{z}) Q_j(\bar{z}) \end{pmatrix}, \tag{2.35}$$

$$\mathcal{J} - U_l^{-*}(z) \mathcal{J} U_l^{-1}(z) = i(\bar{z} - z) \sum_{j=0}^{l} \begin{pmatrix} Q_j^*(z) Q_j(z) & Q_j^*(z) P_j(z) \\ P_j^*(z) Q_j(z) & P_j^*(z) P_j(z) \end{pmatrix} \tag{2.36}$$

hold true.

Proof First we show that for all $l \geq 0$ we have

$$U_l(z) \begin{pmatrix} P_{l+1}^*(x_0) \\ -Q_{l+1}^*(x_0) \end{pmatrix} = \begin{pmatrix} P_{l+1}^*(z) \\ -Q_{l+1}^*(z) \end{pmatrix}. \tag{2.37}$$

Indeed,

$$\begin{aligned} & U_l(z) \begin{pmatrix} P_{l+1}^*(x_0) \\ -Q_{l+1}^*(x_0) \end{pmatrix} \\ &= \begin{pmatrix} I_m - (z - x_0) \sum_{j=0}^{l} P_j(\bar{z})^* Q_j(x_0) & -(z - x_0) \sum_{j=0}^{l} P_j(\bar{z})^* P_j(x_0) \\ (z - x_0) \sum_{j=0}^{l} Q_j(\bar{z})^* Q_j(x_0) & I_m + (z - x_0) \sum_{j=0}^{l} Q_j(\bar{z})^* P_j(x_0) \end{pmatrix} \\ &\quad \times \begin{pmatrix} P_{l+1}^*(x_0) \\ -Q_{l+1}^*(x_0) \end{pmatrix} \\ &= \begin{pmatrix} -P_{l+1}^*(\bar{z}) A_l^* Q_l(x_0) + P_l^*(\bar{z}) A_l Q_{l+1}(x_0) \\ +Q_{l+1}^*(\bar{z}) A_l^* Q_l(x_0) - Q_l^*(\bar{z}) A_l Q_{l+1}(x_0) \end{pmatrix. \\ &\qquad\qquad \begin{matrix} -P_{l+1}^*(\bar{z}) A_l^* Q_l(x_0) + P_l^*(\bar{z}) A_l Q_{l+1}(x_0) \\ +Q_{l+1}^*(\bar{z}) A_l^* Q_l(x_0) - Q_l^*(\bar{z}) A_l Q_{l+1}(x_0) \end{matrix} \Bigg) \begin{pmatrix} P_{l+1}^*(x_0) \\ -Q_{l+1}^*(x_0) \end{pmatrix} \\ &= \begin{pmatrix} P_{l+1}^*(\bar{z}) A_l^* \big(- Q_l(x_0) P_{l+1}^*(x_0) + P_l(x_0) Q_{l+1}^*(x_0) \big) \\ Q_{l+1}^*(\bar{z}) A_l^* \big(Q_l(x_0) P_{l+1}^*(x_0) - P_l(x_0) Q_{l+1}^*(x_0) \big) \end{pmatrix} = \begin{pmatrix} P_{l+1}^*(z) \\ -Q_{l+1}^*(z) \end{pmatrix}. \end{aligned}$$

In this calculation, the second equality follows from (2.17)–(2.20), the third equality follows from (2.15), whereas the fourth one follows from (2.16). We prove formula (2.34) via induction on l. For $l = 0$ we have

$$U_0(z) = b_0(z) = I_{2m} - i(z - x_0)\big(P_0(z) - Q_0(z)\big)^* \big(P_0(x) - Q_0(x)\big)\mathcal{J}.$$

Now suppose that for a fixed $l \geq 0$ the identity (2.34) is satisfied for all $k \leq l$. Then using (2.37) we get

$$\begin{aligned}
&U_l(z)\cdot b_{l+1}(z)\\
&\quad= U_l(z)\cdot\left(I_{2m} - i(z-x_0)\begin{pmatrix} P^*_{l+1}(x_0)\\ -Q^*_{l+1}(x_0)\end{pmatrix}\big(P_{l+1}(x_0)\ -Q_{l+1}(x_0)\big)\mathcal{J}\right)\\
&\quad= U_l(z) - i(z-x_0)U_l(z)\begin{pmatrix} P^*_{l+1}(x_0)\\ -Q^*_{l+1}(x_0)\end{pmatrix}\big(P_{l+1}(x_0)\ -Q_{l+1}(x_0)\big)\mathcal{J}\\
&\quad= U_l(z) - i(z-x_0)\begin{pmatrix} P^*_{l+1}(\bar z)\\ -Q^*_{l+1}(\bar z)\end{pmatrix}\big(P_{l+1}(x_0)\ -Q_{l+1}(x_0)\big)\mathcal{J} = U_{l+1}(z).
\end{aligned}$$

Formula (2.34) is proved.

Now we prove formula (2.35) via induction on l. If $l = 0$ our formula immediately follows from (2.25) and (2.27). Now suppose that for a fixed $l \geq 0$ the identity (2.35) is satisfied for all $k \leq l$. Then

$$\begin{aligned}
&\mathcal{J} - U_{l+1}(z)\mathcal{J}U^*_{l+1}(z)\\
&\quad= \mathcal{J} - \left(U_l(z) - i(z-x_0)\begin{pmatrix} P^*_{l+1}(\bar z)\\ -Q^*_{l+1}(\bar z)\end{pmatrix}\big(P_{l+1}(x_0)\ -\ Q_{l+1}(x_0)\big)\mathcal{J}\right)\\
&\quad\times \mathcal{J}\left(U_l(z) - i(z-x_0)\begin{pmatrix} P^*_{l+1}(\bar z)\\ -Q^*_{l+1}(\bar z)\end{pmatrix}\big(P_{l+1}(x_0)\ -\ Q_{l+1}(x_0)\big)\mathcal{J}\right)^*\\
&\quad= \mathcal{J} - U_l(z)\mathcal{J}U^*_l(z) + i(z-x_0)\begin{pmatrix} P^*_{l+1}(\bar z)\\ -Q^*_{l+1}(\bar z)\end{pmatrix}\big(P_{l+1}(x_0)\ -\ Q_{l+1}(x_0)\big)U^*_l(z)\\
&\quad- i(\bar z - x_0)U_l(z)\begin{pmatrix} P^*_{l+1}(x_0)\\ -Q^*_{l+1}(x_0)\end{pmatrix}\big(P_{l+1}(\bar z)\ -\ Q_{l+1}(\bar z)\big)\\
&\quad- |z-x_0|^2\begin{pmatrix} P^*_{l+1}(\bar z)\\ -Q^*_{l+1}(\bar z)\end{pmatrix}\big(P_{l+1}(x_0)\ -\ Q_{l+1}(x_0)\big)\\
&\quad\times \mathcal{J}\begin{pmatrix} P^*_{l+1}(x_0)\\ -Q^*_{l+1}(x_0)\end{pmatrix}\big(P_{l+1}(\bar z)\ -\ Q_{l+1}(\bar z)\big)
\end{aligned}$$

$$= i(z-\bar{z})\sum_{j=0}^{l}\begin{pmatrix} P_j^*(\bar{z})P_j(\bar{z}) & -P_j^*(\bar{z})Q_j(\bar{z}) \\ -Q_j^*(\bar{z})P_j(\bar{z}) & Q_j^*(\bar{z})Q_j(\bar{z}) \end{pmatrix}$$

$$+ i(z-x_0)\begin{pmatrix} P_{l+1}^*(\bar{z}) \\ -Q_{l+1}^*(\bar{z}) \end{pmatrix}\big(P_{l+1}(\bar{z}) \ - Q_{l+1}(\bar{z})\big) - i(\bar{z}-x_0)\begin{pmatrix} P_{l+1}^*(\bar{z}) \\ -Q_{l+1}^*(\bar{z}) \end{pmatrix}$$

$$\times \big(P_{l+1}(\bar{z}) \ - Q_{l+1}(\bar{z})\big) = i(z-\bar{z})\sum_{j=0}^{l+1}\begin{pmatrix} P_j^*(\bar{z})P_j(\bar{z}) & -P_j^*(\bar{z})Q_j(\bar{z}) \\ -Q_j^*(\bar{z})P_j(\bar{z}) & Q_j^*(\bar{z})Q_j(\bar{z}) \end{pmatrix}$$

In this calculation the third equality follows from (2.37) and (2.15). Formula (2.35) is proved. Now we prove formula (2.36). Using the principle of symmetry (2.23), we get

$$\mathcal{J}\Big(\mathcal{J} - U_l(\bar{z})\mathcal{J}U_l^*(\bar{z})\Big)\mathcal{J} = \mathcal{J} - \mathcal{J}U_l(\bar{z})\mathcal{J}\mathcal{J}\mathcal{J}U_l^*(\bar{z})\mathcal{J} = \mathcal{J} - U_l^{-*}(z)\mathcal{J}U_l^{-1}(z).$$

Consequently,

$$\mathcal{J} - U_l^{-*}(z)\mathcal{J}U_l^{-1}(z) = \mathcal{J}\Big(\mathcal{J} - U_l(\bar{z})\mathcal{J}U_l^*(\bar{z})\Big)\mathcal{J}$$

$$= i(\bar{z}-z)\mathcal{J}\left(\sum_{j=0}^{l}\begin{pmatrix} P_j^*(z)P_j(z) & -P_j^*(z)Q_j(z) \\ -Q_j^*(z)P_j(z) & Q_j^*(z)Q_j(z) \end{pmatrix}\right)\mathcal{J}$$

$$= -i(\bar{z}-z)\sum_{j=0}^{l}\begin{pmatrix} Q_j^*(z)Q_j(z) & Q_j^*(z)P_j(z) \\ P_j^*(z)Q_j(z) & P_j^*(z)P_j(z) \end{pmatrix}.$$

Lemma 2.10 is proved. □

Lemma 2.11 *Suppose that for some* $z_0 = x_0 + iy_0,\ y_0 \neq 0$ *one of the series*

$$\sum_{j=0}^{\infty} P_j^*(z_0)P_j(z_0), \qquad \sum_{j=0}^{\infty} Q_j^*(z_0)Q_j(z_0) \tag{2.38}$$

converges. Then both series

$$\sum_{j=0}^{\infty} P_j^*(x_0)P_j(x_0), \qquad \sum_{j=0}^{\infty} Q_j^*(x_0)Q_j(x_0) \tag{2.39}$$

converge. Furthermore, for all $z \in \mathbb{C}$ both series

$$\sum_{j=0}^{\infty} P_j^*(z) P_j(z), \qquad \sum_{j=0}^{\infty} Q_j^*(z) Q_j(z) \tag{2.40}$$

converge.

Proof From the assumption and Lemma 2.3 it follows that all of the series

$$\sum_{j=0}^{\infty} P_j^*(z_0) P_j(z_0), \qquad \sum_{j=0}^{\infty} Q_j^*(z_0) Q_j(z_0), \tag{2.41}$$

$$\sum_{j=0}^{\infty} P_j^*(\bar{z}_0) P_j(\bar{z}_0), \qquad \sum_{j=0}^{\infty} Q_j^*(\bar{z}_0) Q_j(\bar{z}_0) \tag{2.42}$$

converge. In view of Lemma 2.2 the series

$$\sum_{j=0}^{\infty} P_j^*(z_0) Q_j(z_0), \quad \sum_{j=0}^{\infty} Q_j^*(z_0) P_j(z_0), \tag{2.43}$$

$$\sum_{j=0}^{\infty} P_j^*(\bar{z}_0) Q_j(\bar{z}_0), \quad \sum_{j=0}^{\infty} Q_j^*(\bar{z}_0) P_j(\bar{z}_0) \tag{2.44}$$

converge. We consider the sequence of products of matrix-valued Blaschke-Potapov factors (see (2.31))

$$U_l(z_0) = \overrightarrow{\prod_{j=0}^{l}} \exp\Big(y_0 E_j(x_0)\Big), \quad l \geq 0.$$

From the convergence of the series (2.41)–(2.44) it follows that the right sides of the identities (2.35) and (2.36) converge in the point z_0 for $l \to \infty$. Hence, the left sides of these identities also converge for $l \to +\infty$. Thus, there exists a constant $C > 0$ such that for all $l \geq 0$ the inequalities

$$\|\mathcal{J} - U_l(z_0)\mathcal{J}U_l^*(z_0)\| < C, \quad \|\mathcal{J} - U_l^{-*}(z_0)\mathcal{J}U_l^{-1}(z_0)\| < C$$

are satisfied. In view of Theorem 2.9 the series

$$\sum_{j=0}^{\infty} y_0 E_j(x_0) = y_0 \sum_{j=0}^{\infty} \begin{pmatrix} P_j^*(x_0)P_j(x_0) & -P_j^*(x_0)Q_j(x_0) \\ -Q_j^*(x_0)P_j(x_0) & Q_j^*(x_0)Q_j(x_0) \end{pmatrix}$$

converges. This implies the convergence of both series (2.39).

Suppose now that for some real x_0 both series (2.39) converge. Then Lemma 2.2 implies the convergence of the series

$$\sum_{j=0}^{\infty} Q_j^*(x_0)P_j(x_0), \quad \sum_{j=0}^{\infty} P_j^*(x_0)Q_j(x_0).$$

Thus, the matricial series

$$\sum_{j=0}^{\infty} E_j(x_0) = \sum_{j=0}^{\infty} \begin{pmatrix} P_j^*(x_0)P_j(x_0) & -P_j^*(x_0)Q_j(x_0) \\ -Q_j^*(x_0)P_j(x_0) & Q_j^*(x_0)Q_j(x_0) \end{pmatrix}$$

converges, too. Consequently, Theorem 2.8 implies the existence of the limit

$$\lim_{l\to\infty} U_l(z) = \overrightarrow{\prod_{j=0}^{\infty}} \exp\Big(-i(z-x_0)E_j(x_0)\Big)$$

of the sequence of products of matrix-valued Blaschke-Potapov factors for all complex z. Thus, there exists the limit for $l \to \infty$ of the left side of the equalities (2.35) for all $z \in \mathbb{C}$. This implies the convergence of both series

$$\sum_{j=0}^{\infty} P_j^*(\bar{z})P_j(\bar{z}), \quad \sum_{j=0}^{\infty} Q_j^*(\bar{z})Q_j(\bar{z})$$

for all $z \in \mathbb{C}$.

Lemma 2.11 is proved. □

3 Proof of the Main Theorem

Now we are able to present the proof of Theorem 1.3, which can be considered as a partial generalization of [15, Theorem 3] to the matrix case. Several steps of B. Simon's proof can be more or less adapted directly to the matrix case. However, the proof of the implication "(3) ⇒ (4)" requires new ideas. Our method to overcome these difficulties was based on using methods of the J-theory due to V. P. Potapov. More precisely, Lemma 2.11 (which is a consequence of Theorem 2.9) occupies a key role in our strategy.

Proof First we show the equivalence of statements (1)–(5) in Theorem 1.3.

(1)⇒(2) Suppose that the moment problem (1.9) is completely indeterminate. In this case the deficiency numbers of the operator $\mathbf{L}$ are maximal, i.e., $m_+ = m_- = m$. Let $z_0 \in \mathbb{C} \setminus \mathbb{R}$ be fixed. In view of Lemma 2.1 the series

$\sum_{j=0}^{\infty} P_j^*(z_0)P_j(z_0)$ converges and the matricial column $\pi(z_0)$ belongs to $\ell^2(\mathbb{C}^{m\times m})$.

(2)⇒(3) Suppose that for some $z_0 \in \mathbb{C} \setminus \mathbb{R}$ the matricial column $\pi(z_0)$ belongs to $\ell^2(\mathbb{C}^{m\times m})$. Then the series $\sum_{j=0}^{\infty} P_j^*(z_0)P_j(z_0)$ converges. In view of Lemma 2.3 the series $\sum_{j=0}^{\infty} Q_j^*(z_0)Q_j(z_0)$ converges and the matricial column $\xi(z_0)$ belongs to $\ell^2(\mathbb{C}^{m\times m})$.

(3)⇒(4) Suppose that for some non-real $z_0 = x_0 + iy_0,\ y_0 \neq 0$ the matricial column $\xi(z_0)$ belongs to $\ell^2(\mathbb{C}^{m\times m})$. In view of Lemma 2.11 the two series

$$\sum_{j=0}^{\infty} P_j^*(x_0)P_j(x_0) \quad \text{and} \quad \sum_{j=0}^{\infty} Q_j^*(x_0)Q_j(x_0)$$

converge. Thus, the matricial columns $\pi(x_0)$ and $\xi(x_0)$ belong to $\ell^2(\mathbb{C}^{m\times m})$.

(4)⇒(5) Suppose that for some real x_0 the matricial columns $\pi(x_0)$ and $\xi(x_0)$ belong to $\ell^2(\mathbb{C}^{m\times m})$. Then both series

$$\sum_{j=0}^{\infty} P_j^*(x_0)P_j(x_0) \quad \text{and} \quad \sum_{j=0}^{\infty} Q_j^*(x_0)Q_j(x_0)$$

converge. In view of Lemma 2.11, for all $z \in \mathbb{C}$ both series

$$\sum_{j=0}^{\infty} P_j^*(z)P_j(z) \qquad \sum_{j=0}^{\infty} Q_j^*(z)Q_j(z)$$

converge. Hence, for all $z \in \mathbb{C}$ the matricial columns $\pi(z)$ and $\xi(z)$ belong to $\ell^2(\mathbb{C}^{m\times m})$.

(5)⇒(1) Suppose that for all $z \in \mathbb{C}$ both matricial columns $\pi(z)$ and $\xi(z)$ belong to $\ell^2(\mathbb{C}^{m\times m})$. In particular, then for all non-real z the series $\sum_{j=0}^{\infty} P_j^*(z)P_j(z)$ converge. In view of Lemma 2.1, then the deficiency numbers of the operator $\mathbf{L}$ are maximal, i.e., $m_+ = m_- = m$ and the matricial Hamburger moment problem (1.9) is completely indeterminate.

The equivalence of statements (1)–(5) is proved. We prove statements (6) and (7) analogously as the corresponding statements were proved in [15].

We prove assertion (6). We write the identity (1.19) in the form $z\pi(z) = \mathbf{J}\pi(z)$. From this it follows the equality $z\frac{d\pi}{dz}(z) + \pi(z) = \mathbf{J}\frac{d\pi}{dz}(z)$. Suppose that $x_0 \in \mathbb{R}$ and the non-zero vector $\phi \in \mathbb{C}^m$ are chosen such that both infinite vector columns from (1.26) belong to $\ell^2(\mathbb{C}^{m\times m})$. Then

$$x_0\pi(x_0)\phi = \mathbf{J}\pi(x_0)\phi, \qquad \pi(x_0)\phi = \mathbf{J}\frac{d\pi}{dx}(x_0)\phi - x_0\frac{d\pi}{dx}(x_0)\phi.$$

From this and the choice of x_0 and ϕ it follows that $\pi(x_0)\phi$ and $\frac{d\pi}{dx}(x_0)\phi$ belong to the domain of the adjoint operator $\mathbf{L}^*$. For this reason, the last relations can be written as

$$(\mathbf{L}^* - x_0\mathbf{I})\pi(x_0)\phi = \mathbf{0}, \qquad \pi(x_0)\phi = (\mathbf{L}^* - x_0\mathbf{I})\frac{d\pi}{dx}(x_0)\phi.$$

Here $\mathbf{I}$ denotes the identity operator in $\ell^2(\mathbb{C}^{m\times m})$, whereas $\mathbf{0}$ denotes the null vector in $\ell^2(\mathbb{C}^{m\times m})$.

Our proof will be done in an indirect way. For this reason, we suppose that the matricial Hamburger moment problem (1.9) is completely determinate, i.e., the operator $\mathbf{L}$ is Hermitian, which means $\mathbf{L} = \mathbf{L}^*$. We have

$$\big(\pi(x_0)\phi, \pi(x_0)\phi\big) = \sum_{j=0}^{\infty} \phi^* P_j^*(x_0) P_j(x_0)\phi \geq \phi^* P_0^*(x_0) P_0(x_0)\phi = \phi^* s_0^{-1}\phi > 0.$$

On the other side,

$$\begin{aligned}\big(\pi(x_0)\phi,\ \pi(x_0)\phi\big) &= \Big(\pi(x_0)\phi,\ (\mathbf{L}^* - x_0\mathbf{I})\frac{d\pi}{dx}(x_0)\phi\Big)\\ &= \Big((\mathbf{L}^* - x_0\mathbf{I})\pi(x_0)\phi,\ \frac{d\pi}{dx}(x_0)\phi\Big) = 0.\end{aligned}$$

This contradiction proves the statement (6) of Theorem 1.3.

Finally, we prove assertion (7). We write the relation (1.20) in the form $z\xi(z) + \delta_0 s_0^{1/2} = \mathbf{J}\xi(z)$ where $\delta_0 = \mathrm{col}\,(I, O, O, \ldots) \in \ell^2(\mathbb{C}^{m\times m})$. From this it follows the identity $z\frac{d\xi}{dz}(z) + \xi(z) = \mathbf{J}\frac{d\xi}{dz}(z)$.

Let $x_0 \in \mathbb{R}$ and the non-zero vector $\phi \in \mathbb{C}^m$ be such that both vectors $\xi(x_0)\phi$ and $\frac{d\xi}{dx}(x_0)\phi$ belong to $\ell^2(\mathbb{C}^m)$. Then

$$x_0\xi(x_0)\phi + \delta_0 s_0^{1/2}\phi = \mathbf{J}\xi(x_0)\phi, \qquad \xi(x_0)\phi = \mathbf{J}\frac{d\xi}{dx}(x_0)\phi - x_0\frac{d\xi}{dx}(x_0)\phi.$$

From this it follows that $\xi(x_0)\phi$ and $\frac{d\xi}{dx}(x_0)\phi$ belong to the domain of the adjoint operator $\mathbf{L}^*$. For this reason, the last relations can be written in operator form as

$$(\mathbf{L}^* - x_0\mathbf{I})\xi(x_0)\phi = \delta_0 s_0^{1/2}\phi, \qquad \xi(x_0)\phi = (\mathbf{L}^* - x_0\mathbf{I})\frac{d\xi}{dx}(x_0)\phi.$$

Our proof will be again be done in an indirect way. For this reason, we suppose that the matricial Hamburger moment problem (1.9) is completely determinate, i.e., the operator $\mathbf{L}$ is Hermitian, which means $\mathbf{L} = \mathbf{L}^*$. We have

$$\big(\xi(x_0)\phi, \xi(x_0)\phi\big) = \sum_{j=0}^{\infty} \phi^* Q_j^*(x_0) Q_j(x_0)\phi \geq \phi^* Q_1^*(x_0) Q_1(x_0)\phi = \phi^* s_0 \widehat{H}_1^{-1} s_0\phi > 0.$$

On the other side,

$$\big(\xi(x_0)\phi,\ \xi(x_0)\phi\big) = \Big(\xi(x_0)\phi,\ (\mathbf{L}^* - x_0\mathbf{I})\frac{d\xi}{dx}(x_0)\phi\Big) = \Big((\mathbf{L}^* - x_0\mathbf{I})\xi(x_0)\phi,\ \frac{d\xi}{dx}(x_0)\phi\Big)$$
$$= \Big(\delta_0 s_0^{1/2}\phi,\ \frac{d\xi}{dx}(x_0)\phi\Big) = \phi^* s_0^{1/2}\frac{dQ_0}{dx}(x_0)\phi = 0.$$

This contradiction proves the assertion (7). Theorem 1.3 is proved. □

Remark 3.1 Suppose that the matricial Hamburger moment problem (1.9) is completely indeterminate. In view of Theorem 1.3, for all $z \in \mathbb{C}$ both matricial columns

$$\pi(z) = \operatorname{col}(P_0(z), P_1(z), P_2(z), \ldots), \qquad \xi(z) = \operatorname{col}(Q_0(z), Q_1(z), Q_2(z), \ldots)$$

belong to $\ell^2(\mathbb{C}^{m\times m})$. In view of Lemma 2.2, for fixed $x_0 \in \mathbb{R}$ and all $z \in \mathbb{C}$ the MF

$$U(z) = I_{2m} - i(z - x_0)\sum_{j=0}^{\infty}\begin{pmatrix} P_j^*(\bar z)P_j(x_0) & -P_j^*(\bar z)Q_j(x_0) \\ -Q_j^*(\bar z)P_j(x_0) & Q_j^*(\bar z)Q_j(x_0)\end{pmatrix}\mathcal{J} \tag{3.1}$$

is correctly defined. Clearly, the MF U is entire. It is called resolvent matrix of the completely indeterminate matricial Hamburger moment problem (see [9]). In the classical scalar case of the moment problem the resolvent matrix is also called Nevanlinna matrix (see, e.g., [3]).

Remark 3.2 From (2.31), (2.34) and (3.1) it follows that the resolvent matrix of the completely indeterminate matricial Hamburger moment problem (1.9) admits the representation

$$U(z) = \prod_{j=0}^{\overrightarrow{\infty}} \exp\Big(-i(z - x_0)E_j(x_0)\Big).$$

This representation enables to estimate the exponential type of the resolvent matrix (see [9]). Namely, for each $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that for all $z \in \mathbb{C}$ the inequality

$$\|U(z)\| \le C_\varepsilon \exp(\varepsilon|z|)$$

is satisfied.

At the end of the paper it should be mentioned that B. Fritzsche, B. Kirstein and C. Mädler developed in [25] a Schur analysis approach to develop the matricial Hamburger moment problem in the most general case.

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References

1. D. Damanik, A. Pushnitski, B. Simon, *The Analytic Theory of Matrix Orthogonal Polynomials*, Surv. Approx. Theory, **4** (2008), 1–85.
2. C. Berg, *The matrix moment problem*, in Coimbra Lecture Notes on Orthogonal Polynomials, Editors: A. J. P. L. Branquinho and A. P. Foulquie Moreno, Nova Science Publishers, New York (2008), 1–57.
3. N. S. Akhiezer, *The classical moment problem and some related questions in analysis*, Translated by Kemmer, Hafner Publishing Co., New York (1965). English transl.: Translations of Mathematical Monographs, **92**, American Mathematical Society, Providence, RI (1991).
4. M. G. Krein, *The fundamental propositions of the theory of Hermitian operators with deficiency index* (m, m) (Russian), Ukrain. Mat. Žurnal, **1** (1949), no. 2, 3–66.
5. E. M. Nikishin, V. N. Sorokin, *Rational Approximations and orthogonality* (Russian), Nauka, Moscow (1988).
6. M. G. Krein, *Infinite J-matrices and the moment problem* (Russian), Dokl. AN USSR, 69:3 (1949), 125–128. English transl. in American Mathematical Society Translation, Series 2, 97 (1970), 75–143.
7. Yu. M. Berezanskii, *Expansions in Eigenfunctions of Selfadjoint Operators* (Russian), Naukova Dumka, Kiev (1965).
8. A. I. Aptekarev, E. M. Nikishin, *The scattering problem for a discrete Sturm-Liouville operator* (Russian), Mat. Sb. 121.3 (1983), 327–358. English transl. Mathematics of the USSR-Sbornik, **49** (1984), no. 2, 325–355.
9. I. V. Kovalishina, *Analytic theory of a class of interpolation problems* (Russian), Izv. Akad. Nauk SSSR Ser. Mat., **47** (1983), no. 3, 455–497. English transl. Math. USSR Izvestija, **22** (1984), 419–463.
10. Yu. M. Dyukarev, *Criterion for complete indeterminacy of limiting interpolation problem of Stieltjes type in terms of orthonormal matrix functions* (Russian), Izv. Vuzov Mat., **4** (2015), 3–16. English transl. Russian Math. (Iz. VUZ), **59** (2015), no. 4, 1–12.
11. Yu. M. Dyukarev, *Geometric and operator measures of the degeneracy of the set of the solutions of Stieltjes matrix moment problem* (Russian), Mat. Sb., **207** (2016), no. 4, 47–64. English transl. Sb. Math., **207** (2016), no. 3–4, 510–536.
12. V. I. Kogan, *On operators that are generated by I_p-matrices in the case of maximal deficiency indices* (Russian), Teor. Funktsii. Funktsional. Anal. i. Prilozhen. **207** (1970), no. 11, 103–107.
13. Yu. M. Dyukarev, *Deficiency numbers of symmetric operators generated by block Jacobi matrices* (Russian), Mat. Sb., **197** (2015), no.8, 73–100. English transl. Sb. Math., **197** (2006), no. 7–8, 1177–1203.
14. Yu. M. Dyukarev, *Examples of block Jacobi matrices that generate symmetric operators with arbitrary possible deficiency numbers* (Russian), Mat. Sb., **201** (2010), no. 12, 83–92. English transl. Sb. Math., **201** (2010), no. 11–12, 1791–1800.
15. B. Simon, *The classical moment problem as a self-adjoint finite difference operator*, Advances in Mathematics, **137** (1998), 82–203.
16. V. P. Potapov, *Multiplicative structure of J-contractive matrix functions* (Russian), Trudy, Moskov. Mat. Obšč., **4** (1955), 125–236.
17. A. V. Efimov, V. P. Potapov, *J-expanding matrix-valued functions and their role in the analytical theory of electrical circuits* (Russian), Uspeki Mat. Nauk, **28** (1973), no. 1 (169), 65–130. English transl. Russian Math. Surveys, **28** (1973), 69–140.
18. I. V. Kovalishina, V. P. Potapov, *Seven Papers Translated from the Russian*, American Mathematical Society Translations, Series 2, **138** (1988).

19. V.K Dubovoj , B. Fritzsche , B. Kirstein, *Matricial Version of the Classical Schur Problem*, B. G. Teubner, Stuttgart (1992).
20. Yu. M. Dyukarev, V. E. Katsnelson, *Multiplicative and additive Stieltjes classes of analytic matrix-valued functions and interpolation problems connected with them. I.* (Russian), Teor. Funktsii. Funktsional. Anal. i. Prilozhen. **36** (1981), 13–27.
21. Yu. M. Dyukarev, *Indeterminacy criteria for the Stieltjes matrix moment problem* (Russian), Mat. Zametki, **75** (2004), no. 1, 71–88. English transl. Math. Notes, **75** (2004), no. 1–2, 66–82.
22. Yu. M. Dyukarev, *On the indeterminacy of interpolation problems in the Stieltjes class* (Russian), Mat. Sb., **196** (2005), no. 3, 61–88. English transl. Sb. Math., **196** (2005), no. 3–4, 367–393.
23. Yu. M. Dyukarev, *A generalized Stieltjes criterion for the complete indeterminacy of interpolation problems* (Russian), Mat. Zametki, **84** (2008), no. 1, 23–39. English transl. Math. Notes, **84** (2008), no. 1–2, 22–37.
24. D.Z. Arov, H. Dym, *J-Contractive Matrix Valued Functions and Related Topics*, Encyclopedia Math. Appl, Cambridge Univ. Press, Cambridge, **116** (2008).
25. B. Fritzsche, B. Kirstein, C. Mädler, *On a simultaneous approach to the even and odd truncated matricial Hamburger moment problems*, in Recent Advances in Inverse Scattering, Schur Analysis and Stochastic Processes (Eds.: D. Alpay, B. Kirstein), Operator Theory: Advances and Application, **244**, Springer, Cham (2015), p. 181–285.
26. V. P. Potapov, *Theorem on modulus* (Russian), Teor. Funktsii. Funktsional. Anal. i. Prilozhen., Part I: **38** (1982), 91–103; Part II: **39** (1983), 95–106. English transl. in [18], Part I: 55–65; Part II: 67–77.

On a Blaschke-Type Condition for Subharmonic Functions with Two Sets of Singularities on the Boundary

S. Favorov and L. Golinskii

To Victor Katsnelson on occasion of his 75th anniversary

Abstract Given two compact sets, E and F, on the unit circle, we study the class of subharmonic functions on the unit disk which can grow at the direction of E and F (sets of singularities) at different rate. The main result concerns the Blaschke-type condition for the Riesz measure of such functions. The optimal character of this condition is demonstrated.

Keywords Subharmonic functions · Riesz measure · Harmonic majorant · The Green's function · Layer cake representation · Harmonic measure

Mathematics Subject Classification (2000) 30D50, 31A05

1 Introduction

In 1915, around a century ago, a seminal paper (6-pages note!) [2] by W. Blaschke came out. A condition widely known nowadays as the *Blaschke condition* for zeros of bounded analytic functions on the unit disk $\mathbb{D}$

$$\sum_{\lambda \in Z(f)} (1 - |\lambda|) < \infty \tag{1.1}$$

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was announced in this gem of Complex Analysis. Around 50 years ago both the authors learned about the Blaschke condition from VK, being his graduate students.

It is not our intention reviewing a vast literature with various refinements and far reaching extensions of (1.1), which appeared since then. We mention only that in all such extensions the majorants of the (unbounded) functions in question were *radial*, that is, they depended on the *absolute value* of the independent variable. In other words, the function was allowed to grow uniformly near the unit circle $\mathbb{T}$.

We came across functions with *non-radial* growth for the first time in a result of Killip and Simon [12, Theorem 2.8], where this bound looked

$$\log |L(z, J)| \le \frac{C}{|z^2-1|^2}, \qquad z \in \mathbb{D}. \tag{1.2}$$

In the spectral theory setting of this paper the function L (the perturbation determinant) turned out to belong to the Nevanlinna class, so its zeros satisfied (1.1).

The question arose naturally what one could say about the zeros of a generic function which can grow at the directions toward some selected compact sets on $\mathbb{T}$ (we refer to these sets as the *sets of singularities*). For example, in (1.2) this set is $E = \{\pm 1\}$. The study of such functions and their zero sets was initiated in [3, 4] for analytic functions, and in [6, 7] for subharmonic functions on $\mathbb{D}$. To remain closer to the main subject of our paper—functions with two sets of singularities on $\mathbb{T}$—we mention two results from the preceding papers.

Given a compact set $F \subset \mathbb{T}$, denote by $\rho_F(w)$ the Euclidian distance from a point $w \in \mathbb{C}$ to the set F. Recall the following quantitative characteristic of F known as the *Ahern–Clark type* [1]

$$\alpha(F) := \sup\{\alpha \in \mathbb{R} : \ m\big(\zeta \in \mathbb{T} : \rho_F(\zeta) < x\big) = O(x^\alpha), \quad x \to +0\}, \tag{1.3}$$

$m(A)$ is the normalized Lebesgue measure of a set A.

The first aforementioned result is a particular case of [4, Theorem 0.3].

Theorem A *Given a compact set $F \subset \mathbb{T}$, let an analytic function f on $\mathbb{D}$, $|f(0)| = 1$, satisfy the growth condition*

$$\log |f(z)| \le \frac{M}{(1-|z|)^p \, \rho_F^q(z)}, \qquad z \in \mathbb{D}, \quad M, p, q > 0.$$

Then for each $\varepsilon > 0$ there is a positive number $C = C(F, p, q, \varepsilon)$ so that the Blaschke-type condition holds for the zero set $Z(f)$ of f

$$\sum_{\lambda \in Z(f)} (1-|\lambda|)^{p+1+\varepsilon} \, \rho_F^{(q-\alpha(F)+\varepsilon)_+}(\lambda) \le CM, \qquad (x)_+ := \max(x, 0).$$

As it was pointed out in [6], the natural setting of the problem in question is the set of subharmonic functions of special growth. The analogue of the

Blaschke condition involves then the Riesz measure (generalized Laplacian) of the corresponding function.

The second result is a particular case $n = 2$ of [7, Theorem 5].

Let E and F be two arbitrary compact sets on $\mathbb{T}$. We define a class $\mathcal{S}_{p,q}(E, F)$ of subharmonic on $\mathbb{D}$ functions v, which satisfy

$$v(z) \le \frac{M}{\rho_E^p(z)\, \rho_F^q(z)}, \qquad M, p, q > 0. \tag{1.4}$$

Theorem B *Given two disjoint compact sets $E, F \subset \mathbb{T}$, let a subharmonic function $v \in \mathcal{S}_{p,q}(E, F)$. Then for each $\varepsilon > 0$ the following Blaschke-type condition holds for the Riesz measure μ of v*

$$\int_{\mathbb{D}} (1 - |\lambda|)\, \rho_E^{(p-\alpha(E)+\varepsilon)_+}(\lambda)\, \rho_F^{(q-\alpha(F)+\varepsilon)_+}(\lambda)\, \mu(d\lambda) < \infty.$$

Both the above results actually deal with two sets of singularities, and each case is extreme in a sense. Precisely, such sets are $E = \mathbb{T}$ and F in Theorem A, and the *disjoint* sets E and F in Theorem B. The goal of this paper is to study the case of two generic compact sets which come up as the sets of singularities of a subharmonic function v subject to some special growth condition.

We impose certain restrictions on E and F in the form of "integrability" of the products

$$\|\rho_E^{-a}\rho_F^{-b}\|_1 = \int_{\mathbb{T}} \frac{m(d\zeta)}{\rho_E^a(\zeta)\rho_F^b(\zeta)} < \infty, \qquad a, b \ge 0. \tag{1.5}$$

Here is our main result.

Theorem 1.1 *Given two compact sets E and F on $\mathbb{T}$ subject to (1.5), let a subharmonic function v, $v(0) \ge 0$, with the Riesz measure μ, belong to $\mathcal{S}_{p,q}(E, F)$.*

(i) *If both $0 \le a < p$ and $0 \le b < q$ hold, then for each $\varepsilon > 0$ there is a constant $C = C(p, q, a, b, \varepsilon)$ so that*

$$\int_{\mathbb{D}} \rho_E^{p-a+\varepsilon}(\lambda)\, \rho_F^{q-b+\varepsilon}(\lambda)(1 - |\lambda|)\, \mu(d\lambda) \le CM\, \|\rho_E^{-a}\, \rho_F^{-b}\|_1. \tag{1.6}$$

(ii) *If $0 \le a < p$, $b \ge q$ $\;(0 \le b < q$, $a \ge p)$, then for each $\varepsilon > 0$ there is a constant $C = C(p, q, a, \varepsilon)$ $\;(C = C(p, q, b, \varepsilon))$ so that*

$$\begin{gathered}\int_{\mathbb{D}} \rho_E^{p-a+\varepsilon}(\lambda)\, (1 - |\lambda|)\, \mu(d\lambda) \le CM\, \|\rho_E^{-a}\rho_F^{-q}\|_1, \\ \Big(\int_{\mathbb{D}} \rho_F^{q-b+\varepsilon}(\lambda)\, (1 - |\lambda|)\, \mu(d\lambda) \le CM\, \|\rho_E^{-p}\rho_F^{-b}\|_1.\Big)\end{gathered} \tag{1.7}$$

The procedure we suggest for solving the problem under consideration is pursued in three steps.

Step 1 Given a function $v \in \mathcal{S}_{p,q}(E, F)$, we find a domain $\Omega \subset \mathbb{D}$ so that v has a harmonic majorant, i.e., the harmonic function U exists with $v \le U$ on Ω. By the Green representation, see, e.g., [14, Theorem 4.5.4], which will feature prominently in what follows,

$$v(z) = u(z) - \int_\Omega G_\Omega(z, \lambda)\, \mu(d\lambda), \qquad z \in \Omega. \tag{1.8}$$

Here u is the least harmonic majorant for v, μ the Riesz measure of v, G_Ω the Green's function for Ω

$$G_\Omega(z, \lambda) := \log \frac{1}{|z - \lambda|} - h_\Omega(z, \lambda), \qquad z, \lambda \in \Omega,$$

h_Ω is the solution to the Dirichlet problem on Ω for the boundary value

$$h_\Omega(z, \xi) = \log \frac{1}{|z - \xi|}, \qquad \xi \in \partial\Omega.$$

If Ω contains the origin, and $v(0) \ge 0$, we have from (1.8) with $z = 0$

$$\int_\Omega G_\Omega(0, \lambda)\, \mu(d\lambda) \le u(0) \le U(0). \tag{1.9}$$

Step 2 We apply the lower bound for the Green's function of the type

$$G_\Omega(0, \lambda) \ge c(1 - |\lambda|), \qquad \lambda \in \Omega' \subset \Omega$$

to obtain

$$\int_{\Omega'} (1 - |\lambda|)\, \mu(d\lambda) \le U(0).$$

Step 3 To go over to the integration over the whole unit disk, we invoke a new two-dimensional version of the well-known "layer cake representation" (LCR) theorem, see Proposition 2.8.

In the simplest case when $\Omega = \mathbb{D}$ (see Theorem 3.1 below) the Green's function is

$$G_{\mathbb{D}}(z, \lambda) = \log \left| \frac{1 - \bar{\lambda} z}{z - \lambda} \right|,$$

so we come to the Blaschke condition for μ of the form

$$\int_{\mathbb{D}} (1-|\lambda|)\,\mu(d\lambda) \le \int_{\mathbb{D}} \log\frac{1}{|\lambda|}\,\mu(d\lambda) \le U(0) \tag{1.10}$$

in one step.

We proceed as follows. In Sect. 2 we gather a collection of auxiliary facts on the harmonic measure and majorants, the bounds from below for the Green's function and LCR theorems. The main result is proved in Sect. 3. We also demonstrate its optimal character in Theorem 3.7.

The case of more general conditions on a function v and its associated measure was considered in the papers [10, 11], but these conditions (as well as conclusions) do not look as clear as ours.

2 Preliminaries

2.1 Bounds for the Harmonic Measure

Let $\gamma = [e^{i\theta_1}, e^{i\theta_2}]$ be a closed arc on the unit circle $\mathbb{T}$. For the harmonic measure of this arc with respect to the unit disk $\mathbb{D}$ the explicit expression is known [9, p. 26]

$$\omega(\lambda,\gamma;\mathbb{D}) = \frac{2\alpha-(\theta_2-\theta_1)}{2\pi}, \qquad \lambda\in\mathbb{D},$$

where α is the angle subtended at λ by the arc γ.

Let $\zeta'\in\mathbb{T}$, and $0<t<1$. We put

$$\begin{aligned} \gamma = \gamma_t(\zeta') &:= \{\zeta\in\mathbb{T}:\ |\zeta-\zeta'|\le t\},\\ \Gamma = \Gamma_t(\zeta') &:= \{z\in\mathbb{D}:\ |z-\zeta'| = t\}. \end{aligned} \tag{2.1}$$

It is clear, that ω is constant on Γ. An elementary geometry provides the formula

$$\omega(\lambda,\gamma_t(\zeta');\mathbb{D}) = \frac12 - \frac1\pi \arcsin\frac t2, \qquad \lambda\in\Gamma_t(\zeta').$$

So, there is a uniform bound from below for the harmonic measure of γ on Γ

$$\omega(\lambda,\gamma_t(\zeta');\mathbb{D}) \ge \frac13, \qquad \lambda\in\Gamma_t(\zeta'). \tag{2.2}$$

To proceed further, given a compact set $K\subset\mathbb{T}$, denote by

$$\rho(w) = \rho_K(w) := \operatorname{dist}(w,K), \quad w\in\mathbb{C},$$

the Euclidian distance from w to K. Consider the sets on the unit circle

$$K_t := \{\zeta \in \mathbb{T} : \ \rho_K(\zeta) \le t\}, \qquad K_t' := \{\zeta \in \mathbb{T} : \ \rho_K(\zeta) \ge t\} = \overline{\mathbb{T}\backslash K_t}, \tag{2.3}$$

and the set in $\mathbb{D}$

$$\Gamma_t(K) := \{z \in \mathbb{D} : \ \rho_K(z) = t\}. \tag{2.4}$$

Note that K_t and K_t' are finite unions of disjoint closed arcs.

For each $\lambda \in \Gamma_t(K)$ there is $\zeta' \in K$, such that $|\lambda - \zeta'| = \rho_K(\lambda) = t$, so $\lambda \in \Gamma_t(\zeta')$. If follows from relation (2.2) that

$$\omega(\lambda, \gamma_t(\zeta'); \mathbb{D}) \ge \frac{1}{3}, \qquad \lambda \in \Gamma_t(K).$$

But, by definition, $K_t \supset \gamma_t(\zeta')$ for each $\zeta' \in K$, so monotonicity of the harmonic measure yields

$$\omega(\lambda, K_t; \mathbb{D}) \ge \omega(\lambda, \gamma_t(\zeta'); \mathbb{D}).$$

Proposition 2.1 *Given a compact set $K \subset \mathbb{T}$, let K_t be its closed neighborhood* (2.3)*. Then*

$$\omega(\lambda, K_t; \mathbb{D}) \ge \frac{1}{3}, \qquad \lambda \in \Gamma_t(K). \tag{2.5}$$

Let us now turn to the upper bounds for the harmonic measure of K_t. For a compact set K on $\mathbb{T}$ and $0 < t < 1$, the open set

$$D_t(K) := \{w \in \mathbb{D} : \ \rho_K(w) > \ t\} \tag{2.6}$$

can be disconnected even for simple K. We denote by $\Omega_t(K)$ the connected component of $D_t(K)$ that contains the origin. Clearly, $\Omega_t(K) = \emptyset$ for $t \ge 1$.

In view of connectedness, it is easy to verify that $\Omega_t(K) \supset \Omega_\tau(K)$ for $\tau > t$. It is also important that

$$\partial\Omega_t(K) \subset \partial D_t(K) = \Gamma_t(K) \cup K_t'. \tag{2.7}$$

The following result will be helpful later on.

Proposition 2.2 *Given a compact set $K \subset \mathbb{T}$, and $s > 0$, one has*

$$\{w \in \mathbb{D} : \ \rho_K(w) > s\} \subset \Omega_{s/2}(K). \tag{2.8}$$

Proof Clearly,

$$\{w \in \mathbb{D} : \rho_K(w) > s\} \subset \left\{w \in \mathbb{D} : \rho_K(w) > \frac{s}{2}\right\},$$

and we wish to show that the set on the left side is actually a subset of the connected component of the set on the right side that contains the origin. The argument relies on a simple inequality, which we apply repeatedly throughout the paper

$$\rho_K(z) \le 2\rho_K(rz), \qquad z \in \overline{\mathbb{D}}, \quad 0 \le r \le 1. \tag{2.9}$$

Indeed, by the triangle inequality $\rho_K(z) \le \rho_K(rz) + |rz - z|$, and so

$$\rho_K(z) \le \rho_K(rz) + (1 - |rz|) \le 2\rho_K(rz),$$

as claimed.

It follows from (2.9) that $\rho_K(rz) > s/2$ for all $0 \le r \le 1$ as soon as $\rho_K(z) > s$. In other words, the whole closed interval

$$[0, z] \subset \{w \in \mathbb{D} : \rho_K(w) > s/2\},$$

and so $z \in \Omega_{s/2}(K)$, as needed. □

Proposition 2.3 *Given a number $l \in (0, 1)$, put $k := 2\pi l^{-1} + 1$. Then the following inequality holds for $t < k^{-1}$*

$$\omega(\lambda, K_t; \mathbb{D}) \le \frac{l}{t}(1 - |\lambda|), \qquad \lambda \in \Omega_{kt}(K). \tag{2.10}$$

Proof If $1 - |\lambda| > tl^{-1}$, inequality (2.10) obviously holds. So we assume in what follows that

$$1 - |\lambda| \le \frac{t}{l}, \qquad |\lambda| \ge 1 - \frac{t}{l} > 1 - \frac{1}{kl} = \frac{kl - 1}{kl}. \tag{2.11}$$

For $\lambda = |\lambda|e^{\imath\theta} \in \Omega_{kt}(K)$, and $\zeta = e^{i\varphi} \in K_t$, the Poisson integral representation for the harmonic measure reads

$$\omega(\lambda, K_t; \mathbb{D}) = \int_{K_t} \frac{1 - |\lambda|^2}{|\zeta - \lambda|^2} m(d\zeta) = \frac{(1 - |\lambda|^2)}{2\pi} \int_{e^{i\varphi} \in K_t} \frac{d\varphi}{(1 - |\lambda|)^2 + 4|\lambda| \sin^2 \frac{\varphi - \theta}{2}}.$$

Take $\zeta_1 \in K$ such that $\rho_K(e^{i\theta}) = |e^{i\theta} - \zeta_1|$. Then, in view of (2.10),

$$\begin{aligned} \rho_K(e^{i\theta}) &= |e^{i\theta} - \zeta_1| = |\lambda - \zeta_1 + e^{i\theta} - \lambda| \\ &\ge \rho_K(\lambda) - (1 - |\lambda|) > kt - (1 - |\lambda|). \end{aligned} \tag{2.12}$$

Take $\zeta_2 \in K$ such that $\rho_K(e^{i\varphi}) = |e^{i\varphi} - \zeta_2| \le t$, so, by (2.12),

$$|\theta - \varphi| \ge \left|e^{i\theta} - e^{i\varphi}\right| = \left|e^{i\theta} - \zeta_2 + \zeta_2 - e^{i\varphi}\right| \\ \ge \rho_K(e^{i\theta}) - t > (k-1)t - (1 - |\lambda|).$$

Hence (2.11) implies

$$\pi \ge |\theta - \varphi| \ge (k-1)t - \frac{t}{l} = k_1 t, \quad k_1 := k - 1 - \frac{1}{l} = \frac{2\pi - 1}{l}.$$

Going back to the Poisson integral, we see that

$$\omega(\lambda, K_t; \mathbb{D}) \le \frac{1 - |\lambda|}{4\pi|\lambda|} \int\limits_{k_1 t \le |\varphi - \theta| \le \pi} \frac{d\varphi}{\sin^2 \frac{\varphi - \theta}{2}} \le \frac{\pi(1 - |\lambda|)}{4|\lambda|} \int\limits_{k_1 t \le |\varphi - \theta| \le \pi} \frac{d\varphi}{(\varphi - \theta)^2},$$

or, in view of (2.11),

$$\omega(\lambda, K_t; \mathbb{D}) \le \frac{\pi(1 - |\lambda|)}{2} \frac{kl}{kl - 1} \int_{k_1 t}^{\pi} \frac{dx}{x^2} \le \frac{\pi(1 - |\lambda|)}{2t} \frac{kl}{kl - 1} \frac{l}{2\pi - 1}.$$

An elementary calculation shows that for $l \in (0, 1)$

$$\frac{\pi}{2(2\pi - 1)} \frac{kl}{kl - 1} = \frac{\pi}{2(2\pi - 1)} \frac{l + 2\pi}{l + 2\pi - 1} < 1,$$

and (2.10) follows. □

2.2 Lower Bounds for Green's Functions

Under a *Green's function* of the domain $\Omega_t(K)$ with singularity z we mean a nonnegative function of the form

$$G_t(z, \lambda) = G_{\Omega_t(K)}(z, \lambda) := \log \frac{1}{|z - \lambda|} - h_t(z, \lambda), \qquad z, \lambda \in \Omega_t(K), \tag{2.13}$$

where h_t is the solution to the Dirichlet problem on $\Omega_t(K)$ for the boundary value

$$h_t(z, \xi) = \log \frac{1}{|z - \xi|}, \qquad \xi \in \partial\Omega_t(K). \tag{2.14}$$

Such function exists and is unique, as the boundary $\partial\Omega_t(K)$ is a non-polar set, see, e.g., [14]. The problem we address here is to obtain a lower bound for $G_t(0, \cdot)$ in a smaller domain $\Omega_\tau(K)$ with an appropriate $\tau > t$.

Proposition 2.4 *The Green's function $G_t(0, \cdot)$ for the domain $\Omega_t(K)$ with singularity at the origin and $0 < t < (24\pi + 1)^{-1}$ admits the lower bound*

$$G_t(0, \lambda) \geq \frac{1 - |\lambda|}{2}, \qquad \lambda \in \Omega_{(24\pi+1)t}(K). \tag{2.15}$$

Proof Since

$$1 - |\xi| \leq \rho_K(\xi) = t, \quad |\xi| \geq 1 - t > \frac{1}{2}, \qquad \xi \in \Gamma_t(K),$$

one has

$$h_t(0, \xi) = \log \frac{1}{|\xi|} \leq \log \frac{1}{1 - t} \leq 2t, \quad \xi \in \partial\Omega_t(K) \cap \Gamma_t(K).$$

Next, $h_t(0, \xi) = 0$ for $\xi \in \partial\Omega_t(K) \cap K'_t$, so, by Proposition 2.1 and the Maximum Principle,

$$h_t(0, \lambda) \leq 6t\omega(\lambda, K_t; \mathbb{D}), \qquad \lambda \in \Omega_t(K).$$

Now, the upper bound (2.10) with $l = 1/12$ and $k = 24\pi + 1$ yields

$$h_t(0, \lambda) \leq \frac{1 - |\lambda|}{2}, \qquad \lambda \in \Omega_{(24\pi+1)t}(K),$$

and so

$$G_t(0, \lambda) \geq \log \frac{1}{|\lambda|} - \frac{1 - |\lambda|}{2} \geq \frac{1 - |\lambda|}{2}, \qquad \lambda \in \Omega_{(24\pi+1)t}(K),$$

as needed. □

So far we have been dealing with one compact set K. Keeping in mind the main topic of the paper, consider the intersection

$$D_{t,s}(E, F) := \{w \in \mathbb{D} : \rho_E(w) > t, \rho_F(w) > s\} = D_t(E) \cap D_s(F),$$

where E and F are compact sets on the unit circle, $0 < t, s < 1$. Denote by $\Omega_{t,s}$ the connected component of this open set (or, that is the same, the connected component of $\Omega_t(E) \cap \Omega_s(F)$) so that $0 \in \Omega_{t,s}$. Clearly, $\Omega_{t,s} = \emptyset$ for $\max\{t, s\} \geq 1$. It is not hard to check that

$$\begin{aligned} &\partial\Omega_{t,s} \subset W_1 \cup W_2 \cup W_3, \qquad W_3 := E'_t \cap F'_s, \\ &W_1 := \{w \in \mathbb{D} : \rho_E(w) = t, \ \rho_F(w) \geq s\}, \\ &W_2 := \{w \in \mathbb{D} : \rho_E(w) \geq t, \ \rho_F(w) = s\}. \end{aligned} \tag{2.16}$$

In particular,

$$\partial\Omega_{t,s} \subset \Gamma_t(E) \cup \Gamma_s(F) \cup (E'_t \cap F'_s). \tag{2.17}$$

The inclusion

$$D_{t,s}(E, F) \subset \Omega_{t/2,s/2} \tag{2.18}$$

can be verified in exactly the same way as (2.8) in Proposition 2.2.

We complete with the lower bound for the Green's function $G_{t,s} := G_{\Omega_{t,s}}$.

Proposition 2.5 *The Green's function $G_{t,s}(0, \cdot)$ for the domain $\Omega_{t,s}$ with singularity at the origin and $0 < t, s < (48\pi + 1)^{-1}$ admits the lower bound*

$$G_{t,s}(0, \lambda) \ge \frac{1 - |\lambda|}{2}, \qquad \lambda \in \Omega_{(48\pi+1)t,(48\pi+1)s}. \tag{2.19}$$

Proof We follow the argument from the proof of Proposition 2.4. Write

$$G_{t,s}(0, \lambda) = \log\frac{1}{|\lambda|} - h(\lambda), \quad h(\zeta) = \log\frac{1}{|\zeta|}, \quad \zeta \in \partial\Omega_{t,s},$$

so $h(\zeta) = 0$ for $\zeta \in \partial\Omega_{t,s} \cap \mathbb{T}$. Since

$$|\zeta| \ge 1 - t > 1/2, \quad \zeta \in \Gamma_t(E), \qquad |\zeta| \ge 1 - s > 1/2, \quad \zeta \in \Gamma_s(F),$$

we have

$$\begin{aligned} \log\frac{1}{|\zeta|} &\le 2(1 - |\zeta|) \le 2t, \qquad \zeta \in \Gamma_t(E), \\ \log\frac{1}{|\zeta|} &\le 2(1 - |\zeta|) \le 2s, \qquad \zeta \in \Gamma_s(F). \end{aligned} \tag{2.20}$$

In view of (2.17), (2.20) and Proposition 2.1, it follows from the Maximum Principle that

$$h(\lambda) \le 6t\omega(\lambda, E_t; \mathbb{D}) + 6s\omega(\lambda, F_s; \mathbb{D}), \qquad \lambda \in \Omega_{t,s}.$$

We apply the upper bound for the harmonic measure (2.10)

$$t\omega(\lambda, E_t; \mathbb{D}) + s\omega(\lambda, F_s; \mathbb{D}) \le 2l(1 - |\lambda|), \qquad \lambda \in \Omega_{t,s},$$

so for $l = 1/24$, $k = 48\pi + 1$ we come to

$$h(\lambda) \le \frac{1 - |\lambda|}{2} \le \frac{1}{2}\log\frac{1}{|\lambda|} \Rightarrow G_{t,s}(0, \lambda) \ge \frac{1}{2}\log\frac{1}{|\lambda|} \ge \frac{1 - |\lambda|}{2},$$

as claimed. □

2.3 Harmonic Majorant

The result below concerns particular subharmonic functions and their harmonic majorants.

Proposition 2.6 *Given two compact sets E and F on the unit circle, and $a, b \geq 0$, assume that $\rho_E^{-a}\rho_F^{-b} \in L^1(\mathbb{T})$. Then the function*

$$v_{a,b}(z) := \frac{1}{\rho_E^a(z)\,\rho_F^b(z)}, \qquad z \in \mathbb{D}, \tag{2.21}$$

is subharmonic and admits the harmonic majorant

$$v_{a,b}(z) \leq P_{a,b}(z) := \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2}\,\frac{m(d\zeta)}{\rho_E^a(\zeta)\,\rho_F^b(\zeta)}\,. \tag{2.22}$$

Proof The case $a = b = 0$ is trivial, so let $a + b > 0$. By [14, Theorem 2.4.7], the function

$$v_{a,b}(z) = \sup_{\xi\in E,\,\eta\in F} |(z-\xi)^{-a}(z-\eta)^{-b}|$$

is subharmonic. The inequality (2.9) implies

$$v_{a,b}(r\zeta) \leq 2^{a+b} v_{a,b}(\zeta), \qquad \zeta \in \mathbb{T}, \quad 0 \leq r \leq 1. \tag{2.23}$$

The standard Maximum Principle states that

$$v_{a,b}(rz) \leq \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2}\, v_{a,b}(r\zeta)\, m(d\zeta), \qquad z \in \mathbb{D}, \quad r < 1.$$

The bound (2.22) is now immediate from the latter inequality as $r \to 1-0$ due to (2.23) and the Lebesgue Dominated convergence theorem. □

Remark 2.7 As a matter of fact, $P_{a,b}$ is the *least* harmonic majorant for $v_{a,b}$, see, e.g., [8, pp. 36–37].

2.4 Layer Cake Representation

A key ingredient in our argument is the fundamental result in Analysis, known as the “layer cake representation” (LCR) see, e.g., [13, Theorem 1.13].

Theorem LCR Let (Λ, ν) be a measure space, and $h \geq 0$ a measurable function on Λ. Then for $c > 0$ the equality holds

$$\int_\Lambda h^c(\tau)\,\nu(d\tau) = c\int_0^\infty x^{c-1}\nu(\{\tau : h(\tau) > x\})\,dx. \tag{2.24}$$

In what follows we make use of the two-dimensional analogue of this result.

Proposition 2.8 *Let $f, g \geq 0$ be measurable functions on the measure space (Λ, σ), and $\alpha, \beta > 0$. Then*

$$\begin{aligned} I &:= \int_\Lambda f^\alpha(\tau) g^\beta(\tau)\,\sigma(d\tau) \\ &= \alpha\beta \int_0^\infty\int_0^\infty x^{\alpha-1}y^{\beta-1}\sigma(\{\tau : f(\tau) > x,\ g(\tau) > y\})\,dx\,dy. \end{aligned} \tag{2.25}$$

Proof We apply the LCR (2.24) twice. Put $\nu(d\tau) := g^\beta\sigma(d\tau)$, so

$$I = \int_\Lambda f^\alpha(\tau)\,\nu(d\tau) = \alpha\int_0^\infty x^{\alpha-1}\nu(\{\tau : f(\tau) > x\})\,dx.$$

Write $\Lambda_x := \{\tau : f(\tau) > x\}$, and apply (2.24) once again

$$\begin{aligned} \nu(\Lambda_x) &= \int_{\Lambda_x} g^\beta(\tau)\,\sigma(d\tau) = \beta\int_0^\infty y^{\beta-1}\nu(\{\tau \in \Lambda_x : g(\tau) > y\})\,dy \\ &= \beta\int_0^\infty y^{\beta-1}\nu(\{\tau \in \Lambda : f(\tau) > x,\ g(\tau) > y\})\,dy, \end{aligned}$$

so Fubini's theorem completes the proof. □

3 Problem with Two Compact Sets

3.1 Main Results

Let us go back to our main problem concerning the Blaschke-type condition for the Riesz measure of the subharmonic function which can grow at the direction of two sets of singularities on the unit circle.

As a warm-up, we prove the following result.

Theorem 3.1 *Assume that E and F are two compact sets on $\mathbb{T}$ so that (1.5) holds with $a = p$, $b = q$. For each subharmonic function $v \in \mathcal{S}_{p,q}(E, F)$, $v(0) \geq 0$, with* *the Riesz measure μ, the Blaschke condition holds*

$$\int_{\mathbb{D}} (1-|\lambda|)\,\mu(d\lambda) \le M\|\rho_E^{-p}\rho_F^{-q}\|_1. \tag{3.1}$$

Proof By Proposition 2.6, v admits the harmonic majorant $U = MP_{p,q}$ with $U(0) = M\|\rho_E^{-p}\rho_F^{-q}\|_1$. Relation (1.10) completes the proof. □

The case when $\min(p,q) = 0$, so we actually have one compact set, was elaborated in [6].

The main result of the paper, Theorem 1.1, concerns the rest of the values for a and b, that is, either $0 \le a < p$ or $0 \le b < q$.

Proof of Theorem 1.1

(i) We proceed in three steps, following the procedure outlined in Introduction.

Step 1. Write the hypothesis (1.4) as

$$v(z) \le v_{a,b}(z)\,\frac{M}{\rho_E^{p-a}(z)\rho_F^{q-b}(z)}, \qquad z \in \mathbb{D}.$$

In view of (2.16), Proposition 2.6, and the Maximum Principle, we come to the bound

$$v(z) \le U(z) = P_{a,b}(z)\,\frac{M}{t^{p-a}\,s^{q-b}}, \qquad z \in \Omega_{t,s}.$$

Step 2. Relation (1.9) now reads

$$\int_{\Omega_{t,s}} G_{t,s}(0,\lambda)\,\mu(d\lambda) \le u(0) \le U(0) = \frac{M}{t^{p-a}\,s^{q-b}}\left\|\rho_E^{-a}\,\rho_F^{-b}\right\|_1.$$

By Proposition 2.5 with $\kappa = 48\pi + 1$, one has

$$\int_{\Omega_{\kappa t,\kappa s}} (1-|\lambda|)\,\mu(d\lambda) \le \frac{2M}{t^{p-a}\,s^{q-b}}\left\|\rho_E^{-a}\,\rho_F^{-b}\right\|_1.$$

By (2.18), $D_{2\kappa t,2\kappa s}(E,F) \subset \Omega_{\kappa t,\kappa s}$, so putting

$$\xi := 2\kappa t, \quad \eta := 2\kappa s, \qquad 0 \le t, s \le \frac{1}{2\kappa},$$

we end up with the bound

$$\int_{D_{\xi,\eta}(E,F)} (1-|\lambda|)\,\mu(d\lambda) \le 2(2\kappa)^{p+q-a-b}\,\frac{M}{\xi^{p-a}\,\eta^{q-b}}\,\big\|\rho_E^{-a}\,\rho_F^{-b}\big\|_1. \tag{3.2}$$

Step 3. The LCR theorem comes into play here. By Proposition 2.8 with

$$\Lambda=\mathbb{D},\ \sigma=(1-|\lambda|)\mu,\ f=\rho_E,\ g=\rho_F,\ \alpha=p-a+\varepsilon,\ \beta=q-b+\varepsilon,$$

we see that

$$\int_{\mathbb{D}} \rho_E^{\alpha}(\lambda)\rho_F^{\beta}(\lambda)\sigma(d\lambda)=\alpha\beta\int_0^2\int_0^2 \xi^{\alpha-1}\eta^{\beta-1}\sigma\Big(\{\lambda:\rho_E(\lambda)>\xi,\,\rho_F(\lambda)>\eta\}\Big)d\xi\,d\eta.$$

But, due to (3.2),

$$\begin{aligned}\sigma\Big(\{\lambda:\rho_E(\lambda)>\xi,\,\rho_F(\lambda)>\eta\}\Big)&=\int_{D_{\xi,\eta}(E,F)}(1-|\lambda|)\,\mu(d\lambda)\\ &\le\frac{CM}{\xi^{p-a}\,\eta^{q-b}}\,\big\|\rho_E^{-a}\,\rho_F^{-b}\big\|_1,\end{aligned}$$

so, finally,

$$\int_{\mathbb{D}} \rho_E^{\alpha}(\lambda)\rho_F^{\beta}(\lambda)(1-|\lambda|)\,\mu(d\lambda)\le\alpha\beta\,CM\,\big\|\rho_E^{-a}\,\rho_F^{-b}\big\|_1\int_0^2\xi^{\varepsilon-1}d\xi\int_0^2\eta^{\varepsilon-1}d\eta,$$

and the first statement is proved.

(ii) Assume now that $0\le a<p$ and $b\ge q$. The argument is the same but simpler, as we appeal to the domain $\Omega_t(E)$ and the standard one-dimensional LCR theorem (2.24). Indeed, as in Step 1, we have

$$v(z)\le U(z)=P_{a,b}(z)\,\frac{2^{b-q}M}{t^{p-a}}\,,\qquad z\in\Omega_t(E).$$

Next, relation (1.9) provides

$$\int_{\Omega_t(E)} G_t(0,\lambda)\,\mu(d\lambda)\le\frac{2^{b-q}M}{t^{p-a}}\,\big\|\rho_E^{-a}\,\rho_F^{-b}\big\|_1,$$

so, by Proposition 2.4 with $\kappa = 24\pi + 1$,

$$\int\limits_{\Omega_{\kappa t}(E)} (1-|\lambda|)\,\mu(d\lambda) \le \frac{2^{b-q+1}M}{t^{p-a}}\,\big\|\rho_E^{-a}\,\rho_F^{-b}\big\|_1.$$

By (2.8), $D_{2\kappa t}(E) \subset \Omega_{\kappa t}(E)$, and so for $\xi = 2\kappa t$ we have

$$\int\limits_{D_\xi(E)} (1-|\lambda|)\,\mu(d\lambda) \le (2\kappa)^{p-a}\,\frac{2^{b-q+1}M}{\xi^{p-a}}\,\big\|\rho_E^{-a}\,\rho_F^{-b}\big\|_1.$$

An application of LCR theorem in the form (2.24) with

$$\Lambda = \mathbb{D}, \quad \nu(d\lambda) = (1-|\lambda|)\,\mu(d\lambda), \quad h = \rho_E, \quad c = p-a+\varepsilon$$

leads to the first Blaschke-type condition in (1.7). The proof of the second one is identical. □

The case $a = b = 0$ is important, for there are no integrability assumptions whatsoever.

Corollary 3.2 *Given two compact sets E, F on $\mathbb{T}$, let a subharmonic function v, $v(0) \ge 0$, belong to $\mathcal{S}_{p,q}(E,F)$. Then for each $\varepsilon > 0$ there is a constant $C = C(p,q,\varepsilon)$ so that*

$$\int_{\mathbb{D}} \rho_E^{p+\varepsilon}(\lambda)\,\rho_F^{q+\varepsilon}(\lambda)(1-|\lambda|)\,\mu(d\lambda) \le CM. \tag{3.3}$$

The results of Theorem 1.1 can be extended to the case of n compact sets on the unit circle with no additional efforts.

Theorem 3.3 *Let $K_1,\dots,K_n$ be compact subsets of $\mathbb{T}$, and let v be a subharmonic function on $\mathbb{D}$ with Riesz measure μ such that $v(0) \ge 0$ and*

$$v(z) \le M\rho_{K_1}^{-p_1}(z)\cdots\rho_{K_n}^{-p_n}(z), \qquad z \in \mathbb{D}.$$

Suppose that

$$\rho_{K_1}^{-a_1}(\zeta)\cdots\rho_{K_n}^{-a_n}(\zeta) \in L^1(\mathbb{T})$$

for some

$$a_1 < p_1,\dots,a_k < p_k,\ a_{k+1} \ge p_{k+1},\dots,a_n \ge p_n, \qquad 1 \le k \le n.$$

Then for each $\varepsilon > 0$ *there is a constant* $C = C(p_1, \dots, p_n, a_1, \dots, a_k, \varepsilon)$ *so that*

$$\int_{\mathbb{D}} \rho_{K_1}^{p_1-a_1+\varepsilon}(\lambda) \cdots \rho_{K_k}^{p_k-a_k+\varepsilon}(\lambda)(1-|\lambda|)\, d\mu(\lambda) \le CM \Big\| \prod_{j=1}^{k} \rho_{K_j}^{-a_j} \prod_{i=k+1}^{n} \rho_{K_i}^{-p_i} \Big\|_1.$$

In view of further applications, let us mention a special case of subharmonic functions $v = \log|f|$ with f analytic on the unit disk.

Corollary 3.4 *Let an analytic function* f, $|f(0)| \ge 1$, *satisfy the growth condition*

$$\log|f(z)| \le \frac{M}{\rho_E^p(z)\, \rho_F^q(z)}, \qquad M, p, q > 0, \tag{3.4}$$

with two compact sets E, F *on the unit circle. Assume that the relation* (1.5) *holds for some* $0 \le a < p$ *and* $0 \le b < q$. *Then for each* $\varepsilon > 0$ *there is a constant* $C = C(p, q, a, b, \varepsilon)$ *so that*

$$\sum_{n=1}^{\infty} (1-|\lambda_n|)\, \rho_E^{p-a+\varepsilon}(\lambda_n)\, \rho_F^{q-b+\varepsilon}(\lambda_n) \le CM \left\| \rho_E^{-a} \rho_F^{-b} \right\|_1,$$

where $\{\lambda_n\}_{n\ge1}$ *are the zeros of* f *counting multiplicity.*

Next, we consider the situation where the integrability assumptions are imposed on ρ_E and ρ_F separately. At the moment the following partial result is available.

Proposition 3.5 *Let a subharmonic function* v, $v(0) \ge 0$, *belong to* $\mathcal{S}_{p,q}(E, F)$. *Assume that*

$$\left\| \rho_E^{-p} \right\|_1 = \int_{\mathbb{T}} \frac{m(d\zeta)}{\rho_E^p(\zeta)} < \infty, \quad \left\| \rho_F^{-q} \right\|_1 = \int_{\mathbb{T}} \frac{m(d\zeta)}{\rho_F^q(\zeta)} < \infty. \tag{3.5}$$

Let p', q' *be nonnegative constants such that* $p' + q' > \max(p, q)$. *Then there is a constant* $C = C(p, q, p', q')$ *so that*

$$\int_{\mathbb{D}} \rho_E^{p'}(\lambda)\, \rho_F^{q'}(\lambda)(1-|\lambda|)\, \mu(d\lambda) \le CM \left(\left\| \rho_E^{-p} \right\|_1 + \left\| \rho_F^{-q} \right\|_1 \right). \tag{3.6}$$

Proof We focus on two particular cases of Theorem 1.1, namely, $a = 0, b = q$ and $a = p, b = 0$. The corresponding conditions (1.5) agree with (3.5). It follows from (1.7) that

$$\int_{\mathbb{D}} \left(\rho_E^{p+\varepsilon}(\lambda) + \rho_F^{q+\varepsilon}(\lambda) \right) (1-|\lambda|)\, \mu(d\lambda) \le CM \left(\left\| \rho_E^{-p} \right\|_1 + \left\| \rho_F^{-q} \right\|_1 \right) \tag{3.7}$$

for arbitrary $\varepsilon > 0$. We choose this parameter from the condition

$$0 < \varepsilon < \frac{p' + q' - \max(p, q)}{2}. \tag{3.8}$$

The argument below is quite elementary. Let $0 \le x, y \le 2$. If $y \le x$, we have, by (3.8),

$$x^{p'} y^{q'} = x^{p'+q'} \le 2^{p'+q'-p-\varepsilon} x^{p+\varepsilon}.$$

Similarly, for $x \le y$

$$x^{p'} y^{q'} = y^{p'+q'} \le 2^{p'+q'-q-\varepsilon} y^{q+\varepsilon}.$$

So, for each $0 \le x, y \le 2$ we have

$$x^{p'} y^{q'} \le C\left(x^{p+\varepsilon} + y^{q+\varepsilon}\right), \qquad C = 2^{p'+q'-\min(p,q)-2\varepsilon}.$$

It remains only to put $x := \rho_E(\lambda)$, $y := \rho_F(\lambda)$ and make use of (3.7). The proof is complete. □

Remark 3.6 In some instances the assumption $v(0) \ge 0$ looks somewhat restrictive. If $-\infty < v(0) < 0$, one can apply the above results to the function $v_1(z) = v(z) - v(0)$, which belongs to the same class $\mathcal{S}_{p,q}(E, F)$. But now the constant M depends on v, so we actually have quantitative Blaschke-type conditions. For example,

$$\int_{\mathbb{D}} \rho_E^{p-a+\varepsilon}(\lambda)\, \rho_F^{q-b+\varepsilon}(\lambda)(1 - |\lambda|)\, \mu(d\lambda) < \infty \tag{3.9}$$

holds in place of (1.6).

If $v(0) = -\infty$, consider the Poisson integral in the disk $|z| < 1/2$ with the boundary value v

$$h(z) := \int_{\mathbb{T}} \frac{1 - |2z|^2}{|\zeta - 2z|^2}\, v(\zeta/2) m(d\zeta).$$

Since v is upper semicontinuous, we see that $\lim_{z' \to z} h(z') \le v(z)$ for each z with $|z| = 1/2$. By [14, Theorem 2.4.5] the function

$$v_1(z) = \begin{cases} \max(v(z), h(z)) & \text{for } |z| < 1/2, \\ v(z) & \text{for } |z| \ge 1/2 \end{cases}$$

is subharmonic in $\mathbb{D}$, and the restriction of its Riesz measure μ_1 on the set $\{z \in \mathbb{D} : |z| > 1/2\}$ agrees with μ. Therefore,

$$\int_{\mathbb{D}} \rho_E^{p-a}(\lambda)\rho_F^{q-b}(\lambda)(1-|\lambda|)\Big(\mu_1(d\lambda)-\mu(d\lambda)\Big) = O(1).$$

Since $v_1(0) > -\infty$, we again get the conclusions of the quantitative type similar to (3.9).

3.2 Optimality of the Bounds

We complete the paper with the results which demonstrate the optimal character of the bound (1.6) in Theorem 1.1 and the disjointness of the compact sets in Theorem B.

Given a compact set $K \subset \mathbb{T}$, define the value

$$\delta(K) := \sup\{\tau \ge 0 : \ \rho_K^{-\tau} \in L^1(\mathbb{T})\}.$$

It is clear that $0 \le \delta(K) \le 1$. The equality

$$\int_{\mathbb{T}} \frac{m(d\zeta)}{\rho_K^{\tau}(\zeta)} = 2^{-\tau} + \tau I(\tau, K), \quad \tau > 0, \qquad I(\tau, K) := \int_0^2 \frac{m(K_t)}{t^{\tau+1}}\,dt \tag{3.10}$$

follows easily from the LCR theorem (2.24), see [6, formula (15)]. The characteristic $I(\tau, K)$ appeared already in [5]. So,

$$\delta(K) = \sup\{\tau \ge 0 : \ I(\tau, K) < \infty\}.$$

By [7, Proposition 1], the equality $\delta(K) = \alpha(K)$ holds, $\alpha(K)$ is the Ahern–Clark type (1.3).

Choose two disjoint compact sets E and F with $\delta(E) > 0$, $\delta(F) > 0$. By the definition,

$$\rho_E^{-\delta(E)+\varepsilon} \in L^1(\mathbb{T}), \quad \rho_F^{-\delta(F)+\varepsilon} \in L^1(\mathbb{T}), \quad 0 < \varepsilon < \min(\delta(E), \delta(F)),$$

and so (1.5) holds with $a = \delta(E) - \varepsilon$, $b = \delta(F) - \varepsilon$ (E and F are disjoint). On the other hand,

$$\rho_E^{-\delta(E)-\varepsilon} \notin L^1(\mathbb{T}), \quad \rho_F^{-\delta(F)-\varepsilon} \notin L^1(\mathbb{T}),$$

and, by (3.10),

$$I(\delta(E)+\varepsilon, E) = I(\delta(F)+\varepsilon, F) = +\infty.$$

In notation (2.6) we take t, s small enough so that

$$D_t^c(E) \cap D_s^c(F) = \emptyset, \quad D_t^c(K) := \mathbb{D} \backslash D_t(K). \tag{3.11}$$

Let $p > \delta(E)$, $q > \delta(F)$, and consider the function

$$v_0(z) = v_E(z) + v_F(z) = \frac{1}{\rho_E^p(z)} + \frac{1}{\rho_F^q(z)}, \quad v_0 \in \mathcal{S}_{p,q}(E, F).$$

Denote by μ_E (μ_F) the Riesz measure of the subharmonic function v_E (v_F). The result in Theorem 1.1, (i), states that

$$\int_{\mathbb{D}} \rho_E^{p-\delta(E)+2\varepsilon}(\lambda)\, \rho_F^{q-\delta(F)+2\varepsilon}(\lambda)\, (1 - |\lambda|)\, \mu_0(d\lambda) < \infty, \quad \mu_0 := \mu_E + \mu_F$$

is the Riesz measure of v_0 and $\varepsilon > 0$ is small enough.

Theorem 3.7 *For* $0 < \varepsilon < \min(p - \delta(E), q - \delta(F))$ *the relation holds*

$$I := \int_{\mathbb{D}} \rho_E^{p-\delta(E)-\varepsilon}(\lambda)\, \rho_F^{q-\delta(F)-\varepsilon}(\lambda)\, (1 - |\lambda|)\, \mu_0(d\lambda) = +\infty. \tag{3.12}$$

Proof We bound the integral I from below in a few steps. Clearly,

$$I \ge \int\limits_{D_t^c(E)} \rho_E^{p-\delta(E)-\varepsilon}(\lambda)\, \rho_F^{q-\delta(F)-\varepsilon}(\lambda)\, (1 - |\lambda|)\, \mu_0(d\lambda)$$

$$\ge \int\limits_{D_t^c(E)} \rho_E^{p-\delta(E)-\varepsilon}(\lambda)\, \rho_F^{q-\delta(F)-\varepsilon}(\lambda)\, (1 - |\lambda|)\, \mu_E(d\lambda) = I_1.$$

By (3.11), one has $\rho_F(\lambda) > s$ as long as $\lambda \in D_t^c(E)$, so

$$I_1 \ge s^{q-\delta(F)-\varepsilon} \int\limits_{D_t^c(E)} \rho_E^{p-\delta(E)-\varepsilon}(\lambda)\, (1 - |\lambda|)\, \mu_E(d\lambda). \tag{3.13}$$

We apply [6, Theorem 2], which claims that now

$$\int_{\mathbb{D}} \rho_E^{p-\delta(E)-\varepsilon}(\lambda)\, (1 - |\lambda|)\, \mu_E(d\lambda) = \int\limits_{D_t(E)} \rho_E^{p-\delta(E)-\varepsilon}(\lambda)\, (1 - |\lambda|)\, \mu_E(d\lambda)$$

$$+ \int\limits_{D_t^c(E)} \rho_E^{p-\delta(E)-\varepsilon}(\lambda)\, (1 - |\lambda|)\, \mu_E(d\lambda) = I_2 + I_2^c = +\infty.$$

But $I_2 < \infty$ thanks to the property of the Riesz measure, so $I_2^c = +\infty$. The relation (3.12) follows now from (3.13). ◻

Next, take compact sets $E = F$ such that $\delta(E) = \delta(F) > 0$ and

$$\varepsilon = (1/2)\delta(F), \qquad p > \delta(E), \qquad q > \delta(F).$$

Using (3.10) and the equality $\alpha(E) = \delta(E)$, we get $I(p + q - \varepsilon, E) = \infty$. Hence, by [6, Theorem 2] applied to the set E and the function

$$\tilde{v}(z) := \rho_E^{-(p+q)}(z) = \rho_E^{-p}(z)\rho_F^{-q}(z),$$

the Riesz measure $\tilde{\mu}$ of $\tilde{v}$ satisfies the condition

$$\int_{\mathbb{D}} \rho_E^{p-\alpha(E)+\varepsilon}(\lambda)\, \rho_F^{q-\alpha(F)+\varepsilon}(\lambda)(1-|\lambda|)\, \tilde{\mu}(d\lambda) = \int_{\mathbb{D}} \rho_E^{p+q-\alpha(E)}(\lambda)(1-|\lambda|)\, \tilde{\mu}(d\lambda) = \infty.$$

On the other hand, for the different and disjoint compact sets E and F such that $\alpha(E) = \alpha(F) > 0$, Theorem B implies

$$\int_{\mathbb{D}} \rho_E^{p-\alpha(E)+\varepsilon}(\lambda)\, \rho_F^{q-\alpha(F)+\varepsilon}(\lambda)(1 - |\lambda|)\, \tilde{\mu}(d\lambda) < \infty.$$

References

1. P. Ahern, D. Clark, On inner functions with B^p derivatives, Michigan Math. J. **23** (1976), 107–118.
2. W. Blaschke, Eine Erweiterung des Satzes von Vitali über Folgen analytischer Funktionen, S.-B. Säcks Akad. Wiss. Leipzig Math.-Natur. KI. **67** (1915), 194–200.
3. A. Borichev, L. Golinskii, S. Kupin, A Blaschke-type condition and its application to complex Jacobi matrices, Bull. Lond. Math. Soc. **41** (2009), 117–123.
4. A. Borichev, L. Golinskii, S. Kupin, On zeros of analytic functions satisfying non-radial growth conditions, Rev. Mat. Iberoam., **34**, no. 3 (2018), 1153–1176.
5. L. Carleson, Sets of uniqueness for functions analytic in the unit disc, Acta Math., **87** (1952), 325–345.
6. S. Favorov, L. Golinskii, A Blaschke-Type condition for Analytic and Subharmonic Functions and Application to Contraction Operators, Amer. Math Soc. Transl. **226** (2009), 37–47.
7. S. Favorov, L. Golinskii, Blaschke-type conditions for analytic and subharmonic functions in the unit disk: local analogs and inverse problems, Comput. Methods Funct. Theory, **12** (2012) no. 1, 151–166.
8. J. Garnett, *Bounded analytic functions*, Graduate Texts in Mathematics, **236**, Springer, New York, 2007.
9. J. Garnett, D. Marshall, *Harmonic measure*, Cambridge University Press, Cambridge, 2005.
10. B. Khabibullin, Z. Abdullina, A. Rozit, A uniqueness theorem and subharmonic test functions, Algebra I Analiz, **30** (2018) no. 2, 318–334.
11. B. Khabibullin, N. Tamindarova, Subharmonic test functions and the distribution of zero sets of holomorphic functions, Lobachevskii Journal of Math., **38** (2017) no. 1, 70–79.

12. R. Killip, B. Simon, Sum rules for Jacobi matrices and their applications to spectral theory, Ann. Math., **158** (2003), 253–321.
13. E. Lieb, M. Loss, *Analysis*, Graduate Studies in Mathematics, vol. 14, AMS, Providence, RI, 1997.
14. T. Ransford, *Potential Theory in the Complex Plane*, London Math. Soc. Student Texts, vol. 28, Cambridge University Press, 1995.

Exponential Taylor Domination

Omer Friedland, Gil Goldman, and Yosef Yomdin

Abstract Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be an analytic function in a disk D_R of radius $R > 0$, and assume that f is p-valent in D_R, i.e. it takes each value $c \in \mathbb{C}$ at most p times in D_R. We consider its Borel transform

$$B(f)(z) = \sum_{k=0}^{\infty} \frac{a_k}{k!} z^k,$$

which is an entire function, and show that, for any $R > 1$, the valency of the Borel transform $B(f)$ in D_R is bounded in terms of p, R. We give examples, showing that our bounds, provide a reasonable envelope for the expected behavior of the valency of $B(f)$. These examples also suggest some natural questions, whose expected answer will strongly sharper our estimates.

We present a short overview of some basic results on multi-valent functions, in connection with "Taylor domination", which, for $f(z) = \sum_{k=0}^{\infty} a_k z^k$, is a bound of all its Taylor coefficients a_k through the first few of them. Taylor domination is our main technical tool, so we also discuss shortly some recent results in this direction.

1 Introduction

"Taylor domination" for an analytic function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is an explicit bound of all its Taylor coefficients a_k through the first few of them. This property was classically studied, in particular, in relation with the Bieberbach conjecture, which

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was finally proved in [8]: For a univalent function f we always have $|a_k| \leq k|a_1|$ (see [5, 6, 10, 14] and references therein).

To give an accurate definition, let us assume the radius of convergence of the Taylor series for f is $\widehat{R}$ (for $0 < \widehat{R} \leq +\infty$).

Definition 1.1 ([2]) Let $0 < R < \widehat{R}$, $N \in \mathbb{N}$, and let $S(k)$ be a positive sequence of a subexponential growth. A function f has an $(N, R, S(k))$-Taylor domination property if for any $k \geq N + 1$ we have

$$|a_k|R^k \leq S(k) \max_{0 \leq i \leq N} |a_i|R^i.$$

For a constant sequence $S(k) \equiv C$, we simply denote this property by (N, R, C)-Taylor domination.

The parameters N, R, and $S(k)$ of the Taylor domination are not defined uniquely. In fact, each nonzero analytic function f has this property, with N being the index of its first nonzero Taylor coefficient a_k (see e.g. [2, Proposition 1.2]). Consequently, the property of Taylor domination becomes interesting only for *specific classes* of analytic functions, for which we can specify the parameters N, R, $S(k)$ in an explicit and uniform way.

1.1 Some Classical Examples

One of the most important examples is provided by p-valent functions: Here Taylor domination with explicit parameters is a difficult result of geometric function theory, which is closely related to the Bieberbach conjecture. Since our main result is in this direction, we provide below some background (see Sect. 2).

Another striking example, of roughly the same period (1930th) is Bautin's discovery (see, [3, 4], and, e.g. [12, 21], and references therein), which is one of the most important sources of uniform Taylor domination: The Taylor coefficients of the function in question are polynomials (analytic functions) in a finite number of the problem's parameters. Taylor domination in this case is (formally) a consequence of Hilbert's finiteness theorem, and of its "quantitative" extensions (Hironaka's division algorithm, see e.g. [12] and references therein). Of course, besides discovering a general approach, Bautin provided explicit, and highly non-trivial, calculations for the plane vector fields of degree two, thus obtaining one of the most important results in the second part of Hilbert's 16th problem up today: At most three limit cycles can bifurcate from a center in a quadratic plane vector field.

One more classical result, which we mention, concerns Taylor domination, with explicit parameters, for rational functions. In a sense, this is a special case of p-valent functions, but the known results for rational functions are much sharper. Here Taylor domination is, essentially, equivalent to the important and widely applied "Turan's lemma" (see [17, 18], and, e.g. [2], and references therein).

1.2 Some Recent Developments

Recently, Taylor domination was also investigated in some additional situations:

1. *Linear recurrence relations.* Functions whose Taylor coefficients satisfy linear recurrence relations, in particular, of Poincaré-Perron type, possess an explicit Taylor domination (see [2]).
2. (s, p)-*Valent functions.* Functions which preserve some valency bounds after subtracting from them any polynomial of degree s. A complete characterization of such functions through linear recurrence relations for Taylor coefficients was obtained in [13].
3. *Remez-type inequalities.* These inequalities bound $|f|$ on a disk D through the bound on $|f|$ on a certain subset Ω of D. In [13] a rather accurate Remez-type inequality was obtained for (s, p)-valent functions.
4. *Bautin-type results.* Providing Taylor domination through an explicit description of the Bautin ideals in certain specific cases. Many important results in this direction were obtained in the modern analytic theory of ODE's ([15] and references therein). Beyond the field of analytic theory of ODE's, some (initial) general results were given in [9, 12, 19], while certain specific problems were treated, via direct calculations, in [20, 21].
5. *Efficiently transcendental functions.* In [11] we investigate analytic functions f such that a result of a substitution of f as y into a polynomial $P(z, y)$, i.e. $g(z) = P(z, f(z))$ preserves, for any P, Taylor domination with explicit parameters, depending only on the degree of P. Our main tool is "linear Bautin ideals", as in [20]. These results are applied in [11], via the Pila-Wilkie approach, to bounding the number of rational points on the graph $y = f(x)$.

1.3 The Scope and the Goals of the Paper

As it is clear from the above, the notion of Taylor domination was historically considered mostly for functions $f(z)$ with a finite radius of convergence. The Borel transform [7] maps such a function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ into an entire function

$$g(z) = B(f)(z) = \sum_{k=0}^{\infty} \frac{a_k}{k!} z^k.$$

The class of all the images $B(f)(z)$ of functions $f(z)$ having a nonzero radius of convergence, can be easily described explicitly: It consists of all the entire functions $g(z) = \sum_{k=0}^{\infty} b_k z^k$ with $k!b_k$ growing at most exponentially. There is an integral expression for the inverse of the Borel transform: $f(z) = \int_0^{\infty} e^{-t} g(tz)dt$, which plays important role in Borel's summation method for divergent series. We expect this formula to be important in our line of research, but we do not use it in the present paper.

Of course, the main applications of the Borel transform are in summation of divergent series. However, also its action on regular (but not extendable to the entire complex plane) functions f was extensively studied. Still, to the best of our knowledge, the problems of the behavior of the "valency", and the corresponding problems of Taylor domination, did not get an adequate attention.

The goal of the present paper is to present some initial results in this direction. Given a p-valent function f, we estimate the valency of $B(f)$ on the disks D_R, for given $R > 0$ (see Theorem 3.1 below). We also provide some examples, which outline the degree of the (non)-sharpness of our bounds, and suggest some related questions.

Finally, let us mention some recent observations from [1], concerning the moment and Fourier reconstruction of the "spike-train signals" $F(x) = \sum_{j=1}^{d} a_j \delta(x - x_j)$. The Fourier transform of F is an exponential polynomial $\mathbb{F}(s) = \sum_{j=1}^{d} a_j e^{-2\pi i x_j s}$, while its Stieltjes transform $\mathbb{S}(z) = \sum_{j=1}^{d} \frac{a_j}{1-zx_j}$ is a rational function with the poles at x_j. It is easy to see that the Taylor coefficient at zero of $\mathbb{F}(s)$ are $\frac{m_k}{k!}$, while the Taylor coefficient at zero of $\mathbb{S}(z)$ are m_k. Here

$$m_k = \int x^k F(x) dx$$

are the consecutive moments of our signal $F(x)$. In particular, $\mathbb{F}(s)$ is the Borel transform of $\mathbb{S}(z)$. Specifically, we show in [1], using Taylor domination, that if $\mathbb{F}(s)$ is small on a certain real interval, then all its Taylor coefficients are small. This fact is crucial for comparing accuracy of Fourier and moment reconstructions of spike-train signals. We expect that this result of [1] can be generalized to Borel transforms of general functions, having a Taylor domination property.

2 Some Background on Taylor Domination and Counting Zeroes

By definition, Taylor domination allows us to compare the behavior of $f(z)$ with the behavior of the polynomial $P_N(z) = \sum_{k=0}^{N} a_k z^k$. In particular, the number of zeroes of f, in an appropriate disk, can be easily bounded in this way (see below for more details). However, the opposite direction (bounding zeros implies Taylor domination) is a deep and difficult classical results of geometric function theory, closely related to the Bieberbach conjecture, and going back at least, to Ahlfors. We state here one prominent classical result in this direction [6]. To formulate it accurately, we need the following definition (see [14] and references therein).

Definition 2.1 Let f be a regular in a domain $\Omega \subset \mathbb{C}$. The function f is called p-valent in Ω, if for any $c \in \mathbb{C}$ the number of solutions in Ω of the equation $f(z) = c$ does not exceed p.

Theorem 2.2 (Biernacki [6]) *If f is p-valent in the disk D_R of radius R centered at $0 \in \mathbb{C}$ then for any $k \geq p+1$*

$$|a_k|R^k \leq A(p)k^{2p-1} \max_{1\leq i\leq p} |a_i|R^i,$$

where $A(p)$ is a constant depending only on p.

In our notations, Theorem 2.2 claims that a function f, which is p-valent in D_R, has a $(p, R, A(p)k^{2p-1})$-Taylor domination property. For univalent functions, i.e. for $p = 1$, $R = 1$, Theorem 2.2 gives $|a_k| \leq A(1)k|a_1|$ for any k, while the sharp bound of the Bieberbach conjecture is $|a_k| \leq k|a_1|$.

Various forms of inverse results to Theorem 2.2 are known (the reference list is long, and we skip it here). In particular, an explicit, and reasonably accurate, bound on the number of zeroes of f having a Taylor domination property is given in [16, Lemma 2.2.3]:

Theorem 2.3 *Let f possess an (N, R, C)-Taylor domination. Then, for any $R' < \frac{1}{4}R$, the function f has at most $5N + 5\log(C+2)$ zeros in $D_{R'}$.*

We can replace the bound on the number of zeroes of f by the bound on its valency, if we exclude a_0 in the definition of Taylor domination (or, alternatively, if we consider the derivative f' instead of f).

Remark 2.4 It is natural to ask whether functions f having a (p, R, k^{2p-1})-Taylor domination property (like p-valent functions, according to Biernacki's Theorem 2.2), are, at least, Kp-valent, in, say, $D_{\frac{1}{2}R}$. Since Theorem 2.3 concerns only the case of the constant sequence $S(k) \equiv C$, it is not sharp enough to answer this question. Calculating an optimal C and using Theorem 2.3, gives only an order of $p \log p$ bound on the valency of f (compare with the discussion in Sect. 4 below).

3 Main Result

Theorem 3.1 *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be a p-valent function on the unit disk D_1. Then, for any $R > 1$, the Borel transform $B(f)(z) = \sum_{k=0}^{\infty} \frac{a_k}{k!} z^k$ is q-valent on $D_{R'}$, where $R' < R$, and*

$$q \lessapprox (1 + \log p + \log R)p + R,$$

where $\lessapprox$ means up to universal constants.

Proof Let us start with the following simple remark, which allows us to eliminate the parameter R in a Taylor domination, just by a proper scaling of the independent variable. Indeed, the function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ has an $(N, R, S(k))$-Taylor domination property if and only if the scaled function

$$\widehat{f}(z) := f(Rz) = \sum_{k=0}^{\infty} a_k R^k z^k, \tag{3.1}$$

defined on the unit disk, has an $(N, 1, S(k))$-Taylor domination property.

By assumption, the function f is p-valent on the unit disk D_1, and therefore, by Theorem 2.2, for any $k \geq p+1$ we have

$$|a_k| \leq A(p) k^{2p-1} \max_{1 \leq i \leq p} |a_i|,$$

where $A(p)$ is a constant depending only on p. Hence, for any $R > 1$ and for any $k \geq p+1$, we have

$$|a_k| R^k / k! \leq A(p) k^{2p-1} \max_{1 \leq i \leq p} |a_i| R^k / k! \leq A(p) \eta \max_{1 \leq i \leq p} |a_i|, \tag{3.2}$$

where $\eta = \max_{k \geq p+1} \frac{k^{2p-1} R^k}{k!}$.

Now, since, by our assumptions, $R \geq 1$, the numbers $\frac{R^i}{i!}$ for $1 \leq i \leq p$, grow till $i = [R]$, and then start to decrease, where $[R]$ is the integer part of R. Hence, the minimum of these numbers is achieved either with the first of them $i = 1$, i.e. R, or with the last one $i = p$, which is $\frac{R^p}{p!}$. Therefore, we have

$$\max_{1 \leq i \leq p} \frac{|a_i| R^i}{i!} \geq \nu \max_{1 \leq i \leq p} |a_i|,$$

where $\nu = \min\{R, \frac{R^p}{p!}\}$.

Plugging, the above estimate, in (3.2), we immediately get that, for any $k \geq p+1$,

$$\frac{|a_k| R^k}{k!} \leq \frac{A(p)\eta}{\nu} \max_{1 \leq i \leq p} \frac{|a_i| R^i}{i!}. \tag{3.3}$$

By re-scaling the restriction of $B(f)$ to D_R to the unit disk (i.e. the Borel transform of the re-scaled function in (3.1)), we get

$$B(\widehat{f})(z) = \sum_{k=0}^{\infty} \frac{a_k R^k}{k!} z^k,$$

and thus, inequality (3.3) provides a bound on all the Taylor coefficients $\frac{a_k R^k}{k!}$ of $B(\widehat{f})$ through its first p ones (excluding the constant term), that is, $B(\widehat{f})$ has a $(p, 1, A(p)\eta/\nu)$-Taylor domination property, which, by the above simple remark, also implies that

$$B(f) \text{ has a } \big(p, R, (A(p)\eta/\nu)\big)\text{-Taylor domination property.}$$

Now, we use Theorem 2.3 for the function $B(f)$ with $N = p$ and $C = A(p)\eta/\nu$, which yields the following bound on the valency q of $B(f)$ in $D_{R'}$, for $R' \le R$:

$$\begin{aligned} q &\le 5N + 5\log(C+2) \\ &\le 5p + 5\log(A(p)\eta/\nu + 2). \end{aligned} \tag{3.4}$$

To complete the proof of Theorem 3.1, we need to get an explicit bound on η. Recall,

$$\eta = \max_{k \ge p+1} \frac{k^{2p-1}R^k}{k!}.$$

We shall bound η by considering two cases:

$$\eta = \max\{\eta_1, \eta_2\},$$

where

$$\eta_1 := \max_{p+1 \le k \le 3p} \frac{k^{2p-1}R^k}{k!}, \quad \eta_2 := \max_{k \ge 3p+1} \frac{k^{2p-1}R^k}{k!}.$$

For η_1 we use an immediate estimate

$$\eta_1 \le \frac{(3p)^{2p-1}R^{3p}}{(p+1)!},$$

just taking the maximal possible numerator, and the minimal possible denominator.

In order to estimate η_2, we proceed as follows: We divide the numerator and the denominator by k^{2p-1}, and write $k!$ as

$$k! = (k-2p-1)!\zeta\ , \quad \text{where } \zeta = \prod_{j=1}^{2p}(1 - \frac{2p-1-j}{k}) \ge \frac{1}{3^{2p}}.$$

Consequently we get

$$\eta_2 = \max_{k \ge 3p+1} \frac{k^{2p-1}R^k}{k!} \le 3^{2p}\frac{R^k}{(k-2p-1)!} = 3^{2p}R^{2p-1}\frac{R^{k-2p-1}}{(k-2p-1)!}.$$

It remains to notice, that the last expression, as a function of k, decreases, starting with $k - 2p - 1 = [R]$. Hence, its maximal value is achieved for $k = [R] + 2p - 1$.

Using Stirling formula, we have $\frac{R^R}{R!} \leq e^R$, and hence $\eta_2 \leq 3^{2p} R^{2p-1} e^R$. Thus, we conclude

$$\eta \leq \max\left\{\frac{(3p)^{2p-1} R^{3p}}{(p+1)!}, 3^{2p} R^{2p-1} e^R\right\}.$$

Recall also that $\nu = \min\{R, \frac{R^p}{p!}\}$, and considering (3.4), we conclude

$$\begin{aligned} q &\leq 5p + 5\log(A(p)\eta/\nu + 2) \\ &\lessapprox p + p\log p + p\log R + R, \end{aligned}$$

which completes the proof. □

4 Some Examples

In this section we give some examples, illustrating Theorem 3.1. These examples motivate some natural questions (presumably, open), which we discuss below.

1. As the first example, consider the function $f(z) = 1 + z + z^2 + \cdots = \frac{1}{1-z}$. This function is univalent in D_1, and its Borel transform is e^z. The solutions of the equation $e^z = c$ are all the points $\log c + 2\pi ki$, with whatever brunch of log chosen. Clearly, in the disk D_R there are at most $\frac{R}{2\pi}$ such points, and for some c this bound is achieved. Therefore, e^z in D_R is p-valent, with $p = \frac{R}{2\pi}$.
2. Let us now consider the case of larger p's. Let

$$f_p(z) = (z^p - 1)(e^z - 1) = -\sum_{k=1}^{p} \frac{z^k}{k!} + \sum_{k=p+1}^{\infty} [\frac{1}{(k-p)!} - \frac{1}{k!}] z^k.$$

Clearly, $f_p(z)$ is at least $p + \frac{R}{2\pi}$-valent inD_R, for any R. Indeed, the roots of the two factors in $f_p(z)$ provide the required number of solution of $f_p(z) = 0$. On the other side, $f_p(z)$ is the Borel transform of

$$\tilde{f}_p(z) = -\sum_{k=1}^{p} z^k + \sum_{k=p+1}^{\infty} [\frac{k!}{(k-p)!} - 1] z^k.$$

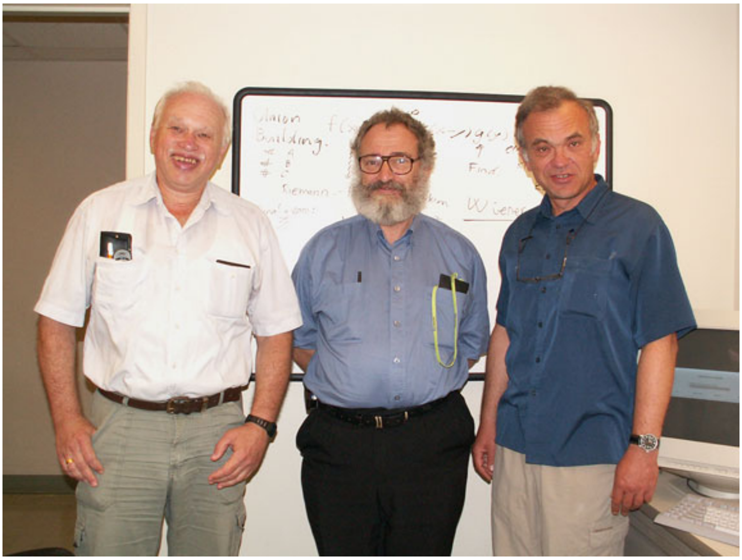
We see that $\tilde{f}_p(z)$ has a $(p, 1, S(k))$-Taylor domination, with $S(k) \sim k^p$. We would expect such functions to be p-valent, at least in smaller disks, but with the tools in our possession we can prove only that $\tilde{f}_p(z)$ is $\sim p\log p$-valent in any disk D_ρ, $\rho < 1$ (just estimate the maximum in k of $k^p \rho^k$ and use Theorem 2.3). We are not aware of "counting zeroes" results, sharp enough to provide Kp-

valency of a function, having $(p, 1, Ck^p)$-Taylor domination. Such a result would be an accurate inversion to the Biernaczki's one (Theorem 2.2 above). Thus, a natural question is *whether a function, having* $(p, 1, Ck^p)$*-Taylor domination is* $K(C)p$*-valent in a disk* $D_{\frac{1}{2}}$*, with the constant* $K(C)$ *depending only on* C*?* Compare Remark 2.4 in Sect. 2 above.

3. Finally, consider $h(z) = e^{z^p} = \sum_{l=1}^{\infty} \frac{z^{lp}}{l!}$. The entire function $h(z)$ is the Borel transform of a *formal power series*

$$\widehat{h}(z) = \sum_{l=1}^{\infty} \frac{z^{lp}(lp)!}{l!},$$

which is divergent for any $z \neq 0$. It is easy to see that $h(z)$ is $\sim p \cdot R^p$-valent in the disk D_R for any R. Indeed, to solve the equation $h(z) = e^{z^p} = c$, we put $u = z^p$, and first solve $e^u = c$, which gives the solutions $u_k = \log c + 2\pi k i$. Then, from each u_k we get p solutions $v_{k,l} = (u_k)^{\frac{1}{p}}$. Notice that these solutions are of absolute value $\sim k^{\frac{1}{p}}$ for k big, and hence $\sim p \cdot R^p$ of them are inside the disk D_R. We conclude that $h(z)$ is $\sim p \cdot R^p$-valent. This example suggests the following question: *Is it possible to extend the results above to the Borel transforms of divergent series?* This would require extending to some divergent series a notion of Taylor domination, and we expect such an extension to be productive in many other questions in this line.



In honor of Victor Katsnelson, Y. Yomdin, VK and A.E. Eremenko

References

1. D. Batenkov, G. Goldman, and Y. Yomdin, *Super-resolution of near-colliding point sources*, arXiv:1904.09186 (2019).
2. D. Batenkov and Y. Yomdin, *Taylor domination, Turá n lemma, and Poincaré -Perron sequences*, Nonlinear analysis and optimization, Contemp. Math., vol. 659, Amer. Math. Soc., Providence, RI, 2016, pp. 1–15.
3. N. Bautin, *Du nombre de cycles limites naissant en cas de variation des coefficients d'un état d'équilibre du type foyer ou centre*, C. R. (Doklady) Acad. Sci. URSS (N. S.) **24** (1939), 669–672 (French).
4. ——, *On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or center type*, American Math. Soc. Translation **1954** (1954), no. 100, 19.
5. L. Bieberbach, *Analytische Fortsetzung*, Ergebnisse der Mathematik und ihrer Grenzgebiete (N.F.), Heft 3, Springer-Verlag, Berlin-Gö ttingen-Heidelberg, 1955 (German).
6. M. Biernacki, *Sur les fonctions multivalentes d'ordre p*, CR Acad. Sci. Paris **203** (1936), 449–451.
7. E. Borel, *Mémoire sur les séries divergentes*, Ann. Sci. École Norm. Sup. (3) **16** (1899), 9–131 (French).
8. L. de Branges, *A proof of the Bieberbach conjecture*, Acta Math. **154** (1985), no. 1–2, 137–152.
9. M. Briskin and Y. Yomdin, *Algebraic families of analytic functions. I*, J. Differential Equations **136** (1997), no. 2, 248–267.
10. M. L. Cartwright, *Some inequalities in the theory of functions*, Math. Ann. **111** (1935), no. 1, 98–118.
11. G. Comte and Y. Yomdin, *Zeroes and rational points of analytic functions*, Ann. Inst. Fourier (Grenoble) **68** (2018), no. 6, 2445–2476 (English, with English and French summaries).
12. J.-P. Francoise and Y. Yomdin, *Bernstein inequalities and applications to analytic geometry and differential equations*, J. Funct. Anal. **146** (1997), no. 1, 185–205.
13. O. Friedland and Y. Yomdin, *(s, p)-valent functions*, Geometric aspects of functional analysis, Lecture Notes in Math., vol. 2169, Springer, Cham, 2017, pp. 123–136.
14. W. K. Hayman, *Multivalent functions*, 2nd ed., Cambridge Tracts in Mathematics, vol. 110, Cambridge University Press, Cambridge, 1994.
15. R. Roussarie, *Bifurcations of planar vector fields and Hilbert's sixteenth problem*, Modern Birkhäuser Classics, Birkhäuser/Springer, Basel, 1998. [2013] reprint of the 1998 edition [MR1628014].
16. N. Roytwarf and Y. Yomdin, *Bernstein classes*, Ann. Inst. Fourier (Grenoble) **47** (1997), no. 3, 825–858 (English, with English and French summaries).
17. P. Turán, *Eine neue Methode in der Analysis und deren Anwendungen*, Akadémiai Kiadó, Budapest, 1953 (German).
18. ——, *On a new method of analysis and its applications*, Pure and Applied Mathematics (New York), John Wiley & Sons, Inc., New York, 1984. With the assistance of G. Halász and J. Pintz; With a foreword by Vera T. Sós; A Wiley-Interscience Publication.
19. Y. Yomdin, *Global finiteness properties of analytic families and algebra of their Taylor coefficients*, The Arnoldfest (Toronto, ON, 1997), Fields Inst. Commun., vol. 24, Amer. Math. Soc., Providence, RI, 1999, pp. 527–555.
20. ——, *Oscillation of analytic curves*, Proc. Amer. Math. Soc. **126** (1998), no. 2, 357–364.
21. ——, *Bautin ideals and Taylor domination*, Publ. Mat. **58** (2014), no. suppl., 529–541.

A Closer Look at the Solution Set of the Truncated Matricial Moment Problem $\mathsf{M}[[\alpha, \infty); (s_j)_{j=0}^m, \preccurlyeq]$

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Dedicated to V. E. Katsnelson on the occasion of his 75th birthday

Abstract In our recent paper Fritzsche et al. (Linear Algebra Appl 544:30–114, 2018) we obtained via $[\alpha, \infty)$-Stieltjes transformation a complete description of the solution set of a matricial truncated Stieltjes-type power moment problem. In this paper we continue these investigations by working out the distinguished role of two particular molecular solutions which were found on a purely algebraic way. It turns out that via $[\alpha, \infty)$-Stieltjes transform these two solutions occupy an extremal position within the set of all solutions. More precisely, if we fix an arbitrary point on the interval $(-\infty, \alpha)$ values at the point x of the $[\alpha, \infty)$-Stieltjes transforms of all solutions fill out a closed matricial interval the endpoints of which are determined by the values at x of the $[\alpha, \infty)$-Stieltjes transforms of the two distinguished molecular solutions. Furthermore, we determine which solutions correspond to remarkable choices of the parameters such as proper pairs. Our method is based on the Schur analysis approach worked out in Fritzsche et al. (Linear Algebra Appl 544:30–114, 2018) and uses interrelations to orthogonal $q \times q$ matrix polynomials.

Keywords Truncated matricial Stieltjes moment problem · Schur–Stieltjes-type algorithm · Orthogonal matrix polynomials · Matricial Weyl intervals

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1 A Personal Note

The systematic treatment of matricial versions of the classical Stieltjes moment problems in combination with related interpolation problems for associated classes of holomorphic matrix-valued functions started with the pioneering work of V. E. Katsnelson in collaboration with his doctorate student Yu. M. Dyukarev in the early 1980s (see [6, 8, 9]).

Their investigations were based on the application of the method of fundamental matrix inequalities (FMI method) created by V. P. Potapov. V. E. Katsnelson was the first who extended the FMI method to continuous problems of analysis including integral representations of matrix-valued non-negative definite kernels (see e. g. [31–36]). The first and second author strongly benefited over many years from countless discussions with Viktor Emmanuilovich who essentially influenced the direction of their mathematical research. The origins of our interest in studying matricial versions of classical moment problems can be traced back to many inspiring working meetings with Viktor Emmanuilovich.

We wholeheartedly want to thank Viktor Emmanuilovich for continuous support over 30 years.

2 Introduction

This paper is a continuation of our investigations on the matricial power moment problem $\mathsf{M}[[\alpha, \infty); (s_j)_{j=0}^{m}, \preccurlyeq]$ which were done in [23]. Moreover, it is intimately connected to [20, 21] where the related problem $\mathsf{M}[[\alpha, \infty); (s_j)_{j=0}^{m}, =]$ was treated. Using a Schur analysis technique we found in [23] via Stieltjes transformation a parametrization of the solution set described by a linear fractional transformation the generating matrix-valued function of which is a $2q \times 2q$ matrix polynomial built from the sequence $(s_j)_{j=0}^{m}$ of prescribed power moments. The set of parameters is a special class of equivalence classes of ordered pairs of $q \times q$ matrix-valued functions which are meromorphic in the domain $\mathbb{C} \setminus [\alpha, \infty)$. The set consisting of these equivalence classes can be interpreted as a projective extension of the Stieltjes class $\mathcal{S}_{q,[\alpha,\infty)}$, which is one of the basic analytic objects in our approach. Amongst this parameter set there is a particularly interesting subset, namely the equivalence classes of so-called proper pairs. By this we mean those equivalence classes which can be represented in a special way by a function belonging to the class $\mathcal{S}_{q,[\alpha,\infty)}$. There arises naturally the question to characterize all solutions which correspond to equivalence classes of proper pairs. This topic is one of the central themes of this paper (see Theorem 13.3).

A second main theme is concerned with the construction of two distinguished molecular solutions of the original moment problem. (A measure is said to be molecular, if it is concentrated only on a finite set.) These measures, which were already investigated in our former work (see [10, 11]), are constructed more or less by algebraic tools (see Definition 6.11). One of the main goals of this paper is to demonstrate which important role these molecular solutions occupy in the whole solution set. For this reason, we look at the $[\alpha, \infty)$-Stieltjes transforms of these two molecular solutions. More precisely, we fix a point x belonging to the interval $(-\infty, \alpha)$. Then it turns out that the values at x of the $[\alpha, \infty)$-Stieltjes transforms of all solutions of the moment problem fill a closed matricial interval the endpoints of which are just the values at x of the $[\alpha, \infty)$-Stieltjes transforms of the two distinguished molecular solutions mentioned above (see Theorem 17.16). In this way, we obtain a far-reaching generalization of a deep result due to Yu. M. Dyukarev [7] who considered the case $\alpha = 0$ in the completely non-degenerate situation. The strategy of our approach is completely different from that applied by Yu. M. Dyukarev. In the heart of our approach stands the interplay of the two related Schur-type algorithms worked out in [20, 21]. More precisely, our construction is based on a careful analysis of the elementary step of the Schur-type algorithm. Namely, it turns out that this elementary step can be written as a composition mapping of four basic transformations for complex $q \times q$ matrices. Then it is shown that all these basic transformations map closed intervals onto closed intervals. The combination of these observations leads us to Proposition B.5 which provides one of the key instruments for the proof of Theorem 17.16.

3 Preliminaries

In order to describe more concretely the central topics studied in this paper, we introduce some notation. Throughout this paper, let p and q be positive integers. Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ be the set of all positive integers, the set of all non-negative integers, the set of all integers, the set of all real numbers, and the set of all complex numbers, respectively. For every choice of $\rho, \kappa \in \mathbb{R} \cup \{-\infty, \infty\}$, let $\mathbb{Z}_{\rho,\kappa} := \{k \in \mathbb{Z} : \rho \leq k \leq \kappa\}$. We will write $\mathbb{C}^{p\times q}$, $\mathbb{C}_{\mathrm{H}}^{q\times q}$, $\mathbb{C}_{\succcurlyeq}^{q\times q}$, and $\mathbb{C}_{\succ}^{q\times q}$ for the set of all complex $p \times q$ matrices, the set of all Hermitian complex $q \times q$ matrices, the set of all non-negative Hermitian complex $q \times q$ matrices, and the set of all positive Hermitian complex $q \times q$ matrices, respectively. We will use $\mathfrak{B}_{\mathbb{R}}$ to denote the σ-algebra of all Borel subsets of $\mathbb{R}$. For each $\Omega \in \mathfrak{B}_{\mathbb{R}} \setminus \{\emptyset\}$, let $\mathfrak{B}_{\Omega} := \mathfrak{B}_{\mathbb{R}} \cap \Omega$. Furthermore, for each $\Omega \in \mathfrak{B}_{\mathbb{R}} \setminus \{\emptyset\}$, we will write $\mathcal{M}_q^{\succcurlyeq}(\Omega)$ to designate the set of all non-negative Hermitian $q \times q$ measures defined on $\mathfrak{B}_{\Omega}$, i. e., the set of all σ-additive mappings $\mu : \mathfrak{B}_{\Omega} \to \mathbb{C}_{\succcurlyeq}^{q\times q}$. We will use the integration theory with respect to non-negative Hermitian $q \times q$ measures, which was worked out independently by I. S. Kats [29] and M. Rosenberg [39]. In particular, for each $\sigma \in \mathcal{M}_q^{\succcurlyeq}(\Omega)$, we will use the notion $\mathcal{L}^1(\Omega, \mathfrak{B}_{\Omega}, \mu; \mathbb{C})$ to denote the space of all Borel measurable functions $f : \Omega \to \mathbb{C}$ for which the integral $\int_{\Omega} f \mathrm{d}\mu$ exists. For every choice of $\Omega \in \mathfrak{B}_{\mathbb{R}} \setminus \{\emptyset\}$ and $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, we will use $\mathcal{M}_{q,\kappa}^{\succcurlyeq}(\Omega)$ to denote the set of all $\sigma \in \mathcal{M}_q^{\succcurlyeq}(\Omega)$ such that the integral

$$s_j^{(\sigma)} := \int_{\Omega} x^j \sigma(\mathrm{d}x)$$

exists for all $j \in \mathbb{Z}_{0,\kappa}$. In this case, $(s_j^{(\sigma)})^* = s_j^{(\sigma)}$ obviously holds true for all $k \in \mathbb{Z}_{0,\kappa}$. Observe that $\mathcal{M}_{q,\ell}^{\succcurlyeq}(\Omega) \subseteq \mathcal{M}_{q,k}^{\succcurlyeq}(\Omega)$ for all $k, \ell \in \mathbb{N}_0$ with $k < \ell$.

We are going to study several aspects of the following two moment problems:

Problem ($\mathsf{M}[[\alpha, \infty); (s_j)_{j=0}^m, \preccurlyeq]$) Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$ and let $(s_j)_{j=0}^m$ be a sequence of complex $q \times q$ matrices. Discuss the set $\mathcal{M}_q^{\succcurlyeq}[[\alpha, \infty); (s_j)_{j=0}^m, \preccurlyeq]$ of all $\sigma \in \mathcal{M}_{q,m}^{\succcurlyeq}([\alpha, \infty))$ for which the matrix $s_m - s_m^{(\sigma)}$ is non-negative Hermitian and in the case $m \geq 1$, moreover $s_j^{(\sigma)} = s_j$ is satisfied for each $j \in \mathbb{Z}_{0,m-1}$.

$\mathsf{M}[[\alpha, \infty); (s_j)_{j=0}^m, \preccurlyeq]$ is intimately connected with the following:

Problem ($\mathsf{M}[[\alpha, \infty); (s_j)_{j=0}^m, =]$) Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m$ be a sequence of complex $q \times q$ matrices. Discuss the set $\mathcal{M}_q^{\succcurlyeq}[[\alpha, \infty); (s_j)_{j=0}^m, =]$ of all $\sigma \in \mathcal{M}_{q,m}^{\succcurlyeq}([\alpha, \infty))$ for which $s_j^{(\sigma)} = s_j$ is fulfilled for all $j \in \mathbb{Z}_{0,m}$.

In the case that a sequence $(s_j)_{j=0}^m$ of complex $q \times q$ matrices is given for which the set $\mathcal{M}_q^{\succcurlyeq}[[\alpha, \infty); (s_j)_{j=0}^m, =]$ is non-empty, we obtained in [21, Theorem 13.1] a complete parametrization of this set via a linear fractional transformation of matrices the generating function of which is a $2q \times 2q$ matrix polynomial built from the sequence $(s_j)_{j=0}^m$ of the original data. In [23], a description of the set $\mathcal{M}_q^{\succcurlyeq}[[\alpha, \infty); (s_j)_{j=0}^m, \preccurlyeq]$ is given.

It seems to be useful to recall the notion of two types of sequences of matrices. If $n \in \mathbb{N}_0$ and if $(s_j)_{j=0}^{2n}$ is a sequence of complex $q \times q$ matrices, then $(s_j)_{j=0}^{2n}$ is called *Hankel non-negative definite* if the block Hankel matrix $H_n := [s_{j+k}]_{j,k=0}^n$ is non-negative Hermitian. A sequence $(s_j)_{j=0}^\infty$ of complex $q \times q$ matrices is called *Hankel non-negative definite* if $(s_j)_{j=0}^{2n}$ is Hankel non-negative definite for all $n \in \mathbb{N}_0$. For all $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, we will write $\mathcal{H}_{q,2\kappa}^{\succcurlyeq}$ for the set of all Hankel non-negative definite sequences $(s_j)_{j=0}^{2\kappa}$ of complex $q \times q$ matrices. Furthermore, for all $n \in \mathbb{N}_0$, let $\mathcal{H}_{q,2n}^{\succcurlyeq,\mathrm{e}}$ be the set of all sequences $(s_j)_{j=0}^{2n}$ of complex $q \times q$ matrices for which there exist complex $q \times q$ matrices s_{2n+1} and s_{2n+2} such that $(s_j)_{j=0}^{2(n+1)} \in \mathcal{H}_{q,2(n+1)}^{\succcurlyeq}$, whereas $\mathcal{H}_{q,2n+1}^{\succcurlyeq,\mathrm{e}}$ stands for the set of all sequences $(s_j)_{j=0}^{2n+1}$ of complex $q \times q$ matrices for which there exist some $s_{2n+2} \in \mathbb{C}^{q \times q}$ such that $(s_j)_{j=0}^{2(n+1)} \in \mathcal{H}_{q,2(n+1)}^{\succcurlyeq}$. For each $m \in \mathbb{N}_0$, the elements of the set $\mathcal{H}_{q,m}^{\succcurlyeq,\mathrm{e}}$ are called *Hankel non-negative definite extendable sequences*. For technical reason, we set $\mathcal{H}_{q,\infty}^{\succcurlyeq,\mathrm{e}} := \mathcal{H}_{q,\infty}^{\succcurlyeq}$. Besides the just introduced classes of sequences of complex $q \times q$ matrices, we will use analogous classes of sequences of complex $q \times q$ matrices, which take into account the influence of the prescribed number $\alpha \in \mathbb{R}$. We will introduce several classes of finite or infinite sequences of complex $q \times q$ matrices, which are characterized by properties of the sequences $(s_j)_{j=0}^\kappa$ and $(-\alpha s_j + s_{j+1})_{j=0}^{\kappa-1}$.

Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. Then, for all $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$, we define the block Hankel matrix $K_n := [s_{j+k+1}]_{j,k=0}^{n}$. Let $\alpha \in \mathbb{R}$. Let $\mathcal{K}^{\succcurlyeq}_{q,0,\alpha} := \mathcal{H}^{\succcurlyeq}_{q,0}$, and, for all $n \in \mathbb{N}$, let $\mathcal{K}^{\succcurlyeq}_{q,2n,\alpha}$ be the set of all sequences $(s_j)_{j=0}^{2n}$ of complex $q \times q$ matrices for which the block Hankel matrices H_n and $-\alpha H_{n-1} + K_{n-1}$ are both non-negative Hermitian, i. e.,

$$\mathcal{K}^{\succcurlyeq}_{q,2n,\alpha} = \left\{ (s_j)_{j=0}^{2n} \in \mathcal{H}^{\succcurlyeq}_{q,2n} \colon (-\alpha s_j + s_{j+1})_{j=0}^{2(n-1)} \in \mathcal{H}^{\succcurlyeq}_{q,2(n-1)} \right\}.$$

For each $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, let $\mathcal{F}_{q,\kappa}$ be the set of all sequences $(s_j)_{j=0}^{\kappa}$ of complex $q \times q$ matrices. Furthermore, for all $n \in \mathbb{N}_0$, let $\mathcal{K}^{\succcurlyeq}_{q,2n+1,\alpha}$ be the set of all sequences $(s_j)_{j=0}^{2n+1} \in \mathcal{F}_{q,2n+1}$ for which the block Hankel matrices H_n and $-\alpha H_n + K_n$ are both non-negative Hermitian, i. e.,

$$\mathcal{K}^{\succcurlyeq}_{q,2n+1,\alpha} := \left\{ (s_j)_{j=0}^{2n+1} \in \mathcal{F}_{q,2n+1} \colon \left\{ (s_j)_{j=0}^{2n}, (-\alpha s_j + s_{j+1})_{j=0}^{2n} \right\} \subseteq \mathcal{H}^{\succcurlyeq}_{q,2n} \right\}.$$

Remark 3.1 Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m} \in \mathcal{K}^{\succcurlyeq}_{q,m,\alpha}$. Then one can easily see that $(s_j)_{j=0}^{\ell} \in \mathcal{K}^{\succcurlyeq}_{q,\ell,\alpha}$ for all $\ell \in \mathbb{Z}_{0,m}$. Thus, let $\mathcal{K}^{\succcurlyeq}_{q,\infty,\alpha}$ be the set of all sequences $(s_j)_{j=0}^{\infty}$ of complex $q \times q$ matrices such that $(s_j)_{j=0}^{m} \in \mathcal{K}^{\succcurlyeq}_{q,m,\alpha}$ for all $m \in \mathbb{N}_0$.

Lemma 3.2 (cf. [16, Lemma 2.9]) *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}^{\succcurlyeq}_{q,\kappa,\alpha}$. Then $s_j \in \mathbb{C}^{q \times q}_{\mathrm{H}}$ for all $j \in \mathbb{Z}_{0,\kappa}$ and $s_{2k} \in \mathbb{C}^{q \times q}_{\succcurlyeq}$ for all $k \in \mathbb{N}_0$ with $2k \leq \kappa$.*

For each $m \in \mathbb{N}_0$, let $\mathcal{K}^{\succcurlyeq,\mathrm{e}}_{q,m,\alpha}$ be the set of all sequences $(s_j)_{j=0}^{m}$ of complex $q \times q$ matrices for which there exists an $s_{m+1} \in \mathbb{C}^{q \times q}$ such that $(s_j)_{j=0}^{m+1}$ belongs to $\mathcal{K}^{\succcurlyeq}_{q,m+1,\alpha}$. Furthermore, let $\mathcal{K}^{\succcurlyeq,\mathrm{e}}_{q,\infty,\alpha} := \mathcal{K}^{\succcurlyeq}_{q,\infty,\alpha}$.

Remark 3.3 Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}^{\succcurlyeq}_{q,\kappa,\alpha}$. Then $(s_j)_{j=0}^{\ell} \in \mathcal{K}^{\succcurlyeq,\mathrm{e}}_{q,\ell,\alpha}$ for all $\ell \in \mathbb{Z}_{0,\kappa-1}$.

Remark 3.4 Let $\alpha \in \mathbb{R}$ and let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$. Then $\mathcal{K}^{\succcurlyeq,\mathrm{e}}_{q,\kappa,\alpha} \subseteq \mathcal{K}^{\succcurlyeq}_{q,\kappa,\alpha}$. Furthermore, if $\kappa \geq 1$ and if $(s_j)_{j=0}^{\kappa} \in \mathcal{K}^{\succcurlyeq}_{q,\kappa,\alpha}$, then $(s_j)_{j=0}^{\ell} \in \mathcal{K}^{\succcurlyeq,\mathrm{e}}_{q,\ell,\alpha}$ for all $\ell \in \mathbb{Z}_{0,\kappa-1}$.

Let $m \in \mathbb{N}_0$. Then we call a sequence $(s_j)_{j=0}^{m}$ of complex $q \times q$ matrices *right-sided α-Stieltjes non-negative definite* if it belongs to $\mathcal{K}^{\succcurlyeq}_{q,m,\alpha}$ and *right-sided α-Stieltjes non-negative definite extendable* if it belongs to $\mathcal{K}^{\succcurlyeq}_{q,m,\alpha}$. (Note that left versions of this notions are considered in [16].) The following result indicates the importance of the sets $\mathcal{K}^{\succcurlyeq}_{q,m,\alpha}$ and $\mathcal{K}^{\succcurlyeq,\mathrm{e}}_{q,m,\alpha}$ for the above formulated truncated matricial Stieltjes-type moment problems.

Theorem 3.5 ([10, Theorems 1.3 and 1.4]) *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m$ be a sequence of complex $q \times q$ matrices. Then:*

(a) $\mathcal{M}_q^{\succcurlyeq}[[\alpha, \infty); (s_j)_{j=0}^m, =] \neq \emptyset$ *if and only if* $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$.

(b) $\mathcal{M}_q^{\succcurlyeq}[[\alpha, \infty); (s_j)_{j=0}^m, \preccurlyeq] \neq \emptyset$ *if and only if* $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq}$.

Corollary 3.6 ([23, Corollary 1.9]) *If $\alpha \in \mathbb{R}$, if $m \in \mathbb{N}_0$, and if $\sigma \in \mathcal{M}_{q,m}^{\succcurlyeq}([\alpha, \infty))$, then $(s_j^{(\sigma)})_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$.*

We introduce now a further class of sequences of complex $q \times q$ matrices which plays an important role in the study of the moment problems under consideration. Let $\alpha \in \mathbb{R}$. For all $n \in \mathbb{N}_0$, let $\mathcal{K}_{q,2n,\alpha}^{\succ}$ be the set of all sequences $(s_j)_{j=0}^{2n}$ of complex $q \times q$ matrices for which H_n is positive Hermitian and, in the case $n \geq 1$, furthermore $-\alpha H_{n-1} + K_{n-1}$ is positive Hermitian. For all $n \in \mathbb{N}_0$, let $\mathcal{K}_{q,2n+1,\alpha}^{\succ}$ be the set of all sequences $(s_j)_{j=0}^{2n+1}$ of complex $q \times q$ matrices for which the block Hankel matrices H_n and $-\alpha H_n + K_n$ are positive Hermitian. Furthermore, let $\mathcal{K}_{q,\infty,\alpha}^{\succ}$ be the set of all sequences $(s_j)_{j=0}^{\infty}$ of complex $q \times q$ matrices such that $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succ}$ for all $m \in \mathbb{N}_0$.

Proposition 3.7 ([16, Proposition 2.20]) *Let $\alpha \in \mathbb{R}$ and let $m \in \mathbb{N}_0$. Then $\mathcal{K}_{q,m,\alpha}^{\succ} \subseteq \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$.*

The combination of Proposition 3.7 with Theorem 3.5 and Remark 3.4 shows that both moment problems under consideration in this paper are solvable for sequences belonging to $\mathcal{K}_{q,m,\alpha}^{\succ}$. This is the so-called non-degenerate case which was studied by Yu. M. Dyukarev [7]. The following result reflects an interrelation between the sets $\mathcal{K}_{q,m,\alpha}^{\succcurlyeq}$ and $\mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$, which strongly influences our following considerations:

Theorem 3.8 ([10, Theorem 5.2]) *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq}$. Then there is a unique sequence $(\tilde{s}_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$ such that*

$$\mathcal{M}_q^{\succcurlyeq}[[\alpha, \infty); (s_j)_{j=0}^m, \preccurlyeq] = \mathcal{M}_q^{\succcurlyeq}[[\alpha, \infty); (\tilde{s}_j)_{j=0}^m, \preccurlyeq]. \tag{3.1}$$

It should be mentioned that in [19] a concrete general principle to describe the sequence $(\tilde{s}_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$ satisfying (3.1) is obtained.

Definition 3.9 (cf. [10, Section 5]) Let $m \in \mathbb{N}_0$ and let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq}$. Then the unique sequence $(\tilde{s}_j)_{j=0}^m$ belonging to $\mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$ for which (3.1) holds true is said to be the *right-sided α-Stieltjes non-negative definite extendable sequence equivalent to* $(s_j)_{j=0}^m$.

Theorem 3.8 is essential for the realization of the above formulated basic strategy of our approach, because, it is namely possible to restrict our considerations to the case that the given sequence $(s_j)_{j=0}^m$ belongs to the subclass $\mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$ of $\mathcal{K}_{q,m,\alpha}^{\succcurlyeq}$.

Lemma 3.10 *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}$, and let $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq}$. Denote by $(\tilde{s}_j)_{j=0}^{m}$ the right-sided α-Stieltjes non-negative definite extendable sequence equivalent to $(s_j)_{j=0}^{m}$. Then $s_m - \tilde{s}_m \in \mathbb{C}_{\succcurlyeq}^{q\times q}$. If $m \geq 1$, then $s_j = \tilde{s}_j$ for all $j \in \mathbb{Z}_{0,m-1}$. Moreover, $(s_j)_{j=0}^{m} = (\tilde{s}_j)_{j=0}^{m}$ if and only if $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$.*

Proof By construction, we have $(\tilde{s}_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$ and (3.1). Theorem 3.5(a) yields $\mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty);(\tilde{s}_j)_{j=0}^{m},=] \neq \emptyset$. Let $\sigma \in \mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty);(\tilde{s}_j)_{j=0}^{m},=]$. Then $s_m^{(\sigma)} = \tilde{s}_m$. Consequently, $\sigma \in \mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty);(\tilde{s}_j)_{j=0}^{m},\preccurlyeq]$. In particular, (3.1) implies $\sigma \in \mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty);(s_j)_{j=0}^{m},\preccurlyeq]$. Thus, $s_m - s_m^{(\sigma)} \in \mathbb{C}_{\succcurlyeq}^{q\times q}$. Because of $s_m^{(\sigma)} = \tilde{s}_m$, we get $s_m - \tilde{s}_m \in \mathbb{C}_{\succcurlyeq}^{q\times q}$. The remaining assertions are immediate consequences of Theorem 3.8. □

Similar as in [20, 21, 23], we reformulate the original truncated matricial moment problem via Stieltjes transform into an equivalent problem of prescribed asymptotic expansions for particular classes of matrix-valued functions, which are holomorphic in $\mathbb{C} \setminus [\alpha,\infty)$. The key for the success of our approach is caused by the fact that the Schur–Stieltjes transform for $q \times q$ matrix-valued holomorphic functions in $\mathbb{C} \setminus [\alpha,\infty)$, which we worked out in [21], is also compatible with the problem under consideration in this paper.

This paper is organized as follows. In Sect. 4, we present some material on the intrinsic structure of sequences belonging to $\mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$. In particular, we draw our attention to the role of α-Stieltjes completely degenerate sequences within the concept of α-Stieltjes non-negative definite extension of α-Stieltjes non-negative definite extendable sequences. In Sect. 5, we recall some basic facts on right - Stieltjes parametrization of sequences from $\mathbb{C}^{q\times q}$. This material is mostly taken from [10, 16]. In Sect. 6, using purely algebraic methods we construct two distinguished molecular solutions $\underline{\sigma}_m$ and $\overline{\sigma}_m$ of the truncated matricial $[\alpha,\infty)$-moment problem. These solutions play a key role in the course of this paper. It turns out that they occupy an extremal position within the whole solution set. Our main strategy to demonstrate that is based on the use of Schur analysis methods. The algebraic aspect of our Schur analysis approach is handled in Sect. 7. This material is mostly taken from [20]. To prepare the function-theoretic aspect of our Schur analysis method, we introduce in Sect. 8 a class $\mathcal{S}_{q,[\alpha,\infty)}$ of in $\mathbb{C} \setminus [\alpha,\infty)$ holomorphic $q \times q$ matrix-valued functions, which can be characterized by an important integral representation (see Theorem 8.2). Due to this fact it will be possible to reformulate the original moment problem as an equivalent problem of finding a prescribed asymptotic expansion in an open sector of the open upper half plane Π_+ of $\mathbb{C}$. This leads us to certain subclasses of the class $\mathcal{S}_{q,[\alpha,\infty)}$, which are determined by mild growth conditions ensuring the integral representation (9.4) which is also called $[\alpha,\infty)$-*Stieltjes transform*. In Sect. 10, via $[\alpha,\infty)$-Stieltjes transform the original moment problem is reformulated as a problem of finding a prescribed asymptotic expansion for in $\mathbb{C} \setminus [\alpha,\infty)$ holomorphic $q \times q$ matrix-valued functions belonging to some subclass of $\mathcal{S}_{q,[\alpha,\infty)}$. The solution set of this reformulated problem is described by a linear fractional transformation of matrices the generating matrix-valued function of it is a $2q \times 2q$ matrix polynomial only built from $(s_j)_{j=0}^{m}$ and α. The parameter set of this linear fractional transformation is a set of equivalence classes of pairs of in $\mathbb{C}\setminus[\alpha,\infty)$ meromorphic $q \times q$ matrix-valued functions. Section 11 contains a detailed description of these pairs under the light of this paper. In Sect. 12, we investigate a coupled pair of Schur–Stieltjes transforms. The first component of this pair realizes the elementary step of our function-theoretic Schur algorithm whereas the second component is connected with the elementary step of the inverse $[\alpha,\infty)$-Schur–Stieltjes algorithm. This theme was opened in [23]. We obtain new insights about the finer structure of the set $\mathcal{S}_{0,q,[\alpha,\infty)}[(s_j)_{j=0}^{0}, \preccurlyeq]$ (see Propositions 12.9, 12.10, and 12.11). The main theme of Sect. 13 is a detailed investigation of some remarkable subsets of the set $\mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^{m}, \preccurlyeq]$ of $[\alpha,\infty)$-Stieltjes transforms of the moment problem $\mathsf{M}[[\alpha,\infty); (s_j)_{j=0}^{m}, \preccurlyeq]$. In particular, we investigate the $[\alpha,\infty)$-Stieltjes transforms $\underline{S}_m$ and $\overline{S}_m$ of the distinguished molecular solutions $\underline{\sigma}_m$ and $\overline{\sigma}_m$ of the truncated $[\alpha,\infty)$-moment problem, respectively, which were found in Sect. 6. We call $\overline{S}_m$ and $\underline{S}_m$ the *upper and lower $\mathcal{S}_{q,[\alpha,\infty)}$-functions associated with a sequence* $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$. In Sect. 14, we introduce via appropriate recurrence formulas two interrelated sequences of $q \times q$ matrix polynomials which turn out to be intimately connected with monic right orthogonal systems of matrix polynomials. In Sect. 15, we show that the canonical $q \times q$ blocks of the resolvent matrix which generates the linear fractional transformation parametrizing the set $\mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^{m}, \preccurlyeq]$ are determined by the $q \times q$ matrix polynomials which are recursively constructed in Sect. 14. In Sect. 16, we express the functions $\underline{S}_m$ and $\overline{S}_m$ in terms of the $q \times q$ matrix polynomials introduced in Sect. 14. In Sect. 17, we fix a sequence $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$ and a number $x \in (-\infty,\alpha)$. Then we determine the set $\mathcal{I}[\alpha, x, (s_j)_{j=0}^{m}; \preccurlyeq]$ of the values in the point x of the $[\alpha,\infty)$-Stieltjes transforms of the measures belonging to $\mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty); (s_j)_{j=0}^{m}, \preccurlyeq]$. This set proves to be a closed matricial interval the endpoints of which are determined by the values $\underline{S}_m(x)$ and $\overline{S}_m(x)$ (see Theorem 17.16). In Sect. 18, we study the relation between consecutive matricial intervals of the type occurring in Theorem 17.16. In the final section Sect. 19, we study the asymptotic behavior of the matricial Weyl intervals. In this way, we are able to generalize some results due to Yu. M. Dyukarev [7] who handled a particular case.

4 On the Structure of Finite α-Stieltjes Non-Negative Definite Extendable Sequences of Complex $q \times q$ Matrices

In this section, we introduce two classes of finite sequences of complex $q \times q$ matrices, which prove to be right-sided α-Stieltjes non-negative definite extendable. We start with some notation. By I_p and $0_{p\times q}$ we designate the unit matrix in $\mathbb{C}^{p\times p}$ and the null matrix in $\mathbb{C}^{p\times q}$, respectively. If the size of a unit matrix and a null matrix is obvious, then we will also omit the indexes. For each $A \in \mathbb{C}^{q\times q}$, let $\operatorname{tr} A$ be the trace of A. If $A \in \mathbb{C}^{p\times q}$, then we denote by $\mathcal{N}(A)$ and $\mathcal{R}(A)$ the null space of A and the column space of A, respectively, and we will use $\operatorname{rank} A$ to denote the rank of A. For every choice of $x, y \in \mathbb{C}^q$, the notation $\langle x, y\rangle_{\mathrm{E}}$ stands for the (left) Euclidean inner product. For each $A \in \mathbb{C}^{p\times q}$, let $\|A\|_{\mathrm{E}} := \sqrt{\operatorname{tr}(A^*A)}$ be the Euclidean norm of A, whereas $\|A\|_{\mathrm{S}}$ stands for the operator norm of A. If $\mathcal{M}$ is a non-empty subset of $\mathbb{C}^q$, then $\mathcal{M}^\perp$ stands for the (left) orthogonal complement of $\mathcal{M}$. If $\mathcal{U}$ is a linear subspace of $\mathbb{C}^q$, then let $\mathbb{P}_{\mathcal{U}}$ be the orthogonal projection matrix onto $\mathcal{U}$, i. e., $\mathbb{P}_{\mathcal{U}}$ is the unique complex $q \times q$ matrix P that fulfills the three conditions $P^2 = P$, $P^* = P$, and $\mathcal{R}(P) = \mathcal{U}$. We will often use the Moore-Penrose inverse of a complex $p \times q$ matrix A. This is the unique complex $q \times p$ matrix X such that the four equations $AXA = A$, $XAX = X$, $(AX)^* = AX$, and $(XA)^* = XA$ hold true (see, e. g. [5, Proposition 1.1.1]). As usual, we will write $A^\dagger$ for this matrix X. If $n \in \mathbb{N}$, if $(p_j)_{j=1}^n$ is a sequence of positive integers, and if $A_j \in \mathbb{C}^{p_j\times q}$ for all $j \in \mathbb{Z}_{1,n}$, then let

$$\operatorname{col}(A_j)_{j=1}^n := \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix}.$$

Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. We will associate with $(s_j)_{j=0}^\kappa$ several matrices, which we will often need in our subsequent considerations: For all $l, m \in \mathbb{N}_0$ with $l \le m \le \kappa$, let

$$y_{l,m}^{\langle s\rangle} := \operatorname{col}(s_j)_{j=l}^m \qquad \text{and} \qquad z_{l,m}^{\langle s\rangle} := [s_l, s_{l+1}, \dots, s_m]. \tag{4.1}$$

Let

$$H_n^{\langle s\rangle} := [s_{j+k}]_{j,k=0}^n \qquad \text{for all } n \in \mathbb{N}_0 \text{ with } 2n \le \kappa, \tag{4.2}$$

$$K_n^{\langle s\rangle} := [s_{j+k+1}]_{j,k=0}^n \qquad \text{for all } n \in \mathbb{N}_0 \text{ with } 2n+1 \le \kappa. \tag{4.3}$$

Let

$$L_0^{\langle s\rangle} := s_0, \qquad L_n^{\langle s\rangle} := s_{2n} - z_{n,2n-1}^{\langle s\rangle}(H_{n-1}^{\langle s\rangle})^\dagger y_{n,2n-1}^{\langle s\rangle} \tag{4.4}$$

for all $n \in \mathbb{N}$ with $2n \le \kappa$. Let

$$\Theta_0^{\langle s\rangle} := 0_{p\times q} \qquad \text{and let} \qquad \Theta_n^{\langle s\rangle} := z_{n,2n-1}^{\langle s\rangle}(H_{n-1}^{\langle s\rangle})^\dagger y_{n,2n-1}^{\langle s\rangle} \tag{4.5}$$

for all $n \in \mathbb{N}$ with $2n - 1 \leq \kappa$. In situations in which it is obvious which sequence $(s_j)_{j=0}^{\kappa}$ of complex matrices is meant, we will write $y_{l,m}$, $z_{l,m}$, H_n, K_n, L_n, and Θ_n instead of $y_{j,k}^{\langle s\rangle}$, $z_{j,k}^{\langle s\rangle}$, $H_n^{\langle s\rangle}$, $K_n^{\langle s\rangle}$, $L_n^{\langle s\rangle}$, and $\Theta_n^{\langle s\rangle}$, respectively.

Let $\alpha \in \mathbb{C}$ and let $\kappa \in \mathbb{N} \cup \{\infty\}$. Then the sequence $(a_j)_{j=0}^{\kappa-1}$ given by

$$a_j := s_{\alpha\triangleright j} \qquad \text{and} \qquad s_{\alpha\triangleright j} := -\alpha s_j + s_{j+1} \tag{4.6}$$

for all $j \in \mathbb{Z}_{0,\kappa-1}$ plays a key role in our following considerations. We define

$$\begin{aligned}
&\Theta_{\alpha\triangleright n} := \Theta_n^{\langle a\rangle} && && \text{for all } n \in \mathbb{N}_0 \text{ with } 2n \leq \kappa, \\
&H_{\alpha\triangleright n} := H_n^{\langle a\rangle}, && L_{\alpha\triangleright n} := L_n^{\langle a\rangle} && \text{for all } n \in \mathbb{N}_0 \text{ with } 2n+1 \leq \kappa, \\
&K_{\alpha\triangleright n} := K_n^{\langle a\rangle} && && \text{for all } n \in \mathbb{N}_0 \text{ with } 2n+2 \leq \kappa,
\end{aligned} \tag{4.7}$$

and $y_{\alpha\triangleright l,m} := y_{l,m}^{\langle a\rangle}$ and $z_{\alpha\triangleright l,m} := z_{l,m}^{\langle a\rangle}$ for all $l, m \in \mathbb{N}_0$ with $l \leq m \leq \kappa$. Obviously, $-\alpha H_n + K_n = H_{\alpha\triangleright n}$ for all $n \in \mathbb{N}_0$ with $2n + 1 \leq \kappa$.

Remark 4.1 If $(s_j)_{j=0}^{\kappa}$ is a sequence of complex $p \times q$ matrices, then $H_n = \begin{bmatrix} H_{n-1} & y_{n,2n-1} \\ z_{n,2n-1} & s_{2n} \end{bmatrix}$ for all $n \in \mathbb{N}$ with $2n \leq \kappa$.

Let

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{4.8}$$

be the block representation of a complex $(p+q) \times (r+s)$ matrix M with $p \times r$ block A. Then we consider the *Schur complement*

$$M/A := D - CA^{\dagger}B. \tag{4.9}$$

In view of Remark 4.1 and (4.4), we have then:

Remark 4.2 Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. For all $n \in \mathbb{N}$ with $2n \leq \kappa$, then L_n is the Schur complement H_n/H_{n-1}.

Now we turn our attention to an important subclass of Hankel non-negative definite sequences. For all $n \in \mathbb{N}_0$, let $\mathcal{H}_{q,2n}^{\succcurlyeq,\mathrm{cd}} := \{(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\succcurlyeq} \colon L_n^{\langle s\rangle} = 0_{q\times q}\}$. The elements of the set $\mathcal{H}_{q,2n}^{\succcurlyeq,\mathrm{cd}}$ are called *Hankel completely degenerate*.

Let $(s_j)_{j=0}^{\infty} \in \mathcal{H}_{q,\infty}^{\succcurlyeq}$ and let $n \in \mathbb{N}_0$. Then $(s_j)_{j=0}^{\infty}$ is called *Hankel completely degenerate of order n* if $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\succcurlyeq,\mathrm{cd}}$. The symbol $\mathcal{H}_{q,\infty}^{\succcurlyeq,\mathrm{cd},n}$ stands for the set of all Hankel completely degenerate sequences of order n. A sequence $(s_j)_{j=0}^{\infty}$ belonging to $\mathcal{H}_{q,\infty}^{\succcurlyeq}$ is called *Hankel completely degenerate* if there exists an $n \in \mathbb{N}_0$

such that $(s_j)_{j=0}^{\infty}$ is Hankel completely degenerate of order n. Obviously, $\mathcal{H}_{q,\infty}^{\succcurlyeq,\mathrm{cd},n} \subseteq \mathcal{H}_{q,\infty}^{\succcurlyeq,\mathrm{cd},\ell}$ for all integers ℓ and n with $0 \leq n \leq \ell$.

For every choice of $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}_0$, let $\mathcal{K}_{q,2n,\alpha}^{\succcurlyeq,\mathrm{cd}} := \mathcal{K}_{q,2n,\alpha}^{\succcurlyeq} \cap \mathcal{H}_{q,2n}^{\succcurlyeq,\mathrm{cd}}$ and let $\mathcal{K}_{q,2n+1,\alpha}^{\succcurlyeq,\mathrm{cd}} := \{(s_j)_{j=0}^{2n+1} \in \mathcal{K}_{q,2n+1,\alpha}^{\succcurlyeq} \colon (s_{\alpha\triangleright j})_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\succcurlyeq,\mathrm{cd}}\}$.

Example 4.3 Let $\alpha \in \mathbb{R}$. Then $\mathcal{K}_{q,0,\alpha}^{\succcurlyeq,\mathrm{cd}} = \{(0_{q\times q})_{j=0}^{0}\}$ and $(0_{q\times q})_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{cd}}$ for all $m \in \mathbb{N}_0$.

Let $\alpha \in \mathbb{R}$ and let $(s_j)_{j=0}^{\infty} \in \mathcal{K}_{q,\infty,\alpha}^{\succcurlyeq}$. For each $m \in \mathbb{N}_0$, then $(s_j)_{j=0}^{\infty}$ is called *α-Stieltjes completely degenerate of order m* if $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{cd}}$. The sequence $(s_j)_{j=0}^{\infty}$ is called *α-Stieltjes completely degenerate* if there exists an $m \in \mathbb{N}_0$ such that $(s_j)_{j=0}^{\infty}$ is α-Stieltjes completely degenerate of order m. For each $m \in \mathbb{N}_0$, we denote by $\mathcal{K}_{q,\infty,\alpha}^{\succcurlyeq,\mathrm{cd},m}$ the set of all sequences $(s_j)_{j=0}^{\infty} \in \mathcal{K}_{q,\infty,\alpha}^{\succcurlyeq}$ which are α-Stieltjes completely degenerate of order m. Furthermore, we will write $\mathcal{K}_{q,\infty,\alpha}^{\succcurlyeq,\mathrm{cd}}$ for the set of all sequences $(s_j)_{j=0}^{\infty} \in \mathcal{K}_{q,\infty,\alpha}^{\succcurlyeq}$ which are α-Stieltjes completely degenerate. Obviously, $\bigcup_{m=0}^{\infty} \mathcal{K}_{q,\infty,\alpha}^{\succcurlyeq,\mathrm{cd},m} = \mathcal{K}_{q,\infty,\alpha}^{\succcurlyeq,\mathrm{cd}}$.

Example 4.4 Let $\alpha \in \mathbb{R}$. Then Example 4.3 shows that $\mathcal{K}_{q,\infty,\alpha}^{\succcurlyeq,\mathrm{cd},0} = \{(0_{q\times q})_{j=0}^{\infty}\}$.

Proposition 4.5 ([16, Proposition 5.9]) *Let $\alpha \in \mathbb{R}$ and $m \in \mathbb{N}_0$. Then $\mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{cd}} \subseteq \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$.*

Now we turn our attention to the elements of sequences $(s_j)_{j=0}^{m}$ belonging to $\mathcal{K}_{q,m,\alpha}^{\succcurlyeq}$. First we describe, for an arbitrary sequence $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$, the set of all $s_{m+1} \in \mathbb{C}^{q\times q}$ such that $(s_j)_{j=0}^{m+1} \in \mathcal{K}_{q,m+1,\alpha}^{\succcurlyeq}$.

Proposition 4.6 ([10, Proposition 4.9]) *Let $\alpha \in \mathbb{R}$, let $n \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{2n} \in \mathcal{K}_{q,2n,\alpha}^{\succcurlyeq,\mathrm{e}}$. Further, let $s_{2n+1} \in \mathbb{C}^{q\times q}$. Then the following statements hold true:*

(a) *The sequence $(s_j)_{j=0}^{2n+1}$ belongs to $\mathcal{K}_{q,2n+1,\alpha}^{\succcurlyeq}$ if and only if there exists a matrix $G \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ such that $s_{2n+1} = \Theta_{\alpha\triangleright n} + \alpha s_{2n} + G$.*
(b) *The sequence $(s_j)_{j=0}^{2n+1}$ belongs to $\mathcal{K}_{q,2n+1,\alpha}^{\succcurlyeq,\mathrm{e}}$ if and only if there exists a matrix $G \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ such that $s_{2n+1} = \Theta_{\alpha\triangleright n} + \alpha s_{2n} + L_n L_n^{\dagger} G L_n L_n^{\dagger}$.*

Proposition 4.7 ([10, Proposition 4.13]) *Let $\alpha \in \mathbb{R}$, let $n \in \mathbb{N}$, and let $(s_j)_{j=0}^{2n-1} \in \mathcal{K}_{q,2n-1,\alpha}^{\succcurlyeq,\mathrm{e}}$. Furthermore, let $s_{2n} \in \mathbb{C}^{q\times q}$. Then the following statements hold true:*

(a) *The sequence $(s_j)_{j=0}^{2n}$ belongs to $\mathcal{K}_{q,2n,\alpha}^{\succcurlyeq}$ if and only if there exists a matrix $G \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ such that $s_{2n} = \Theta_n + G$.*

(b) *The sequence* $(s_j)_{j=0}^{2n}$ *belongs to* $\mathcal{K}_{q,2n,\alpha}^{\succcurlyeq,\mathrm{e}}$ *if and only if there exists a matrix* $G \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ *such that* $s_{2n} = \Theta_n + L_{\alpha\triangleright n-1} L_{\alpha\triangleright n-1}^{\dagger} G L_{\alpha\triangleright n-1} L_{\alpha\triangleright n-1}^{\dagger}$.

A closer look at Propositions 4.6 and 4.7 leads us to the following notion.

Remark 4.8 Let $\alpha \in \mathbb{C}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m}$ be a sequence from $\mathbb{C}^{p\times q}$. Then

$$\mathfrak{a}_m := \begin{cases} \Theta_{\alpha\triangleright n} + \alpha s_{2n} & \text{if } m = 2n \text{ with some } n \in \mathbb{N}_0 \\ \Theta_n & \text{if } m = 2n-1 \text{ with some } n \in \mathbb{N} \end{cases} \tag{4.10}$$

is called the *α-Stieltjes minimal element associated with* $(s_j)_{j=0}^{m}$. Obviously, one can see that $s_{2n+1} - \mathfrak{a}_{2n} = L_{\alpha\triangleright n}$ for all $n \in \mathbb{N}_0$ and $s_{2n} - \mathfrak{a}_{2n-1} = L_n$ for all $n \in \mathbb{N}$.

Example 4.9 If $\alpha \in \mathbb{C}$, if $m \in \mathbb{N}_0$, and if $s_j := 0_{p\times q}$ for all $j \in \mathbb{Z}_{0,m}$, then $\mathfrak{a}_m = 0_{p\times q}$.

The following remark shows why the notion "α-Stieltjes minimal element associated with $(s_j)_{j=0}^{m}$" was chosen in Remark 4.8.

Remark 4.10 Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$. Denote by $\mathfrak{a}_m$ the α-Stieltjes minimal element associated with $(s_j)_{j=0}^{m}$. Let $s_{m+1} \in \mathbb{C}^{q\times q}$ be such that $(s_j)_{j=0}^{m+1} \in \mathcal{K}_{q,m+1,\alpha}^{\succcurlyeq}$. In view of Propositions 4.6(a) and 4.7(a), then $s_{m+1} - \mathfrak{a}_m \in \mathbb{C}_{\succcurlyeq}^{q\times q}$.

Proposition 4.11 *Let* $\alpha \in \mathbb{R}$*, let* $m \in \mathbb{N}_0$*, and let* $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{cd}}$*. Denote by* $\mathfrak{a}_m$ *the α-Stieltjes minimal element associated with* $(s_j)_{j=0}^{m}$*. Then*

$$\left\{s_{m+1} \in \mathbb{C}^{q\times q} : (s_j)_{j=0}^{m+1} \in \mathcal{K}_{q,m+1,\alpha}^{\succcurlyeq,\mathrm{e}}\right\} = \{\mathfrak{a}_m\}.$$

Proof Proposition 4.5 shows that $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$. Because of $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{cd}}$, we infer $L_n = 0_{q\times q}$ if $m = 2n$ with some $n \in \mathbb{N}_0$ and $L_{\alpha\triangleright n-1} = 0_{q\times q}$ if $m = 2n-1$ with some $n \in \mathbb{N}$. Applying Propositions 4.6(b) and 4.7(b) and Remark 4.8 completes the proof. □

Proposition 4.12 *Let* $\alpha \in \mathbb{R}$*, let* $m \in \mathbb{N}_0$*, and let* $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$*. Denote by* $\mathfrak{a}_m$ *the α-Stieltjes minimal element associated with* $(s_j)_{j=0}^{m}$*. Further, let* $s_{m+1} \in \mathbb{C}^{q\times q}$*. Then* $(s_j)_{j=0}^{m+1} \in \mathcal{K}_{q,m+1,\alpha}^{\succcurlyeq,\mathrm{cd}}$ *if and only* $s_{m+1} = \mathfrak{a}_m$*.*

Proof If $(s_j)_{j=0}^{m+1} \in \mathcal{K}_{q,m+1,\alpha}^{\succcurlyeq,\mathrm{cd}}$ then $s_{m+1} = \mathfrak{a}_m$ because of Remark 4.8. Conversely, suppose $s_{m+1} = \mathfrak{a}_m$. Then Propositions 4.6(a) and 4.7(a) yield $(s_j)_{j=0}^{m+1} \in \mathcal{K}_{q,m+1,\alpha}^{\succcurlyeq}$. Consequently, because of $s_{m+1} = \mathfrak{a}_m$ and Remark 4.8, we get $(s_j)_{j=0}^{m+1} \in \mathcal{K}_{q,m+1,\alpha}^{\succcurlyeq,\mathrm{cd}}$. □

Proposition 4.13 *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, let $m \in \mathbb{Z}_{0,\kappa}$, and let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}^{\succcurlyeq}_{q,\kappa,\alpha}$ be such that $(s_j)_{j=0}^{m} \in \mathcal{K}^{\succcurlyeq,\mathrm{cd}}_{q,m,\alpha}$. Then $(s_j)_{j=0}^{\ell} \in \mathcal{K}^{\succcurlyeq,\mathrm{cd}}_{q,\ell,\alpha}$ for all $\ell \in \mathbb{Z}_{m,\kappa}$.*

Proof It is sufficient to consider the case $m < \kappa$. In view of $(s_j)_{j=0}^{\kappa} \in \mathcal{K}^{\succcurlyeq}_{q,\kappa,\alpha}$, we infer from Remarks 3.1 and 3.3 that $(s_j)_{j=0}^{m+1} \in \mathcal{K}^{\succcurlyeq,\mathrm{e}}_{q,m+1,\alpha}$. Thus, the combination of Propositions 4.11 and 4.12 yields $(s_j)_{j=0}^{m+1} \in \mathcal{K}^{\succcurlyeq,\mathrm{cd}}_{q,m+1,\alpha}$. By induction, we obtain then that $(s_j)_{j=0}^{\ell} \in \mathcal{K}^{\succcurlyeq,\mathrm{cd}}_{q,\ell,\alpha}$ for all $\ell \in \mathbb{Z}_{m,\kappa}$. □

Corollary 4.14 *Let $\alpha \in \mathbb{R}$, let $(s_j)_{j=0}^{\infty} \in \mathcal{K}^{\succcurlyeq}_{q,\infty,\alpha}$, and let $m \in \mathbb{N}_0$. If $(s_j)_{j=0}^{\infty}$ is α-Stieltjes completely degenerate of order m, then $(s_j)_{j=0}^{\infty}$ is α-Stieltjes completely degenerate of order ℓ for all $\ell \in \mathbb{Z}_{m,\infty}$.*

Proof This follows from Proposition 4.13. □

Now we iterate the construction from Remark 4.8.

Definition 4.15 Let $\alpha \in \mathbb{C}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m}$ be a sequence from $\mathbb{C}^{p\times q}$. Further, let $(s_j)_{j=m+1}^{\infty}$ be a sequence from $\mathbb{C}^{p\times q}$. Then $(s_j)_{j=0}^{\infty}$ is called the *α-Stieltjes minimal extension of* $(s_j)_{j=0}^{m}$ if, for all $\ell \in \mathbb{Z}_{m+1,\infty}$, the matrix s_ℓ coincides with the α-Stieltjes minimal element associated with $(s_j)_{j=0}^{\ell-1}$.

Example 4.16 Let $\alpha \in \mathbb{C}$, let $m \in \mathbb{N}_0$, and let $s_j := 0_{p\times q}$ for all $j \in \mathbb{Z}_{0,m}$. In view of Example 4.9, the α-Stieltjes minimal extension $(s_j)_{j=0}^{\infty}$ of $(s_j)_{j=0}^{m}$ fulfills $s_j = 0_{p\times q}$ for all $j \in \mathbb{Z}_{m+1,\infty}$.

Proposition 4.17 *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m} \in \mathcal{K}^{\succcurlyeq,\mathrm{e}}_{q,m,\alpha}$. Then there is a unique sequence $(s_j)_{j=m+1}^{\infty}$ from $\mathbb{C}^{q\times q}$ such that $(s_j)_{j=0}^{\infty} \in \mathcal{K}^{\succcurlyeq,\mathrm{cd},m+1}_{q,\infty,\alpha}$, namely, that sequence $(s_j)_{j=m+1}^{\infty}$, for which $(s_j)_{j=0}^{\infty}$ is the α-Stieltjes minimal extension of $(s_j)_{j=0}^{m}$.*

Proof Apply Propositions 4.12 and 4.5 and use Definition 4.15. □

As an immediate consequence of Proposition 4.17 and $\mathcal{K}^{\succcurlyeq,\mathrm{cd}}_{q,\infty,\alpha} \subseteq \mathcal{K}^{\succcurlyeq}_{q,\infty,\alpha}$ for all $\alpha \in \mathbb{R}$, we obtain the following result, which also follows from Proposition 4.6(b):

Corollary 4.18 *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m} \in \mathcal{K}^{\succcurlyeq,\mathrm{e}}_{q,m,\alpha}$. Then there exists a sequence $(s_j)_{j=m+1}^{\infty}$ of complex $q \times q$ matrices such that $(s_j)_{j=0}^{\infty} \in \mathcal{K}^{\succcurlyeq}_{q,\infty,\alpha}$.*

Corollary 4.19 *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m} \in \mathcal{K}^{\succcurlyeq,\mathrm{cd}}_{q,m,\alpha}$. Then there is a unique sequence $(s_j)_{j=m+1}^{\infty}$ from $\mathbb{C}^{q\times q}$ such that $(s_j)_{j=0}^{\infty} \in \mathcal{K}^{\succcurlyeq}_{q,\infty,\alpha}$, namely, that sequence $(s_j)_{j=m+1}^{\infty}$ for which $(s_j)_{j=0}^{\infty}$ is the α-Stieltjes minimal extension of $(s_j)_{j=0}^{m}$.*

Proof Combine Propositions 4.11 and 4.17. □

Proposition 4.17 leads us to the following notion.

Definition 4.20 Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$. Further, let $(s_j)_{j=m+1}^{\infty}$ be the unique sequence from $\mathbb{C}^{q\times q}$ such that $(s_j)_{j=0}^{\infty} \in \mathcal{K}_{q,\infty,\alpha}^{\succcurlyeq,\mathrm{cd},m+1}$. Then $(s_j)_{j=0}^{\infty}$ is called the *α-Stieltjes completely degenerate sequence associated with* $(s_j)_{j=0}^{m}$.

Example 4.21 Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $s_j := 0_{q\times q}$ for $j \in \mathbb{Z}_{0,m}$. Then $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{cd}}$. In view of Proposition 4.17 and Example 4.16, then the α-Stieltjes completely degenerate sequence $(s_j)_{j=0}^{\infty}$ associated with $(s_j)_{j=0}^{m}$ fulfills $s_j = 0_{q\times q}$ for $j \in \mathbb{Z}_{m+1,\infty}$.

5 Right α-Stieltjes Parametrization

In this section, we recall some basic facts on right α-Stieltjes parametrization from [10, 16]. We use the Löwner semi-ordering in $\mathbb{C}_{\mathrm{H}}^{q\times q}$, i. e., we write $A \succcurlyeq B$ or $B \preccurlyeq A$ in order to indicate that A and B are Hermitian complex matrices such that the matrix $A - B$ is non-negative Hermitian. Before introducing the central notion of this section, we note that we again use the matrices introduced in Sect. 4.

Definition 5.1 ([16, Definition 4.2]) Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. Then the sequence $(Q_j)_{j=0}^{\kappa}$ given by $Q_{2k} := L_k$ for all $k \in \mathbb{N}_0$ with $2k \le \kappa$ and by $Q_{2k+1} := L_{\alpha\triangleright k}$ for all $k \in \mathbb{N}_0$ with $2k + 1 \le \kappa$ is called the *right α-Stieltjes parametrization of* $(s_j)_{j=0}^{\kappa}$. In the case $\alpha = 0$, the sequence $(Q_j)_{j=0}^{\kappa}$ is simply said to be the *right Stieltjes parametrization of* $(s_j)_{j=0}^{\kappa}$.

According to (4.4), (4.6), (4.1), and (4.2), we have in particular

$$Q_0 = s_0, \qquad Q_1 = s_{\alpha\triangleright 0} = s_1 - \alpha s_0, \qquad Q_2 = s_2 - s_1 s_0^{\dagger} s_1, \tag{5.1}$$

and

$$Q_3 = s_{\alpha\triangleright 2} - s_{\alpha\triangleright 1} s_{\alpha\triangleright 0}^{\dagger} s_{\alpha\triangleright 1} = s_3 - \alpha s_2 - (s_2 - \alpha s_1)(s_1 - \alpha s_0)^{\dagger}(s_2 - \alpha s_1).$$

Remark 5.2 ([16, Remark 4.3]) Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(Q_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. Then it can be immediately checked by induction that there is a unique sequence $(s_j)_{j=0}^{\kappa}$ of complex $p \times q$ matrices such that $(Q_j)_{j=0}^{\kappa}$ is the right α-Stieltjes parametrization of $(s_j)_{j=0}^{\kappa}$, namely the sequence $(s_j)_{j=0}^{\kappa}$ recursively given by $s_{2k} = \Theta_k + Q_{2k}$ for all $k \in \mathbb{N}_0$ with $2k \le \kappa$ and $s_{2k+1} = \alpha s_{2k} + \Theta_{\alpha\triangleright k} + Q_{2k+1}$ for all $k \in \mathbb{N}_0$ with $2k + 1 \le \kappa$.

From Remarks 5.2 and 4.8 we get the following observation.

Remark 5.3 Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence from $\mathbb{C}^{p\times q}$ with right α-Stieltjes parametrization $(Q_j)_{j=0}^{\kappa}$. For each $m \in \mathbb{Z}_{0,\kappa}$, let $\mathfrak{a}_m$ be the α-Stieltjes minimal element associated with $(s_j)_{j=0}^{m}$. Furthermore, let $\mathfrak{a}_{-1} := 0_{p\times q}$. For all $j \in \mathbb{Z}_{0,\kappa}$, then $Q_j = s_j - \mathfrak{a}_{j-1}$.

Using Remark 4.2, we obtain:

Remark 5.4 Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. For all $n \in \mathbb{N}$ with $2n \leq \kappa$, then Q_{2n} is the Schur complement H_n/H_{n-1}. For all $n \in \mathbb{N}$ with $2n+1 \leq \kappa$, furthermore Q_{2n+1} is the Schur complement $H_{\alpha\triangleright n}/H_{\alpha\triangleright n-1}$.

Remark 5.5 ([16, Remark 4.8]) Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence from $\mathbb{C}^{p\times q}$. Denote by $(Q_j)_{j=0}^{\kappa}$ the right α-Stieltjes parametrization of $(s_j)_{j=0}^{\kappa}$. For all $m \in \mathbb{Z}_{0,\kappa}$, then $(Q_j)_{j=0}^{m}$ is exactly the right α-Stieltjes parametrization of $(s_j)_{j=0}^{m}$.

The following result shows that the membership of a sequence $(s_j)_{j=0}^{\kappa}$ of complex $q \times q$ matrices to one of the classes $\mathcal{K}_{q,\kappa,\alpha}^{\succcurlyeq}$, $\mathcal{K}_{q,\kappa,\alpha}^{\succcurlyeq,\mathrm{e}}$, and $\mathcal{K}_{q,\kappa,\alpha}^{\succ}$ can be characterized in terms of its right α-Stieltjes parametrization.

Theorem 5.6 ([16, Theorem 4.12]) *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices with right α-Stieltjes parametrization $(Q_j)_{j=0}^{\kappa}$. Then:*

(a) *The sequence $(s_j)_{j=0}^{\kappa}$ belongs to $\mathcal{K}_{q,\kappa,\alpha}^{\succcurlyeq}$ if and only if $Q_j \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ for all $j \in \mathbb{Z}_{0,\kappa}$ and, in the case $\kappa \geq 2$, furthermore $\mathcal{N}(Q_j) \subseteq \mathcal{N}(Q_{j+1})$ for all $j \in \mathbb{Z}_{0,\kappa-2}$.*

(b) *The sequence $(s_j)_{j=0}^{\kappa}$ belongs to $\mathcal{K}_{q,\kappa,\alpha}^{\succcurlyeq,\mathrm{e}}$ if and only if $Q_j \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ for all $j \in \mathbb{Z}_{0,\kappa}$ and, in the case $\kappa \geq 1$, furthermore $\mathcal{N}(Q_j) \subseteq \mathcal{N}(Q_{j+1})$ for all $j \in \mathbb{Z}_{0,\kappa-1}$.*

(c) *The sequence $(s_j)_{j=0}^{\kappa}$ belongs to $\mathcal{K}_{q,\kappa,\alpha}^{\succ}$ if and only if $Q_j \in \mathbb{C}_{\succ}^{q\times q}$ for all $j \in \mathbb{Z}_{0,\kappa}$.*

Against to the background of Remark 5.4, Theorem 5.6 tells us that the membership of a sequence $(s_j)_{j=0}^{\kappa}$ of complex $q \times q$ matrices to one of the classes $\mathcal{K}_{q,\kappa,\alpha}^{\succcurlyeq}$ or $\mathcal{K}_{q,\kappa,\alpha}^{\succcurlyeq,\mathrm{e}}$ can be characterized by the interplay of consecutive Schur complements in the block Hankel matrices H_n and $H_{\alpha\triangleright n}$. More precisely, the interplay is described by a successive inclusion of the null spaces of consecutive Schur complements in H_n and $H_{\alpha\triangleright n}$. This observation is very essential for the rest of this paper.

Corollary 5.7 *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$ with right α-Stieltjes parametrization $(Q_j)_{j=0}^{m}$. Then the following statements are equivalent:*

(i) $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succ}$.

(ii) $Q_m \in \mathbb{C}_{\succ}^{q\times q}$.

(iii) $\det Q_m \neq 0$.

Proof Because of $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$, Theorem 5.6(b) implies:

(I) $(Q_j)_{j=0}^{m}$ is a sequence from $\mathbb{C}_{\succcurlyeq}^{q \times q}$.
(II) In the case $m \geq 1$, for all $j \in \mathbb{Z}_{0,m-1}$ it holds $\mathcal{N}(Q_j) \subseteq \mathcal{N}(Q_{j+1})$.

"(i)⇒(ii)": This follows from Theorem 5.6(c).
"(ii)⇒(iii)": This is trivial.
"(iii)⇒(i)": In view of (iii) we have $\mathcal{N}(Q_m) = \{0_{q\times 1}\}$. Thus, (II) implies for all $j \in \mathbb{Z}_{0,m}$ that $\mathcal{N}(Q_j) = \{0_{q\times 1}\}$. Combining this with (I) we infer that $(Q_j)_{j=0}^{m}$ is a sequence from $\mathbb{C}_{\succ}^{q\times q}$. Thus, Theorem 5.6(c) implies (i). □

Proposition 5.8 ([16, Proposition 5.3]) *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq}$ with right α-Stieltjes parametrization $(Q_j)_{j=0}^{m}$. Then $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{cd}}$ if and only if $Q_m = 0_{q\times q}$.*

Corollary 5.9 *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{\infty} \in \mathcal{K}_{q,\infty,\alpha}^{\succcurlyeq,\mathrm{cd},m}$. Let $(Q_j)_{j=0}^{\infty}$ be the right α-Stieltjes parametrization of $(s_j)_{j=0}^{\infty}$. For each $\ell \in \mathbb{Z}_{m,\infty}$, then $Q_\ell = 0_{q\times q}$.*

Proof Let $\ell \in \mathbb{Z}_{m,\infty}$. From Corollary 4.14 we infer then $(s_j)_{j=0}^{\ell} \in \mathcal{K}_{q,\ell,\alpha}^{\succcurlyeq,\mathrm{cd}}$, whereas Remark 5.5 yields that $(Q_j)_{j=0}^{\ell}$ is the right α-Stieltjes parametrization of $(s_j)_{j=0}^{\ell}$. Thus, Proposition 5.8 implies $Q_\ell = 0_{q\times q}$. □

Lemma 5.10 *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{\infty} \in \mathcal{K}_{q,\infty,\alpha}^{\succcurlyeq,\mathrm{cd},m}$. Then:*

(a) *If $m = 2n$ with some $n \in \mathbb{N}_0$, then the sequences $(s_j)_{j=0}^{\infty}$ and $(s_{\alpha\triangleright j})_{j=0}^{\infty}$ are both Hankel completely degenerate of order n.*
(b) *If $m = 2n + 1$ with some $n \in \mathbb{N}_0$, then $(s_{\alpha\triangleright j})_{j=0}^{\infty}$ is Hankel completely degenerate of order n and $(s_j)_{j=0}^{\infty}$ is Hankel completely degenerate of order $n + 1$.*

Proof First observe that the sequences $(s_j)_{j=0}^{\infty}$ and $(s_{\alpha\triangleright j})_{j=0}^{\infty}$ both belong to $\mathcal{H}_{q,\infty}^{\succcurlyeq}$ by virtue of the definitions of $\mathcal{K}_{q,\infty,\alpha}^{\succcurlyeq}$ and $\mathcal{H}_{q,\infty}^{\succcurlyeq}$. Let $(Q_j)_{j=0}^{\infty}$ be the right α-Stieltjes parametrization of $(s_j)_{j=0}^{\infty}$. Because of the choice of $(s_j)_{j=0}^{\infty}$, we obtain from Corollary 5.9 that $Q_m = 0_{q\times q}$ and $Q_{m+1} = 0_{q\times q}$. In view of Definition 5.1, the proof is complete. □

Remark 5.11 Let $\alpha \in \mathbb{R}$. Then $\mathcal{K}_{q,\infty,\alpha}^{\succcurlyeq,\mathrm{cd}} \subseteq \mathcal{H}_{q,\infty}^{\succcurlyeq,\mathrm{cd}}$ by virtue of Lemma 5.10.

Remark 5.12 Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m}$ be a sequence of complex $q \times q$ matrices. Denote by $(\overline{s}_j)_{j=0}^{\infty}$ the α-Stieltjes minimal extension of $(s_j)_{j=0}^{m}$.

In view of Proposition 4.17 and Remark 3.4, then the following statements are equivalent:

(i) $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$.
(ii) $(\overline{s}_j)_{j=0}^{\infty} \in \mathcal{K}_{q,\infty,\alpha}^{\succcurlyeq,\mathrm{cd},m+1}$.
(iii) $(\overline{s}_j)_{j=0}^{\infty} \in \mathcal{K}_{q,\infty,\alpha}^{\succcurlyeq}$.

Definition 5.13 Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq}$. If $m \geq 1$, then let $(\underline{s}_j)_{j=0}^{\infty}$ be the α-Stieltjes minimal extension of $(s_j)_{j=0}^{m-1}$. If $m = 0$, then let $(\underline{s}_j)_{j=0}^{\infty}$ be given by $\underline{s}_j := 0_{q\times q}$ for each $j \in \mathbb{N}_0$. Then the sequence $(\underline{s}_j)_{j=0}^{\infty}$ is called the *lower α-Stieltjes completely degenerate sequence associated with* $(s_j)_{j=0}^{m}$.

The choice of the notion introduced in Definition 5.13 is caused by the following:

Lemma 5.14 *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}$, and let $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq}$. Further, let $(\underline{s}_j)_{j=0}^{\infty}$ be the lower α-Stieltjes completely degenerate sequence associated with $(s_j)_{j=0}^{m}$. Then $(\underline{s}_j)_{j=0}^{\infty} \in \mathcal{K}_{q,\infty,\alpha}^{\succcurlyeq,\mathrm{cd},m}$ and $s_m - \underline{s}_m \in \mathbb{C}_{\succcurlyeq}^{q\times q}$.*

Proof First we consider the case $m \geq 1$. In view of $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq}$, Remark 3.4 yields $(s_j)_{j=0}^{m-1} \in \mathcal{K}_{q,m-1,\alpha}^{\succcurlyeq,\mathrm{e}}$. Thus, Definition 5.13 and Remark 5.12 give $(\underline{s}_j)_{j=0}^{\infty} \in \mathcal{K}_{q,\infty,\alpha}^{\succcurlyeq,\mathrm{cd},m}$. By construction (see Definition 4.15 and Remark 4.8), the matrix $\underline{s}_m$ is the α-Stieltjes minimal element associated with $(s_j)_{j=0}^{m-1}$. Thus, taking into account $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq}$, we infer from Remark 4.10 that $s_m - \underline{s}_m \in \mathbb{C}_{\succcurlyeq}^{q\times q}$. If $m = 0$, then Example 4.4 and Remarks 3.3 and 3.4 show that $(\underline{s}_j)_{j=0}^{\infty}$ belongs to $\mathcal{K}_{q,\infty,\alpha}^{\succcurlyeq}$ and that $s_0 - \underline{s}_0 \in \mathbb{C}_{\succcurlyeq}^{q\times q}$. □

Definition 5.15 Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$. Then the α-Stieltjes minimal extension $(\overline{s}_j)_{j=0}^{\infty}$ of $(s_j)_{j=0}^{m}$ is called the *upper α-Stieltjes completely degenerate sequence associated with* $(s_j)_{j=0}^{m}$.

Remark 5.16 Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}$, and let $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$. Then the lower α-Stieltjes completely degenerate sequence $(\overline{s}_j)_{j=0}^{\infty}$ associated with $(s_j)_{j=0}^{m}$ is exactly the upper α-Stieltjes completely degenerate sequence associated with $(s_j)_{j=0}^{m-1}$.

The following result shows why we chosen the notion introduced in Definition 5.15.

Proposition 5.17 *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$. Then:*

(a) *$(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq}$ and the upper α-Stieltjes completely degenerate sequence $(\overline{s}_j)_{j=0}^{\infty}$ associated with $(s_j)_{j=0}^{m}$ belongs to $\mathcal{K}_{q,\infty,\alpha}^{\succcurlyeq,\mathrm{cd},m+1}$. Furthermore, each $s_{m+1} \in \mathbb{C}^{q\times q}$ such that $(s_j)_{j=0}^{m+1} \in \mathcal{K}_{q,m+1,\alpha}^{\succcurlyeq}$ fulfills $s_{m+1} - \overline{s}_{m+1} \in \mathbb{C}_{\succcurlyeq}^{q\times q}$.*

(b) *Let* $\mathfrak{a}_{-1} := 0_{q\times q}$ *and, for each* $m \in \mathbb{N}$, *let* $\mathfrak{a}_{m-1}$ *be the* α*-Stieltjes minimal element associated with* $(s_j)_{j=0}^{m-1}$. *Let* $(\underline{s}_j)_{j=0}^{\infty}$ *be the lower* α*-Stieltjes completely degenerate sequence associated with* $(s_j)_{j=0}^{m}$ *and let* $(Q_j)_{j=0}^{m}$ *be the right* α*-Stieltjes parametrization of* $(s_j)_{j=0}^{m}$. *Then the following statements are equivalent:*

(i) $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{cd}}$.
(ii) $(\overline{s}_j)_{j=0}^{\infty} = (\underline{s}_j)_{j=0}^{\infty}$.
(iii) $s_m = \mathfrak{a}_{m-1}$.
(iv) $Q_m = 0_{q\times q}$.

Proof

(a) Use Remarks 5.12, 4.10, and 3.4.
(b) "(i)⇔(iii)": In the case $m = 0$, this equivalence follows from Example 4.3. In the case $m \in \mathbb{N}$, the application of Proposition 4.12 yields the equivalence of (i) and (iii).
"(ii)⇔(iii)": From the construction of the sequences $(\overline{s}_j)_{j=0}^{\infty}$ and $(\underline{s}_j)_{j=0}^{\infty}$ we see that (ii) is equivalent to $\overline{s}_m = \underline{s}_m$. Further, by construction of $\overline{s}_m$ and $\underline{s}_m$, we have $\overline{s}_m = s_m$ and $\underline{s}_m = \mathfrak{a}_{m-1}$. Thus, (ii) and (iii) are equivalent.
"(iii)⇔(iv)": This follows from Remark 5.3.

□

6 On Distinguished Molecular Solutions of the Truncated Matricial $[\alpha, \infty)$-Stieltjes Moment Problems

In the preceding section, we have seen that the α-Stieltjes completely degenerate sequences of complex $q \times q$ matrices occupy a distinguished role within the set of all right-sided α-Stieltjes non-negative definite sequences of complex $q \times q$ matrices. In this section, we will discuss α-Stieltjes completely degenerate sequences under the view of truncated matricial Stieltjes moment problems. We will see that such sequences are intimately connected to molecular solutions having certain extremal properties.

For arbitrarily given $\alpha \in \mathbb{R}$ and $m \in \mathbb{N}_0$, we consider an arbitrary sequence $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{cd}}$. In view of Proposition 4.5, then $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$. Thus, Theorem 3.5(a) tells us that $\mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty); (s_j)_{j=0}^{m}, =] \neq \emptyset$. In particular, $\mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty); (s_j)_{j=0}^{m}, \preccurlyeq] \neq \emptyset$. In [10, Section 6], we took a closer look at these sets. For the convenience of the reader, we recall some essential features of this topic.

We start with an observation on the matricial Hamburger moment problem. If $\Omega \in \mathfrak{B}_{\mathbb{R}} \setminus \{\emptyset\}$ and if $\sigma \in \mathcal{M}_q^{\succcurlyeq}(\Omega)$ then σ is said to be *molecular* if there exists a finite subset N of Ω such that $\sigma(\Omega \setminus N) = 0_{q\times q}$.

Theorem 6.1 ([11, Propositions 2.38 and 4.9]) *Let $n \in \mathbb{N}$ and let $(s_j)_{j=0}^{2n-1} \in \mathcal{H}_{q,2n-1}^{\succcurlyeq,\mathrm{e}}$. Then:*

(a) *There is a unique sequence $(s_k)_{k=2n}^{\infty}$ of complex $q \times q$ matrices such that $(s_j)_{j=0}^{\infty}$ is a Hankel completely degenerate Hankel non-negative definite sequence of order n.*
(b) *The set $\mathcal{M}_q^{\succcurlyeq}[\mathbb{R}; (s_j)_{j=0}^{\infty}, =]$ contains exactly one element σ_n.*
(c) *The measure σ_n is molecular. In particular, $\sigma_n \in \mathcal{M}_{q,\infty}^{\succcurlyeq}(\mathbb{R})$.*

Definition 6.2 ([11, Definition 4.10]) Let $n \in \mathbb{N}$ and let $(s_j)_{j=0}^{2n-1} \in \mathcal{H}_{q,2n-1}^{\succcurlyeq,\mathrm{e}}$. Then the measure σ_n given via Theorem 6.1(b) is called the *Hankel completely degenerate non-negative Hermitian measure* (short *CD-measure*) associated with $(s_j)_{j=0}^{2n-1}$.

In the situation, that an odd number of prescribed matricial moments are given, a characterization of the case that the set $\mathcal{M}_q^{\succcurlyeq}[\mathbb{R}; (s_j)_{j=0}^{2n}, =]$ consists of exactly one element μ is stated in [11, Theorem 8.7]. For additional information, we refer to [11, Chs. 4 and 5].

Theorem 6.3 ([10, Theorems 6.1 and 6.3]) *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m}$ be a sequence from $\mathbb{C}^{q\times q}$. Then:*

(a) *The set $\mathcal{M}_q^{\succcurlyeq}[[\alpha, \infty); (s_j)_{j=0}^{m}, =]$ consists of exactly one element μ if and only if $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{cd}}$.*
(b) *Let $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{cd}}$. Then:*
  (b1) *If $m = 0$, then μ is the $q \times q$ zero measure on $\mathfrak{B}_{[\alpha,\infty)}$.*
  (b2) *If $m = 2n$ with some $n \in \mathbb{N}$, then $(s_j)_{j=0}^{2n-1} \in \mathcal{H}_{q,2n-1}^{\succcurlyeq,\mathrm{e}}$ and μ is the restriction of the CD-measure σ_n associated with $(s_j)_{j=0}^{2n-1}$ onto $\mathfrak{B}_{[\alpha,\infty)}$.*
  (b3) *If $m = 2n + 1$ with some $n \in \mathbb{N}_0$, then $(s_j)_{j=0}^{2n+1} \in \mathcal{H}_{q,2n+1}^{\succcurlyeq,\mathrm{e}}$ and μ is the restriction of the CD-measure σ_{n+1} associated with $(s_j)_{j=0}^{2n+1}$ onto $\mathfrak{B}_{[\alpha,\infty)}$.*

Theorem 6.3 leads us to the following notion.

Definition 6.4 Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{cd}}$. Then the unique measure $\mu \in \mathcal{M}_q^{\succcurlyeq}([\alpha, \infty))$ which satisfies $\mathcal{M}_q^{\succcurlyeq}[[\alpha, \infty); (s_j)_{j=0}^{m}, =] = \{\mu\}$ is called the $[\alpha, \infty)$*-measure associated with* $(s_j)_{j=0}^{m}$. It will be also denoted by $\mu_{(s_j)_{j=0}^{m}}$.

Remark 6.5 Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{cd}}$. Then the combination of Definition 6.2, Theorems 6.3(b), and 6.1(c) shows that the measure $\mu_{(s_j)_{j=0}^{m}}$ is molecular and belongs to $\mathcal{M}_{q,\infty}^{\succcurlyeq}([\alpha, \infty))$.

Remark 6.6 Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}$, let $\ell \in \mathbb{Z}_{0,m-1}$, and let $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq}$ be such that $(s_j)_{j=0}^{\ell} \in \mathcal{K}_{q,\ell,\alpha}^{\succcurlyeq,\mathrm{cd}}$. Then Proposition 4.13 shows that $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{cd}}$. From Theorem 6.3 and Definition 6.4 then one can see that $\mu_{(s_j)_{j=0}^{m}} = \mu_{(s_j)_{j=0}^{\ell}}$.

Example 6.7 Let $\alpha \in \mathbb{R}$ and let $s_0 := 0_{q\times q}$. Then Example 4.3 shows that $(s_j)_{j=0}^{0} \in \mathcal{K}_{q,0,\alpha}^{\succcurlyeq,\mathrm{cd}}$. Thus $\mu_{(s_j)_{j=0}^{0}}$ is exactly the $q \times q$ zero measure on $\mathfrak{B}_{[\alpha,\infty)}$.

The following result can be considered as an extended version of [10, Theorem 6.4].

Theorem 6.8 *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq}$. Let $\mathfrak{a}_{-1} := 0_{q\times q}$ and, in the case $m \geq 1$, let $\mathfrak{a}_{m-1}$ be the α-Stieltjes minimal element associated with $(s_j)_{j=0}^{m-1}$. Further, let $(\tilde{s}_j)_{j=0}^{m}$ be the right-sided α-Stieltjes non-negative definite extendable sequence equivalent to $(s_j)_{j=0}^{m}$. Then:*

(a) *The following statements are equivalent:*

  (i) *The set $\mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty); (s_j)_{j=0}^{m}, \preccurlyeq]$ consists of exactly one element μ.*
  (ii) $\tilde{s}_m = \mathfrak{a}_{m-1}$.
  (iii) $(\tilde{s}_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{cd}}$.

(b) *Let (iii) be satisfied. Then $\mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty); (s_j)_{j=0}^{m}, \preccurlyeq] = \{\mu_{(\tilde{s}_j)_{j=0}^{m}}\}$, where $\mu_{(\tilde{s}_j)_{j=0}^{m}}$ is the $[\alpha,\infty)$-measure associated with $(\tilde{s}_j)_{j=0}^{m}$.*

(c) *Let (iii) be satisfied and let $m = 2n+1$ with some $n \in \mathbb{N}_0$. Then $(\tilde{s}_j)_{j=0}^{2n+1} \in \mathcal{H}_{q,2n+1}^{\succcurlyeq,\mathrm{e}}$ and μ is the restriction of the CD-measure associated with $(\tilde{s}_j)_{j=0}^{2n+1}$ onto $\mathfrak{B}_{[\alpha,\infty)}$.*

Proof

(a) "(i)⇔(ii)": This follows from [10, Theorem 6.4].
"(ii)⇔(iii)": This follows from Example 4.3 and Proposition 4.12.

(b) From the choice of $(\tilde{s}_j)_{j=0}^{m}$ we have (3.1). In view of (iii), Theorem 6.3 yields $\mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty); (\tilde{s}_j)_{j=0}^{m}, =] = \{\mu_{(\tilde{s}_j)_{j=0}^{m}}\}$. Thus, $\mu_{(\tilde{s}_j)_{j=0}^{m}} \in \mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty); (\tilde{s}_j)_{j=0}^{m}, \preccurlyeq]$. Consequently, (3.1) yields $\mu_{(\tilde{s}_j)_{j=0}^{m}} \in \mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty); (s_j)_{j=0}^{m}, \preccurlyeq]$. From (iii) and (a) we get that the set $\mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty); (s_j)_{j=0}^{m}, \preccurlyeq]$ contains exactly one element. Therefore, (b) is proved.

(c) This follows from [10, Theorem 6.4].

□

Proposition 6.9 *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{cd}}$. Denote by $\mu_{(s_j)_{j=0}^{m}}$ the $[\alpha,\infty)$-measure associated with $(s_j)_{j=0}^{m}$. Then $\mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty); (s_j)_{j=0}^{m}, \preccurlyeq] = \{\mu_{(s_j)_{j=0}^{m}}\}$.*

Proof In view of Proposition 4.5, we have $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$. Thus, if we denote by $(\tilde{s}_j)_{j=0}^m$ the right-sided α-Stieltjes non-negative definite extendable sequence equivalent to $(s_j)_{j=0}^m$, then Theorem 3.8 yields $(\tilde{s}_j)_{j=0}^m = (s_j)_{j=0}^m$. In particular, $(\tilde{s}_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{cd}}$. Thus, Theorem 6.8(b) implies $\mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty); (s_j)_{j=0}^m, \preccurlyeq] = \{\mu_{(\tilde{s}_j)_{j=0}^m}\} = \{\mu_{(s_j)_{j=0}^m}\}$. □

Remark 6.10 Let $\alpha \in \mathbb{R}$ and let $m \in \mathbb{N}_0$.

(a) Let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$ and let $(\overline{s}_j)_{j=0}^\infty$ be the upper α-Stieltjes completely degenerate sequence associated with $(s_j)_{j=0}^m$. Then Definition 5.15 and Proposition 5.17(a) show that $(\overline{s}_j)_{j=0}^{m+1} \in \mathcal{K}_{q,m+1,\alpha}^{\succcurlyeq,\mathrm{cd}}$.

(b) Let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq}$ and let $(\underline{s}_j)_{j=0}^\infty$ be the lower α-Stieltjes completely degenerate sequence associated with $(s_j)_{j=0}^m$. Then Definition 5.13 and Lemma 5.14 yield $(\underline{s}_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{cd}}$.

Remark 6.10 leads us to the following notions.

Definition 6.11 Let $\alpha \in \mathbb{R}$ and let $m \in \mathbb{N}_0$.

(a) Let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$ and let $(\overline{s}_j)_{j=0}^\infty$ be the upper α-Stieltjes completely degenerate sequence associated with $(s_j)_{j=0}^m$. By the *upper CD-measure* $\overline{\sigma}_{(s_j)_{j=0}^m}$ *associated with* $(s_j)_{j=0}^m$ *and* $[\alpha,\infty)$ we mean the $[\alpha,\infty)$-measure $\mu_{(\overline{s}_j)_{j=0}^{m+1}}$ associated with $(\overline{s}_j)_{j=0}^{m+1}$.

(b) Let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq}$ and let $(\underline{s}_j)_{j=0}^\infty$ be the lower α-Stieltjes completely degenerate sequence associated with $(s_j)_{j=0}^m$. By the *lower CD-measure* $\underline{\sigma}_{(s_j)_{j=0}^m}$ *associated with* $(s_j)_{j=0}^m$ *and* $[\alpha,\infty)$ we mean the $[\alpha,\infty)$-measure $\mu_{(\underline{s}_j)_{j=0}^m}$ associated with $(\underline{s}_j)_{j=0}^m$.

In the case where it is clear which sequence $(s_j)_{j=0}^m$ is meant, we will also write $\overline{\sigma}_m$ and $\underline{\sigma}_m$ instead of $\overline{\sigma}_{(s_j)_{j=0}^m}$ and $\underline{\sigma}_{(s_j)_{j=0}^m}$.

Example 6.12 Let $\alpha \in \mathbb{R}$ and let $(s_j)_{j=0}^0 \in \mathcal{K}_{q,0,\alpha}^{\succcurlyeq}$. Then by construction $\underline{s}_0 = 0_{q\times q}$ and, consequently, the lower CD-measure $\underline{\sigma}_0$ associated with $(s_j)_{j=0}^0$ and $[\alpha,\infty)$ is the $q \times q$ zero measure defined on $\mathfrak{B}_{[\alpha,\infty)}$.

Now we are going to describe first aspects of the distinguished role which the measures introduced in Definition 6.11 occupy in the framework of truncated $[\alpha,\infty)$-Stieltjes matrix moment problems. First we consider the case $\alpha \in \mathbb{R}$, $m \in \mathbb{N}_0$, and $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$. Then we will see that the upper CD-measure $\overline{\sigma}_m$ associated with $(s_j)_{j=0}^m$ and $[\alpha,\infty)$ is the unique measure $\sigma \in \mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty); (s_j)_{j=0}^m, =] \cap \mathcal{M}_{q,m+1}^{\succcurlyeq}([\alpha,\infty))$ for which the matrix $s_{m+1}^{(\sigma)}$ is minimal with respect to the Löwner semi-ordering in $\mathbb{C}_{\mathrm{H}}^{q\times q}$:

Proposition 6.13 *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$. Denote by $\overline{\sigma}_m$ the upper CD-measure associated with $(s_j)_{j=0}^m$ and $[\alpha,\infty)$. Then the measure $\overline{\sigma}_m$ is molecular and belongs to $\mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty); (s_j)_{j=0}^m, =] \cap \mathcal{M}_{q,\infty}^{\succcurlyeq}([\alpha,\infty))$. Furthermore, if $\sigma \in \mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty); (s_j)_{j=0}^m, =] \cap \mathcal{M}_{q,m+1}^{\succcurlyeq}([\alpha,\infty))$, then $s_{m+1}^{(\sigma)} - s_{m+1}^{(\overline{\sigma}_m)} \succcurlyeq 0_{q\times q}$ with equality if and only if $\sigma = \overline{\sigma}_m$.*

Proof Remark 6.10, Definition 6.11, and Remark 6.5 show that $\overline{\sigma}_m$ is molecular and belongs to $\mathcal{M}_{q,\infty}^{\succcurlyeq}([\alpha,\infty))$. Denote by $(\overline{s}_j)_{j=0}^\infty$ the upper α-Stieltjes completely degenerate sequence associated with $(s_j)_{j=0}^m$. By construction, then $\mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty); (\overline{s}_j)_{j=0}^{m+1}, =] = \{\overline{\sigma}_m\}$ and $(\overline{s}_j)_{j=0}^m = (s_j)_{j=0}^m$. In particular, $\overline{\sigma}_m \in \mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty); (s_j)_{j=0}^m, =]$. Now we consider an arbitrary $\sigma \in \mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty); (s_j)_{j=0}^m, =] \cap \mathcal{M}_{q,m+1}^{\succcurlyeq}([\alpha,\infty))$. From Remark 6.10(a) we get $(\overline{s}_j)_{j=0}^{m+1} \in \mathcal{K}_{q,m+1,\alpha}^{\succcurlyeq,\mathrm{cd}}$. In view of Corollary 3.6 and Remark 3.4, we get $(s_j^{(\sigma)})_{j=0}^{m+1} \in \mathcal{K}_{q,m+1,\alpha}^{\succcurlyeq}$. Because of the choice of σ, we have $(s_j^{(\sigma)})_{j=0}^m = (s_j)_{j=0}^m$. Thus, Proposition 5.17(a) yields $s_{m+1}^{(\sigma)} - \overline{s}_{m+1} \succcurlyeq 0_{q\times q}$. Now suppose that $s_{m+1}^{(\sigma)} = \overline{s}_{m+1}$. From the choice of σ and $\mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty); (\overline{s}_j)_{j=0}^{m+1}, =] = \{\overline{\sigma}_m\}$ we get $\sigma \in \mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty); (\overline{s}_j)_{j=0}^{m+1}, =]$ and $\sigma = \overline{\sigma}_m$. □

Now we consider arbitrary $\alpha \in \mathbb{R}$, $m \in \mathbb{N}_0$, and $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq}$. We will see that the lower CD-measure $\underline{\sigma}_m$ associated with $(s_j)_{j=0}^m$ and $[\alpha,\infty)$ is characterized by an extremal property amongst all measures belonging to $\mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty); (s_j)_{j=0}^m, \preccurlyeq]$.

Proposition 6.14 *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq}$. Denote by $\underline{\sigma}_m$ the lower CD-measure associated with $(s_j)_{j=0}^m$ and $[\alpha,\infty)$. Then:*

(a) *The measure $\underline{\sigma}_m$ is molecular and belongs to $\mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty); (s_j)_{j=0}^m, \preccurlyeq] \cap \mathcal{M}_{q,\infty}^{\succcurlyeq}([\alpha,\infty))$.*
(b) *Let $\sigma \in \mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty); (s_j)_{j=0}^m, \preccurlyeq]$. Then $s_m^{(\sigma)} - s_m^{(\underline{\sigma}_m)} \succcurlyeq 0_{q\times q}$ with equality if and only if $\sigma = \underline{\sigma}_m$.*

Proof

(a) Remark 6.10, Definition 6.11, and Remark 6.5 show that $\underline{\sigma}_m$ is molecular and belongs to $\mathcal{M}_{q,\infty}^{\succcurlyeq}([\alpha,\infty))$. Denote by $(\underline{s}_j)_{j=0}^\infty$ the lower α-Stieltjes completely degenerate sequence associated with $(s_j)_{j=0}^m$. From the construction of $\underline{\sigma}_m$, we have

$$\mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty); (\underline{s}_j)_{j=0}^m, =] = \{\underline{\sigma}_m\}. \tag{6.1}$$

From Lemma 5.14 we know that $s_m - \underline{s}_m \in \mathbb{C}_{\succcurlyeq}^{q\times q}$. If $m \geq 1$, then by construction (see Definition 5.13) we also have $(\underline{s}_j)_{j=0}^{m-1} = (s_j)_{j=0}^{m-1}$. Consequently, from (6.1), we get $\underline{\sigma}_m \in \mathcal{M}_q^{\succcurlyeq}[[\alpha, \infty); (s_j)_{j=0}^{m}, \preccurlyeq]$.

(b) First we consider the case $m = 0$. Then Example 6.12 shows that $\underline{\sigma}_m$ is the $q \times q$ zero measure on $\mathfrak{B}_{[\alpha,\infty)}$. In particular, $s_0^{(\underline{\sigma}_m)} = 0_{q\times q}$. Thus, in view of the choice of σ we have $s_0^{(\sigma)} - s_0^{(\underline{\sigma}_m)} = \sigma([\alpha, \infty)) \succcurlyeq 0_{q\times q}$. If $s_0^{(\sigma)} - s_0^{(\underline{\sigma}_m)} = 0_{q\times q}$ then $\sigma([\alpha, \infty)) = s_0^{(\sigma)} - s_0^{(\underline{\sigma}_m)} = 0_{q\times q}$, which implies that σ is the $q \times q$ zero measure on $\mathfrak{B}_{[\alpha,\infty)}$. Now let $m \in \mathbb{N}$. By the choice of σ, we have

$$\sigma \in \mathcal{M}_{q,m}^{\succcurlyeq}([\alpha, \infty)) \qquad \text{and} \qquad (s_j^{(\sigma)})_{j=0}^{m-1} = (s_j)_{j=0}^{m-1}. \tag{6.2}$$

Corollary 3.6 shows then that $(s_j^{(\sigma)})_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$. Thus, Remark 3.4 yields $(s_j^{(\sigma)})_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq}$. Taking into account Lemma 5.14, we see that $s_m^{(\sigma)} - \underline{s}_m \in \mathbb{C}_{\succcurlyeq}^{q\times q}$. Combining this with (6.1), we conclude $s_m^{(\sigma)} - s_m^{(\underline{\sigma}_m)} \in \mathbb{C}_{\succcurlyeq}^{q\times q}$. If $s_m^{(\sigma)} = s_m^{(\underline{\sigma}_m)}$, then from (6.2) we get $(s_j^{(\sigma)})_{j=0}^{m} = (s_j^{(\underline{\sigma}_m)})_{j=0}^{m}$. Consequently, (6.1) implies $\sigma = \underline{\sigma}_m$. Conversely, if $m \in \mathbb{N}_0$ and if $\sigma = \underline{\sigma}_m$, then $s_m^{(\sigma)} - s_m^{(\underline{\sigma}_m)} = 0_{q\times q}$.

□

Proposition 6.15 *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$. Denote by $\underline{\sigma}_m$ (resp. $\overline{\sigma}_m$) the lower (resp. upper) CD-measure associated with $(s_j)_{j=0}^{m}$ and $[\alpha, \infty)$ and by $(\underline{s}_j)_{j=0}^{\infty}$ (resp. $(\overline{s}_j)_{j=0}^{\infty}$) the lower (resp. upper) α-Stieltjes completely degenerate sequence associated with $(s_j)_{j=0}^{m}$. Then $\mathcal{M}_q^{\succcurlyeq}[[\alpha, \infty); (\underline{s}_j)_{j=0}^{m}, =] = \{\underline{\sigma}_m\}$ and $\mathcal{M}_q^{\succcurlyeq}[[\alpha, \infty); (\overline{s}_j)_{j=0}^{m+1}, =] = \{\overline{\sigma}_m\}$.*

Proof In view of Remark 6.10, we have $(\overline{s}_j)_{j=0}^{m+1} \in \mathcal{K}_{q,m+1,\alpha}^{\succcurlyeq,\mathrm{cd}}$ and $(\underline{s}_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{cd}}$. Thus, the assertions follow from Theorem 6.3 and Definitions 6.4 and 6.11.
□

The following result complements Proposition 6.14.

Theorem 6.16 ([10, Theorem 5.4]) *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq}$. Denote by $(\tilde{s}_j)_{j=0}^{m}$ the right-sided α-Stieltjes non-negative definite extendable sequence equivalent to $(s_j)_{j=0}^{m}$. Let $\mathfrak{a}_{-1} := 0_{q\times q}$ and, in the case $m \in \mathbb{N}$, let $\mathfrak{a}_{m-1}$ be the α-Stieltjes minimal element associated with $(s_j)_{j=0}^{m-1}$. Then*

$$\left\{s_m^{(\sigma)} : \sigma \in \mathcal{M}_q^{\succcurlyeq}[[\alpha, \infty); (s_j)_{j=0}^{m}, \preccurlyeq]\right\} = \{s \in \mathbb{C}_{\mathrm{H}}^{q\times q} : \mathfrak{a}_{m-1} \preccurlyeq s \preccurlyeq \tilde{s}_m\}.$$

We consider the set $\mathcal{M}_q^{\succcurlyeq}[[\alpha, \infty); (s_j)_{j=0}^m, \preccurlyeq]$ now under the aspect of right α-Stieltjes parametrization.

Proposition 6.17 *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$ with right α-Stieltjes parametrization $(Q_j)_{j=0}^m$. For each $\sigma \in \mathcal{M}_q^{\succcurlyeq}[[\alpha, \infty); (s_j)_{j=0}^m, \preccurlyeq]$, let $(Q_j^{(\sigma)})_{j=0}^m$ be the right α-Stieltjes parametrization of $(s_j^{(\sigma)})_{j=0}^m$. Then:*

(a) $\{Q_m^{(\sigma)} : \sigma \in \mathcal{M}_q^{\succcurlyeq}[[\alpha, \infty); (s_j)_{j=0}^m, \preccurlyeq]\} = \{Q \in \mathbb{C}_{\mathrm{H}}^{q \times q} : 0_{q \times q} \preccurlyeq Q \preccurlyeq Q_m\}$.

(b) *Suppose $m \in \mathbb{N}$ and let $\sigma \in \mathcal{M}_q^{\succcurlyeq}[[\alpha, \infty); (s_j)_{j=0}^m, \preccurlyeq]$. Then $Q_j^{(\sigma)} = Q_j$ for each $j \in \mathbb{Z}_{0,m-1}$.*

Proof Using Remark 5.3 and the notations given there, we get then $Q_m = s_m - \mathfrak{a}_{m-1}$. We consider an arbitrary $\sigma \in \mathcal{M}_q^{\succcurlyeq}[[\alpha, \infty); (s_j)_{j=0}^m, \preccurlyeq]$. In particular, we conclude $s_m - s_m^{(\sigma)} \in \mathbb{C}_{\succcurlyeq}^{q \times q}$ and, if $m \geq 1$, furthermore $s_j^{(\sigma)} = s_j$ for each $j \in \mathbb{Z}_{0,m-1}$. We set $\mathfrak{a}_{-1}^{(\sigma)} := 0_{q \times q}$. If $m \in \mathbb{N}$, then let $\mathfrak{a}_{m-1}^{(\sigma)}$ be the α-Stieltjes minimal element associated with $(s_j^{(\sigma)})_{j=0}^{m-1}$. Then $\mathfrak{a}_{m-1}^{(\sigma)} = \mathfrak{a}_{m-1}$ and, applying Remark 5.3 again, we conclude that $Q_m^{(\sigma)} = s_m^{(\sigma)} - \mathfrak{a}_{m-1}^{(\sigma)}$. Consequently, $Q_m - Q_m^{(\sigma)} = s_m - \mathfrak{a}_{m-1} - (s_m^{(\sigma)} - \mathfrak{a}_{m-1}^{(\sigma)}) = s_m - s_m^{(\sigma)}$. Using $s_m - s_m^{(\sigma)} \in \mathbb{C}_{\succcurlyeq}^{q \times q}$ it follows $Q_m - Q_m^{(\sigma)} \in \mathbb{C}_{\succcurlyeq}^{q \times q}$. Since σ belongs to $\mathcal{M}_q^{\succcurlyeq}[[\alpha, \infty); (s_j)_{j=0}^m, \preccurlyeq]$, from Corollary 3.6 we get $(s_j^{(\sigma)})_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$. Thus, Theorem 5.6(b) yields $Q_m^{(\sigma)} \in \mathbb{C}_{\succcurlyeq}^{q \times q}$. Hence,

$$\left\{Q_m^{(\sigma)} : \sigma \in \mathcal{M}_q^{\succcurlyeq}[[\alpha, \infty); (s_j)_{j=0}^m, \preccurlyeq]\right\} \subseteq \{Q \in \mathbb{C}_{\mathrm{H}}^{q \times q} : 0_{q \times q} \preccurlyeq Q \preccurlyeq Q_m\}. \tag{6.3}$$

Conversely, let $Q \in \mathbb{C}_{\mathrm{H}}^{q \times q}$ be such that $0_{q \times q} \preccurlyeq Q \preccurlyeq Q_m$ holds true. Consequently,

$$Q \in \mathbb{C}_{\succcurlyeq}^{q \times q} \qquad \text{and} \qquad Q_m - Q \in \mathbb{C}_{\succcurlyeq}^{q \times q}. \tag{6.4}$$

Let

$$Q'_m := Q. \tag{6.5}$$

If $m \in \mathbb{N}$, then let

$$Q'_j := Q_j \qquad \text{for each } j \in \mathbb{Z}_{0,m-1}. \tag{6.6}$$

In view of Remark 5.2, then there exists a unique sequence $(s'_j)_{j=0}^m$ from $\mathbb{C}^{q \times q}$ such that $(Q'_j)_{j=0}^m$ is the right α-Stieltjes parametrization of $(s'_j)_{j=0}^m$. Let $\mathfrak{a}'_{-1} := 0_{q \times q}$. If $m \in \mathbb{N}$, then we denote by $\mathfrak{a}'_{m-1}$ the α-Stieltjes minimal element associated with $(s'_j)_{j=0}^{m-1}$. According to Remark 5.5 and (6.6), the sequence $(Q_j)_{j=0}^{m-1}$ is the right

Stieltjes parametrization of $(s'_j)_{j=0}^{m-1}$. Consequently, Remark 5.2 yields that

$$s_j = s'_j \qquad \text{for each } j \in \mathbb{Z}_{0,m-1}. \tag{6.7}$$

Thus, (4.10) shows that $\mathfrak{a}'_{m-1} = \mathfrak{a}_{m-1}$. Because of Remark 5.3, we get then $Q'_m = s'_m - \mathfrak{a}_{m-1}$. Combining this with $Q_m = s_m - \mathfrak{a}_{m-1}$ and (6.5), we obtain

$$s_m - s'_m = (Q_m + \mathfrak{a}_{m-1}) - (Q'_m + \mathfrak{a}_{m-1}) = Q_m - Q'_m = Q_m - Q.$$

Thus, (6.4) provides us $s_m - s'_m \in \mathbb{C}_{\succcurlyeq}^{q\times q}$. If $m = 0$, then, in view of (4.4), Definition 5.1, and (6.5), we have $s'_0 = Q'_0 = Q$ and because of (6.4), hence, $s'_0 \in \mathbb{C}_{\succcurlyeq}^{q\times q}$. Thus, Theorem 5.6(b) implies $(s'_j)_{j=0}^{0} \in \mathcal{K}_{q,0,\alpha}^{\succcurlyeq,\mathrm{e}}$. In view of (6.4) and (6.5), Lemma A.17 yields $\mathcal{N}(Q_m) \subseteq \mathcal{N}(Q'_m)$. Now let $m \in \mathbb{N}$. Because of $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$, Theorem 5.6(b) yields $Q_j \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ for all $j \in \mathbb{Z}_{0,m}$ and $\mathcal{N}(Q_j) \subseteq \mathcal{N}(Q_{j+1})$ for all $j \in \mathbb{Z}_{0,m-1}$. Consequently, using additionally (6.5), (6.6), and (6.4), we obtain $Q'_j \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ for all $j \in \mathbb{Z}_{0,m}$ and, in view of (6.6) and $\mathcal{N}(Q_m) \subseteq \mathcal{N}(Q'_m)$, furthermore

$$\mathcal{N}(Q'_j) \subseteq \mathcal{N}(Q'_{j+1}) \qquad \text{for all } j \in \mathbb{Z}_{0,m-1}.$$

Since $(Q'_j)_{j=0}^{m}$ is the right α-Stieltjes parametrization of $(s'_j)_{j=0}^{m}$, from Theorem 5.6(b) we get then

$$(s'_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}. \tag{6.8}$$

hence, (6.8) is proved for each $m \in \mathbb{N}_0$. Now we consider again the case that m is a non-negative integer. Taking into account that (6.8) is true, Theorem 3.5(b) yields $\mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty); (s'_j)_{j=0}^{m}, =] \neq \emptyset$. Let $\sigma \in \mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty); (s'_j)_{j=0}^{m}, =]$. In particular, $s_j^{(\sigma)} = s'_j$ for all $j \in \mathbb{Z}_{0,m}$. Consequently, $Q_j^{(\sigma)} = Q'_j$ for all $j \in \mathbb{Z}_{0,m}$. In view of (6.6), part (b) is proved. Furthermore, (6.5) shows that

$$Q_m^{(\sigma)} = Q'_m = Q. \tag{6.9}$$

Combining $s_j^{(\sigma)} = s'_j$ for all $j \in \mathbb{Z}_{0,m}$ and $s_m - s'_m \in \mathbb{C}_{\succcurlyeq}^{q\times q}$, we infer $s_m - s_m^{(\sigma)} \in \mathbb{C}_{\succcurlyeq}^{q\times q}$. If $m \in \mathbb{N}$, then, using (6.7), we get furthermore $s_j^{(\sigma)} = s_j$ for all $j \in \mathbb{Z}_{0,m-1}$. Thus, σ belongs to $\mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty); (s_j)_{j=0}^{m}, \preccurlyeq]$. Hence, (6.9) provides us

$$\{Q \in \mathbb{C}_{\succcurlyeq}^{q\times q} : Q_m - Q \in \mathbb{C}_{\succcurlyeq}^{q\times q}\} \subseteq \left\{Q_m^{(\sigma)} : \sigma \in \mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty); (s_j)_{j=0}^{m}, \preccurlyeq]\right\}. \tag{6.10}$$

Because of (6.3) and (6.10), the proof of part (a) is also complete. □

Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq}$. Then Theorem 6.16 leads us to an explicit description of the set of all matrices $s'_m \in \mathbb{C}_{\mathrm{H}}^{q\times q}$ which satisfy $s_m - s'_m \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ and $(s'_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$ where in the case $m \geq 1$ we have set $s'_j := s_j$ for all $j \in \mathbb{Z}_{0,m-1}$.

Proposition 6.18 *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq}$. Denote by $(\tilde{s}_j)_{j=0}^m$ the right-sided α-Stieltjes non-negative definite extendable sequence equivalent to $(s_j)_{j=0}^m$. Let $\mathfrak{a}_{-1} := 0_{q\times q}$. If $m \geq 1$, then let $\mathfrak{a}_{m-1}$ be the α-Stieltjes minimal element associated with $(s_j)_{j=0}^{m-1}$. Then the set $\mathcal{E}((s_j)_{j=0}^m)$ of all $s \in \mathbb{C}_{\mathrm{H}}^{q\times q}$ for which the sequence $(s'_j)_{j=0}^m$ given by*

$$s'_j := \begin{cases} s_j & \text{if } j \in \mathbb{Z}_{0,m-1} \\ s & \text{if } j = m \end{cases} \tag{6.11}$$

fulfills $s_m - s'_m \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ and $(s'_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$ admits the representation

$$\mathcal{E}\Big((s_j)_{j=0}^m\Big) = \{s \in \mathbb{C}_{\mathrm{H}}^{q\times q} : \mathfrak{a}_{m-1} \preccurlyeq s \preccurlyeq \tilde{s}_m\}. \tag{6.12}$$

In particular, if $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$, then $\mathcal{E}((s_j)_{j=0}^m) = \{s \in \mathbb{C}_{\mathrm{H}}^{q\times q} : \mathfrak{a}_{m-1} \preccurlyeq s \preccurlyeq s_m\}$.

Proof Let $s \in \mathbb{C}_{\mathrm{H}}^{q\times q}$ be such that $\mathfrak{a}_{m-1} \preccurlyeq s \preccurlyeq \tilde{s}_m$ is satisfied. We consider the sequence $(s'_j)_{j=0}^m$ defined by (6.11). Then from Theorem 6.16 we infer the existence of a measure $\sigma \in \mathcal{M}_{q,m}^{\succcurlyeq}([\alpha,\infty))$ which fulfills $s_j^{(\sigma)} = s'_j$ for each $j \in \mathbb{Z}_{0,m}$. Corollary 3.6 implies $(s_j^{(\sigma)})_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$. Consequently, $(s'_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$. Furthermore, Lemma 3.10 shows that $s_m - s'_m = s_m - s \succcurlyeq s_m - \tilde{s}_j \succcurlyeq 0_{q\times q}$. Thus,

$$\{s \in \mathbb{C}_{\mathrm{H}}^{q\times q} : \mathfrak{a}_{m-1} \preccurlyeq s \preccurlyeq \tilde{s}_m\} \subseteq \mathcal{E}\Big((s_j)_{j=0}^m\Big). \tag{6.13}$$

Conversely, now we consider an arbitrary $s \in \mathcal{E}((s_j)_{j=0}^m)$. Then $(s'_j)_{j=0}^m$ defined by (6.11) satisfies

$$s_m - s'_m \in \mathbb{C}_{\succcurlyeq}^{q\times q} \qquad \text{and} \qquad (s'_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq}. \tag{6.14}$$

Theorem 3.5(a) provides us then $\mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty);(s'_j)_{j=0}^m,=] \neq \emptyset$. Let $\mu \in \mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty);(s'_j)_{j=0}^m,=]$. In view of (6.14), then $\mu \in \mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty);(s_j)_{j=0}^m,\preccurlyeq]$ and $s_m^{(\sigma)} = s'_m = s$. Now Theorem 6.16 implies $s \in \mathbb{C}_{\mathrm{H}}^{q\times q}$ and $\mathfrak{a}_{m-1} \preccurlyeq s \preccurlyeq \tilde{s}_m$. Hence, $\mathcal{E}((s_j)_{j=0}^m) \subseteq \{r \in \mathbb{C}_{\mathrm{H}}^{q\times q} : \mathfrak{a}_{m-1} \preccurlyeq r \preccurlyeq \tilde{s}_m\}$. Therefore, from (6.13) we get (6.12). If $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$, then Lemma 3.10 yields $(s_j)_{j=0}^{m} = (\tilde{s}_j)_{j=0}^{m}$. This completes the proof. □

7 A Schur-Type Algorithm for Sequences of Complex Matrices

The basic object of this section was introduced in [20]. We want to recall its definition. For this reason, we start with the following notion:

Definition 7.1 ([25, Definition 4.13]) Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. The sequence $(s_j^{\sharp})_{j=0}^{\kappa}$ given by $s_0^{\sharp} := s_0^{\dagger}$ and $s_j^{\sharp} := -s_0^{\dagger} \sum_{l=0}^{j-1} s_{j-l} s_l^{\sharp}$ for all $j \in \mathbb{Z}_{1,\kappa}$ is said to be the *reciprocal sequence corresponding to* $(s_j)_{j=0}^{\kappa}$.

Definition 7.2 ([20, Definition 4.1]) Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. Then we call the sequence $(s_j^{[+,\alpha]})_{j=0}^{\kappa}$ given, for all $j \in \mathbb{Z}_{0,\kappa}$, by $s_j^{[+,\alpha]} := -\alpha s_{j-1} + s_j$ where $s_{-1} := 0_{p\times q}$, the $[+,\alpha]$-*transform of* $(s_j)_{j=0}^{\kappa}$.

Obviously, the $[+,\alpha]$-transform of $(s_j)_{j=0}^{\kappa}$ is connected with the sequence $(s_{\alpha\triangleright j})_{j=0}^{\kappa-1}$ given in (4.6) via $s_{j+1}^{[+,\alpha]} = s_{\alpha\triangleright j}$ for all $j \in \mathbb{Z}_{0,\kappa-1}$. Furthermore, we have $s_0^{[+,\alpha]} = s_0$.

Let $\alpha \in \mathbb{C}$. In order to prepare the basic construction in Sect. 12, we study the reciprocal sequence corresponding to the $[+,\alpha]$-transform of a sequence. Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices with $[+,\alpha]$-transform $(u_j)_{j=0}^{\kappa}$. Then we define $(s_j^{[\sharp,\alpha]})_{j=0}^{\kappa}$ by $s_j^{[\sharp,\alpha]} := u_j^{\sharp}$ for all $j \in \mathbb{Z}_{0,\kappa}$, i. e., $(s_j^{[\sharp,\alpha]})_{j=0}^{\kappa}$ is defined to be the reciprocal sequence corresponding to the $[+,\alpha]$-transform of $(s_j)_{j=0}^{\kappa}$.

Definition 7.3 ([20, Definition 7.1]) Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. Then the sequence $(s_j^{[1,\alpha]})_{j=0}^{\kappa-1}$ defined by $s_j^{[1,\alpha]} := -s_0 s_{j+1}^{[\sharp,\alpha]} s_0$ for all $j \in \mathbb{Z}_{0,\kappa-1}$ is called the *first* α*-Schur transform* (or short the *first* α*-S-transform*) *of* $(s_j)_{j=0}^{\kappa}$.

Theorem 7.4 ([20, Theorem 7.21(b)]) *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}$, and let $(s_j)_{j=0}^{m}$ be a sequence of complex $q \times q$ matrices with first α-S-transform $(s_j^{[1,\alpha]})_{j=0}^{m-1}$. If $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$, then $(s_j^{[1,\alpha]})_{j=0}^{m-1} \in \mathcal{K}_{q,m-1,\alpha}^{\succcurlyeq,\mathrm{e}}$.*

Definition 7.5 ([20, Definition 10.1]) Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, let $(t_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices, and let A be a complex $p \times q$ matrix. The sequence $(t_j^{[-1,\alpha,A]})_{j=0}^{\kappa+1}$ recursively defined by $t_0^{[-1,\alpha,A]} := A$ and

$$t_j^{[-1,\alpha,A]} := \alpha^j A + \sum_{l=1}^{j} \alpha^{j-l} A A^\dagger \left[\sum_{k=0}^{l-1} t_{l-k-1} A^\dagger (t_k^{[-1,\alpha,A]})^{[+,\alpha]} \right]$$

for all $j \in \mathbb{Z}_{1,\kappa+1}$ is called the *first inverse α-S-transform corresponding to* $[(t_j)_{j=0}^{\kappa}, A]$.

The α-Schur transform for sequences of complex $p \times q$ matrices introduced above generates in a natural way a corresponding algorithm for (finite and infinite) sequences of complex $q \times q$ matrices. In generalization of Definition 7.3, we introduced the following:

Definition 7.6 ([20, Definition 8.1]) Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. The sequence $(s_j^{[0,\alpha]})_{j=0}^{\kappa}$ given by $s_j^{[0,\alpha]} := s_j$ for all $j \in \mathbb{Z}_{0,\kappa}$ is called the 0*-th α-S-transform of* $(s_j)_{j=0}^{\kappa}$. In the case $\kappa \geq 1$, for all $k \in \mathbb{Z}_{1,\kappa}$, the *$k$-th α-S-transform* $(s_j^{[k,\alpha]})_{j=0}^{\kappa-k}$ *of* $(s_j)_{j=0}^{\kappa}$ is recursively defined by $s_j^{[k,\alpha]} := t_j^{[1,\alpha]}$ for all $j \in \mathbb{Z}_{0,\kappa-k}$, where $(t_j)_{j=0}^{\kappa-(k-1)}$ denotes the $(k-1)$-th α-S-transform of $(s_j)_{j=0}^{\kappa}$.

A comprehensive investigation of this algorithm was carried out in [20].

In the following, we will make essential use of the fact that, for the α-Stieltjes non-negative definite extendable sequences and its distinguished subclasses, there are remarkable connections between their right α-Stieltjes parametrization and the Schur algorithm.

Theorem 7.7 ([20, Theorem 9.15]) *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\succcurlyeq,\mathrm{e}}$. Then $(s_0^{[j,\alpha]})_{j=0}^{\kappa}$ is exactly the right α-Stieltjes parametrization of $(s_j)_{j=0}^{\kappa}$.*

Theorem 7.8 ([20, Theorem 9.26]) *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\succcurlyeq,\mathrm{e}}$, and let $k \in \mathbb{Z}_{0,\kappa}$. Denote by $(t_j)_{j=0}^{\kappa-k}$ the k-th α-S-transform of $(s_j)_{j=0}^{\kappa}$ and by $(Q_j)_{j=0}^{\kappa}$ the right α-Stieltjes parametrization of $(s_j)_{j=0}^{\kappa}$. Then $(Q_{k+j})_{j=0}^{\kappa-k}$ is exactly the right α-Stieltjes parametrization of $(t_j)_{j=0}^{\kappa-k}$.*

8 The Class $\mathcal{S}_{q,[\alpha,\infty)}$

In this section, we summarize some basic facts about the class of $[\alpha,\infty)$-Stieltjes functions of order q, which are mostly taken from our former paper [22]. If $A \in \mathbb{C}^{q\times q}$, then let $\operatorname{Re} A := \frac{1}{2}(A + A^*)$ and $\operatorname{Im} A := \frac{1}{2\mathrm{i}}(A - A^*)$ be the real part and the imaginary part of A, respectively. Let $\Pi_+ := \{z \in \mathbb{C}\colon \operatorname{Im} z > 0\}$ be the open upper half plane of $\mathbb{C}$.

Definition 8.1 Let $\alpha \in \mathbb{R}$ and let $F\colon \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C}^{q\times q}$. Then F is called a $[\alpha, \infty)$-*Stieltjes function of order* q if F satisfies the following three conditions:

(I) F is holomorphic in $\mathbb{C} \setminus [\alpha, \infty)$.
(II) For all $w \in \Pi_+$, the matrix $\operatorname{Im}[F(w)]$ is non-negative Hermitian.
(III) For all $w \in (-\infty, \alpha)$, the matrix $F(w)$ is non-negative Hermitian.

We denote by $\mathcal{S}_{q,[\alpha,\infty)}$ the set of all $[\alpha, \infty)$-Stieltjes functions of order q. For a comprehensive survey on the class $\mathcal{S}_{q,[\alpha,\infty)}$, we refer the reader to [22]. We start with a useful characterization of the membership of a function to the class $\mathcal{S}_{q,[\alpha,\infty)}$. Let $\Pi_- := \{z \in \mathbb{C}\colon \operatorname{Im} z < 0\}$ and let $\mathbb{C}_{\alpha,-} := \{z \in \mathbb{C}\colon \operatorname{Re} z \in (-\infty, \alpha)\}$. We observe that, for all $\mu \in \mathcal{M}_q^{\succcurlyeq}([\alpha, \infty))$ and each $z \in \mathbb{C} \setminus [\alpha, \infty)$, the function $h_{\alpha,z}\colon [\alpha, \infty) \to \mathbb{C}$ defined by $h_{\alpha,z}(t) := (1 + t - \alpha)/(t - z)$ belongs to $\mathcal{L}^1([\alpha, \infty), \mathfrak{B}_{[\alpha,\infty)}, \mu; \mathbb{C})$.

Theorem 8.2 (cf. [22, Theorem 3.6]) *Let $\alpha \in \mathbb{R}$ and let $F\colon \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C}^{q\times q}$. Then:*

(a) *If $F \in \mathcal{S}_{q,[\alpha,\infty)}$, then there are a unique matrix $\gamma \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ and a unique non-negative Hermitian measure $\mu \in \mathcal{M}_q^{\succcurlyeq}([\alpha, \infty))$ such that*

$$F(z) = \gamma + \int_{[\alpha,\infty)} \frac{1 + t - \alpha}{t - z} \mu(\mathrm{d}t) \tag{8.1}$$

holds true for each $z \in \mathbb{C} \setminus [\alpha, \infty)$.

(b) *If there are a matrix $\gamma \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ and a non-negative Hermitian measure $\mu \in \mathcal{M}_q^{\succcurlyeq}([\alpha, \infty))$ such that F can be represented via (8.1) for each $z \in \mathbb{C} \setminus [\alpha, \infty)$, then F belongs to the class $\mathcal{S}_{q,[\alpha,\infty)}$.*

In the special case that $q = 1$ and $\alpha = 0$ hold true, Theorem 8.2 can be found in Krein/Nudelman [37, Appendix]. Furthermore, observe that Theorem 8.2 shows in particular that all constant $q \times q$ matrix-valued functions defined on $\mathbb{C} \setminus [\alpha, \infty)$ which value is a non-negative Hermitian complex matrix belong to $\mathcal{S}_{q,[\alpha,\infty)}$.

Notation 8.3 For all $F \in \mathcal{S}_{q,[\alpha,\infty)}$, we will write (γ_F, μ_F) for the unique pair $(\gamma, \mu) \in \mathbb{C}_{\succcurlyeq}^{q\times q} \times \mathcal{M}_q^{\succcurlyeq}([\alpha, \infty))$ for which the representation (8.1) holds true for all $z \in \mathbb{C} \setminus [\alpha, \infty)$.

In the sequel, we will sometimes meet situations where interrelations between the null space (respectively, column space) of a function $F \in \mathcal{S}_{q,[\alpha,\infty)}$ and the null space (respectively, column space) of a given matrix $A \in \mathbb{C}^{p\times q}$ are of interest.

Notation 8.4 Let $\alpha \in \mathbb{R}$ and let $A \in \mathbb{C}^{q\times q}$. We denote by $\mathcal{S}_{q,[\alpha,\infty)}[A]$ the set of all $F \in \mathcal{S}_{q,[\alpha,\infty)}$ which satisfy $\mathcal{R}(F(z)) \subseteq \mathcal{R}(A)$ for all $z \in \mathbb{C} \setminus [\alpha, \infty)$.

Observe that the fact that a matrix-valued function $F \in \mathcal{S}_{q,[\alpha,\infty)}$ belongs to the subclass $\mathcal{S}_{q,[\alpha,\infty)}[A]$ of the class $\mathcal{S}_{q,[\alpha,\infty)}$ can be characterized by several conditions (see [22, Lemma 3.18]). In particular, we have the following:

Lemma 8.5 ([23, Lemma 4.9]) *Let $\alpha \in \mathbb{R}$, let $F \in \mathcal{S}_{q,[\alpha,\infty)}$, and let $A \in \mathbb{C}^{q\times q}$. Then $F \in \mathcal{S}_{q,[\alpha,\infty)}[A]$ if and only if $\mathcal{R}(\gamma_F) + \mathcal{R}(\mu_F([\alpha,\infty))) \subseteq \mathcal{R}(A)$.*

9 On Some Subclasses of $\mathcal{S}_{q,[\alpha,\infty)}$

In this section, we summarize some basic facts about several subclasses of $\mathcal{S}_{q,[\alpha,\infty)}$, which are characterized by growth properties on the positive imaginary axis. It should be mentioned that scalar versions of the function classes were introduced and studied in Kats/Krein [30]. We recognized in [21] that a detailed analysis of the behavior on the positive imaginary axis of the concrete functions of $F \in \mathcal{S}_{q,[\alpha,\infty)}$ under study is very useful. For this reason, we turn now our attention to some subclasses of $\mathcal{S}_{q,[\alpha,\infty)}$, which are described in terms of their growth on the positive imaginary axis. First we consider the set

$$\mathcal{S}^{\diamond}_{q,[\alpha,\infty)} := \left\{F \in \mathcal{S}_{q,[\alpha,\infty)} \colon \lim_{y\to\infty} \|F(\mathrm{i}y)\|_{\mathrm{S}} = 0\right\}. \tag{9.1}$$

In [21, Section 4], we considered a particular subclass of the class $\mathcal{S}^{\diamond}_{q,[\alpha,\infty)}$ introduced in (9.1). We have seen in [22, Proposition 3.15] that, for an arbitrary function $F \in \mathcal{S}_{q,[\alpha,\infty)}$, the null space of $F(z)$ is independent from the concrete choice of $z \in \mathbb{C} \setminus [\alpha,\infty)$. For the case that F belongs to $\mathcal{S}^{\diamond}_{q,[\alpha,\infty)}$, a complete description of this constant null space was given in [21, Proposition 3.7]. Against to this background, we single out now a special subclass of $\mathcal{S}^{\diamond}_{q,[\alpha,\infty)}$: In view of Notation 8.3, for all $A \in \mathbb{C}^{p\times q}$, let

$$\mathcal{S}^{\diamond}_{q,[\alpha,\infty)}[A] := \left\{F \in \mathcal{S}^{\diamond}_{q,[\alpha,\infty)} \colon \mathcal{N}(A) \subseteq \mathcal{N}(\mu_F([\alpha,\infty)))\right\} \tag{9.2}$$

Observe that the constant function defined on $\mathbb{C}\setminus[\alpha,\infty)$ with value $0_{q\times q}$ belongs to $\mathcal{S}^{\diamond}_{q,[\alpha,\infty)}[A]$ for all $A \in \mathbb{C}^{p\times q}$. In [21], the role of the matrix A was taken by a matrix which is generated from the sequence of the given data of the moment problem via a Schur-type algorithm.

An important subclass of the class $\mathcal{S}_{q,[\alpha,\infty)}$ is the set

$$\mathcal{S}_{0,q,[\alpha,\infty)} := \left\{F \in \mathcal{S}_{q,[\alpha,\infty)} \colon \sup_{y\in[1,\infty)} y\|F(\mathrm{i}y)\|_{\mathrm{S}} < \infty\right\}. \tag{9.3}$$

Let Ω be a non-empty closed subset of $\mathbb{R}$ and let $\sigma \in \mathcal{M}^{\succcurlyeq}_q(\Omega)$. Then, in view of [22, Lemma A.4], for each $z \in \mathbb{C} \setminus \Omega$, the function $f_z \colon \Omega \to \mathbb{C}$ defined by

$f_z(t) := (t-z)^{-1}$ belongs to $\mathcal{L}^1(\Omega, \mathfrak{B}_\Omega, \sigma; \mathbb{C})$. In particular, for each $\alpha \in \mathbb{R}$ and each $\sigma \in \mathcal{M}_q^{\succcurlyeq}([\alpha, \infty))$, the matrix-valued function $S_\sigma : \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C}^{q \times q}$ given by

$$S_\sigma(z) := \int_{[\alpha,\infty)} \frac{1}{t-z} \sigma(\mathrm{d}t) \tag{9.4}$$

is well defined and it is called $[\alpha, \infty)$-*Stieltjes transform of* σ.

There is an important characterization of the set of all $[\alpha, \infty)$-Stieltjes transforms of measures belonging to $\mathcal{M}_q^{\succcurlyeq}([\alpha, \infty))$:

Theorem 9.1 ([21, Theorem 3.2]) *Let $\alpha \in \mathbb{R}$. The mapping $\sigma \mapsto S_\sigma$ is a bijective correspondence between $\mathcal{M}_q^{\succcurlyeq}([\alpha, \infty))$ and $\mathcal{S}_{0,q,[\alpha,\infty)}$. In particular, $\mathcal{S}_{0,q,[\alpha,\infty)} = \{S_\sigma : \sigma \in \mathcal{M}_q^{\succcurlyeq}([\alpha, \infty))\}$.*

For each $F \in \mathcal{S}_{0,q,[\alpha,\infty)}$, the unique measure $\sigma \in \mathcal{M}_q^{\succcurlyeq}([\alpha, \infty))$ satisfying $S_\sigma = F$ is called the $[\alpha, \infty)$-*Stieltjes measure of* F and we will also write σ_F for σ. Theorem 9.1 and (9.3) indicate that the $[\alpha, \infty)$-Stieltjes transform S_σ of a measure $\sigma \in \mathcal{M}_q^{\succcurlyeq}([\alpha, \infty))$ is characterized by a particular mild growth on the positive imaginary axis.

In view of Theorem 9.1, the Problems $\mathsf{M}[[\alpha, \infty); (s_j)_{j=0}^m, \preccurlyeq]$ and $\mathsf{M}[[\alpha, \infty); (s_j)_{j=0}^m, =]$ can be reformulated as an equivalent problem in the class $\mathcal{S}_{0,q,[\alpha,\infty)}$ as follows:

Problem ($\mathsf{S}[[\alpha, \infty); (s_j)_{j=0}^m, \preccurlyeq]$) Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m$ be a sequence of complex $q \times q$ matrices. Describe the set $\mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^m, \preccurlyeq]$ of all $F \in \mathcal{S}_{0,q,[\alpha,\infty)}$ the $[\alpha, \infty)$-Stieltjes measure of which belongs to $\mathcal{M}_q^{\succcurlyeq}[[\alpha, \infty); (s_j)_{j=0}^m, \preccurlyeq]$.

Problem ($\mathsf{S}[[\alpha, \infty); (s_j)_{j=0}^m, =]$) Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m$ be a sequence of complex $q \times q$ matrices. Describe the set $\mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^m, =]$ of all $F \in \mathcal{S}_{0,q,[\alpha,\infty)}$ the $[\alpha, \infty)$-Stieltjes measure of which belongs to $\mathcal{M}_q^{\succcurlyeq}[[\alpha, \infty); (s_j)_{j=0}^m, =]$.

In [21, Section 6], we stated a reformulation of the original power moment problem $\mathsf{M}[[\alpha, \infty); (s_j)_{j=0}^\kappa, =]$ as an equivalent problem of finding a prescribed asymptotic expansion in a sector of the open upper half plane Π_+.

For all $\alpha \in \mathbb{R}$ and all $\kappa \in \mathbb{N} \cup \{\infty\}$, we now consider the class

$$\mathcal{S}_{\kappa,q,[\alpha,\infty)} := \left\{F \in \mathcal{S}_{0,q,[\alpha,\infty)} : \sigma_F \in \mathcal{M}_{q,\kappa}^{\succcurlyeq}([\alpha, \infty))\right\}. \tag{9.5}$$

10 The Classes $\mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^m, \preccurlyeq]$ and $\mathcal{S}_{\kappa,q,[\alpha,\infty)}[(s_j)_{j=0}^\kappa, =]$

In [21, Section 4], we studied the particular subclasses of the class $\mathcal{S}_{\kappa,q,[\alpha,\infty)}$, which was introduced in (9.3) for $\kappa = 0$ and in (9.5) for $\kappa \in \mathbb{N} \cup \{\infty\}$. In view of Theorem 9.1, for each function F belonging to one of the classes $\mathcal{S}_{\kappa,q,[\alpha,\infty)}$ with some $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, we can consider the $[\alpha,\infty)$-Stieltjes measure σ_F of F. Now we turn our attention to subclasses of functions $F \in \mathcal{S}_{\kappa,q,[\alpha,\infty)}$ with prescribed first $\kappa+1$ power moments of the $[\alpha,\infty)$-Stieltjes measure σ_F.

For all $\alpha \in \mathbb{R}$, all $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and each sequence $(s_j)_{j=0}^\kappa$ of complex $q \times q$ matrices, now we consider the class

$$\mathcal{S}_{\kappa,q,[\alpha,\infty)}[(s_j)_{j=0}^\kappa, =] := \left\{F \in \mathcal{S}_{\kappa,q,[\alpha,\infty)} : \sigma_F \in \mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty); (s_j)_{j=0}^\kappa, =]\right\}. \tag{10.1}$$

Now we characterize those sequences for which the sets defined in (10.1) are non-empty.

Theorem 10.1 ([21, Theorem 5.3]) *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $q \times q$ matrices. Then $\mathcal{S}_{\kappa,q,[\alpha,\infty)}[(s_j)_{j=0}^\kappa, =] \neq \emptyset$ if and only if $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa,\alpha}^{\succcurlyeq,\mathrm{e}}$.*

Remark 10.2 Let $\alpha \in \mathbb{R}$ and let $F : \mathbb{C} \setminus [\alpha,\infty) \to \mathbb{C}^{q\times q}$ be a function. Then:

(a) If $F \in \mathcal{S}_{0,q,[\alpha,\infty)}$, then $F \in \mathcal{S}_{0,q,[\alpha,\infty)}[(s_j)_{j=0}^0, =]$ with $s_0 := \sigma_F([\alpha,\infty))$.
(b) Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa,\alpha}^{\succcurlyeq,\mathrm{e}}$. If $F \in \mathcal{S}_{\kappa,q,[\alpha,\infty)}[(s_j)_{j=0}^\kappa, =]$, then $F \in \mathcal{S}_{0,q,[\alpha,\infty)}$ and $\sigma_F([\alpha,\infty)) = s_0$.

Now we state a useful characterization of the set of functions given in (10.1).

Theorem 10.3 ([21, Theorem 5.4]) *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $q \times q$ matrices. In view of (9.4), then*

$$\mathcal{S}_{\kappa,q,[\alpha,\infty)}[(s_j)_{j=0}^\kappa, =] = \left\{S_\sigma : \sigma \in \mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty); (s_j)_{j=0}^\kappa, =]\right\}.$$

Theorem 10.3 shows that $\mathcal{S}_{\kappa,q,[\alpha,\infty)}[(s_j)_{j=0}^\kappa, =]$ coincides with the solution set of Problem $\mathsf{S}[[\alpha,\infty); (s_j)_{j=0}^\kappa, =]$, which is via $[\alpha,\infty)$-Stieltjes transform equivalent to the original Problem $\mathsf{M}[[\alpha,\infty); (s_j)_{j=0}^\kappa, =]$. Thus, the investigation of the set $\mathcal{S}_{\kappa,q,[\alpha,\infty)}[(s_j)_{j=0}^\kappa, =]$ is a central theme of our further considerations.

Now we consider special subclasses of the class $\mathcal{S}_{m,q,[\alpha,\infty)}$, which was introduced in (9.3) for $m = 0$ and in (9.5) for each $m \in \mathbb{N}$. For all $\alpha \in \mathbb{R}$, all $m \in \mathbb{N}_0$, and each sequence $(s_j)_{j=0}^m$ of complex $q \times q$ matrices, we consider the class

$$\mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^m, \preccurlyeq] := \left\{F \in \mathcal{S}_{m,q,[\alpha,\infty)} : \sigma_F \in \mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty); (s_j)_{j=0}^m, \preccurlyeq]\right\}. \tag{10.2}$$

Remark 10.4 Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}$, and let $(s_j)_{j=0}^m$ be a sequence from $\mathbb{C}^{q\times q}$. Then $\mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^m, \preccurlyeq] \subseteq \mathcal{S}_{\ell,q,[\alpha,\infty)}[(s_j)_{j=0}^\ell, =]$ for all $\ell \in \mathbb{Z}_{0,m-1}$.

Now we characterize those sequences for which the sets defined in (10.2) are non-empty.

Theorem 10.5 ([23, Theorem 6.4]) *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m$ be a sequence from $\mathbb{C}^{q\times q}$. Then $\mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^m, \preccurlyeq] \neq \emptyset$ if and only if $(s_j)_{j=0}^m \in \mathcal{K}^{\succcurlyeq}_{q,m,\alpha}$. Furthermore, in view of (9.4), if $(s_j)_{j=0}^m \in \mathcal{K}^{\succcurlyeq}_{q,m,\alpha}$, then $\mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^m, \preccurlyeq] = \{S_\sigma : \sigma \in \mathcal{M}^{\succcurlyeq}_q[[\alpha,\infty); (s_j)_{j=0}^m, \preccurlyeq]\}$.*

The following result should be compared with [21, Proposition 5.5].

Proposition 10.6 *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}$, let $(s_j)_{j=0}^m \in \mathcal{K}^{\succcurlyeq}_{q,m,\alpha}$, and let $F \in \mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^m, \preccurlyeq]$. For all $z \in \mathbb{C} \setminus [\alpha,\infty)$, the equations $\mathcal{R}(F(z)) = \mathcal{R}(s_0)$, $\mathcal{N}(F(z)) = \mathcal{N}(s_0)$, $[F(z)][F(z)]^\dagger = s_0 s_0^\dagger$, and $[F(z)]^\dagger[F(z)] = s_0^\dagger s_0$ hold true. Moreover, the function F belongs to the class $\mathcal{S}_{q,[\alpha,\infty)}$ with $\mathcal{R}(s_0) = \mathcal{R}(\gamma_F) + \mathcal{R}(\mu_F([\alpha,\infty)))$ and $\mathcal{N}(s_0) = \mathcal{N}(\gamma_F) \cap \mathcal{N}(\mu_F([\alpha,\infty)))$.*

Proof In view of Remark 3.4, we have $(s_j)_{j=0}^{m-1} \in \mathcal{K}^{\succcurlyeq,e}_{q,m-1,\alpha}$, whereas Remark 10.4 yields $F \in \mathcal{S}_{m-1,q,[\alpha,\infty)}[(s_j)_{j=0}^{m-1}, =]$. Thus, the application [21, Proposition 5.5] completes the proof. □

Proposition 10.7 ([23, Proposition 6.6]) *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m \in \mathcal{K}^{\succcurlyeq}_{q,m,\alpha}$. Then $\mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^m, \preccurlyeq] \subseteq \mathcal{S}_{q,[\alpha,\infty)}[s_0]$.*

11 Stieltjes Pairs of Meromorphic $q \times q$ Matrix-Valued Functions in $\mathbb{C} \setminus [\alpha,\infty)$

In this section, we consider an arbitrary $\alpha \in \mathbb{R}$. Then the set $\mathbb{C} \setminus [\alpha,\infty)$ is clearly a region in $\mathbb{C}$. We consider a class of ordered pairs of $q \times q$ matrix-valued meromorphic functions in $\mathbb{C} \setminus [\alpha,\infty)$ which turns out to be closely related to the class $\mathcal{S}_{q,[\alpha,\infty)}$ introduced in Definition 8.1. This set of ordered pairs of $q \times q$ matrix-valued meromorphic functions in $\mathbb{C} \setminus [\alpha,\infty)$ plays an important role in our subsequent considerations. Indeed, this set acts as the set of parameters in our description of the set of Stieltjes transforms of the solutions of our original truncated matricial moment problem at the interval $[\alpha,\infty)$.

Before we introduce the central object of this section, we make some preliminaries. A subset $\mathcal{D}$ of $\mathbb{C}$ is called *discrete* if for every bounded subset $\mathcal{B}$ of $\mathbb{C}$ the intersection $\mathcal{D} \cap \mathcal{B}$ only contains a finite number of points. By a *region* we mean an open connected subset of $\mathbb{C}$. Let $\mathcal{G}$ be a region in $\mathbb{C}$ and let f be a complex function in $\mathcal{G}$. Then f is called *meromorphic in $\mathcal{G}$* if there exists a discrete subset $\mathbb{P}_f$ of $\mathcal{G}$

such that f is holomorphic in $\mathbb{H}_f := \mathcal{G} \setminus \mathbb{P}_f$ whereas f has a pole in each point of $\mathbb{P}_f$. We denote by $\mathcal{M}(\mathcal{G})$ the set of all meromorphic functions in $\mathcal{G}$. The notation $\mathcal{H}(\mathcal{G})$ stands for the set of all complex-valued holomorphic functions in $\mathcal{G}$.

Now we extend these notions to matrix-valued functions. Let $\mathcal{G}$ be a region in $\mathbb{C}$ and let $r, s \in \mathbb{N}$. Let $f = [f_{jk}]_{\substack{j=1,\dots,r\\k=1,\dots,s}} \in [\mathcal{M}(\mathcal{G})]^{r\times s}$. Then the sets $\mathbb{H}_f := \bigcap_{j=1}^r \bigcap_{k=1}^s \mathbb{H}_{f_{jk}}$ and $\mathbb{P}_f := \bigcup_{j=1}^r \bigcup_{k=1}^s \mathbb{P}_{f_{jk}}$ are called the *holomorphicity set* of f and the *pole set of* f, respectively. Then one can easily see that $\mathbb{P}_f$ is a discrete subset of $\mathcal{G}$ and that $\mathbb{H}_f \cup \mathbb{P}_f = \mathcal{G}$ and $\mathbb{H}_f \cap \mathbb{P}_f = \emptyset$ hold true. We consider an $f \in [\mathcal{M}(\mathcal{G})]^{r\times s}$ also as a mapping f between the sets $\mathbb{H}_f$ and $\mathbb{C}^{r\times s}$. In the following, we use the particular signature matrices

$$\tilde{J}_q := \begin{bmatrix} 0_{q\times q} & -\mathrm{i}I_q \\ \mathrm{i}I_q & 0_{q\times q} \end{bmatrix} \qquad \text{and} \qquad J_q := \begin{bmatrix} 0_{q\times q} & -I_q \\ -I_q & 0_{q\times q} \end{bmatrix}.$$

Remark 11.1 For all $A, B \in \mathbb{C}^{q\times q}$, the equations $\begin{bmatrix} A \\ B \end{bmatrix}^*(-\tilde{J}_q)\begin{bmatrix} A \\ B \end{bmatrix} = 2\,\mathrm{Im}(B^*A)$ and $\begin{bmatrix} A \\ B \end{bmatrix}^*(-J_q)\begin{bmatrix} A \\ B \end{bmatrix} = 2\,\mathrm{Re}(B^*A)$ hold true. In particular, $\begin{bmatrix} A \\ I_q \end{bmatrix}^*(-\tilde{J}_q)\begin{bmatrix} A \\ I_q \end{bmatrix} = 2\,\mathrm{Im}\,A$ and $\begin{bmatrix} A \\ I_q \end{bmatrix}^*(-J_q)\begin{bmatrix} A \\ I_q \end{bmatrix} = 2\,\mathrm{Re}\,A$ are valid for each $A \in \mathbb{C}^{q\times q}$.

Definition 11.2 ([23, Definitions 7.1 and 7.2]) Let $\alpha \in \mathbb{R}$. Let $\phi, \psi \in [\mathcal{M}(\mathbb{C} \setminus [\alpha,\infty))]^{q\times q}$. Then (ϕ, ψ) is called a $q \times q$ *Stieltjes pair in* $\mathbb{C} \setminus [\alpha, \infty)$ if there exists a discrete subset $\mathcal{D}$ of $\mathbb{C} \setminus [\alpha,\infty)$ such that the following three conditions are fulfilled:

(i) ϕ are ψ are holomorphic in $\mathbb{C} \setminus ([\alpha,\infty) \cup \mathcal{D})$.
(ii) $\operatorname{rank}\begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} = q$ for each $z \in \mathbb{C} \setminus ([\alpha,\infty) \cup \mathcal{D})$.
(iii) $\begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix}^*\left(\frac{-\tilde{J}_q}{2\,\mathrm{Im}\,z}\right)\begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ and $\begin{bmatrix} (z-\alpha)\phi(z) \\ \psi(z) \end{bmatrix}^*\left(\frac{-\tilde{J}_q}{2\,\mathrm{Im}\,z}\right)\begin{bmatrix} (z-\alpha)\phi(z) \\ \psi(z) \end{bmatrix} \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ for every choice of $z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{D})$.

The set of all $q \times q$ Stieltjes pairs in $\mathbb{C} \setminus [\alpha,\infty)$ will be denoted by $\mathcal{P}^{(q,q)}_{-\tilde{J}_q,\succcurlyeq}(\mathbb{C} \setminus [\alpha,\infty))$. A pair $(\phi,\psi) \in \mathcal{P}^{(q,q)}_{-\tilde{J}_q,\succcurlyeq}(\mathbb{C} \setminus [\alpha,\infty))$ is said to be a *proper* $q \times q$ *Stieltjes pair in* $\mathbb{C} \setminus [\alpha,\infty)$ if $\det\psi$ does not vanish identically in $\mathbb{C} \setminus [\alpha,\infty)$. The set of all proper $q \times q$ Stieltjes pairs in $\mathbb{C} \setminus [\alpha,\infty)$ will be denoted by $\tilde{\mathcal{P}}^{(q,q)}_{-\tilde{J}_q,\succcurlyeq}(\mathbb{C} \setminus [\alpha,\infty))$.

Remark 11.3 ([23, Remarks 7.3 and 7.5, Definition 7.4]) Let $\alpha \in \mathbb{R}$, let $(\phi,\psi) \in \mathcal{P}^{(q,q)}_{-\tilde{J}_q,\succcurlyeq}(\mathbb{C} \setminus [\alpha,\infty))$, and let g be a $q \times q$ matrix-valued function which is meromorphic in $\mathbb{C} \setminus [\alpha,\infty)$ such that $\det g$ does not vanish identically. Then it is readily checked that $(\phi g, \psi g) \in \mathcal{P}^{(q,q)}_{-\tilde{J}_q,\succcurlyeq}(\mathbb{C} \setminus [\alpha,\infty))$. Stieltjes pairs (ϕ_1,ψ_1) and (ϕ_2,ψ_2) are called *equivalent* if there is a function $\theta \in [\mathcal{M}(\mathbb{C} \setminus [\alpha,\infty))]^{q\times q}$ such that $\det\theta$ does not identically vanish in $\mathbb{C} \setminus [\alpha,\infty)$ and that $\phi_2 = \phi_1\theta$ and $\psi_2 = \phi_2\theta$ are satisfied. It is easily checked that this relation is an equivalence relation on the set $\mathcal{P}^{(q,q)}_{-\tilde{J}_q,\succcurlyeq}(\mathbb{C} \setminus [\alpha,\infty))$. For each $(\phi,\psi) \in \mathcal{P}^{(q,q)}_{-\tilde{J}_q,\succcurlyeq}(\mathbb{C} \setminus [\alpha,\infty))$, we denote by $\langle(\phi, \psi)\rangle$ the equivalence class generated by (ϕ, ψ). Furthermore, we write $\langle\mathcal{P}^{(q,q)}_{-\tilde{J}_q,\succcurlyeq}(\mathbb{C} \setminus [\alpha, \infty))\rangle$ for the set of all these equivalence classes.

Let $\mathfrak{I}_q$ and $\mathfrak{O}_q$ be the constant functions (defined on $\mathbb{C} \setminus [\alpha, \infty)$) with value I_q and $0_{q\times q}$, respectively. From [23, Proposition 7.7] one can see that the class $\mathcal{P}^{(q,q)}_{-\tilde{J}_q,\succcurlyeq}(\mathbb{C}\setminus[\alpha, \infty))$ can be considered as a projective extension of the class $\mathcal{S}_{q,[\alpha,\infty)}$.

Now we turn our attention to a particular subclass of $\mathcal{P}^{(q,q)}_{-\tilde{J}_q,\succcurlyeq}(\mathbb{C} \setminus [\alpha, \infty))$, which was introduced in [23, Notation 7.13].

Notation 11.4 Let $\alpha \in \mathbb{R}$ and let $A \in \mathbb{C}^{q\times p}$. We denote by $\mathcal{P}_{q,\alpha}[A]$ the set of all $(\phi, \psi) \in \mathcal{P}^{(q,q)}_{-\tilde{J}_q,\succcurlyeq}(\mathbb{C} \setminus [\alpha, \infty))$ such that $\mathcal{R}(\phi(z)) \subseteq \mathcal{R}(A)$ for all points $z \in \mathbb{C} \setminus [\alpha, \infty)$ which are points of holomorphicity of ϕ. Further, let $\tilde{\mathcal{P}}_{q,\alpha}[A] := \mathcal{P}_{q,\alpha}[A] \cap \tilde{\mathcal{P}}^{(q,q)}_{-\tilde{J}_q,\succcurlyeq}(\mathbb{C} \setminus [\alpha, \infty))$.

Remark 11.5 ([23, Remark 7.15]) Let $\alpha \in \mathbb{R}$. Further, let A be a non-singular complex $q \times q$ matrix. Then $\mathcal{P}_{q,\alpha}[A] = \mathcal{P}^{(q,q)}_{-\tilde{J}_q,\succcurlyeq}(\mathbb{C} \setminus [\alpha, \infty))$.

Example 11.6 Let $\alpha \in \mathbb{R}$, let $Q \in \mathbb{C}^{q\times q}$ be such that $\mathcal{R}(Q^*) = \mathcal{R}(Q)$, and let $\eta \in \mathbb{C} \setminus \{0\}$. Let $\phi, \psi : \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C}^{q\times q}$ be defined by $\phi(z) := \eta Q$ and $\psi(z) := \mathbb{P}_{\mathcal{N}(Q)}$. In view of Remark A.6, then it is readily checked that (ϕ, ψ) belongs to $\mathcal{P}_{q,\alpha}[Q]$ and that conditions (i)–(iii) of Definition 11.2 are fulfilled with the discrete subset $\mathcal{D} := \emptyset$ of $\mathbb{C} \setminus [\alpha, \infty)$.

Example 11.7 Let $\alpha \in \mathbb{R}$ and let $A \in \mathbb{C}^{q\times q}$. Then $(\mathfrak{O}_q, \mathfrak{I}_q) \in \mathcal{P}_{q,\alpha}[A]$ and conditions (i)–(iii) of Definition 11.2 are fulfilled with the discrete subset $\mathcal{D} := \emptyset$ of $\mathbb{C} \setminus [\alpha, \infty)$.

The procedure of constructing subclasses of $\mathcal{P}^{(q,q)}_{-\tilde{J}_q,\succcurlyeq}(\mathbb{C} \setminus [\alpha, \infty))$ via Notation 11.4 stands in full harmony with the equivalence relation in $\mathcal{P}^{(q,q)}_{-\tilde{J}_q,\succcurlyeq}(\mathbb{C}\setminus[\alpha, \infty))$ introduced in Remark 11.3:

Lemma 11.8 ([23, Lemma 7.17]) *Let $\alpha \in \mathbb{R}$, let $A \in \mathbb{C}^{q\times q}$, and let $(\phi_1, \psi_1) \in \mathcal{P}_{q,\alpha}[A]$. Further, let $(\phi_2, \psi_2) \in \mathcal{P}^{(q,q)}_{-\tilde{J}_q,\succcurlyeq}(\mathbb{C} \setminus [\alpha, \infty))$ be such that $\langle(\phi_1, \psi_1)\rangle = \langle(\phi_2, \psi_2)\rangle$. Then $(\phi_2, \psi_2) \in \mathcal{P}_{q,\alpha}[A]$.*

The following result shows that the class $\mathcal{P}_{q,\alpha}[A]$ can be considered as a projective extension of the class $\mathcal{S}_{q,[\alpha,\infty)}[A]$ introduced in Notation 8.4.

Remark 11.9 ([23, Remark 7.18]) Let $\alpha \in \mathbb{R}$ and let $f \in \mathcal{S}_{q,[\alpha,\infty)}$. According to [23, Proposition 7.7], then $(f, \mathfrak{I}_q)$ belongs to $\tilde{\mathcal{P}}^{(q,q)}_{-\tilde{J}_q,\succcurlyeq}(\mathbb{C} \setminus [\alpha, \infty))$. Furthermore, if $A \in \mathbb{C}^{q\times q}$ is given, then $f \in \mathcal{S}_{q,[\alpha,\infty)}[A]$ if and only if $(f, \mathfrak{I}_q) \in \mathcal{P}^{(q,q)}_{-\tilde{J}_q,\succcurlyeq}(\mathbb{C} \setminus [\alpha, \infty))$.

12 On a Coupled Pair of Schur–Stieltjes-Type Transforms

Now we recall some aspects of the elementary step of our Schur-type algorithm for the class $\mathcal{S}_{q,[\alpha,\infty)}$, used in [21, Section 9]. We will be led to a situation which, roughly speaking, looks as follows: Let $\alpha \in \mathbb{R}$, let $A \in \mathbb{C}^{p\times q}$ and let $F : \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C}^{p\times q}$. Then the matrix-valued functions $F^{[+,\alpha,A]} : \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C}^{p\times q}$ and $F^{[-,\alpha,A]} : \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C}^{p\times q}$ which are defined by

$$F^{[+,\alpha,A]}(z) := -A\Big(I_q + (z-\alpha)^{-1}[F(z)]^\dagger A\Big) \tag{12.1}$$

and

$$F^{[-,\alpha,A]}(z) := -(z-\alpha)^{-1}A\Big[I_q + A^\dagger F(z)\Big]^\dagger, \tag{12.2}$$

respectively, will be central objects in our further considerations. The matrix-valued functions $F^{[+,\alpha,A]}$ and $F^{[-,\alpha,A]}$ are called the (α, A)-*Schur–Stieltjes transform* of F and the *inverse* (α, A)-*Schur–Stieltjes transform* of F, respectively. The generic case studied here concerns the situation where $p = q$, A is a complex $q \times q$ matrix with later specified properties, and $F \in \mathcal{S}_{q,[\alpha,\infty)}$. In essential cases, the formulas (12.1) and (12.2) can be rewritten as linear fractional transformations with appropriately chosen generating matrix-valued functions (see [21, Section 9]). The role of these generating functions will be played by the matrix polynomials $W_{\alpha,A}$ and $V_{\alpha,A}$ which are given as follows:

Remark 12.1 Let $\alpha \in \mathbb{R}$ and let $A \in \mathbb{C}^{p\times q}$. Then $V_{\alpha,A} : \mathbb{C} \to \mathbb{C}^{(p+q)\times(p+q)}$ and $W_{\alpha,A} : \mathbb{C} \to \mathbb{C}^{(p+q)\times(p+q)}$ given by

$$V_{\alpha,A}(z) := \begin{bmatrix} 0_{p\times p} & -A \\ (z-\alpha)A^\dagger & (z-\alpha)I_q \end{bmatrix}, \quad W_{\alpha,A}(z) := \begin{bmatrix} (z-\alpha)I_p & A \\ -(z-\alpha)A^\dagger & I_q - A^\dagger A \end{bmatrix} \tag{12.3}$$

are linear $(p+q)\times(p+q)$ matrix polynomials and, in particular, holomorphic in $\mathbb{C}$.

The use of the matrix polynomial $V_{\alpha,A}$ was inspired by some constructions in the paper [26]. In particular, we mention [26, formula (2.3)]. In their constructions, Hu and Chen used Drazin inverses instead of Moore-Penrose inverses of matrices. Since both types of generalized inverses coincide for Hermitian matrices (see, e. g. [24, Proposition A.2]), we can conclude that in the generic case the matrix polynomials $V_{\alpha,A}$ coincide for $\alpha = 0$ with the functions used in [26].

Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices, and let $m \in \mathbb{Z}_{0,\kappa}$. For all $l \in \mathbb{Z}_{0,m}$, let $(s_j^{[l,\alpha]})_{j=0}^{\kappa-l}$ be the l-th α-

S-transform of $(s_j)_{j=0}^{\kappa}$ (see Definition 7.6). Let the sequence $(V_{\alpha,s_0^{[j,\alpha]}})_{j=0}^{m}$ be given via (12.3), let

$$\mathfrak{V}^{[\alpha,(s_j)_{j=0}^m]} := V_{\alpha,s_0^{[0,\alpha]}} V_{\alpha,s_0^{[1,\alpha]}} \cdots V_{\alpha,s_0^{[m-1,\alpha]}} V_{\alpha,s_0^{[m,\alpha]}}, \tag{12.4}$$

and let

$$\mathfrak{V}^{[\alpha,(s_j)_{j=0}^m]} = \begin{bmatrix} \mathfrak{v}_{11}^{[\alpha,(s_j)_{j=0}^m]} & \mathfrak{v}_{12}^{[\alpha,(s_j)_{j=0}^m]} \\ \mathfrak{v}_{21}^{[\alpha,(s_j)_{j=0}^m]} & \mathfrak{v}_{22}^{[\alpha,(s_j)_{j=0}^m]} \end{bmatrix} \tag{12.5}$$

be the $q \times q$ block representation of $\mathfrak{V}^{[\alpha,(s_j)_{j=0}^m]}$ with $p \times p$ block $\mathfrak{v}_{11}^{[\alpha,(s_j)_{j=0}^m]}$. Furthermore, let the sequence $(W_{\alpha,s_0^{[j,\alpha]}})_{j=0}^{m}$ be given via (12.3), let

$$\mathfrak{W}^{[\alpha,(s_j)_{j=0}^m]} := W_{\alpha,s_0^{[0,\alpha]}} W_{\alpha,s_0^{[1,\alpha]}} \cdots W_{\alpha,s_0^{[m-1,\alpha]}} W_{\alpha,s_0^{[m,\alpha]}}, \tag{12.6}$$

and let

$$\mathfrak{W}^{[\alpha,(s_j)_{j=0}^m]} = \begin{bmatrix} \mathfrak{w}_{11}^{[\alpha,(s_j)_{j=0}^m]} & \mathfrak{w}_{12}^{[\alpha,(s_j)_{j=0}^m]} \\ \mathfrak{w}_{21}^{[\alpha,(s_j)_{j=0}^m]} & \mathfrak{w}_{22}^{[\alpha,(s_j)_{j=0}^m]} \end{bmatrix}$$

be the $q \times q$ block representation of $\mathfrak{W}^{[\alpha,(s_j)_{j=0}^m]}$ with $p \times p$ block $\mathfrak{w}_{11}^{[\alpha,(s_j)_{j=0}^m]}$.

Remark 12.2 Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. For all $m \in \mathbb{Z}_{1,\kappa}$ and all $l \in \mathbb{Z}_{0,m-1}$, one can see then from (12.4), (12.6), and [20, Remark 8.3] that

$$\mathfrak{V}^{[\alpha,(s_j^{[l,\alpha]})_{j=0}^{m-l}]} = \mathfrak{V}^{[\alpha,(s_j^{[l,\alpha]})_{j=0}^{m-(l+1)}]} V_{\alpha,s_0^{[m,\alpha]}},$$

$$\mathfrak{V}^{[\alpha,(s_j^{[l,\alpha]})_{j=0}^{m-l}]} = V_{\alpha,s_0^{[l,\alpha]}} \mathfrak{V}^{[\alpha,(t_j^{[l,\alpha]})_{j=0}^{m-(l+1)}]},$$

$$\mathfrak{W}^{[\alpha,(s_j^{[l,\alpha]})_{j=0}^{m-l}]} = \mathfrak{W}^{[\alpha,(s_j^{[l,\alpha]})_{j=0}^{m-(l+1)}]} W_{\alpha,s_0^{[m,\alpha]}},$$

and

$$\mathfrak{W}^{[\alpha,(s_j^{[l,\alpha]})_{j=0}^{m-l}]} = W_{\alpha,s_0^{[l,\alpha]}} \mathfrak{W}^{[\alpha,(t_j^{[l,\alpha]})_{j=0}^{m-(l+1)}]}$$

hold true, where $t_j := s_j^{[l+1,\alpha]}$ for all $j \in \mathbb{Z}_{0,m-(l+1)}$.

Now we are going to consider the situation which will turn out to be typical for larger parts of our future considerations. Let $A \in \mathbb{C}^{q\times q}_{\succcurlyeq}$ and let $G \in \mathcal{S}_{q,[\alpha,\infty)}[A]$ where the class $\mathcal{S}_{q,[\alpha,\infty)}[A]$ was introduced in Notation 8.4. Our aim is then to investigate the function $G^{[-,\alpha,A]}$ given by (12.2). We begin by rewriting formula (12.2) as linear fractional transformation. In the sequel, we will often use the fact that, for each $G \in \mathcal{S}_{q,[\alpha,\infty)}$, the matrix γ_G given via Notation 8.3 is non-negative Hermitian.

Lemma 12.3 *Let $\alpha \in \mathbb{R}$, let $A \in \mathbb{C}^{q\times q}_{\succcurlyeq}$, and let $G \in \mathcal{S}_{q,[\alpha,\infty)}[A]$. For all $z \in \mathbb{C}\setminus[\alpha,\infty)$, then $G(z) \in \mathcal{Q}_{[(z-\alpha)A^\dagger,(z-\alpha)I_q]}$ and $G^{[-,\alpha,A]}(z) = \mathcal{S}^{(q,q)}_{V_{\alpha,A}(z)}(G(z))$.*

Proof In view of Lemma 8.5, the assertion follows immediately from [21, Lemma 9.8]. □

Assuming the situation of Lemma 12.3, now we obtain useful insights into the structure of the inverse (α, A)-Schur–Stieltjes transform of F.

Proposition 12.4 *Let $A \in \mathbb{C}^{q\times q}_{\succcurlyeq}$, let $\alpha \in \mathbb{R}$, let $G \in \mathcal{S}_{q,[\alpha,\infty)}[A]$, and let $u_0 := A(A+\gamma_G)^\dagger A$. Then $G^{[-,\alpha,A]}\colon \mathbb{C}\setminus[\alpha,\infty) \to \mathbb{C}^{q\times q}$ given by (12.2) belongs to $\mathcal{S}_{0,q,[\alpha,\infty)}[(u_j)_{j=0}^0, =]$ and fulfills $\mathcal{R}(G^{[-,\alpha,A]}(z)) = \mathcal{R}(A)$ and $\mathcal{N}(G^{[-,\alpha,A]}(z)) = \mathcal{N}(A)$ for all $z \in \mathbb{C}\setminus[\alpha,\infty)$.*

Proof Combine Lemma 8.5 with [21, Proposition 9.10]. □

We have verified in [21, Section 9] that under appropriate conditions the equations

$$(F^{[+,\alpha,A]})^{[-,\alpha,A]} = F \qquad \text{and} \qquad (G^{[-,\alpha,A]})^{[+,\alpha,A]} = G \tag{12.7}$$

hold true. The formulas in (12.7) show that the functions $F^{[+,\alpha,A]}$ and $G^{[-,\alpha,A]}$ form indeed a coupled pair of transformations.

In [21, Section 10], we studied the following situation: Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m \in \mathcal{K}^{\succcurlyeq,\mathrm{e}}_{q,m,\alpha}$. Then Theorem 10.1 yields that the class $\mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^m, =]$ is non-empty. If $F \in \mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^m, =]$, then our interest in [21, Section 10] was concentrated on the (α, s_0)-Schur–Stieltjes transform $F^{[+,\alpha,s_0]}$ of F.

We will consider now a function $F \in \mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^m, \preccurlyeq]$. The following result provides essential information about the (α, s_0)-Schur–Stieltjes transform $F^{[+,\alpha,s_0]}$ of F.

Theorem 12.5 *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}$, let $(s_j)_{j=0}^m \in \mathcal{K}^{\succcurlyeq}_{q,m,\alpha}$ with α-S-transform $(s_j^{[1,\alpha]})_{j=0}^{m-1}$, and let $F \in \mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^m, \preccurlyeq]$. Then $F^{[+,\alpha,s_0]}$ belongs to $\mathcal{S}_{m-1,q,[\alpha,\infty)}[(s_j^{[1,\alpha]})_{j=0}^{m-1}, \preccurlyeq]$.*

Proof Because of (10.2), we have $F \in \mathcal{S}_{m,q,[\alpha,\infty)}$ and $\sigma_F \in \mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty)]; (s_j)_{j=0}^m, \preccurlyeq$. In particular, $\sigma_F \in \mathcal{M}_{q,m}^{\succcurlyeq}([\alpha,\infty))$. We set

$$t_j := s_j^{(\sigma_F)} \qquad \text{for all } j \in \mathbb{Z}_{0,m}. \tag{12.8}$$

Because of (12.8), the application of Corollary 3.6 yields $(t_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$. Furthermore, from (12.8) and (10.1) we get $F \in \mathcal{S}_{m,q,[\alpha,\infty)}[(t_j)_{j=0}^m, =]$. From $(t_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$ and Remark 3.4 we get $(t_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq}$. Because of this and $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq}$, from [16, Lemma 2.9(a), (b)] we see that $s_j \in \mathbb{C}_{\mathrm{H}}^{q\times q}$ and $t_j \in \mathbb{C}_{\mathrm{H}}^{q\times q}$ for all $j \in \mathbb{Z}_{0,m}$ hold true. Because of (12.8) and the choice of F, we infer $t_j = s_j$ for all $j \in \mathbb{Z}_{0,m-1}$ and $t_m \preccurlyeq s_m$. In particular, $t_0 = s_0$. If $(s_j^{[1,\alpha]})_{j=0}^{m-1}$ and $(t_j^{[1,\alpha]})_{j=0}^{m-1}$ are the α-S-transforms of $(s_j)_{j=0}^m$ and $(t_j)_{j=0}^m$, respectively, the application of [23, Lemma 3.6] yields then that $(s_j^{[1,\alpha]})_{j=0}^{m-1}$ and $(t_j^{[1,\alpha]})_{j=0}^{m-1}$ are sequences from $\mathbb{C}_{\mathrm{H}}^{q\times q}$ which satisfy $t_{m-1}^{[1,\alpha]} \preccurlyeq s_{m-1}^{[1,\alpha]}$ and, in the case $m \geq 2$, moreover

$$s_j^{[1,\alpha]} = t_j^{[1,\alpha]} \qquad \text{for all } j \in \mathbb{Z}_{0,m-2}. \tag{12.9}$$

In view of $(t_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$ and $F \in \mathcal{S}_{m,q,[\alpha,\infty)}[(t_j)_{j=0}^m, =]$, [21, Theorem 10.3] yields $F^{[+,\alpha,t_0]} \in \mathcal{S}_{m-1,q,[\alpha,\infty)}[(t_j^{[1,\alpha]})_{j=0}^{m-1}, =]$. Combining this with $s_0 = t_0$, we get $F^{[+,\alpha,s_0]} \in \mathcal{S}_{m-1,q,[\alpha,\infty)}[(t_j^{[1,\alpha]})_{j=0}^{m-1}, =]$. In view of $t_{m-1}^{[1,\alpha]} \preccurlyeq s_{m-1}^{[1,\alpha]}$ and (12.9), this implies $F^{[+,\alpha,s_0]} \in \mathcal{S}_{m-1,q,[\alpha,\infty)}[(s_j^{[1,\alpha]})_{j=0}^{m-1}, \preccurlyeq]$. □

In [21, Section 11] we considered the following situation: Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}$, let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$ with first α-S-transform $(s_j^{[1,\alpha]})_{j=0}^{m-1}$, and let $F \in \mathcal{S}_{m-1,q,[\alpha,\infty)}[(s_j^{[1,\alpha]})_{j=0}^{m-1}, =]$. Then our interest in [21, Section 11] was concentrated on the inverse (α, s_0)-Schur–Stieltjes transform $F^{[-,\alpha,s_0]}$ of F. The following result on this theme is of fundamental importance.

Theorem 12.6 ([21, Theorem 11.3]) *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}$, let $(s_j)_{j=0}^m \in \mathcal{K}_{q,\kappa,\alpha}^{\succcurlyeq,\mathrm{e}}$ with first α-S-transform $(s_j^{[1,\alpha]})_{j=0}^{m-1}$, and let $F \in \mathcal{S}_{m-1,q,[\alpha,\infty)}[(s_j^{[1,\alpha]})_{j=0}^{m-1}, =]$. Then $F^{[-,\alpha,s_0]}$ belongs to $\mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^m, =]$.*

Now we consider the following situation: Let $s_0 \in \mathbb{C}_{\mathrm{H}}^{q\times q}$. Further, let $M \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ be such that the conditions $s_0 - M \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ and $\operatorname{rank} M = \operatorname{rank} s_0$ are satisfied. Then we will show that the function $G\colon \mathbb{C}\setminus[\alpha,\infty) \to \mathbb{C}^{q\times q}$ which is defined by $G(z) := s_0 M^\dagger(s_0 - M)$ belongs to $\mathcal{S}_{q,[\alpha,\infty)}[s_0]$ and if $S := G^{[-,\alpha,s_0]}$ stands for the inverse (α, s_0)-Schur–Stieltjes transform of G then S belongs to $\mathcal{S}_{0,q,[\alpha,\infty)}[(s_j)_{j=0}^0, \preccurlyeq]$ and satisfies $\sigma_S([\alpha,\infty)) = M$.

Proposition 12.7 *Let* $s_0 \in \mathbb{C}_{\mathrm{H}}^{q\times q}$ *and* $M \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ *be such that* $s_0 - M \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ *and* $\operatorname{rank} M = \operatorname{rank} s_0$*, and let* $\alpha \in \mathbb{R}$*. Then* $G\colon \mathbb{C}\setminus[\alpha,\infty) \to \mathbb{C}^{q\times q}$ *defined by* $G(z) := s_0 M^\dagger(s_0 - M)$ *belongs to* $\mathcal{S}_{q,[\alpha,\infty)}[s_0]$ *and the inverse* (α, s_0)*-Schur–Stieltjes transform* $S := G^{[-,\alpha,s_0]}$ *of* G *fulfills* $S \in \mathcal{S}_{0,q,[\alpha,\infty)}[(s_j)_{j=0}^0, \preccurlyeq]$ *and* $\sigma_S([\alpha,\infty)) = M$*. Moreover,* $S \in \mathcal{S}_{0,q,[\alpha,\infty)}[(s_j)_{j=0}^0, =]$ *if and only if* $M = s_0$*. If* $M = s_0$*, then* $S(z) = (\alpha - z)^{-1} s_0$ *for all* $z \in \mathbb{C}\setminus[\alpha,\infty)$ *and* $\sigma_S = s_0\delta_\alpha$*, where* δ_α *is the Dirac measure on* $([\alpha,\infty), \mathfrak{B}_{[\alpha,\infty)})$ *with unit mass at* α*.*

Proof In view of Lemma A.20(b), we have $s_0 M^\dagger(s_0 - M) \in \mathbb{C}_{\succcurlyeq}^{q\times q}$. Thus, Theorem 8.2 yields $G \in \mathcal{S}_{q,[\alpha,\infty)}$ and

$$(\gamma_G, \mu_G) = \left(s_0 M^\dagger(s_0 - M), o_q\right), \tag{12.10}$$

where $o_q\colon \mathfrak{B}_{[\alpha,\infty)} \to \mathbb{C}^{q\times q}$ is given by $o_q(B) := 0_{q\times q}$. Because of $s_0 s_0^\dagger[s_0 M^\dagger(s_0 - M)] = s_0 M^\dagger(s_0 - M)$, we get $s_0 s_0^\dagger G = G$. Hence, $G \in \mathcal{S}_{q,[\alpha,\infty)}[s_0]$. Lemma A.20(a) yields $s_0 \in \mathbb{C}_{\succcurlyeq}^{q\times q}$, $s_0 s_0^\dagger = M M^\dagger$, and $s_0^\dagger s_0 = M^\dagger M$. Consequently, (12.10) shows that

$$s_0 + \gamma_G = s_0 + s_0 M^\dagger(s_0 - M) = s_0 + s_0 M^\dagger s_0 - s_0 M^\dagger M = s_0 M^\dagger s_0.$$

Thanks to $s_0 \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ and Proposition 12.4, we get $S \in \mathcal{S}_{0,q,[\alpha,\infty)}[(u_j)_{j=0}^0, =]$, where $u_0 := s_0(s_0 + \gamma_G)^\dagger s_0$. In view of (12.10) and Lemma A.20(c) we obtain

$$u_0 = s_0(s_0 + \gamma_G)^\dagger s_0 = s_0\left[s_0 + s_0 M^\dagger(s_0 - M)\right]^\dagger s_0 = M. \tag{12.11}$$

From $S \in \mathcal{S}_{0,q,[\alpha,\infty)}[(u_j)_{j=0}^0, =]$ and (12.11) we conclude $\sigma_S([\alpha,\infty)) = M$ and that S belongs to $\mathcal{S}_{0,q,[\alpha,\infty)}[(s_j)_{j=0}^0, =]$ if and only if $M = s_0$. Furthermore, because of $s_0 - M \in \mathbb{C}_{\succcurlyeq}^{q\times q}$, $S \in \mathcal{S}_{0,q,[\alpha,\infty)}[(u_j)_{j=0}^0, =]$, and (12.11), we get $S \in \mathcal{S}_{0,q,[\alpha,\infty)}[(s_j)_{j=0}^0, \preccurlyeq]$. If $M = s_0$, then, for all $z \in \mathbb{C}\setminus[\alpha,\infty)$, we get $G(z) = 0_{q\times q}$ and, in view of (12.2), furthermore

$$S(z) = G^{[-,\alpha,s_0]}(z) = -(z-\alpha)^{-1} s_0\left[I_q + s_0^\dagger G(z)\right]^\dagger = \frac{1}{\alpha - z} s_0.$$

Setting $\sigma := s_0\delta_\alpha$, from (9.4) we get in the case $M = s_0$ then

$$S_\sigma(z) = \int_{[\alpha,\infty)} \frac{1}{t-z}\sigma(\mathrm{d}t) = \int_{[\alpha,\infty)} \frac{1}{t-z}(s_0\delta_\alpha)(\mathrm{d}t) = \frac{1}{\alpha - z} s_0 = S(z)$$

for all $z \in \mathbb{C}\setminus[\alpha,\infty)$. By virtue of Theorem 9.1, hence $\sigma_S = \sigma$. □

Remark 12.8 Let $\alpha \in \mathbb{R}$, let $s_0 \in \mathbb{C}_{\mathrm{H}}^{q\times q}$, and $M \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ be such that $s_0 - M \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ and $\operatorname{rank} M = \operatorname{rank} s_0$. Then Proposition 12.7 shows that there exists some constant function $G \in \mathcal{S}_{q,[\alpha,\infty)}[s_0]$ such that $S := G^{[-,\alpha,s_0]}$ satisfies $S \in \mathcal{S}_{0,q,[\alpha,\infty)}[(s_j)_{j=0}^{0}, \preccurlyeq]$ and $\sigma_S([\alpha,\infty)) = M$.

Our following consideration complements the topic of Proposition 12.7.

Proposition 12.9 *Let $\alpha \in \mathbb{R}$, let $s_0 \in \mathbb{C}_{\succcurlyeq}^{q\times q}$, and let $G \in \mathcal{S}_{q,[\alpha,\infty)}[s_0]$. Then $S := G^{[-,\alpha,s_0]}$ belongs to $\mathcal{S}_{0,q,[\alpha,\infty)}[(s_j)_{j=0}^{0}, \preccurlyeq]$ and* $\operatorname{rank} \sigma_S([\alpha,\infty)) = \operatorname{rank} s_0$.

Proof The application of Proposition 12.4 yields that S belongs to $\mathcal{S}_{0,q,[0,\infty)}[(u_j)_{j=0}^{0}, =]$, where $u_0 := s_0(s_0+\gamma_G)^{\dagger}s_0$. In view of (10.1), this implies $S \in \mathcal{S}_{0,q,[\alpha,\infty)}$ and $\sigma_S([\alpha,\infty)) = u_0 = s_0(s_0+\gamma_G)^{\dagger}s_0$. Because of $\gamma_G \in \mathbb{C}_{\succcurlyeq}^{q\times q}$, we have $s_0 + \gamma_G \succcurlyeq s_0 \succcurlyeq 0_{q\times q}$, which in view of [18, Lemma A.7] implies $s_0 \succcurlyeq s_0(s_0+\gamma_G)^{\dagger}s_0 \succcurlyeq 0_{q\times q}$ and $\mathcal{R}(s_0(s_0+\gamma_G)^{\dagger}s_0) = \mathcal{R}(s_0)$. Thus, we obtain $\sigma_S([\alpha,\infty)) \preccurlyeq s_0$ and $\operatorname{rank}\sigma_S([\alpha,\infty)) = \operatorname{rank} s_0$. □

It should be mentioned that, in view of Remark 11.9 and Lemma 12.3, the assertion $S \in \mathcal{S}_{0,q,[\alpha,\infty)}[(s_j)_{j=0}^{0}, \preccurlyeq]$ in Proposition 12.9 is also a direct consequence of [23, Proposition 11.3]. On the other hand, if $S \in \mathcal{S}_{0,q,[\alpha,\infty)}[(s_j)_{j=0}^{0}, \preccurlyeq]$ fulfills $\operatorname{rank}\sigma_S([\alpha,\infty)) < \operatorname{rank} s_0$, then we see from Proposition 12.9 that there is no $G \in \mathcal{S}_{q,[\alpha,\infty)}[s_0]$ such that $G^{[-,\alpha,s_0]} = S$. However, [21, Proposition 11.2(b), (c)] shows that there exists a $(\phi,\psi) \in \mathcal{P}_{q,\alpha}[s_0]$ such that S is generated by the linear fractional transformation of the pair (ϕ,ψ) with generating matrix-valued function V_{α,s_0}. Because of the above arguments, the pair (ϕ,ψ) is then not proper.

The following result continues the theme of [21, Proposition 11.2].

Proposition 12.10 *Let $\alpha \in \mathbb{R}$, let $s_0 \in \mathbb{C}_{\succcurlyeq}^{q\times q}$, and let $F \in \mathcal{S}_{0,q,[\alpha,\infty)}[(s_j)_{j=0}^{0}, \preccurlyeq]$ be such that* $\operatorname{rank}\sigma_F([\alpha,\infty)) = \operatorname{rank} s_0$ *is satisfied. Further, let W_{α,s_0} be given by* (12.3) *and let $W_{\alpha,s_0}\begin{bmatrix} F \\ \mathfrak{I}_q \end{bmatrix} = \begin{bmatrix} \phi \\ \psi \end{bmatrix}$ be the $q \times q$ block representation of $W_{\alpha,s_0}\begin{bmatrix} F \\ \mathfrak{I}_q \end{bmatrix}$. Then:*

(a) $\det\psi(z) \neq 0$ *for all* $z \in \mathbb{C}\setminus[\alpha,\infty)$.
(b) $G\colon \mathbb{C}\setminus[\alpha,\infty) \to \mathbb{C}^{q\times q}$ *defined by* $G(z) := [\phi(z)][\psi(z)]^{-1}$ *belongs to* $\mathcal{S}_{q,[\alpha,\infty)}[s_0]$.
(c) $F = G^{[-,\alpha,s_0]}$.

Proof

(a) Let $z \in \mathbb{C}\setminus[\alpha,\infty)$. In view of (12.3), we have

$$\begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} = W_{\alpha,s_0}(z)\begin{bmatrix} F(z) \\ I_q \end{bmatrix} = \begin{bmatrix} (z-\alpha)F(z)+s_0 \\ -(z-\alpha)s_0^{\dagger}F(z)+I_q - s_0^{\dagger}s_0 \end{bmatrix}.$$

Consequently,

$$\phi(z) = (z-\alpha)F(z) + s_0 \quad \text{and} \quad \psi(z) = -(z-\alpha)s_0^\dagger F(z) + I_q - s_0^\dagger s_0. \tag{12.12}$$

Now we consider an arbitrary $v \in \mathcal{N}(\psi(z))$. From (12.12) we infer then

$$(I_q - s_0^\dagger s_0)v = (z-\alpha)s_0^\dagger [F(z)]v. \tag{12.13}$$

In view of (12.13) and $s_0(I_q - s_0^\dagger s_0) = 0_{q\times q}$, we conclude $(z-\alpha)s_0 s_0^\dagger [F(z)]v = s_0(I_q - s_0^\dagger s_0)v = 0_{q\times 1}$. Because of $z - \alpha \neq 0$, then $s_0 s_0^\dagger [F(z)]v = 0_{q\times 1}$ follows. Since the assumption $s_0 \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ implies $(s_j)_{j=0}^0 \mathcal{K}_{q,0,\alpha}^{\succcurlyeq}$, from $F \in \mathcal{S}_{0,q,[\alpha,\infty)}[(s_j)_{j=0}^0, \preccurlyeq]$ and Proposition 10.7 we obtain $F \in \mathcal{S}_{q,[\alpha,\infty)}[s_0]$. In view of Notation 8.4 then $\mathcal{R}(F(z)) \subseteq \mathcal{R}(s_0)$. Thus from Remark A.4 then $s_0 s_0^\dagger F(z) = F(z)$. Combining this with $s_0 s_0^\dagger [F(z)]v = 0_{q\times 1}$, we get $[F(z)]v = 0_{q\times 1}$. Hence, from (12.13) it follows $v = s_0^\dagger s_0 v$. Let $M := \sigma_F([\alpha,\infty))$. Therefore, because of the choice of F, then $M \in \mathbb{C}_{\succcurlyeq}^{q\times q}$, $s_0 - M \in \mathbb{C}_{\succcurlyeq}^{q\times q}$, and $\operatorname{rank} M = \operatorname{rank} s_0$. Consequently, we infer from Lemma A.20(a) that $s_0^\dagger s_0 = M^\dagger M$. Combining this with $v = s_0 s_0^\dagger v$, we get $v = M^\dagger M v$. Because of $F \in \mathcal{S}_{0,q,[\alpha,\infty)}$ and $M \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ we obtain from [21, Lemma 3.9] that $\mathcal{N}(F(z)) = \mathcal{N}(M)$ holds true. Thus, using $[F(z)]v = 0_{q\times 1}$, we conclude then $Mv = 0_{q\times 1}$. In view of $v = M^\dagger M v$, we get then $v = 0_{q\times 1}$. Hence, $\mathcal{N}(\psi(z)) = \{0_{q\times 1}\}$. This implies $\det \psi(z) \neq 0$.

(b) In view of [23, Proposition 11.2(b)], the pair (ϕ,ψ) belongs to $\mathcal{P}_{q,\alpha}[s_0]$ and, in particular, to $\mathcal{P}_{-\tilde{J}_q,\succcurlyeq}^{(q,q)}(\mathbb{C} \setminus [\alpha,\infty))$. Part (a) yields that (ϕ,ψ) is proper. Thus, Remark 11.9 provides us that $G \in \mathcal{S}_{q,[\alpha,\infty)}$. If $\mathfrak{I}_q$ denotes the constant function in $\mathbb{C} \setminus [\alpha,\infty)$ with value I_q, then [21, Proposition 7.11] yields $(G,\mathfrak{I}_q) \in \mathcal{P}_{-\tilde{J}_q,\succcurlyeq}^{(q,q)}(\mathbb{C}\setminus[\alpha,\infty))$ and $\langle(\phi,\psi)\rangle = \langle(G,\mathfrak{I}_q)\rangle$. Since (ϕ,ψ) belongs to $\mathcal{P}_{q,\alpha}[s_0]$, Lemma 11.8 shows that $(G,\mathfrak{I}_q)$ belongs to $\mathcal{P}_{q,\alpha}[s_0]$. Since G belongs to $\mathcal{S}_{q,[\alpha,\infty)}$, Remark 11.9 gives $G \in \mathcal{S}_{q,[\alpha,\infty)}[s_0]$.

(c) We consider again an arbitrary $z \in \mathbb{C} \setminus [\alpha,\infty)$. Then [23, Proposition 11.2(c)] yields $\det[(z-\alpha)s_0^\dagger \phi(z) + (z-\alpha)I_q \cdot \psi(z)] \neq 0$ and

$$\big[0_{q\times q}\cdot\phi(z) - s_0\psi(z)\big]\Big[(z-\alpha)s_0^\dagger\phi(z) + (z-\alpha)I_q\cdot\psi(z)\Big]^{-1} = F(z).$$

In view of (12.3), part (a) implies then $\det[(z-\alpha)s_0^\dagger G(z) + (z-\alpha)I_q] \neq 0$ and

$$F(z) = \big[0_{q\times q}\cdot G(z) - s_0\big]\Big[(z-\alpha)s_0^\dagger G(z) + (z-\alpha)I_q\Big]^{-1} = \mathcal{S}_{V_{\alpha,s_0}(z)}^{(q,q)}(G(z)). \tag{12.14}$$

Taking into account $s_0 \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ and part (b), Lemma 12.3 then yields $\mathcal{S}_{V_{\alpha,s_0}(z)}^{(q,q)}(G(z)) = G^{[-,\alpha,s_0]}(z)$. Comparing with (12.14), we get $F = G^{[-,\alpha,s_0]}$. □

A closer look at Propositions 12.9 and 12.10 leads us to the following observation.

Proposition 12.11 *Let $\alpha \in \mathbb{R}$, let $s_0 \in \mathbb{C}_{\succcurlyeq}^{q\times q}$, and let*

$$\tilde{\mathcal{S}}_{0,q,[\alpha,\infty)}[(s_j)_{j=0}^0, \preccurlyeq] := \left\{S \in \mathcal{S}_{0,q,[\alpha,\infty)}[(s_j)_{j=0}^0, \preccurlyeq]\colon \operatorname{rank} \sigma_S([\alpha,\infty)) = \operatorname{rank} s_0\right\}.$$

Then the mapping $T_{[-,\alpha,s_0]}\colon \mathcal{S}_{q,[\alpha,\infty)}[s_0] \to \tilde{\mathcal{S}}_{0,q,[\alpha,\infty)}[(s_j)_{j=0}^0, \preccurlyeq]$ given by $T_{[-,\alpha,s_0]}(G) := G^{[-,\alpha,s_0]}$ is well defined and bijective. The inverse mapping $T_{[-,\alpha,s_0]}^{-1}$ is given, for $F \in \tilde{\mathcal{S}}_{0,q,[\alpha,\infty)}[(s_j)_{j=0}^0, \preccurlyeq]$, by $T_{[-,\alpha,s_0]}^{-1}(F) = F^{[+,\alpha,s_0]}$.

Proof In view of the assumption $s_0 \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ and Proposition 12.9, we obtain $T_{[-,\alpha,s_0]}(\mathcal{S}_{q,[\alpha,\infty)}[s_0]) \subseteq \tilde{\mathcal{S}}_{0,q,[\alpha,\infty)}[(s_j)_{j=0}^0, \preccurlyeq]$. In view of parts (b) and (c) of Proposition 12.10, we get $\tilde{\mathcal{S}}_{0,q,[\alpha,\infty)}[(s_j)_{j=0}^0, \preccurlyeq] \subseteq T_{[-,\alpha,s_0]}(\mathcal{S}_{q,[\alpha,\infty)}[s_0])$. Consequently, $T_{[-,\alpha,s_0]}(\mathcal{S}_{q,[\alpha,\infty)}[s_0]) = \tilde{\mathcal{S}}_{0,q,[\alpha,\infty)}[(s_j)_{j=0}^0, \preccurlyeq]$. Because of $s_0 \in \mathbb{C}_{\succcurlyeq}^{q\times q}$, we have $(s_j)_{j=0}^0 \in \mathcal{K}_{q,0,\alpha}^{\succcurlyeq,\mathrm{e}}$. For each $G \in \mathcal{S}_{q,[\alpha,\infty)}[s_0]$ [21, Corollary 9.14] yields $(G^{[-,\alpha,s_0]})^{[+,\alpha,s_0]} = G$. Thus, $[T_{[-,\alpha,s_0]}(G)]^{[+,\alpha,s_0]} = G$. Hence, $T_{[-,\alpha,s_0]}$ is also injective and the inverse mapping is given, for $F \in \tilde{\mathcal{S}}_{0,q,[\alpha,\infty)}[(s_j)_{j=0}^0, \preccurlyeq]$, by $T_{[-,\alpha,s_0]}^{-1}(F) = F^{[+,\alpha,s_0]}$. □

13 Some Observations on Distinguished Elements of the Set $\mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^m, \preccurlyeq]$

The main goal of this section is a closer look on some distinguished elements of the set $\mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^m, \preccurlyeq]$. Our starting point is the following description of this set which was one of the central results in [23]:

Theorem 13.1 ([23, Theorem 12.3]) *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$. Let $(s_j^{[m,\alpha]})_{j=0}^0$ be the m-th α-S-transform of $(s_j)_{j=0}^m$. Let $\mathfrak{V}^{[\alpha,(s_j)_{j=0}^m]}$ be defined via (12.4) and (12.3). Furthermore, let (12.5) be the $q \times q$ block representation of $\mathfrak{V}^{[\alpha,(s_j)_{j=0}^m]}$. Then the following statements hold true:*

(a) *For each* $(\phi, \psi) \in \mathcal{P}_{q,\alpha}[s_0^{[m,\alpha]}]$, *the function* $\det(\mathfrak{v}_{21}^{[\alpha,(s_j)_{j=0}^m]}\phi + \mathfrak{v}_{22}^{[\alpha,(s_j)_{j=0}^m]}\psi)$ *is meromorphic in* $\mathbb{C} \setminus [\alpha, \infty)$ *and does not vanish identically. Furthermore,*

$$(\mathfrak{v}_{11}^{[\alpha,(s_j)_{j=0}^m]}\phi + \mathfrak{v}_{12}^{[\alpha,(s_j)_{j=0}^m]}\psi)(\mathfrak{v}_{21}^{[\alpha,(s_j)_{j=0}^m]}\phi + \mathfrak{v}_{22}^{[\alpha,(s_j)_{j=0}^m]}\psi)^{-1} \in \mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^m, \preccurlyeq].$$

(b) *For each* $F \in \mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^m, \preccurlyeq]$, *there exists a pair* $(\phi, \psi) \in \mathcal{P}_{q,\alpha}[s_0^{[m,\alpha]}]$ *of* $q \times q$ *matrix-valued functions* ϕ *and* ψ *which are holomorphic in* $\mathbb{C} \setminus [\alpha, \infty)$ *such that, for each* $z \in \mathbb{C} \setminus [\alpha, \infty)$, *the inequality* $\det[\mathfrak{v}_{21}^{[\alpha,(s_j)_{j=0}^m]}(z)\phi(z) + \mathfrak{v}_{22}^{[\alpha,(s_j)_{j=0}^m]}(z)\psi(z)] \neq 0$ *and the representation*

$$F(z) = \left[\mathfrak{v}_{11}^{[\alpha,(s_j)_{j=0}^m]}(z)\phi(z) + \mathfrak{v}_{12}^{[\alpha,(s_j)_{j=0}^m]}(z)\psi(z)\right] \times \left[\mathfrak{v}_{21}^{[\alpha,(s_j)_{j=0}^m]}(z)\phi(z) + \mathfrak{v}_{22}^{[\alpha,(s_j)_{j=0}^m]}(z)\psi(z)\right]^{-1}$$

of F *hold true.*

(c) *Let* $(\phi_1, \psi_1), (\phi_2, \psi_2) \in \mathcal{P}_{q,\alpha}[s_0^{[m,\alpha]}]$. *Then* $\langle(\phi_1, \psi_1)\rangle = \langle(\phi_2, \psi_2)\rangle$ *is fulfilled if and only if the equation* $(\mathfrak{v}_{11}^{[\alpha,(s_j)_{j=0}^m]}\phi_1 + \mathfrak{v}_{12}^{[\alpha,(s_j)_{j=0}^m]}\psi_1)(\mathfrak{v}_{21}^{[\alpha,(s_j)_{j=0}^m]}\phi_1 + \mathfrak{v}_{22}^{[\alpha,(s_j)_{j=0}^m]}\psi_1)^{-1} = (\mathfrak{v}_{11}^{[\alpha,(s_j)_{j=0}^m]}\phi_2 + \mathfrak{v}_{12}^{[\alpha,(s_j)_{j=0}^m]}\psi_2)(\mathfrak{v}_{21}^{[\alpha,(s_j)_{j=0}^m]}\phi_2 + \mathfrak{v}_{22}^{[\alpha,(s_j)_{j=0}^m]})$ $\psi_2{}^{-1}$ *holds true.*

In the following, we will often use an essential fact expressed in Theorem 7.7. Indeed, for each sequence $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$, its right α-Stieltjes parametrization $(Q_j)_{j=0}^m$ can be expressed in terms of the sequences $((s_j^{[k,\alpha]})_{j=0}^{m-k})_{k=0}^m$ of k-th α-S-transforms of $(s_j)_{j=0}^m$.

Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m$ be a sequence of complex $q \times q$ matrices. Denote by $(s_j^{[m,\alpha]})_{j=0}^0$ the m-th α-S-transform of $(s_j)_{j=0}^m$. According to Lemma 11.8, we write $\langle\mathcal{P}_{q,\alpha}[s_0^{[m,\alpha]}]\rangle$ for the set of the equivalence classes the representatives of which belong to $\mathcal{P}_{q,\alpha}[s_0^{[m,\alpha]}]$. From Theorem 13.1 one gets immediately the following result:

Corollary 13.2 *Let* $\alpha \in \mathbb{R}$, *let* $m \in \mathbb{N}_0$, *and let* $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$ *with right* α*-Stieltjes parametrization* $(Q_j)_{j=0}^m$. *Then the mapping* $\Sigma_{m,\preccurlyeq}^{\langle s\rangle} : \langle\mathcal{P}_{q,\alpha}[Q_m]\rangle \to \mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^m, \preccurlyeq]$ *given by*

$$\Sigma_{m,\preccurlyeq}^{\langle s\rangle}(\langle(\phi, \psi)\rangle) := (\mathfrak{v}_{11}^{[\alpha,(s_j)_{j=0}^m]}\phi + \mathfrak{v}_{12}^{[\alpha,(s_j)_{j=0}^m]}\psi)(\mathfrak{v}_{21}^{[\alpha,(s_j)_{j=0}^m]}\phi + \mathfrak{v}_{22}^{[\alpha,(s_j)_{j=0}^m]}\psi)^{-1}$$

is well defined and bijective.

Proof Theorem 7.7 yields $s_0^{[m,\alpha]} = Q_m$, where $(s_j^{[m,\alpha]})_{j=0}^0$ denotes the m-th α-S-transform of $(s_j)_{j=0}^m$. The application of [23, Corollary 12.4] completes the proof. □

Let the assumptions of Corollary 13.2 be satisfied and let the $2q \times 2q$ matrix polynomial $\mathfrak{V}^{[\alpha,(s_j)_{j=0}^m]}$ be defined by (12.3) and (12.4). In view of Theorem 13.1 and Corollary 13.2 the function $\mathfrak{V}^{[\alpha,(s_j)_{j=0}^m]}$ is then also called *resolvent matrix for the problem* $\mathsf{M}[[\alpha,\infty); (s_j)_{j=0}^m, \preccurlyeq]$.

The following two lines of investigations naturally arise as a consequence of Corollary 13.2.

1. Let $\mathfrak{M}$ be a subset of $\langle \mathcal{P}_{q,\alpha}[Q_m] \rangle$ which is of some interest for several reasons. Then determine that subset of $\mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^m, \preccurlyeq]$ which corresponds to the parameter set $\mathfrak{M}$ via Corollary 13.2.
2. Let S be a distinguished element of $\mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^m, \preccurlyeq]$. Then determine that element of $\langle \mathcal{P}_{q,\alpha}[Q_m] \rangle$ which produces S via Corollary 13.2.

We start with a particular question which can be classified under the first case of the just formulated topics.

Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$. Denote by $(Q_j)_{j=0}^m$ the right α-Stieltjes parametrization of $(s_j)_{j=0}^m$. Then our aim is to determine that subset of $\mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^m, \preccurlyeq]$ which corresponds in the sense of Corollary 13.2 to the set of all proper pairs belonging to $\mathcal{P}_{q,\alpha}[Q_m]$. Because of Remark 11.9, these pairs stand in a bijective correspondence to the class $\mathcal{S}_{q,[\alpha,\infty)}[Q_m]$. Caused by this fact we are able to apply immediately our former results from Propositions 12.9 and 12.10 where we have treated the case $m = 0$.

Theorem 13.3 *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$. Denote by $(Q_j)_{j=0}^m$ the right α-Stieltjes parametrization of $(s_j)_{j=0}^m$. Let $\Sigma_{m,\preccurlyeq}^{\langle s \rangle}$ be the bijective correspondence between $\langle \mathcal{P}_{q,\alpha}[Q_m] \rangle$ and $\mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^m, \preccurlyeq]$ defined in Corollary 13.2. Let $F \in \mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^m, \preccurlyeq]$ and let $\langle(\phi,\psi)\rangle := (\Sigma_{m,\preccurlyeq}^{\langle s \rangle})^{-1}(F)$. Denote by $(Q_j^{\langle \sigma_F \rangle})_{j=0}^m$ the right α-Stieltjes parametrization of $(s_j^{(\sigma_F)})_{j=0}^m$. Let V_{α,Q_m} be defined by (12.3) and let $F_m := \tilde{\mathcal{S}}_{V_{\alpha,Q_m}}^{(q,q)}((\phi,\psi))$, where the mapping $\tilde{\mathcal{S}}_{V_{\alpha,Q_m}}^{(q,q)}$ is defined in Notation C.1. Then:*

(a) $F_m \in \mathcal{S}_{0,q,[\alpha,\infty)}[(Q_{m-j})_{j=0}^0, \preccurlyeq]$ *and* $\sigma_{F_m}([\alpha,\infty)) = Q_m^{\langle \sigma_F \rangle}$.

(b) *Denote by $\tilde{\mathcal{P}}_{q,\alpha}[Q_m]$ the subclass of all proper pairs (ϕ,ψ) belonging to $\mathcal{P}_{q,\alpha}[Q_m]$.*

  (b1) *If* $(\Sigma_{m,\preccurlyeq}^{\langle s \rangle})^{-1}(F) \in \tilde{\mathcal{P}}_{q,\alpha}[Q_m]$, *then* $\operatorname{rank} Q_m^{\langle \sigma_F \rangle} = \operatorname{rank} Q_m$.

  (b2) *If* $\operatorname{rank} Q_m^{\langle \sigma_F \rangle} = \operatorname{rank} Q_m$, *then* $(\Sigma_{m,\preccurlyeq}^{\langle s \rangle})^{-1}(F) \in \tilde{\mathcal{P}}_{q,\alpha}[Q_m]$.

Proof Denote by $(s_j^{[m,\alpha]})_{j=0}^{0}$ the m-th α-S-transform of $(s_j)_{j=0}^{m}$. In view of $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$ the application of Theorem 7.7 yields $s_0^{[m,\alpha]} = Q_m$.

(a) In view of Theorem 5.6(b), we have $Q_m \in \mathbb{C}_{\succcurlyeq}^{q\times q}$. Again applying Theorem 5.6(b), we see that $(Q_{m-j})_{j=0}^{0} \in \mathcal{K}_{q,0,\alpha}^{\succcurlyeq,\mathrm{e}}$. Now the application of Theorem 13.1(a) to the sequence $(Q_{m-j})_{j=0}^{0}$ completes the proof of part (a).

(b1) Let $(\Sigma_{m,\preccurlyeq}^{\langle s\rangle})^{-1}(F) \in \tilde{\mathcal{P}}_{q,\alpha}[Q_m]$. Then the function $\det\psi$ does not identically vanish in $\mathbb{C} \setminus [\alpha,\infty)$ and because of [23, Proposition 7.11] and Remark 11.9 the function $G := \phi\psi^{-1}$ satisfies $G \in \mathcal{S}_{q,[\alpha,\infty)}[Q_m]$. From the definitions of the considered mappings and Lemma 12.3 we get $F_m = \mathcal{S}_{V_{\alpha,Q_m}}^{(q,q)}(G) = G^{[-,\alpha,Q_m]}$. Thus, Proposition 12.9 yields $\operatorname{rank}\sigma_{F_m}([\alpha,\infty)) = \operatorname{rank} Q_m$. In view of part (a), then $\operatorname{rank} Q_m^{\langle\sigma_F\rangle} = \operatorname{rank} Q_m$.

(b2) This follows from Proposition 12.10.

□

In the rest of this section, we discuss situations which are associated with the second of the above formulated topics.

Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$. Because of Remark 3.4, then $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq}$. We denote by $\overline{\sigma}_m$ the upper CD-measure associated with $(s_j)_{j=0}^{m}$ and $[\alpha,\infty)$ and by $\underline{\sigma}_m$ the lower CD-measure associated with $(s_j)_{j=0}^{m}$ and $[\alpha,\infty)$ (see Definition 6.11). In Sect. 6 (see in particular Propositions 6.13 and 6.14), we have seen that $\overline{\sigma}_m$ and $\underline{\sigma}_m$ belong to $\mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty); (s_j)_{j=0}^{m}, \preccurlyeq]$ and possess special extremal properties within the elements of this set. Now we are interested in the $[\alpha,\infty)$-Stieltjes transforms $\overline{S}_m$ and $\underline{S}_m$ of $\overline{\sigma}_m$ and $\underline{\sigma}_m$, respectively. We will call $\overline{S}_m$ and $\underline{S}_m$ the *upper and lower* $\mathcal{S}_{q,[\alpha,\infty)}$*-functions associated with* $(s_j)_{j=0}^{m}$. These two functions will play an important role in our subsequent considerations. In Theorem 13.1, we obtained a complete description of the set $\mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^{m}, \preccurlyeq]$ of $[\alpha,\infty)$-Stieltjes transforms of measures belonging to $\mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty); (s_j)_{j=0}^{m}, \preccurlyeq]$. Now we are interested in the position of $\overline{S}_m$ and $\underline{S}_m$ in the set $\mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^{m}, \preccurlyeq]$. In particular, we determine the pairs $(\overline{\phi}_m, \overline{\psi}_m) \in \mathcal{P}_{q,\alpha}[s_0^{[m,\alpha]}]$ and $(\underline{\phi}_m, \underline{\psi}_m) \in \mathcal{P}_{q,\alpha}[s_0^{[m,\alpha]}]$ which correspond to $\overline{S}_m$ and $\underline{S}_m$, respectively, according to Theorem 13.1(b). It can be expected that these pairs possess certain extremal properties within the set $\mathcal{P}_{q,\alpha}[s_0^{[m,\alpha]}]$.

In the first step we express the functions $\underline{S}_m$ and $\overline{S}_m$ explicitly in terms of the sequences $(\underline{s}_j)_{j=0}^{m}$ and $(\overline{s}_j)_{j=0}^{m+1}$, respectively. For this reason, we start with the following observation.

Lemma 13.4 *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, let $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$ with right α-Stieltjes parametrization $(Q_j)_{j=0}^{m}$, and let $\eta \in \mathbb{C} \setminus \{0\}$. Then* $\det[\eta\mathfrak{v}_{21}^{[\alpha,(s_j)_{j=0}^{m}]}(z)Q_m +$

$\mathfrak{v}_{22}^{[\alpha,(s_j)_{j=0}^m]}(z)\mathbb{P}_{\mathcal{N}(Q_m)}] \neq 0$ *and* $\det \mathfrak{v}_{22}^{[\alpha,(s_j)_{j=0}^m]}(z) \neq 0$ *for all* $z \in \mathbb{C} \setminus [\alpha, \infty)$. *In particular,* $\operatorname{rank}[\mathfrak{v}_{21}^{[\alpha,(s_j)_{j=0}^m]}(z), \mathfrak{v}_{22}^{[\alpha,(s_j)_{j=0}^m]}(z)] = q$ *for all* $z \in \mathbb{C} \setminus [\alpha, \infty)$.

Proof From Proposition 5.6(b) we infer $\mathcal{R}(Q_m^*) = \mathcal{R}(Q_m)$. Denote by $(s_j^{[m,\alpha]})_{j=0}^0$ the m-th α-S-transform of $(s_j)_{j=0}^m$. According to Theorem 7.7, then $Q_m = s_0^{[m,\alpha]}$. The combination of [23, Proposition 12.1] with Examples 11.6 and 11.7 completes the proof. □

Proposition 13.5 *Let* $\alpha \in \mathbb{R}$, *let* $m \in \mathbb{N}_0$, *and let* $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$. *Denote by* $(\underline{s}_j)_{j=0}^\infty$ *(resp.* $(\overline{s}_j)_{j=0}^\infty$*) the lower (resp. upper) α-Stieltjes completely degenerate sequence associated with* $(s_j)_{j=0}^m$. *For all* $z \in \mathbb{C} \setminus [\alpha, \infty)$, *then* $\det \mathfrak{v}_{22}^{[\alpha,(\underline{s}_j)_{j=0}^m]}(z) \neq 0$, $\det \mathfrak{v}_{22}^{[\alpha,(\overline{s}_j)_{j=0}^{m+1}]}(z) \neq 0$,

$$\underline{S}_m(z) = \mathfrak{v}_{12}^{[\alpha,(\underline{s}_j)_{j=0}^m]}(z)\left[\mathfrak{v}_{22}^{[\alpha,(\underline{s}_j)_{j=0}^m]}(z)\right]^{-1},$$

and

$$\overline{S}_m(z) = \mathfrak{v}_{12}^{[\alpha,(\overline{s}_j)_{j=0}^{m+1}]}(z)\left[\mathfrak{v}_{22}^{[\alpha,(\overline{s}_j)_{j=0}^{m+1}]}(z)\right]^{-1}.$$

Proof Because of Remark 6.10(a), we have $(\overline{s}_j)_{j=0}^{m+1} \in \mathcal{K}_{q,m+1,\alpha}^{\succcurlyeq,\mathrm{cd}}$, whereas the combination of Remarks 3.4 and 6.10(b) shows that $(\underline{s}_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{cd}}$. According to Proposition 4.5, in particular $(\overline{s}_j)_{j=0}^{m+1} \in \mathcal{K}_{q,m+1,\alpha}^{\succcurlyeq,\mathrm{e}}$ and $(\underline{s}_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$. From Lemma 13.4 we obtain then $\det \mathfrak{v}_{22}^{[\alpha,(\underline{s}_j)_{j=0}^m]}(z) \neq 0$ and $\det \mathfrak{v}_{22}^{[\alpha,(\overline{s}_j)_{j=0}^{m+1}]}(z) \neq 0$ for all $z \in \mathbb{C} \setminus [\alpha, \infty)$. Thus, Proposition 6.15 and [23, Theorem 13.2] yield the remaining identities. □

Now we are going to express the functions $\underline{S}_m$ and $\overline{S}_m$ explicitly in terms of the original sequence $(s_j)_{j=0}^m$.

Lemma 13.6 *Let* $\alpha \in \mathbb{R}$, *let* $m \in \mathbb{N}_0$, *and let* $(s_j)_{j=0}^{m+1} \in \mathcal{K}_{q,m+1,\alpha}^{\succcurlyeq,\mathrm{e}}$ *with right α-Stieltjes parametrization* $(Q_j)_{j=0}^{m+1}$. *Let* $\mathfrak{V}^{[\alpha,(s_j)_{j=0}^m]}$ *and* $\mathfrak{V}^{[\alpha,(s_j)_{j=0}^{m+1}]}$ *be defined via* (12.4) *and* (12.3). *Furthermore, let* (12.5) *and* $\mathfrak{V}^{[\alpha,(s_j)_{j=0}^{m+1}]} = [\mathfrak{v}_{k\ell}^{[\alpha,(s_j)_{j=0}^{m+1}]}]_{k,\ell=1}^2$

be the $q \times q$ block representations of $\mathfrak{V}^{[\alpha,(s_j)_{j=0}^{m}]}$ and $\mathfrak{V}^{[\alpha,(s_j)_{j=0}^{m+1}]}$, respectively. For all $z \in \mathbb{C}$, then

$$\mathfrak{v}_{11}^{[\alpha,(s_j)_{j=0}^{m+1}]}(z) = (z-\alpha)\mathfrak{v}_{12}^{[\alpha,(s_j)_{j=0}^{m}]}(z)Q_{m+1}^{\dagger},$$

$$\mathfrak{v}_{12}^{[\alpha,(s_j)_{j=0}^{m+1}]}(z) = (z-\alpha)\mathfrak{v}_{12}^{[\alpha,(s_j)_{j=0}^{m}]}(z) - \mathfrak{v}_{11}^{[\alpha,(s_j)_{j=0}^{m}]}(z)Q_{m+1},$$

$$\mathfrak{v}_{21}^{[\alpha,(s_j)_{j=0}^{m+1}]}(z) = (z-\alpha)\mathfrak{v}_{22}^{[\alpha,(s_j)_{j=0}^{m}]}(z)Q_{m+1}^{\dagger}, \text{ and}$$

$$\mathfrak{v}_{22}^{[\alpha,(s_j)_{j=0}^{m+1}]}(z) = (z-\alpha)\mathfrak{v}_{22}^{[\alpha,(s_j)_{j=0}^{m}]}(z) - \mathfrak{v}_{21}^{[\alpha,(s_j)_{j=0}^{m}]}(z)Q_{m+1}.$$

Proof Let $(s_j^{[m+1,\alpha]})_{j=0}^{0}$ be the $(m+1)$-th α-S-transform of $(s_j)_{j=0}^{m+1}$. In view of Theorem 7.7, we have $s_0^{[m+1,\alpha]} = Q_{m+1}$. From Remark 12.2 we obtain then $\mathfrak{V}^{[\alpha,(s_j)_{j=0}^{m+1}]} = \mathfrak{V}^{[\alpha,(s_j)_{j=0}^{m}]}V_{\alpha,Q_{m+1}}$. Using the $q \times q$ block partitions of $\mathfrak{V}^{[\alpha,(s_j)_{j=0}^{m+1}]}$ and $\mathfrak{V}^{[\alpha,(s_j)_{j=0}^{m}]}$ as well as (12.3), a straightforward calculation completes the proof. □

Lemma 13.7 *Let $\alpha \in \mathbb{R}$. Then:*

(a) *Let $m \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$. Denote by $(\overline{s}_j)_{j=0}^{\infty}$ the upper α-Stieltjes completely degenerate sequence associated with $(s_j)_{j=0}^{m}$. For all $z \in \mathbb{C}$, then*

$$\mathfrak{v}_{11}^{[\alpha,(\overline{s}_j)_{j=0}^{m+1}]}(z) = 0_{q\times q}, \qquad \mathfrak{v}_{12}^{[\alpha,(\overline{s}_j)_{j=0}^{m+1}]}(z) = (z-\alpha)\mathfrak{v}_{12}^{[\alpha,(s_j)_{j=0}^{m}]}(z),$$

$$\mathfrak{v}_{21}^{[\alpha,(\overline{s}_j)_{j=0}^{m+1}]}(z) = 0_{q\times q}, \quad \text{and} \quad \mathfrak{v}_{22}^{[\alpha,(\overline{s}_j)_{j=0}^{m+1}]}(z) = (z-\alpha)\mathfrak{v}_{22}^{[\alpha,(s_j)_{j=0}^{m}]}(z).$$

(b) *Let $m \in \mathbb{N}$. Denote by $(\underline{s}_j)_{j=0}^{\infty}$ the lower α-Stieltjes completely degenerate sequence associated with $(s_j)_{j=0}^{m}$. For all $z \in \mathbb{C}$, then*

$$\mathfrak{v}_{11}^{[\alpha,(\underline{s}_j)_{j=0}^{m}]}(z) = 0_{q\times q}, \qquad \mathfrak{v}_{12}^{[\alpha,(\underline{s}_j)_{j=0}^{m}]}(z) = (z-\alpha)\mathfrak{v}_{12}^{[\alpha,(s_j)_{j=0}^{m-1}]}(z),$$

$$\mathfrak{v}_{21}^{[\alpha,(\underline{s}_j)_{j=0}^{m}]}(z) = 0_{q\times q}, \quad \text{and} \quad \mathfrak{v}_{22}^{[\alpha,(\underline{s}_j)_{j=0}^{m}]}(z) = (z-\alpha)\mathfrak{v}_{22}^{[\alpha,(s_j)_{j=0}^{m-1}]}(z).$$

Proof Denote by $(\underline{Q}_j)_{j=0}^{\infty}$ (resp. $(\overline{Q}_j)_{j=0}^{\infty}$) the right α-Stieltjes parametrization of $(\underline{s}_j)_{j=0}^{\infty}$ (resp. $(\overline{s}_j)_{j=0}^{\infty}$). In view of Definition 5.15 and Remark 5.12, we have $(\overline{s}_j)_{j=0}^{m+1} \in \mathcal{K}_{q,m+1,\alpha}^{\succcurlyeq,\mathrm{e}}$. If $m \geq 1$, then Definition 5.13 and Remark 5.12 show that $(\underline{s}_j)_{j=0}^{m}$ belongs to $\mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$. According to Definitions 5.13 and 5.15, Remark 5.16, and Proposition 5.17, we have $(\underline{s}_j)_{j=0}^{\infty} \in \mathcal{K}_{q,\infty,\alpha}^{\succcurlyeq,\mathrm{cd},m}$ and $(\overline{s}_j)_{j=0}^{\infty} \in \mathcal{K}_{q,\infty,\alpha}^{\succcurlyeq,\mathrm{cd},m+1}$. From Corollary 5.9 we get then $\underline{Q}_m = 0_{q\times q}$ and $\overline{Q}_{m+1} = 0_{q\times q}$. In view of

Definitions 4.15, 5.15, and 5.13, we have $\overline{s}_j = s_j$ for all $j \in \mathbb{Z}_{0,m}$ and, in the case $m \geq 1$, furthermore $\underline{s}_j = s_j$ for all $j \in \mathbb{Z}_{0,m-1}$. Now, the application of Lemma 13.6 to the sequence $(\overline{s}_j)_{j=0}^{m+1}$, and, in the case $m \geq 1$, to the sequence $(\underline{s}_j)_{j=0}^{m}$ completes the proof. □

Proposition 13.8 *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$. For all $z \in \mathbb{C} \setminus [\alpha, \infty)$, then $\det \mathfrak{v}_{22}^{[\alpha,(s_j)_{j=0}^m]}(z) \neq 0$ and $\overline{S}_m(z) = \mathfrak{v}_{12}^{[\alpha,(s_j)_{j=0}^m]}(z)[\mathfrak{v}_{22}^{[\alpha,(s_j)_{j=0}^m]}(z)]^{-1}$. If $m \geq 1$, then $\det \mathfrak{v}_{22}^{[\alpha,(s_j)_{j=0}^{m-1}]}(z) \neq 0$ and $\underline{S}_m(z) = \mathfrak{v}_{12}^{[\alpha,(s_j)_{j=0}^{m-1}]}(z)[\mathfrak{v}_{22}^{[\alpha,(s_j)_{j=0}^{m-1}]}(z)]^{-1}$ for all $z \in \mathbb{C} \setminus [\alpha, \infty)$.*

Proof Combine Proposition 13.5 with Lemma 13.7. □

Our next step can be described as follows. Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$. Denote by $\underline{\sigma}_m$ (resp. $\overline{\sigma}_m$) the lower (resp. upper) CD-measure associated with $(s_j)_{j=0}^m$ and $[\alpha, \infty)$. Then we are going to determine the position of the $[\alpha, \infty)$-Stieltjes transform $\underline{S}_m$ of $\underline{\sigma}_m$ (resp. $\overline{S}_m$ of $\overline{\sigma}_m$) within the general description of the set $\mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^m, \preccurlyeq]$ of $[\alpha, \infty)$-Stieltjes transforms of the measures belonging to $\mathcal{M}_q^{\succcurlyeq}[[\alpha, \infty); (s_j)_{j=0}^m, \preccurlyeq]$.

Theorem 13.9 *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$ with right α-Stieltjes parametrization $(Q_j)_{j=0}^m$. Let $\mathfrak{V}^{[\alpha,(s_j)_{j=0}^m]}$ be defined via (12.4) and (12.3) and let (12.5) be the $q \times q$ block representation of $\mathfrak{V}^{[\alpha,(s_j)_{j=0}^m]}$. Let $\overline{S}_m$ and $\underline{S}_m$ be the upper and lower $\mathcal{S}_{q,[\alpha,\infty)}$-functions associated with $(s_j)_{j=0}^m$, respectively. Then:*

(a) *The functions $\underline{S}_m$ and $\overline{S}_m$ both belong to $\mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^m, \preccurlyeq]$.*

(b) *Let $\underline{\phi}_m, \underline{\psi}_m \colon \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C}^{q\times q}$ be defined by $\underline{\phi}_m(z) := Q_m$ and $\underline{\psi}_m(z) := \mathbb{P}_{\mathcal{N}(Q_m)}$ where the matrix $\mathbb{P}_{\mathcal{N}(Q_m)}$ is introduced in Remark A.2. Then the pair $(\underline{\phi}_m, \underline{\psi}_m)$ belongs to $\mathcal{P}_{q,\alpha}[Q_m]$. For all $z \in \mathbb{C} \setminus [\alpha, \infty)$, furthermore $\det[\mathfrak{v}_{21}^{[\alpha,(s_j)_{j=0}^m]}(z)\underline{\phi}_m(z) + \mathfrak{v}_{22}^{[\alpha,(s_j)_{j=0}^m]}(z)\underline{\psi}_m(z)] \neq 0$ and*

$$\underline{S}_m(z) = \left[\mathfrak{v}_{11}^{[\alpha,(s_j)_{j=0}^m]}(z)\underline{\phi}_m(z) + \mathfrak{v}_{12}^{[\alpha,(s_j)_{j=0}^m]}(z)\underline{\psi}_m(z)\right] \times \left[\mathfrak{v}_{21}^{[\alpha,(s_j)_{j=0}^m]}(z)\underline{\phi}_m(z) + \mathfrak{v}_{22}^{[\alpha,(s_j)_{j=0}^m]}(z)\underline{\psi}_m(z)\right]^{-1}.$$

(c) *Let $\overline{\phi}_m, \overline{\psi}_m \colon \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C}^{q\times q}$ be defined by $\overline{\phi}_m(z) := 0_{q\times q}$ and $\overline{\psi}_m(z) := I_q$. Then the pair $(\overline{\phi}_m, \overline{\psi}_m)$ belongs to $\mathcal{P}_{q,\alpha}[Q_m]$. For all $z \in \mathbb{C} \setminus [\alpha, \infty)$, furthermore $\det[\mathfrak{v}_{21}^{[\alpha,(s_j)_{j=0}^m]}(z)\overline{\phi}_m(z) + \mathfrak{v}_{22}^{[\alpha,(s_j)_{j=0}^m]}(z)\overline{\psi}_m(z)] \neq 0$ is meromorphic in $\mathbb{C} \setminus [\alpha, \infty)$ and does not vanish identically. Furthermore, $\overline{S}_m$*

can be represented via

$$\overline{S}_m(z) = \left[\mathfrak{v}_{11}^{[\alpha,(s_j)_{j=0}^m]}(z)\overline{\phi}_m(z) + \mathfrak{v}_{12}^{[\alpha,(s_j)_{j=0}^m]}(z)\overline{\psi}_m(z)\right] \times \left[\mathfrak{v}_{21}^{[\alpha,(s_j)_{j=0}^m]}(z)\overline{\phi}_m(z) + \mathfrak{v}_{22}^{[\alpha,(s_j)_{j=0}^m]}(z)\overline{\psi}_m(z)\right]^{-1}.$$

Proof For each $j \in \{1, 2\}$, the functions $\underline{G}_j, \overline{G}_j : \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C}^{q\times q}$ defined by

$$\underline{G}_j(z) := \mathfrak{v}_{j1}^{[\alpha,(s_j)_{j=0}^m]}(z)\underline{\phi}_m(z) + \mathfrak{v}_{j2}^{[\alpha,(s_j)_{j=0}^m]}(z)\underline{\psi}_m(z)$$

and

$$\overline{G}_j(z) := \mathfrak{v}_{j1}^{[\alpha,(s_j)_{j=0}^m]}(z)\overline{\phi}_m(z) + \mathfrak{v}_{j2}^{[\alpha,(s_j)_{j=0}^m]}(z)\overline{\psi}_m(z).$$

admit for each $z \in \mathbb{C} \setminus [\alpha, \infty)$ the representation

$$\underline{G}_j(z) = \mathfrak{v}_{j1}^{[\alpha,(s_j)_{j=0}^m]}(z)Q_m + \mathfrak{v}_{j2}^{[\alpha,(s_j)_{j=0}^m]}(z)\mathbb{P}_{\mathcal{N}(Q_m)}, \quad \overline{G}_j(z) = \mathfrak{v}_{j2}^{[\alpha,(s_j)_{j=0}^m]}(z),$$

resp. In view of Lemma 13.4, we have thus $\det \underline{G}_2(z) \neq 0$ and $\det \overline{G}_2(z) \neq 0$ for all $z \in \mathbb{C} \setminus [\alpha, \infty)$. In particular, the functions $\det \underline{G}_2$ and $\det \overline{G}_2$ does not vanish identically and the functions $\underline{F} := \underline{G}_1\underline{G}_2^{-1}$ and $\overline{F} := \overline{G}_1\overline{G}_2^{-1}$ admit for all $z \in \mathbb{C} \setminus [\alpha, \infty)$ the representations $\underline{F}(z) = \underline{G}_1(z)[\underline{G}_2(z)]^{-1}$ and $\overline{F}(z) = \overline{G}_1(z)[\overline{G}_2(z)]^{-1}$, resp. From Proposition 5.6(b) we infer $\mathcal{R}(Q_m^*) = \mathcal{R}(Q_m)$. Examples 11.6 and 11.7 show then that $(\underline{\phi}_m, \underline{\psi}_m)$ and $(\overline{\phi}_m, \overline{\psi}_m)$ both belong to $\mathcal{P}_{q,\alpha}[Q_m]$. Denote by $(s_j^{[m,\alpha]})_{j=0}^0$ the m-th α-S-transform of $(s_j)_{j=0}^m$. According to Theorem 7.7, we have $Q_m = s_0^{[m,\alpha]}$. Theorem 13.1(a) then yields that $\underline{F}$ and $\overline{F}$ both belong to $\mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^m, \preccurlyeq]$. Using Proposition 13.8, we infer $\overline{S}_m(z) = \overline{F}(z)$ for all $z \in \mathbb{C} \setminus [\alpha, \infty)$. To complete the proof, we show that $\underline{S}_m(z) = \underline{F}(z)$ for all $z \in \mathbb{C} \setminus [\alpha, \infty)$ holds trues. To that end, let $z \in \mathbb{C} \setminus [\alpha, \infty)$ be arbitrary.

First we consider the case $m = 0$. We have

$$\mathfrak{V}^{[\alpha,(s_j)_{j=0}^0]}(z) = V_{\alpha,s_0}(z) = V_{\alpha,Q_0}(z) = \begin{bmatrix} 0_{q\times q} & -Q_0 \\ (z-\alpha)Q_0^\dagger & (z-\alpha)I_q \end{bmatrix}$$

and, in view of (12.5), hence $\underline{G}_1(z) = 0_{q\times q} \cdot Q_0 + (-Q_0)\mathbb{P}_{\mathcal{N}(Q_0)} = 0_{q\times q}$, implying $\underline{F}(z) = 0_{q\times q}$. Since we know from Example 6.12 that $\underline{\sigma}_0$ is the $q \times q$ zero measure on $\mathfrak{B}_{[\alpha,\infty)}$, we get $\underline{F}(z) = \underline{S}_0(z)$.

Now assume $m \geq 1$. In view of $\mathbb{P}_{\mathcal{N}(Q_m)} = I_q - Q_m^\dagger Q_m$, we obtain from Lemma 13.6 then

$$\mathfrak{v}_{11}^{[\alpha,(s_j)_{j=0}^m]}(z)\underline{\phi}_m(z) = (z-\alpha)\mathfrak{v}_{12}^{[\alpha,(s_j)_{j=0}^{m-1}]}(z)Q_m^\dagger Q_m,$$

$$\mathfrak{v}_{12}^{[\alpha,(s_j)_{j=0}^m]}(z)\underline{\psi}_m(z) = (z-\alpha)\mathfrak{v}_{12}^{[\alpha,(s_j)_{j=0}^{m-1}]}(z)(I_q - Q_m^\dagger Q_m),$$

$$\mathfrak{v}_{21}^{[\alpha,(s_j)_{j=0}^m]}(z)\underline{\phi}_m(z) = (z-\alpha)\mathfrak{v}_{22}^{[\alpha,(s_j)_{j=0}^{m-1}]}(z)Q_m^\dagger Q_m, \text{ and}$$

$$\mathfrak{v}_{22}^{[\alpha,(s_j)_{j=0}^m]}(z)\underline{\psi}_m(z) = (z-\alpha)\mathfrak{v}_{22}^{[\alpha,(s_j)_{j=0}^{m-1}]}(z)(I_q - Q_m^\dagger Q_m).$$

Consequently, $\underline{G}_1(z) = (z-\alpha)\mathfrak{v}_{12}^{[\alpha,(s_j)_{j=0}^{m-1}]}(z)$ and $\underline{G}_2(z) = (z-\alpha)\mathfrak{v}_{22}^{[\alpha,(s_j)_{j=0}^{m-1}]}(z)$. Thus, $\underline{F}(z) = \mathfrak{v}_{12}^{[\alpha,(s_j)_{j=0}^{m-1}]}(z)[\mathfrak{v}_{22}^{[\alpha,(s_j)_{j=0}^{m-1}]}(z)]^{-1}$. Hence, $\underline{S}_m(z) = \underline{F}(z)$ by virtue of Proposition 13.8. □

We have seen in Propositions 6.13 and 6.14 that the measures $\overline{\sigma}_m$ and $\underline{\sigma}_m$ are the unique solutions of certain extremal problems within the set $\mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty); (s_j)_{j=0}^m, \preccurlyeq]$. If we look back now to Theorem 13.9 and consider the corresponding pairs $(\overline{\phi}_m, \overline{\psi}_m)$ and $(\underline{\phi}_m, \underline{\psi}_m)$ belonging to $\mathcal{P}_{q,\alpha}[Q_m]$, then it should be mentioned that these pairs consist of constant $\mathbb{C}^{q\times q}$-valued functions in $\mathbb{C} \setminus [\alpha,\infty)$, which have extremal rank properties. Indeed, the function $\underline{\phi}_m$ satisfies $\operatorname{rank} \underline{\phi}_m = \operatorname{rank} Q_m$, which is the maximal possible rank of a $q \times q$ matrix-valued function ϕ with $\mathcal{R}(\phi(z)) \subseteq \mathcal{R}(Q_m)$ for all points $z \in \mathbb{C}\setminus[\alpha,\infty)$ which are points of holomorphy of ϕ, whereas the function $\overline{\phi}_m$ has rank 0 which is clearly the minimal possible rank.

It should be mentioned that a careful study of the particular $[\alpha,\infty)$-Stieltjes transforms $\underline{S}_m$ and $\overline{S}_m$ was initiated by Yu. M. Dyukarev [7], who considered the case of $\alpha = 0$ and a sequence $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,0}^{\succ}$. These investigations were continued by A. E. Choque Rivero [4, Theorem 4.8], who could express the functions $\underline{S}_m$ and $\overline{S}_m$ in terms of orthogonal $q \times q$ matrix polynomials. In his recent PhD thesis [27] (see also [28]), B. Jeschke was able to extend essential results due to Yu. M. Dyukarev [7] to the case of an arbitrary $\alpha \in \mathbb{R}$.

We start our last topic in this section with the following specification of Theorem 6.16.

Theorem 13.10 *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$. Let $\mathfrak{a}_{-1} := 0_{q\times q}$ and, in the case $m \in \mathbb{N}$, let $\mathfrak{a}_{m-1}$ be the α-Stieltjes minimal element associated with $(s_j)_{j=0}^{m-1}$. Then*

$$\left\{s_m^{(\sigma)} : \sigma \in \mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty); (s_j)_{j=0}^m, \preccurlyeq]\right\} = \{s \in \mathbb{C}_{\mathrm{H}}^{q\times q} : \mathfrak{a}_{m-1} \preccurlyeq s \preccurlyeq s_m\}.$$

Proof Denote by $(\tilde{s}_j)_{j=0}^m$ the right-sided α-Stieltjes non-negative definite extendable sequence equivalent to $(s_j)_{j=0}^m$. In view of $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$ we infer from Lemma 3.10 that $(s_j)_{j=0}^m = (\tilde{s}_j)_{j=0}^m$. Thus the application of Theorem 6.16 completes the proof. □

Against to the background of Corollary 13.2, Theorem 13.10 leads us to the following question: Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$ with right α-Stieltjes parametrization $(Q_j)_{j=0}^m$. Let $\mathfrak{a}_{m-1}$ be given as in Theorem 13.10 and let $s \in \mathbb{C}_{\mathrm{H}}^{q\times q}$ be such that $\mathfrak{a}_{m-1} \preccurlyeq s \preccurlyeq s_m$. Then describe the subset of $\langle \mathcal{P}_{q,\alpha}[Q_m] \rangle$, which corresponds via Corollary 13.2 to the set

$$\left\{F \in \mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^m, \preccurlyeq] \colon s_m^{(\sigma_F)} = s\right\}.$$

We start with the treatment of the case $m = 0$.

Remark 13.11 Let $\alpha \in \mathbb{R}$ and let $M, N \in \mathbb{C}^{q\times q}$. Since $\mathcal{S}_{q,[\alpha,\infty)}^{\diamond}[M]$ given by (9.1) is non-empty, the set

$$\begin{aligned}\mathcal{P}_{q,\alpha}[N, M, =] \\ := \left\{(NM^{\dagger}G + N - M, N^{\dagger}M + I_q - N^{\dagger}N) \colon G \in \mathcal{S}_{q,[\alpha,\infty)}^{\diamond}[M]\right\}\end{aligned} \tag{13.1}$$

is non-empty as well.

We continue to use the notation given in Remark 13.11.

Lemma 13.12 *Let $\alpha \in \mathbb{R}$, let $M \in \mathbb{C}_{\mathrm{H}}^{q\times q}$, and let $N \in \mathbb{C}^{q\times q}$. Then*

$$\begin{aligned}&\mathcal{P}_{q,\alpha}[N, M, =] \\ &\quad= \left\{(NM^{\dagger}FM^{\dagger}M + N - M, N^{\dagger}M + I_q - N^{\dagger}N) \colon F \in \mathcal{S}_{q,[\alpha,\infty)}^{\diamond}\right\}.\end{aligned}$$

Proof From [21, Remark 4.6] we get

$$\mathcal{S}_{q,[\alpha,\infty)}^{\diamond}[M] = \{M^{\dagger}MFM^{\dagger}M \colon F \in \mathcal{S}_{q,[\alpha,\infty)}^{\diamond}\}. \tag{13.2}$$

Because of $M^* = M$ and Remark A.6, we have $M^{\dagger}M = MM^{\dagger}$. Thus,

$$\begin{aligned}\{M^{\dagger}GM^{\dagger}M \colon G \in \mathcal{S}_{q,[\alpha,\infty)}^{\diamond}\} &= \{M^{\dagger}MM^{\dagger}GM^{\dagger}MM^{\dagger}M \colon G \in \mathcal{S}_{q,[\alpha,\infty)}^{\diamond}\} \\ &= \left\{M^{\dagger}(M^{\dagger}MGM^{\dagger}M)M^{\dagger}M \colon G \in \mathcal{S}_{q,[\alpha,\infty)}^{\diamond}\right\} \\ &= \left\{M^{\dagger}FM^{\dagger}M \colon F \in \mathcal{S}_{q,[\alpha,\infty)}^{\diamond}[M]\right\}.\end{aligned}$$

Taking into account (13.1), the proof is complete. □

Example 13.13 Let $\alpha \in \mathbb{R}$ and let $N \in \mathbb{C}^{q\times q}$. Then $\mathcal{P}_{q,\alpha}[N, 0_{q\times q}, =] = \{(N, I_q - N^\dagger N)\}$. If $N \in \mathbb{C}_{\mathrm{H}}^{q\times q}$, we see from Lemma 13.12 and formula (13.2) for N instead of M furthermore

$$\mathcal{P}_{q,\alpha}[N, N, =] = \Big\{(NN^\dagger FN^\dagger N, N^\dagger N + I_q - N^\dagger N)\colon F \in \mathcal{S}^{\diamond}_{q,[\alpha,\infty)}\Big\}$$
$$= \Big\{(G, I_q)\colon G \in \mathcal{S}^{\diamond}_{q,[\alpha,\infty)}[N]\Big\}.$$

Proposition 13.14 *Let $\alpha \in \mathbb{R}$, let $s_0 \in \mathbb{C}^{q\times q}_{\succcurlyeq}$, let $\Sigma^{\langle s\rangle}_{0,\preccurlyeq}\colon \langle\mathcal{P}_{q,\alpha}[s_0]\rangle \to \mathcal{S}_{0,q,[\alpha,\infty)}[(s_j)_{j=0}^{0}, \preccurlyeq]$ be the bijection defined in Corollary 13.2, and let $M \in \mathbb{C}^{q\times q}_{\succcurlyeq}$ with $s_0 - M \in \mathbb{C}^{q\times q}_{\succcurlyeq}$. Then $\mathcal{P}_{q,\alpha}[s_0, M, =] \subseteq \mathcal{P}_{q,\alpha}[s_0]$ and*

$$\Sigma^{\langle s\rangle}_{0,\preccurlyeq}\big(\langle\mathcal{P}_{q,\alpha}[s_0, M, =]\rangle\big) = \Big\{S \in \mathcal{S}_{0,q,[\alpha,\infty)}[(s_j)_{j=0}^{0}, \preccurlyeq]\colon \sigma_S([\alpha,\infty)) = M\Big\}.$$

Proof Let $r_0 := M$.

(I) First consider an arbitrary $(\phi_1, \psi_1) \in \mathcal{P}_{q,\alpha}[s_0, r_0, =]$. Using (13.1) we have then $\phi_1 = s_0 r_0^\dagger G + s_0 - r_0$ and $\psi_1 = s_0^\dagger r_0 + I_q - s_0^\dagger s_0$ for some $G \in \mathcal{S}^{\diamond}_{q,[\alpha,\infty)}[r_0]$. According to [21, Theorem 13.1(a)], the function $F := \mathcal{S}^{(q,q)}_{V_{\alpha,r_0}}(G)$ is well defined and belongs to $\mathcal{S}_{0,q,[\alpha,\infty)}[(r_j)_{j=0}^{0}, =]$. Using [21, Proposition 12.13(a)], we see that $X_1, Y_1\colon \mathbb{C}\setminus[\alpha,\infty) \to \mathbb{C}^{q\times q}$ defined by $X_1(z) := -r_0$ and $Y_1(z) := (z-\alpha)[r_0^\dagger G(z) + I_q]$ fulfill $\det Y_1(z) \neq 0$ and $F(z) = [X_1(z)][Y_1(z)]^{-1}$ for all $z \in \mathbb{C}\setminus[\alpha,\infty)$. Observe that $\left[\begin{smallmatrix} X_1(z) \\ Y_1(z)\end{smallmatrix}\right] = [V_{\alpha,r_0}(z)]\left[\begin{smallmatrix} G(z) \\ I_q\end{smallmatrix}\right]$ for all $z \in \mathbb{C}\setminus[\alpha,\infty)$. Furthermore, $s_0 - r_0 \in \mathbb{C}^{q\times q}_{\succcurlyeq}$ yields $F \in \mathcal{S}_{0,q,[\alpha,\infty)}[(s_j)_{j=0}^{0}, \preccurlyeq]$.

(II) By virtue of [23, Theorem 12.3(b)] there exists a pair $(\phi_2, \psi_2) \in \mathcal{P}_{q,\alpha}[s_0]$ such that ϕ_2 and ψ_2 are both holomorphic in $\mathbb{C}\setminus[\alpha,\infty)$, fulfilling $\det Y_2(z) \neq 0$ and $F(z) = [X_2(z)][Y_2(z)]^{-1}$ for all $z \in \mathbb{C}\setminus[\alpha,\infty)$, where $X_2, Y_2\colon \mathbb{C}\setminus[\alpha,\infty) \to \mathbb{C}^{q\times q}$ are defined by $X_2(z) := -s_0\psi_2(z)$ and $Y_2(z) := (z-\alpha)[s_0^\dagger\phi_2(z) + \psi_2(z)]$. Observe that $s_0 s_0^\dagger \phi_2(z) = \phi_2(z)$ and $\left[\begin{smallmatrix} X_2(z) \\ Y_2(z)\end{smallmatrix}\right] = [V_{\alpha,s_0}(z)]\left[\begin{smallmatrix} \phi_2(z) \\ \psi_2(z)\end{smallmatrix}\right]$ for all $z \in \mathbb{C}\setminus[\alpha,\infty)$. Using $s_0 - r_0 \in \mathbb{C}^{q\times q}_{\succcurlyeq}$, Lemma A.17, Remarks A.4(b), and A.6, we have $s_0^\dagger s_0 r_0^\dagger = r_0$. Straightforward calculations yield $[W_{\alpha,s_0}(z)][V_{\alpha,s_0}(z)] = (z-\alpha)\operatorname{diag}(s_0 s_0^\dagger, I_q)$ and

$$\big[W_{\alpha,s_0}(z)\big]\big[V_{\alpha,r_0}(z)\big] = (z-\alpha)\begin{bmatrix} s_0 r_0^\dagger & s_0 - r_0 \\ 0_{q\times q} & s_0^\dagger r_0 + I_q - s_0^\dagger s_0 \end{bmatrix}$$

for all $z \in \mathbb{C} \setminus [\alpha, \infty)$. Hence, we can conclude

$$\begin{bmatrix} W_{\alpha,s_0}(z) \end{bmatrix} \begin{bmatrix} X_2(z) \\ Y_2(z) \end{bmatrix} = (z-\alpha)\operatorname{diag}(s_0 s_0^\dagger, I_q) \begin{bmatrix} \phi_2(z) \\ \psi_2(z) \end{bmatrix} = (z-\alpha) \begin{bmatrix} \phi_2(z) \\ \psi_2(z) \end{bmatrix}$$

and

$$\begin{aligned} \begin{bmatrix} W_{\alpha,s_0}(z) \end{bmatrix} \begin{bmatrix} X_1(z) \\ Y_1(z) \end{bmatrix} &= (z-\alpha) \left[\begin{array}{c:c} s_0 r_0^\dagger & s_0 - r_0 \\ \hdashline 0_{q\times q} & s_0^\dagger r_0 + I_q - s_0^\dagger s_0 \end{array}\right] \begin{bmatrix} G(z) \\ I_q \end{bmatrix} \\ &= (z-\alpha) \begin{bmatrix} \phi_1(z) \\ \psi_1(z) \end{bmatrix} \end{aligned}$$

for all $z \in \mathbb{C} \setminus [\alpha, \infty)$. Consequently,

$$\begin{bmatrix} \phi_2(z) \\ \psi_2(z) \end{bmatrix} [\theta(z)] = \frac{1}{z-\alpha} \begin{bmatrix} W_{\alpha,s_0}(z) \end{bmatrix} \begin{bmatrix} F(z) \\ I_q \end{bmatrix} [Y_1(z)] = \begin{bmatrix} \phi_1(z) \\ \psi_1(z) \end{bmatrix}$$

for all $z \in \mathbb{C} \setminus [\alpha, \infty)$, where $\theta \colon \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C}^{q\times q}$ defined by $\theta(z) := [Y_2(z)]^{-1}[Y_1(z)]$ is holomorphic fulfilling $\det \theta(z) \neq 0$ for all $z \in \mathbb{C} \setminus [\alpha, \infty)$. Using Remark 11.3 and Lemma 11.8, we can infer $\langle(\phi_1, \phi_2)\rangle = \langle(\phi_2, \phi_2)\rangle$ and $(\phi_1, \psi_1) \in \mathcal{P}_{q,\alpha}[s_0]$. Since $\Sigma_{0,\preccurlyeq}^{\langle s\rangle}(\langle(\phi_2, \phi_2)\rangle) = X_2 Y_2^{-1} = F \in \mathcal{S}_{0,q,[\alpha,\infty)}[(r_j)_{j=0}^0, =] \subseteq \mathcal{S}_{0,q,[\alpha,\infty)}[(s_j)_{j=0}^0, \preccurlyeq]$, we have shown $\mathcal{P}_{q,\alpha}[s_0, r_0, =] \subseteq \mathcal{P}_{q,\alpha}[s_0]$ and $\Sigma_{0,\preccurlyeq}^{\langle s\rangle}(\langle \mathcal{P}_{q,\alpha}[s_0, r_0, =]\rangle) \subseteq \{S \in \mathcal{S}_{0,q,[\alpha,\infty)}[(s_j)_{j=0}^0, \preccurlyeq] \colon \sigma_S([\alpha, \infty)) = r_0\}$.

(III) Conversely, now consider an arbitrary $F \in \mathcal{S}_{0,q,[\alpha,\infty)}[(s_j)_{j=0}^0, \preccurlyeq]$ satisfying $\sigma_F([\alpha, \infty)) = r_0$. Then F belongs to $\mathcal{S}_{0,q,[\alpha,\infty)}[(r_j)_{j=0}^0, =]$. By virtue of [21, Theorem 13.1(a)], we have $F = \mathcal{S}_{V_{\alpha,r_0}}^{(q,q)}(G)$ for some $G \in \mathcal{S}_{q,[\alpha,\infty)}^{\diamond}[r_0]$. Let $\phi_1 := s_0 r_0^\dagger G + s_0 - r_0$ and let $\psi_1 := s_0^\dagger r_0 + I_q - s_0^\dagger s_0$. Using (13.1), we see then that $(\phi_1, \psi_1) \in \mathcal{P}_{q,\alpha}[s_0, r_0, =]$. In view of part (II) of the proof, there exists a pair $(\phi_2, \psi_2) \in \mathcal{P}_{q,\alpha}[s_0]$ satisfying $\langle(\phi_1, \phi_2)\rangle = \langle(\phi_2, \phi_2)\rangle$ and $\Sigma_{0,\preccurlyeq}^{\langle s\rangle}(\langle(\phi_2, \phi_2)\rangle) = F$. Consequently, $\{S \in \mathcal{S}_{0,q,[\alpha,\infty)}[(s_j)_{j=0}^0, \preccurlyeq] \colon \sigma_S([\alpha, \infty)) = r_0\} \subseteq \Sigma_{0,\preccurlyeq}^{\langle s\rangle}(\langle \mathcal{P}_{q,\alpha}[s_0, r_0, =]\rangle)$.

□

Corollary 13.15 *Let $\alpha \in \mathbb{R}$, let $s_0 \in \mathbb{C}_{\succcurlyeq}^{q\times q}$, let $r := \operatorname{rank} s_0$, and let $\Sigma_{0,\preccurlyeq}^{\langle s\rangle} \colon \langle \mathcal{P}_{q,\alpha}[s_0]\rangle \to \mathcal{S}_{0,q,[\alpha,\infty)}[(s_j)_{j=0}^0, \preccurlyeq]$ be the bijection defined in Corollary 13.2. If $\ell \in \mathbb{N}_0$, then the set $\mathcal{M}_\ell := \{M \in \mathbb{C}_{\succcurlyeq}^{q\times q} \colon s_0 - M \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ and $\operatorname{rank} M = \ell\}$ is non-empty if and only if $\ell \leq r$. Furthermore, $\bigcup_{\ell=0}^{r} \mathcal{M}_\ell = [0_{q\times q}, s_0]$ and $\langle \bigcup_{\ell=0}^{r} \bigcup_{M\in\mathcal{M}_\ell} \mathcal{P}_{q,\alpha}[s_0, M, =]\rangle = \langle \mathcal{P}_{q,\alpha}[s_0]\rangle$.*

For each $\ell \in \mathbb{Z}_{0,r}$, *the equation* $\Sigma_{0,\preccurlyeq}^{\langle s\rangle}(\langle\bigcup_{M\in\mathcal{M}_\ell}\mathcal{P}_{q,\alpha}[s_0,M,=]\rangle) = \{S \in \mathcal{S}_{0,q,[\alpha,\infty)}[(s_j)_{j=0}^{0},\preccurlyeq]\colon \operatorname{rank}\sigma_S([\alpha,\infty)) = \ell\}$ *holds true. Furthermore,* $\Sigma_{0,\preccurlyeq}^{\langle s\rangle}(\langle\bigcup_{\ell=0}^{r}\bigcup_{M\in\mathcal{M}_\ell}\mathcal{P}_{q,\alpha}[s_0,M,=]\rangle) = \mathcal{S}_{0,q,[\alpha,\infty)}[(s_j)_{j=0}^{0},\preccurlyeq]$.

Proof Use Lemma A.17, Proposition 13.14, and Corollary 13.2. □

Let the assumption of Proposition 13.14 be satisfied. Then $(s_j)_{j=0}^{0} \in \mathcal{K}_{q,0,\alpha}^{\succcurlyeq,\mathrm{e}}$ and for $\sigma \in \mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty);(s_j)_{j=0}^{0},\preccurlyeq]$ we have $s_0^{(\sigma)} = \sigma([\alpha,\infty))$. Thus, Proposition 13.14 completely answers our question in the case $m = 0$.

Now we draw our attention to the remaining case $m \in \mathbb{N}$. If $A \in \mathbb{C}^{q\times q}$ then we use the set $\mathcal{S}_{q,[\alpha,\infty)}^{\diamond}[A]$ introduced in (9.2).

Lemma 13.16 *Let* $\alpha \in \mathbb{R}$, *let* $m \in \mathbb{N}_0$, *and let* $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$ *with right* α*-Stieltjes parametrization* $(Q_j)_{j=0}^{m}$. *Then the mapping* $\Sigma_{m,=}^{\langle s\rangle}\colon \mathcal{S}_{q,[\alpha,\infty)}^{\diamond}[Q_m] \to \mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^{m},=]$ *given by*

$$\Sigma_{m,=}^{\langle s\rangle}(G) := (\mathfrak{v}_{11}^{[\alpha,(s_j)_{j=0}^{m}]}G + \mathfrak{v}_{12}^{[\alpha,(s_j)_{j=0}^{m}]})(\mathfrak{v}_{21}^{[\alpha,(s_j)_{j=0}^{m}]}G + \mathfrak{v}_{22}^{[\alpha,(s_j)_{j=0}^{m}]})^{-1}$$

is well defined and bijective.

Proof Theorem 7.7 yields $s_0^{[m,\alpha]} = Q_m$, where $(s_j^{[m,\alpha]})_{j=0}^{0}$ denotes the m-th α-S-transform of $(s_j)_{j=0}^{m}$. Now the assertion can be seen from [23, Proposition 12.13 and Theorem 12.10]. □

Lemma 13.17 *Let* $\alpha \in \mathbb{R}$, *let* $m \in \mathbb{N}$, *and let* $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$ *with right* α*-Stieltjes parametrization* $(Q_j)_{j=0}^{m}$ *and* m*-th* α*-S-transform* $(s_j^{[m,\alpha]})_{j=0}^{0}$. *Let* $\Sigma_{m-1,=}^{\langle s\rangle}\colon \mathcal{S}_{q,[\alpha,\infty)}^{\diamond}[Q_{m-1}] \to \mathcal{S}_{m-1,q,[\alpha,\infty)}[(s_j)_{j=0}^{m-1},=]$ *be the bijection given in Lemma 13.16. Then* $\mathcal{S}_{0,q,[\alpha,\infty)}[(s_j^{[m,\alpha]})_{j=0}^{0},=] \subseteq \mathcal{S}_{q,[\alpha,\infty)}^{\diamond}[Q_{m-1}]$ *and* $\Sigma_{m-1,=}^{\langle s\rangle}(\mathcal{S}_{0,q,[\alpha,\infty)}[(s_j^{[m,\alpha]})_{j=0}^{0},=]) = \mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^{m},=]$.

Proof Denote by $(t_j)_{j=0}^{0}$ the m-th α-S-transform of $(s_j)_{j=0}^{m}$. From [21, Lemma 5.6] we obtain $\mathcal{S}_{0,q,[\alpha,\infty)}[(t_j)_{j=0}^{0},=] \subseteq \mathcal{S}_{q,[\alpha,\infty)}^{\diamond}[t_0]$. Theorem 7.7 yields $t_0 = Q_m$. According to Theorem 5.6(b), we have $\mathcal{N}(Q_{m-1}) \subseteq \mathcal{N}(Q_m)$. Thus, the application of [21, Remark 4.5] provides us $\mathcal{S}_{q,[\alpha,\infty)}^{\diamond}[Q_m] \subseteq \mathcal{S}_{q,[\alpha,\infty)}^{\diamond}[Q_{m-1}]$. Consequently,

$$\mathcal{S}_{0,q,[\alpha,\infty)}[(t_j)_{j=0}^{0},=] \subseteq \mathcal{S}_{q,[\alpha,\infty)}^{\diamond}[t_0] = \mathcal{S}_{q,[\alpha,\infty)}^{\diamond}[Q_m] \subseteq \mathcal{S}_{q,[\alpha,\infty)}^{\diamond}[Q_{m-1}]. \tag{13.3}$$

In view of Theorem 5.6(b), we have $Q_m \in \mathbb{C}_{\succcurlyeq}^{q\times q}$. Hence, $t_0 \in \mathbb{C}_{\succcurlyeq}^{q\times q}$, i. e., $(t_j)_{j=0}^{0}$ belongs to $\mathcal{K}_{q,0,\alpha}^{\succcurlyeq,\mathrm{e}}$. According to (5.1), the sequence $(t_j)_{j=0}^{0}$ is the right α-Stieltjes parametrization of $(t_j)_{j=0}^{0}$. Let $\Sigma_{0,=}^{\langle t\rangle}\colon \mathcal{S}_{q,[\alpha,\infty)}^{\diamond}[t_0] \to \mathcal{S}_{0,q,[\alpha,\infty)}[(t_j)_{j=0}^{0},=]$ be the bijection defined as in Lemma 13.16. Then $\Sigma_{0,=}^{\langle t\rangle}(\mathcal{S}_{q,[\alpha,\infty)}^{\diamond}[t_0]) = \mathcal{S}_{0,q,[\alpha,\infty)}[(t_j)_{j=0}^{0},=]$. In view of (13.3), consequently

$$\Sigma_{m-1,=}^{\langle s\rangle}\Big(\Sigma_{0,=}^{\langle t\rangle}\Big(\mathcal{S}_{q,[\alpha,\infty)}^{\diamond}[t_0]\Big)\Big) = \Sigma_{m-1,=}^{\langle s\rangle}\Big(\mathcal{S}_{0,q,[\alpha,\infty)}[(t_j)_{j=0}^{0},=]\Big).$$

Let $\Sigma_{m,=}^{\langle s\rangle}\colon \mathcal{S}_{q,[\alpha,\infty)}^{\diamond}[Q_m] \to \mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^{m},=]$ be the bijection defined in Lemma 13.16. In view of (12.4) and (12.5), we can infer from Proposition C.3 that

$$\Sigma_{m-1,=}^{\langle s\rangle}\Big(\Sigma_{0,=}^{\langle t\rangle}(\mathcal{S}_{q,[\alpha,\infty)}^{\diamond}[t_0])\Big) = \Sigma_{m,=}^{\langle s\rangle}\Big(\mathcal{S}_{q,[\alpha,\infty)}^{\diamond}[t_0]\Big)$$

holds true. Furthermore, by virtue of Lemma 13.16, we have

$$\Sigma_{m,=}^{\langle s\rangle}\Big(\mathcal{S}_{q,[\alpha,\infty)}^{\diamond}[Q_m]\Big) = \mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^{m},=].$$

In view of $Q_m = t_0$ and $t_0 = s_0^{[m,\alpha]}$, the proof is complete. □

Observe that we know from Remark 4.10 that under the assumptions of the following proposition the matricial interval $[\mathfrak{a}_{m-1}, s_m]$ is non-empty.

The following result provides a complete answer to the problem under consideration in the case $m \in \mathbb{N}$.

Proposition 13.18 *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}$, let $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$ with right α-Stieltjes parametrization $(Q_j)_{j=0}^{m}$, let $\Sigma_{m,\preccurlyeq}^{\langle s\rangle}\colon \langle\mathcal{P}_{q,\alpha}[Q_m]\rangle \to \mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^{m}, \preccurlyeq]$ be the bijection defined in Corollary 13.2, and let $M \in [\mathfrak{a}_{m-1}, s_m]$, where $\mathfrak{a}_{m-1}$ is given via (4.10). Then $\mathcal{P}_{q,\alpha}[Q_m, M - \mathfrak{a}_{m-1}, =] \subseteq \mathcal{P}_{q,\alpha}[Q_m]$ and*

$$\Sigma_{m,\preccurlyeq}^{\langle s\rangle}\big(\langle\mathcal{P}_{q,\alpha}[Q_m, M-\mathfrak{a}_{m-1},=]\rangle\big) = \left\{S \in \mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^{m}, \preccurlyeq]\colon \int_{[\alpha,\infty)} x^m \sigma_S(\mathrm{d}x) = M\right\}.$$

Proof Denote by $(t_j)_{j=0}^{0}$ the m-th α-S-transform of $(s_j)_{j=0}^{m}$. Then Theorem 7.7 yields $t_0 = Q_m$. In view of Theorem 5.6(b), we have $Q_m \in \mathbb{C}_{\succcurlyeq}^{q\times q}$. Hence, $t_0 \in \mathbb{C}_{\succcurlyeq}^{q\times q}$, i. e., $(t_j)_{j=0}^{0}$ belongs to $\mathcal{K}_{q,0,\alpha}^{\succcurlyeq,\mathrm{e}}$. Obviously $(t_j)_{j=0}^{0}$ is the right α-Stieltjes parametrization of $(t_j)_{j=0}^{0}$. Let $\Sigma_{0,\preccurlyeq}^{\langle t\rangle}\colon \langle\mathcal{P}_{q,\alpha}[t_0]\rangle \to \mathcal{S}_{0,q,[\alpha,\infty)}[(t_j)_{j=0}^{0}, \preccurlyeq]$ be the bijection defined analogously as in Corollary 13.2. Let $N := M - \mathfrak{a}_{m-1}$. Then $N \in \mathbb{C}_{\succcurlyeq}^{q\times q}$. By virtue of Remark 5.3, furthermore

$$Q_m - N = Q_m - (M - \mathfrak{a}_{m-1}) = s_m - \mathfrak{a}_{m-1} - M + \mathfrak{a}_{m-1} = s_m - M \in \mathbb{C}_{\succcurlyeq}^{q\times q},$$

i. e., $t_0 - N \in \mathbb{C}^{q\times q}_{\succcurlyeq}$. The application of Proposition 13.14 yields then $\mathcal{P}_{q,\alpha}[t_0, N, =] \subseteq \mathcal{P}_{q,\alpha}[t_0]$, i. e., $\mathcal{P}_{q,\alpha}[Q_m, M - \mathfrak{a}_{m-1}, =] \subseteq \mathcal{P}_{q,\alpha}[Q_m]$, and

$$\Sigma^{\langle t\rangle}_{0,\preccurlyeq}\big(\langle\mathcal{P}_{q,\alpha}[t_0, N, =]\rangle\big) = \Big\{S \in \mathcal{S}_{0,q,[\alpha,\infty)}[(t_j)^0_{j=0}, \preccurlyeq]\colon \sigma_S([\alpha,\infty)) = N\Big\}.$$

Let the sequence $(r_j)^m_{j=0}$ be given by $r_j := s_j$ for all $j \in \mathbb{Z}_{0,m-1}$ and by $r_m := M$. Denote by $(Q^{\langle r\rangle}_j)^m_{j=0}$ the right α-Stieltjes parametrization of $(r_j)^m_{j=0}$. According to Remark 5.5, then $Q^{\langle r\rangle}_j = Q_j$ for all $j \in \mathbb{Z}_{0,m-1}$. In view of Remark 4.8, (4.7), and (4.5), furthermore $\mathfrak{a}^{\langle r\rangle}_{m-1} = \mathfrak{a}_{m-1}$. Using Remark 5.3, we obtain then

$$Q^{\langle r\rangle}_m = r_m - \mathfrak{a}^{\langle r\rangle}_{m-1} = M - \mathfrak{a}_{m-1} = N.$$

Consequently, $Q^{\langle r\rangle}_m \in \mathbb{C}^{q\times q}_{\succcurlyeq}$ and $Q_m - Q^{\langle r\rangle}_m \in \mathbb{C}^{q\times q}_{\succcurlyeq}$, implying $\mathcal{N}(Q_m) \subseteq \mathcal{N}(Q^{\langle r\rangle}_m)$ due to Lemma A.17. Taking into account Theorem 5.6(b), we can conclude $\mathcal{N}(Q^{\langle r\rangle}_{m-1}) = \mathcal{N}(Q_{m-1}) \subseteq \mathcal{N}(Q_m) \subseteq \mathcal{N}(Q^{\langle r\rangle}_m)$ as well as $\mathcal{N}(Q^{\langle r\rangle}_j) \subseteq \mathcal{N}(Q^{\langle r\rangle}_{j+1})$ for each $j \in \mathbb{Z}_{0,m-2}$, and, hence, $(r_j)^m_{j=0} \in \mathcal{K}^{\succcurlyeq,\mathrm{e}}_{q,m,\alpha}$. Theorem 7.7 yields then $r^{[m,\alpha]}_0 = Q^{\langle r\rangle}_m$. Therefore,

$$\Sigma^{\langle t\rangle}_{0,\preccurlyeq}\big(\langle\mathcal{P}_{q,\alpha}[Q_m, M - \mathfrak{a}_{m-1}, =]\rangle\big) = \mathcal{S}_{0,q,[\alpha,\infty)}[(r^{[m,\alpha]}_j)^0_{j=0}, =].$$

In view of $(r_j)^{m-1}_{j=0} \in \mathcal{K}^{\succcurlyeq,\mathrm{e}}_{q,m-1,\alpha}$ and Remark 5.5, let $\Sigma^{\langle r\rangle}_{m-1,=}\colon \mathcal{S}^{\diamond}_{q,[\alpha,\infty)}[Q^{\langle r\rangle}_{m-1}] \to \mathcal{S}_{m-1,q,[\alpha,\infty)}[(r_j)^{m-1}_{j=0}, =]$ be the bijection defined analogously as in Lemma 13.16. The application of Lemma 13.17 yields $\mathcal{S}_{0,q,[\alpha,\infty)}[(r^{[m,\alpha]}_j)^0_{j=0}, =] \subseteq \mathcal{S}^{\diamond}_{q,[\alpha,\infty)}[Q^{\langle r\rangle}_{m-1}]$ and

$$\Sigma^{\langle r\rangle}_{m-1,=}\Big(\mathcal{S}_{0,q,[\alpha,\infty)}[(r^{[m,\alpha]}_j)^0_{j=0}, =]\Big) = \mathcal{S}_{m,q,[\alpha,\infty)}[(r_j)^m_{j=0}, =].$$

Taking into account the definition of $(r_j)^m_{j=0}$ and $s_m - M \in \mathbb{C}^{q\times q}_{\succcurlyeq}$, we have $\Sigma^{\langle r\rangle}_{m-1,=} = \Sigma^{\langle s\rangle}_{m-1,=}$ and

$$\mathcal{S}_{m,q,[\alpha,\infty)}[(r_j)^m_{j=0}, =] = \Big\{S \in \mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)^m_{j=0}, \preccurlyeq]\colon \int_{[\alpha,\infty)} x^m \sigma_S(\mathrm{d}x) = M\Big\}.$$

In view of (12.4) and (12.5), we can infer from Corollaries 13.2 and 13.16 and Proposition C.4 that

$$\Sigma^{\langle s\rangle}_{m-1,=}\Big(\Sigma^{\langle t\rangle}_{0,\preccurlyeq}\big(\langle\mathcal{P}_{q,\alpha}[Q_m, M-\mathfrak{a}_{m-1},=]\rangle\big)\Big) = \Sigma^{\langle s\rangle}_{m,\preccurlyeq}\big(\langle\mathcal{P}_{q,\alpha}[Q_m, M-\mathfrak{a}_{m-1},=]\rangle\big)$$

holds true. Consequently, we obtain

$$\begin{aligned}&\Sigma^{\langle s\rangle}_{m,\preccurlyeq}\big(\langle\mathcal{P}_{q,\alpha}[Q_m, M-\mathfrak{a}_{m-1},=]\rangle\big)\\&=\Sigma^{\langle r\rangle}_{m-1,=}\Big(\mathcal{S}_{0,q,[\alpha,\infty)}[(r^{[m,\alpha]}_j)^0_{j=0},=]\Big)=\mathcal{S}_{m,q,[\alpha,\infty)}[(r_j)^m_{j=0},=]\\&=\left\{S\in\mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)^m_{j=0},\preccurlyeq]\colon \int_{[\alpha,\infty)}x^m\sigma_S(\mathrm{d}x)=M\right\}.\end{aligned}$$

□

Corollary 13.19 *Let $\alpha\in\mathbb{R}$, let $m\in\mathbb{N}$, let $(s_j)^m_{j=0}\in\mathcal{K}^{\succcurlyeq,\mathrm{e}}_{q,m,\alpha}$ with right α-Stieltjes parametrization $(Q_j)^m_{j=0}$, and let $\Sigma^{\langle s\rangle}_{m,\preccurlyeq}\colon\langle\mathcal{P}_{q,\alpha}[Q_m]\rangle\to\mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)^m_{j=0},\preccurlyeq]$ be the bijection defined in Corollary 13.2. Let $\mathfrak{a}_{m-1}$ be given by (4.10). Let $\ell\in\mathbb{N}_0$. Then the set $\mathcal{M}_\ell:=\{M\in\mathbb{C}^{q\times q}_{\mathrm{H}}\colon\mathfrak{a}_{m-1}\preccurlyeq M\preccurlyeq s_m$ and $\operatorname{rank}M=\ell\}$ is non-empty if and only if $\operatorname{rank}\mathfrak{a}_{m-1}\preccurlyeq\ell\preccurlyeq\operatorname{rank}s_m$. In this case, $\bigcup_{M\in\mathcal{M}_\ell}\mathcal{P}_{q,\alpha}[Q_m,M-\mathfrak{a}_{m-1},=]\subseteq\mathcal{P}_{q,\alpha}[Q_m]$ and $\Sigma^{\langle s\rangle}_{m,\preccurlyeq}(\langle\bigcup_{M\in\mathcal{M}_\ell}\mathcal{P}_{q,\alpha}[Q_m,M-\mathfrak{a}_{m-1},=]\rangle)=\{S\in\mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)^m_{j=0},\preccurlyeq]\colon\operatorname{rank}\int_{[\alpha,\infty)}x^m\sigma_S(\mathrm{d}x)=\ell\}$.*

Proof By virtue of Remark 13.11 the set $\mathcal{P}_{q,\alpha}[Q_m,M-\mathfrak{a}_{m-1},=]$ is non-empty for all $M\in\mathbb{C}^{q\times q}$. Lemma A.17 shows that $\mathcal{M}_\ell\neq\emptyset$ if and only if $\operatorname{rank}\mathfrak{a}_{m-1}\preccurlyeq\ell\preccurlyeq\operatorname{rank}s_m$. Now suppose $\operatorname{rank}\mathfrak{a}_{m-1}\preccurlyeq\ell\preccurlyeq\operatorname{rank}s_m$. Let $M\in\mathcal{M}_\ell$ and let $(\phi,\psi)\in\mathcal{P}_{q,\alpha}[Q_m,M-\mathfrak{a}_{m-1},=]$. According to Proposition 13.18, then $S:=\Sigma^{\langle s\rangle}_{m,\preccurlyeq}(\langle(\phi,\psi)\rangle)$ belongs to $\mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)^m_{j=0},\preccurlyeq]$ and fulfills $\int_{[\alpha,\infty)}x^m\sigma_S(\mathrm{d}x)=M$. In particular, $\operatorname{rank}\int_{[\alpha,\infty)}x^m\sigma_S(\mathrm{d}x)=\ell$.

Conversely, consider a function $S\in\mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)^m_{j=0},\preccurlyeq]$ with $\operatorname{rank}\int_{[\alpha,\infty)}x^m\sigma_S(\mathrm{d}x)=\ell$. Then the Hermitian matrix $M:=\int_{[\alpha,\infty)}x^m\sigma_S(\mathrm{d}x)$ fulfills $s_m-M\in\mathbb{C}^{q\times q}_{\succcurlyeq}$ and $\operatorname{rank}M=\ell$. Using Theorem 6.16, we can conclude furthermore $\mathfrak{a}_{m-1}\preccurlyeq M$. Hence, $M\in\mathcal{M}_\ell$ and, according to Proposition 13.18, there exists a pair $(\phi,\psi)\in\mathcal{P}_{q,\alpha}[Q_m,M-\mathfrak{a}_{m-1},=]$ such that $\Sigma^{\langle s\rangle}_{m,\preccurlyeq}(\langle(\phi,\psi)\rangle)=S$. □

14 Orthogonal Matrix Polynomials and Right α-Stieltjes Parametrization

In this section we treat various aspects of orthogonal matrix polynomials related to Problem $\mathsf{M}[[\alpha, \infty); (s_j)_{j=0}^{m}, \preccurlyeq]$. To be more precise: Let $\alpha \in \mathbb{R}$, let $(s_j)_{j=0}^{2\kappa+1} \in \mathcal{K}_{q,2\kappa+1,\alpha}^{\succcurlyeq}$, and let the sequence $(s_{\alpha\triangleright j})_{j=0}^{2\kappa}$ be given by (4.6). Then we will construct in recursive way two interrelated systems $(\dot{\mathbf{r}}_\ell)_{\ell=0}^{\kappa}$ and $(\dot{\mathbf{t}}_\ell)_{\ell=0}^{\kappa}$ of $q \times q$ matrix polynomials such that $(\dot{\mathbf{r}}_\ell)_{\ell=0}^{\kappa}$ is a monic right orthogonal system with respect to $(s_j)_{j=0}^{2\kappa}$ and $(\dot{\mathbf{t}}_\ell)_{\ell=0}^{\kappa}$ is a monic right orthogonal system with respect to $(s_{\alpha\triangleright j})_{j=0}^{2\kappa}$ (see Proposition 14.7). A similar result will be obtained in Proposition 14.8 for sequences $(s_j)_{j=0}^{2\kappa} \in \mathcal{K}_{q,2\kappa,\alpha}^{\succcurlyeq}$.

The background for the concrete construction can be described as follows: Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$. Then we have seen in Theorem 13.1 and Corollary 13.2 that the $2q \times 2q$ matrix polynomial $\mathfrak{V}^{[\alpha,(s_j)_{j=0}^{m}]}$ defined by (12.4) and (12.3) yields a complete description of the set $\mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^{m}, \preccurlyeq]$ of $[\alpha, \infty)$-Stieltjes transforms of non-negative Hermitian measures belonging to $\mathcal{M}_q^{\succcurlyeq}[[\alpha, \infty); (s_j)_{j=0}^{m}, \preccurlyeq]$. A closer look at the construction of $\mathfrak{V}^{[\alpha,(s_j)_{j=0}^{m}]}$ given in (12.4) shows that this $2q \times 2q$ matrix polynomial is a product of $m + 1$ linear $2q \times 2q$ matrix polynomials. This means that there are recursions for the $q \times q$ blocks of the block representation (12.5) of $\mathfrak{V}^{[\alpha,(s_j)_{j=0}^{m}]}$. Now we are going to discuss more carefully $q \times q$ matrix polynomials defined by recursions of this type. Against to the background of Theorem 7.7 we choose the right α-Stieltjes parametrization $(Q_j)_{j=0}^{\kappa}$ of a given sequence $(s_j)_{j=0}^{\kappa}$ of complex $q \times q$ matrices as coefficients in the construction of the concrete recurrence formulas in the following notation.

Notation 14.1 Denote by $\dot{\mathbf{p}}_0$ and $\dot{\mathbf{q}}_0$ the matrix polynomials given by

$$\dot{\mathbf{p}}_0(z) := I_q \qquad \text{and} \qquad \dot{\mathbf{q}}_0(z) := 0_{q\times q}.$$

Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices with right α-Stieltjes parametrization $(Q_j)_{j=0}^{\kappa}$. Then let the matrix polynomials $\dot{\mathbf{p}}_1$ and $\dot{\mathbf{q}}_1$ be defined by

$$\dot{\mathbf{p}}_1(z) := (z-\alpha)I_q \qquad \text{and} \qquad \dot{\mathbf{q}}_1(z) := Q_0.$$

In the case $\kappa \geq 1$, let the matrix polynomials $\dot{\mathbf{p}}_k$ and $\dot{\mathbf{q}}_k$ for all $k \in \mathbb{Z}_{2,\kappa+1}$ be recursively defined by

$$\dot{\mathbf{p}}_{2\ell}(z) := \dot{\mathbf{p}}_{2\ell-1}(z) - \dot{\mathbf{p}}_{2\ell-2}(z) Q_{2\ell-2}^{\dagger} Q_{2\ell-1}$$

and

$$\dot{\mathbf{q}}_{2\ell}(z) := \dot{\mathbf{q}}_{2\ell-1}(z) - \dot{\mathbf{q}}_{2\ell-2}(z) Q_{2\ell-2}^{\dagger} Q_{2\ell-1},$$

if $k = 2\ell$ with some $\ell \in \mathbb{N}$, and by

$$\dot{\mathbf{p}}_{2\ell+1}(z) := (z-\alpha)\dot{\mathbf{p}}_{2\ell}(z) - \dot{\mathbf{p}}_{2\ell-1}(z)Q_{2\ell-1}^{\dagger}Q_{2\ell}$$

and

$$\dot{\mathbf{q}}_{2\ell+1}(z) := (z-\alpha)\dot{\mathbf{q}}_{2\ell}(z) - \dot{\mathbf{q}}_{2\ell-1}(z)Q_{2\ell-1}^{\dagger}Q_{2\ell},$$

if $k = 2\ell + 1$ with some $\ell \in \mathbb{N}$.

For each $\ell \in \mathbb{N}_0$ let $\epsilon_{2\ell}, \epsilon_{2\ell+1} : \mathbb{C} \to \mathbb{C}$ be defined by

$$\epsilon_{2\ell}(z) := z - \alpha \qquad \text{and} \qquad \epsilon_{2\ell+1}(z) := 1. \tag{14.1}$$

Remark 14.2 Suppose $\kappa \geq 1$. Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices with right α-Stieltjes parametrization $(Q_j)_{j=0}^{\kappa}$ and let $m \in \mathbb{Z}_{1,\kappa}$. Then

$$\dot{\mathbf{p}}_{m+1} = \epsilon_m \dot{\mathbf{p}}_m - \dot{\mathbf{p}}_{m-1}Q_{m-1}^{\dagger}Q_m \quad \text{and} \quad \dot{\mathbf{q}}_{m+1} = \epsilon_m \dot{\mathbf{q}}_m - \dot{\mathbf{q}}_{m-1}Q_{m-1}^{\dagger}Q_m.$$

In view of Remark 5.5, we obtain furthermore:

Remark 14.3 Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices with right α-Stieltjes parametrization $(Q_j)_{j=0}^{\kappa}$. For each $k \in \mathbb{Z}_{1,\kappa+1}$, the matrix polynomials $\dot{\mathbf{p}}_k$ and $\dot{\mathbf{q}}_k$ are built only from the matrices $Q_0, Q_1, \ldots, Q_{k-1}$ and thus only from the matrices $s_0, s_1, \ldots, s_{k-1}$.

Using mathematical induction, we can easily conclude:

Remark 14.4 For each $\ell \in \mathbb{N}_0$ with $2\ell - 1 \leq \kappa$, the function $\dot{\mathbf{p}}_{2\ell}$ is a complex $q \times q$ matrix polynomial with degree ℓ and leading coefficient I_q. For each $\ell \in \mathbb{N}_0$ with $2\ell \leq \kappa$, the function $\dot{\mathbf{p}}_{2\ell+1}$ is a complex $q \times q$ matrix polynomial with degree $\ell + 1$ and leading coefficient I_q, satisfying $\dot{\mathbf{p}}_{2\ell+1}(\alpha) = 0_{q\times q}$.

Regarding Remark 14.4, we are able to define the following matrix polynomials:

Notation 14.5 Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices. For each $\ell \in \mathbb{N}_0$ with $2\ell - 1 \leq \kappa$ let $\dot{\mathbf{r}}_\ell := \dot{\mathbf{p}}_{2\ell}$. For each $\ell \in \mathbb{N}_0$ with $2\ell \leq \kappa$ denote by $\dot{\mathbf{t}}_\ell$ the uniquely determined complex $q \times q$ matrix polynomial, satisfying $\dot{\mathbf{p}}_{2\ell+1}(z) = (z-\alpha)\dot{\mathbf{t}}_\ell(z)$ for all $z \in \mathbb{C}$.

Lemma 14.6 *Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices with right α-Stieltjes parametrization $(Q_j)_{j=0}^{\kappa}$. Using Notation 14.5, for all $z \in \mathbb{C}$, then $\dot{\mathbf{r}}_0(z) = I_q$ and $\dot{\mathbf{t}}_0(z) = I_q$ and, furthermore,*

$$\dot{\mathbf{r}}_\ell(z) = (z-\alpha)\dot{\mathbf{t}}_{\ell-1}(z) - \dot{\mathbf{r}}_{\ell-1}(z)Q_{2\ell-2}^{\dagger}Q_{2\ell-1} \textit{ for each } \ell \in \mathbb{N} \textit{ with } 2\ell - 1 \leq \kappa$$

and

$$\dot{\mathbf{t}}_\ell(z) = \dot{\mathbf{r}}_\ell(z) - \dot{\mathbf{t}}_{\ell-1}(z) Q_{2\ell-1}^\dagger Q_{2\ell} \qquad \textit{for each } \ell \in \mathbb{N} \textit{ with } 2\ell \le \kappa.$$

Proof Consider an arbitrary $w \in \mathbb{C}$. We have $\dot{\mathbf{r}}_0(w) = \dot{\mathbf{p}}_0(w) = I_q$ and $(w - \alpha)\dot{\mathbf{t}}_0(w) = \dot{\mathbf{p}}_1(w) = (w-\alpha)I_q$. Now assume $\kappa \ge 1$ and consider an arbitrary $\ell \in \mathbb{N}$ with $2\ell - 1 \le \kappa$. Then

$$\begin{aligned}\dot{\mathbf{r}}_\ell(w) = \dot{\mathbf{p}}_{2\ell}(w) &= \dot{\mathbf{p}}_{2\ell-1}(w) - \dot{\mathbf{p}}_{2\ell-2}(w) Q_{2\ell-2}^\dagger Q_{2\ell-1} \\ &= (w-\alpha)\dot{\mathbf{t}}_{\ell-1}(w) - \dot{\mathbf{r}}_{\ell-1}(w) Q_{2\ell-2}^\dagger Q_{2\ell-1}.\end{aligned}$$

Now assume $\kappa \ge 2$ and consider an arbitrary $\ell \in \mathbb{N}$ with $2\ell \le \kappa$. Then we have

$$\begin{aligned}(w-\alpha)\dot{\mathbf{t}}_\ell(w) = \dot{\mathbf{p}}_{2\ell+1}(w) &= (w-\alpha)\dot{\mathbf{p}}_{2\ell}(w) - \dot{\mathbf{p}}_{2\ell-1}(w) Q_{2\ell-1}^\dagger Q_{2\ell} \\ &= (w-\alpha)\dot{\mathbf{r}}_\ell(w) - (w-\alpha)\dot{\mathbf{t}}_{\ell-1}(w) Q_{2\ell-1}^\dagger Q_{2\ell} \\ &= (w-\alpha)\Big[\dot{\mathbf{r}}_\ell(w) - \dot{\mathbf{t}}_{\ell-1}(w) Q_{2\ell-1}^\dagger Q_{2\ell}\Big].\end{aligned}$$

Since $w \in \mathbb{C}$ was arbitrarily chosen, the assertion follows. □

Now we consider the case of an α-Stieltjes non-negative definite sequence. Then the following two propositions shed much light to the system of $q \times q$ matrix polynomials introduced in Notation 14.1. Indeed, it will turn out that these $q \times q$ matrix polynomials possess remarkable orthogonality properties.

Proposition 14.7 *Let* $(s_j)_{j=0}^{2\kappa+1} \in \mathcal{K}_{q,2\kappa+1,\alpha}^{\succcurlyeq}$ *with right* α*-Stieltjes parametrization* $(Q_j)_{j=0}^{2\kappa+1}$. *In view of Notation 14.5, then* $(\dot{\mathbf{r}}_\ell)_{\ell=0}^{\kappa}$ *is a monic right orthogonal system with respect to* $(s_j)_{j=0}^{2\kappa}$ *and* $(\dot{\mathbf{t}}_\ell)_{\ell=0}^{\kappa}$ *is a monic right orthogonal system with respect to* $(s_{\alpha\triangleright j})_{j=0}^{2\kappa}$, *where* $(s_{\alpha\triangleright j})_{j=0}^{2\kappa}$ *is given in* (4.6).

Proof The key instrument of our proof will be a twofold application of Proposition D.5, namely once to $(\dot{\mathbf{r}}_\ell)_{\ell=0}^{\kappa}$ and $(s_j)_{j=0}^{2\kappa}$ and a second time to $(\dot{\mathbf{t}}_\ell)_{\ell=0}^{\kappa}$ and $(s_{\alpha\triangleright j})_{j=0}^{2\kappa}$. We use mathematical induction. From Lemma 14.6 we see that $\dot{\mathbf{r}}_0$ and $\dot{\mathbf{t}}_0$ are both complex $q \times q$ matrix polynomials with degree 0 and leading coefficient I_q. Now assume $\kappa \ge 1$. Using Remark A.4, we can infer from Theorem 5.6(a) that $Q_j Q_j^\dagger Q_{j+1} = Q_{j+1}$ holds true for all $j \in \mathbb{Z}_{0,2\kappa-1}$. Observe that the sequences $(s_j)_{j=0}^{2\kappa}$ and $(s_{\alpha\triangleright j})_{j=0}^{2\kappa}$ both belong to $\mathcal{H}_{q,2\kappa}^{\succcurlyeq}$. Hence, for all $\ell \in \mathbb{Z}_{1,\kappa}$, the matrices H_ℓ and $H_{\alpha\triangleright\ell}$ are both non-negative Hermitian. Regarding Remark 4.1, we can apply Lemma A.8 for all $\ell \in \mathbb{Z}_{1,\kappa}$ to conclude $\mathcal{R}(y_{\ell,2\ell-1}) \subseteq \mathcal{R}(H_{\ell-1})$ and $\mathcal{N}(H_{\ell-1}) \subseteq \mathcal{N}(z_{\ell,2\ell-1})$ as well as $\mathcal{R}(y_{\alpha\triangleright\ell,2\ell-1}) \subseteq \mathcal{R}(H_{\alpha\triangleright\ell-1})$ and $\mathcal{N}(H_{\alpha\triangleright\ell-1}) \subseteq$

$\mathcal{N}(z_{\alpha\triangleright\ell,2\ell-1})$. Taking into account $\begin{bmatrix} H_\ell \\ z_{\ell+1,2\ell+1} \end{bmatrix} = \begin{bmatrix} z_{0,\ell} \\ K_\ell \end{bmatrix}$ and $-\alpha H_\ell + K_\ell = H_{\alpha\triangleright\ell} = \begin{bmatrix} H_{\alpha\triangleright\ell-1} & y_{\alpha\triangleright\ell,2\ell-1} \\ * & * \end{bmatrix}$ we obtain

$$\begin{aligned} &\big([0_{\ell q\times q}, I_{\ell q}, 0_{\ell q\times q}] - \alpha[I_{\ell q}, 0_{\ell q\times 2q}]\big)\begin{bmatrix} H_\ell \\ z_{\ell+1,2\ell+1} \end{bmatrix} \\ &= [0_{\ell q\times q}, I_{\ell q}, 0_{\ell q\times q}]\begin{bmatrix} z_{0,\ell} \\ K_\ell \end{bmatrix} - \alpha[I_{\ell q}, 0_{\ell q\times 2q}]\begin{bmatrix} H_\ell \\ z_{\ell+1,2\ell+1} \end{bmatrix} \\ &= [I_{\ell q}, 0_{\ell q\times q}](K_\ell - \alpha H_\ell) = [H_{\alpha\triangleright\ell-1}, y_{\alpha\triangleright\ell,2\ell-1}] \end{aligned} \tag{14.2}$$

for all $\ell \in \mathbb{Z}_{1,\kappa}$. We are now going to show by mathematical induction that, for all $\ell \in \mathbb{Z}_{1,\kappa}$, the following statement holds true:

(I_ℓ) The matrix polynomials $\dot{\mathbf{r}}_\ell$ and $\dot{\mathbf{t}}_\ell$ have degree ℓ and leading coefficient I_q. Furthermore, $H_{\ell-1}r_\ell = y_{\ell,2\ell-1}$ and $H_{\alpha\triangleright\ell-1}t_\ell = y_{\alpha\triangleright\ell,2\ell-1}$, where r_ℓ and t_ℓ are taken from the block representations $Y_\ell(\dot{\mathbf{r}}_\ell) = \begin{bmatrix} -r_\ell \\ I_q \end{bmatrix}$ and $Y_\ell(\dot{\mathbf{t}}_\ell) = \begin{bmatrix} -t_\ell \\ I_q \end{bmatrix}$ of the matrices $Y_\ell(\dot{\mathbf{r}}_\ell)$ and $Y_\ell(\dot{\mathbf{t}}_\ell)$ given via Notation D.1.

According to Lemma 14.6, we have

$$\dot{\mathbf{r}}_1(z) = (z-\alpha)\dot{\mathbf{t}}_0(z) - \dot{\mathbf{r}}_0(z)Q_0^\dagger Q_1 = (z-\alpha)I_q - Q_0^\dagger Q_1 = zI_q - (\alpha I_q + Q_0^\dagger Q_1)$$

and

$$\dot{\mathbf{t}}_1(z) = \dot{\mathbf{r}}_1(z) - \dot{\mathbf{t}}_0(z)Q_1^\dagger Q_2 = \dot{\mathbf{r}}_1(z) - Q_1^\dagger Q_2$$

for all $z \in \mathbb{C}$. Consequently, $\dot{\mathbf{r}}_1$ and $\dot{\mathbf{t}}_1$ are complex $q\times q$ matrix polynomials with degree 1 and leading coefficient I_q and, furthermore, $r_1 = \alpha I_q + Q_0^\dagger Q_1$ and $t_1 = r_1 + Q_1^\dagger Q_2$ hold true. Because of (5.1) and $Q_0Q_0^\dagger Q_1 = Q_1$, we hence obtain

$$\begin{aligned} H_0r_1 = s_0r_1 = Q_0(\alpha I_q + Q_0^\dagger Q_1) &= \alpha Q_0 + Q_0Q_0^\dagger Q_1 \\ &= \alpha Q_0 + Q_1 = \alpha s_0 + s_1 - \alpha s_0 = s_1 = y_{1,1}. \end{aligned}$$

Regarding $\mathcal{R}(y_{1,1}) \subseteq \mathcal{R}(H_0)$ and $\mathcal{N}(H_0) \subseteq \mathcal{N}(z_{1,1})$, we can conclude from Lemma A.9 in combination with Remark 5.4, then $H_1\begin{bmatrix} -r_1 \\ I_q \end{bmatrix} = \begin{bmatrix} 0_{q\times q} \\ Q_2 \end{bmatrix}$. Taking additionally into account (5.1), (14.2), and $Q_1Q_1^\dagger Q_2 = Q_2$, we get

$$\begin{aligned} H_{\alpha\triangleright 0}t_1 &= H_{\alpha\triangleright 0}r_1 + H_{\alpha\triangleright 0}Q_1^\dagger Q_2 = -[H_{\alpha\triangleright 0}, y_{\alpha\triangleright 1,1}]\begin{bmatrix} -r_1 \\ I_q \end{bmatrix} + y_{\alpha\triangleright 1,1} + s_{\alpha\triangleright 0}Q_1^\dagger Q_2 \\ &= -\big([0_{q\times q}, I_q, 0_{q\times q}] - \alpha[I_q, 0_{q\times 2q}]\big)\begin{bmatrix} H_1 \\ z_{2,3} \end{bmatrix}\begin{bmatrix} -r_1 \\ I_q \end{bmatrix} + y_{\alpha\triangleright 1,1} + Q_1Q_1^\dagger Q_2 \\ &= \big(\alpha[I_q, 0_{q\times 2q}] - [0_{q\times q}, I_q, 0_{q\times q}]\big)\begin{bmatrix} 0_{q\times q} \\ Q_2 \\ * \end{bmatrix} + y_{\alpha\triangleright 1,1} + Q_2 = y_{\alpha\triangleright 1,1}. \end{aligned}$$

Consequently, statement (I_1) is valid.

Now assume $\kappa \geq 2$ and suppose that ($I_{\ell-1}$) holds true for some $\ell \in \mathbb{Z}_{2,\kappa}$. By virtue of Lemma 14.6, we have $\dot{\mathbf{r}}_\ell(z) = (z-\alpha)\dot{\mathbf{t}}_{\ell-1}(z) - \dot{\mathbf{r}}_{\ell-1}(z)Q^\dagger_{2\ell-2}Q_{2\ell-1}$ and $\dot{\mathbf{t}}_\ell(z) = \dot{\mathbf{r}}_\ell(z) - \dot{\mathbf{t}}_{\ell-1}(z)Q^\dagger_{2\ell-1}Q_{2\ell}$ for all $z \in \mathbb{C}$. In view of ($I_{\ell-1}$), then $\dot{\mathbf{r}}_\ell$ und $\dot{\mathbf{t}}_\ell$ are complex $q \times q$ matrix polynomials with degree ℓ and leading coefficient I_q and, furthermore,

$$r_\ell = -\begin{bmatrix} 0_{q\times q} \\ -t_{\ell-1} \end{bmatrix} + \alpha \begin{bmatrix} -t_{\ell-1} \\ I_q \end{bmatrix} + \begin{bmatrix} -r_{\ell-1} \\ I_q \end{bmatrix} Q^\dagger_{2\ell-2}Q_{2\ell-1},$$

and

$$t_\ell = r_\ell + \begin{bmatrix} -t_{\ell-1} \\ I_q \end{bmatrix} Q^\dagger_{2\ell-1}Q_{2\ell}$$

hold true. Because of $[H_{\ell-1}, y_{\ell,2\ell-1}] = [y_{0,\ell-1}, K_{\ell-1}]$, we have

$$H_{\ell-1}\begin{bmatrix} 0_{q\times q} \\ -t_{\ell-1} \end{bmatrix} = [H_{\ell-1}, y_{\ell,2\ell-1}]\begin{bmatrix} 0_{q\times q} \\ -t_{\ell-1} \\ I_q \end{bmatrix} - y_{\ell,2\ell-1} = K_{\ell-1}\begin{bmatrix} -t_{\ell-1} \\ I_q \end{bmatrix} - y_{\ell,2\ell-1}.$$

In view of $\mathcal{R}(y_{\ell-1,2\ell-3}) \subseteq \mathcal{R}(H_{\ell-2})$, $\mathcal{N}(H_{\ell-2}) \subseteq \mathcal{N}(z_{\ell-1,2\ell-3})$, and ($I_{\ell-1}$), we can conclude from Lemma A.9 in combination with Remark 5.4, furthermore $H_{\ell-1}\left[\begin{smallmatrix} -r_{\ell-1} \\ I_q \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0_{(\ell-1)q\times q} \\ Q_{2\ell-2} \end{smallmatrix}\right]$. Regarding $\mathcal{R}(y_{\alpha\triangleright\ell-1,2\ell-3}) \subseteq \mathcal{R}(H_{\alpha\triangleright\ell-2})$, $\mathcal{N}(H_{\alpha\triangleright\ell-2}) \subseteq \mathcal{N}(z_{\alpha\triangleright\ell-1,2\ell-3})$, and ($I_{\ell-1}$), we get by Lemma A.9 and Remark 5.4 similarly $H_{\alpha\triangleright\ell-1}\left[\begin{smallmatrix} -t_{\ell-1} \\ I_q \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0_{(\ell-1)q\times q} \\ Q_{2\ell-1} \end{smallmatrix}\right]$. Taking additionally into account $H_{\alpha\triangleright\ell-1} = -\alpha H_{\ell-1} + K_{\ell-1}$ and $Q_{2\ell-2}Q^\dagger_{2\ell-2}Q_{2\ell-1} = Q_{2\ell-1}$, we obtain then

$$\begin{aligned} H_{\ell-1}r_\ell &= -H_{\ell-1}\begin{bmatrix} 0_{q\times q} \\ -t_{\ell-1} \end{bmatrix} + \alpha H_{\ell-1}\begin{bmatrix} -t_{\ell-1} \\ I_q \end{bmatrix} + H_{\ell-1}\begin{bmatrix} -r_{\ell-1} \\ I_q \end{bmatrix} Q^\dagger_{2\ell-2}Q_{2\ell-1} \\ &= -K_{\ell-1}\begin{bmatrix} -t_{\ell-1} \\ I_q \end{bmatrix} + y_{\ell,2\ell-1} + \alpha H_{\ell-1}\begin{bmatrix} -t_{\ell-1} \\ I_q \end{bmatrix} + \begin{bmatrix} 0_{(\ell-1)q\times q} \\ Q_{2\ell-2} \end{bmatrix} Q^\dagger_{2\ell-2}Q_{2\ell-1} \\ &= -H_{\alpha\triangleright\ell-1}\begin{bmatrix} -t_{\ell-1} \\ I_q \end{bmatrix} + y_{\ell,2\ell-1} + \begin{bmatrix} 0_{(\ell-1)q\times q} \\ Q_{2\ell-2}Q^\dagger_{2\ell-2}Q_{2\ell-1} \end{bmatrix} = y_{\ell,2\ell-1}. \end{aligned}$$

Regarding $\mathcal{R}(y_{\ell,2\ell-1}) \subseteq \mathcal{R}(H_{\ell-1})$ and $\mathcal{N}(H_{\ell-1}) \subseteq \mathcal{N}(z_{\ell,2\ell-1})$, we can infer from Lemma A.9 in combination with Remark 5.4, then $H_\ell\left[\begin{smallmatrix} -r_\ell \\ I_q \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0_{\ell q\times q} \\ Q_{2\ell} \end{smallmatrix}\right]$. Taking additionally into account (14.2) and $Q_{2\ell-1}Q_{2\ell-1}^{\dagger}Q_{2\ell} = Q_{2\ell}$, we thus get

$$H_{\alpha\rhd\ell-1}t_\ell = H_{\alpha\rhd\ell-1}r_\ell + H_{\alpha\rhd\ell-1}\begin{bmatrix}-t_{\ell-1}\\ I_q\end{bmatrix}Q_{2\ell-1}^{\dagger}Q_{2\ell}$$

$$= -[H_{\alpha\rhd\ell-1}, y_{\alpha\rhd\ell,2\ell-1}]\begin{bmatrix}-r_\ell\\ I_q\end{bmatrix} + y_{\alpha\rhd\ell,2\ell-1} + \begin{bmatrix}0_{(\ell-1)q\times q}\\ Q_{2\ell-1}\end{bmatrix}Q_{2\ell-1}^{\dagger}Q_{2\ell}$$

$$= -\big([0_{\ell q\times q}, I_{\ell q}, 0_{\ell q\times q}] - \alpha[I_{\ell q}, 0_{\ell q\times 2q}]\big)\begin{bmatrix}H_\ell\\ z_{\ell+1,2\ell+1}\end{bmatrix}\begin{bmatrix}-r_\ell\\ I_q\end{bmatrix}$$

$$+ y_{\alpha\rhd\ell,2\ell-1} + \begin{bmatrix}0_{(\ell-1)q\times q}\\ Q_{2\ell-1}Q_{2\ell-1}^{\dagger}Q_{2\ell}\end{bmatrix}$$

$$= \big(\alpha[I_{\ell q}, 0_{\ell q\times 2q}] - [0_{\ell q\times q}, I_{\ell q}, 0_{\ell q\times q}]\big)\begin{bmatrix}0_{\ell q\times q}\\ Q_{2\ell}\\ *\end{bmatrix} + y_{\alpha\rhd\ell,2\ell-1} + \begin{bmatrix}0_{(\ell-1)q\times q}\\ Q_{2\ell}\end{bmatrix}$$

$$= y_{\alpha\rhd\ell,2\ell-1}.$$

Consequently, statement (I_ℓ) is valid.

Hence, statement (I_ℓ) holds true for all $\ell \in \mathbb{Z}_{1,\kappa}$. Applying Proposition D.5 twice, once to $(s_j)_{j=0}^{2\kappa}$ and $(\dot{\mathbf{r}}_\ell)_{\ell=0}^{\kappa}$ and a second time to $(s_{\alpha\rhd j})_{j=0}^{2\kappa}$ and $(\dot{\mathbf{t}}_\ell)_{\ell=0}^{\kappa}$, completes the proof. □

The following result is the analogue to Proposition 14.7 for sequences $(s_j)_{j=0}^{2\kappa}$ of odd length. It can be proved in a similar way. We omit the details.

Proposition 14.8 *Suppose $\kappa \geq 1$. Let $(s_j)_{j=0}^{2\kappa} \in \mathcal{K}_{q,2\kappa,\alpha}^{\succcurlyeq}$ with right α-Stieltjes parametrization $(Q_j)_{j=0}^{2\kappa}$. Using Notation 14.5, then $(\dot{\mathbf{r}}_\ell)_{\ell=0}^{\kappa}$ is a monic right orthogonal system with respect to $(s_j)_{j=0}^{2\kappa}$ and $(\dot{\mathbf{t}}_\ell)_{\ell=0}^{\kappa-1}$ is a monic right orthogonal system with respect to $(s_{\alpha\rhd j})_{j=0}^{2\kappa-2}$, where $(s_{\alpha\rhd j})_{j=0}^{2\kappa-2}$ is given via (4.6).*

It should be mentioned that, in the particular case $(s_j)_{j=0}^{2\kappa} \in \mathcal{K}_{q,2\kappa,\alpha}^{\succ}$, the left version of Proposition 14.8 is equivalent to [3, Proposition 7.2].

For the case $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\succcurlyeq,\mathrm{e}}$, some information on the localization of zeros of the polynomials $\det \dot{\mathbf{r}}_\ell$ and $\det \dot{\mathbf{t}}_\ell$ will be given by Remark 15.5 below.

Now we are going to explain the role of the polynomials $(\dot{\mathbf{q}}_j)_{j=0}^{\kappa+1}$ introduced in Notation 14.1. It turns out that these $q \times q$ matrix polynomials are exactly the second kind matrix polynomials of the sequence $(\dot{\mathbf{p}}_j)_{j=0}^{\kappa+1}$ with respect to the sequence $(s_j)_{j=0}^{\kappa}$.

Proposition 14.9 *Let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\succcurlyeq}$. In view of Notations 14.1 and E.2, for all $k \in \mathbb{Z}_{0,\kappa+1}$, then $\dot{\mathbf{q}}_k = \dot{\mathbf{p}}_k^{[\![s]\!]}$.*

Proof We use mathematical induction. Consider an arbitrary $z \in \mathbb{C}$. We have $\deg \dot{\mathbf{p}}_0 = 0$. Hence, $\dot{\mathbf{p}}_0^{[\![s]\!]}(z) = 0_{q\times q} = \dot{\mathbf{q}}_0(z)$. Observe that $\dot{\mathbf{p}}_1(w) = (w-\alpha)\dot{\mathbf{p}}_0(w)$ holds true for all $w \in \mathbb{C}$. Using Lemma E.5, we thus obtain $\dot{\mathbf{p}}_1^{[\![s]\!]} = s_0\dot{\mathbf{p}}_0$. Consequently, by virtue of (5.1), then $\dot{\mathbf{p}}_1^{[\![s]\!]}(z) = Q_0 = \dot{\mathbf{q}}_1(z)$ follows.

Now assume $\kappa \geq 1$ and suppose that $\dot{\mathbf{q}}_{2\ell-2} = \dot{\mathbf{p}}_{2\ell-2}^{[\![s]\!]}$ and $\dot{\mathbf{q}}_{2\ell-1} = \dot{\mathbf{p}}_{2\ell-1}^{[\![s]\!]}$ are valid for some $\ell \in \mathbb{N}$ with $2\ell - 1 \leq \kappa$. Regarding Remark 14.4, we have $\deg \dot{\mathbf{p}}_{2\ell-2} = \ell - 1 \leq \kappa + 1$ and $\deg \dot{\mathbf{p}}_{2\ell-1} = \ell \leq \kappa + 1$. Using Notation 14.1 and Remark E.3, we can thus conclude

$$\begin{aligned}\dot{\mathbf{p}}_{2\ell}^{[\![s]\!]} &= (\dot{\mathbf{p}}_{2\ell-1} - \dot{\mathbf{p}}_{2\ell-2}Q_{2\ell-2}^{\dagger}Q_{2\ell-1})^{[\![s]\!]} = \dot{\mathbf{p}}_{2\ell-1}^{[\![s]\!]} - \dot{\mathbf{p}}_{2\ell-2}^{[\![s]\!]}Q_{2\ell-2}^{\dagger}Q_{2\ell-1}\\ &= \dot{\mathbf{q}}_{2\ell-1} - \dot{\mathbf{q}}_{2\ell-2}Q_{2\ell-2}^{\dagger}Q_{2\ell-1} = \dot{\mathbf{q}}_{2\ell}.\end{aligned} \tag{14.3}$$

It remains to consider the case $2\ell \leq \kappa$. In view of Remark 3.1, we have $(s_j)_{j=0}^{2\ell} \in \mathcal{K}_{q,2\ell,\alpha}^{\succcurlyeq}$. Taking into account Remark 14.3, then Proposition 14.8 shows that $(\dot{\mathbf{r}}_k)_{k=0}^{\ell}$ is a monic right orthogonal system with respect to $(s_j)_{j=0}^{2\ell}$. Consequently, the matrix polynomial $\dot{\mathbf{r}}_\ell$ has degree ℓ and leading coefficient I_q. Observe that $(s_j)_{j=0}^{2\ell}$ belongs to $\mathcal{H}_{q,2\ell}^{\succcurlyeq}$. The combination of Lemma E.4 and Proposition D.5 thus yields $P^{[\![s]\!]}(z) = z\dot{\mathbf{r}}_\ell^{[\![s]\!]}(z)$, where the matrix polynomial P is given via $P(w) := w\dot{\mathbf{r}}_\ell(w)$. Let the matrix polynomial Q be defined by $Q(w) := w\dot{\mathbf{q}}_{2\ell}(w)$. Because of $\dot{\mathbf{r}}_\ell = \dot{\mathbf{p}}_{2\ell}$ and (14.3), we obtain then $P^{[\![s]\!]} = Q$. Taking additionally into account

$$(z-\alpha)\dot{\mathbf{p}}_{2\ell}(z) = z\dot{\mathbf{p}}_{2\ell}(z) - \alpha\dot{\mathbf{p}}_{2\ell}(z) = z\dot{\mathbf{r}}_\ell(z) - \alpha\dot{\mathbf{p}}_{2\ell}(z) = P(z) - \alpha\dot{\mathbf{p}}_{2\ell}(z),$$

$$(z-\alpha)\dot{\mathbf{q}}_{2\ell}(z) = z\dot{\mathbf{q}}_{2\ell}(z) - \alpha\dot{\mathbf{q}}_{2\ell}(z) = Q(z) - \alpha\dot{\mathbf{q}}_{2\ell}(z),$$

and, furthermore, $\deg P = \ell + 1 \leq \kappa + 1$ and $\deg \dot{\mathbf{p}}_{2\ell} = \ell \leq \kappa + 1$, we can infer with Notation 14.1 and Remark E.3 therefore

$$\begin{aligned}\dot{\mathbf{p}}_{2\ell+1}^{[\![s]\!]} &= (P - \alpha\dot{\mathbf{p}}_{2\ell} - \dot{\mathbf{p}}_{2\ell-1}Q_{2\ell-1}^{\dagger}Q_{2\ell})^{[\![s]\!]} = P^{[\![s]\!]} - \alpha\dot{\mathbf{p}}_{2\ell}^{[\![s]\!]} - \dot{\mathbf{p}}_{2\ell-1}^{[\![s]\!]}Q_{2\ell-1}^{\dagger}Q_{2\ell}\\ &= Q - \alpha\dot{\mathbf{q}}_{2\ell} - \dot{\mathbf{q}}_{2\ell-1}Q_{2\ell-1}^{\dagger}Q_{2\ell} = \dot{\mathbf{q}}_{2\ell+1}.\end{aligned}$$

Consequently, the assertion is proved by mathematical induction. □

15 Representation of the Resolvent Matrix of Problem M$[[\alpha, \infty); (s_j)_{j=0}^m, \preccurlyeq]$ in Terms of Orthogonal Matrix Polynomials

Our first goal in this section can be described as follows: Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$. Furthermore, let $\mathfrak{V}^{[\alpha,(s_j)_{j=0}^m]}$ be the resolvent matrix for Problem M$[[\alpha, \infty); (s_j)_{j=0}^m, \preccurlyeq]$, which was defined by (12.3) and (12.4). Then we will have a closer look at the canonical $q \times q$ blocks of this $2q \times 2q$ matrix polynomial. In particular, we will see that these $q \times q$ blocks are closely related to the $q \times q$ matrix polynomials introduced in Notation 14.1. The combination of Proposition 15.1 with Propositions 14.7 and 14.8 shows that the lower $q \times q$ blocks of the resolvent matrix contain alternately right orthogonal matrix polynomials with respect to the sequences $(s_j)_{j=0}^{\kappa}$ and $(s_{\alpha \triangleright j})_{j=0}^{\kappa}$, where the latter is introduced in (4.6) and whereas the upper $q \times q$ blocks of the resolvent matrix turn out to be intimately connected with the $q \times q$ matrix polynomials of the second kind (see Appendix E) with respect to the sequence $(s_j)_{j=0}^{\kappa}$.

Proposition 15.1 *Let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\succcurlyeq,\mathrm{e}}$ with right α-Stieltjes parametrization $(Q_j)_{j=0}^{\kappa}$. For all $z \in \mathbb{C}$ and each $n \in \mathbb{N}_0$ with $2n \leq \kappa$, then*

$$\mathfrak{V}^{[\alpha,(s_j)_{j=0}^{2n}]}(z) = (z-\alpha)^n \begin{bmatrix} -(z-\alpha)\dot{\mathbf{q}}_{2n}(z)Q_{2n}^{\dagger} & -\dot{\mathbf{q}}_{2n+1}(z) \\ (z-\alpha)\dot{\mathbf{p}}_{2n}(z)Q_{2n}^{\dagger} & \dot{\mathbf{p}}_{2n+1}(z) \end{bmatrix}.$$

For all $z \in \mathbb{C}$ and each $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$, furthermore

$$\mathfrak{V}^{[\alpha,(s_j)_{j=0}^{2n+1}]}(z) = (z-\alpha)^{n+1} \begin{bmatrix} -\dot{\mathbf{q}}_{2n+1}(z)Q_{2n+1}^{\dagger} & -\dot{\mathbf{q}}_{2n+2}(z) \\ \dot{\mathbf{p}}_{2n+1}(z)Q_{2n+1}^{\dagger} & \dot{\mathbf{p}}_{2n+2}(z) \end{bmatrix}.$$

Proof Because of Theorem 7.7, we have $Q_j = s_0^{[j,\alpha]}$ for all $j \in \mathbb{Z}_{0,\kappa}$. We use mathematical induction. Consider an arbitrary $z \in \mathbb{C}$. Regarding $s_0^{[0,\alpha]} = Q_0$ and Notation 14.1, we see

$$\begin{aligned}\mathfrak{V}^{[\alpha,(s_j)_{j=0}^{0}]}(z) &= V_{\alpha,Q_0}(z) \\ &= \begin{bmatrix} 0_{q\times q} & -Q_0 \\ (z-\alpha)Q_0^{\dagger} & (z-\alpha)I_q \end{bmatrix} = \begin{bmatrix} -(z-\alpha)\dot{\mathbf{q}}_0(z)Q_0^{\dagger} & -\dot{\mathbf{q}}_1(z) \\ (z-\alpha)\dot{\mathbf{p}}_0(z)Q_0^{\dagger} & \dot{\mathbf{p}}_1(z) \end{bmatrix}.\end{aligned}$$

Now assume $\kappa \geq 1$. In view of $s_0^{[m,\alpha]} = Q_m$, we have $\mathfrak{V}^{[\alpha,(s_j)_{j=0}^{m}]} = \mathfrak{V}^{[\alpha,(s_j)_{j=0}^{m-1}]} V_{\alpha,Q_m}$ for all $m \in \mathbb{Z}_{1,\kappa}$. Taking into account Notation 14.1, we have then

$$\begin{aligned}\mathfrak{V}^{[\alpha,(s_j)_{j=0}^{1}]}(z) &= \mathfrak{V}^{[\alpha,(s_j)_{j=0}^{0}]}(z) V_{\alpha,Q_1}(z) \\ &= \begin{bmatrix} -(z-\alpha)\dot{\mathbf{q}}_0(z)Q_0^{\dagger} & -\dot{\mathbf{q}}_1(z) \\ (z-\alpha)\dot{\mathbf{p}}_0(z)Q_0^{\dagger} & \dot{\mathbf{p}}_1(z) \end{bmatrix} \begin{bmatrix} 0_{q\times q} & -Q_1 \\ (z-\alpha)Q_1^{\dagger} & (z-\alpha)I_q \end{bmatrix} \\ &= \left[\begin{array}{c:c} -(z-\alpha)\dot{\mathbf{q}}_1(z)Q_1^{\dagger} & (z-\alpha)\dot{\mathbf{q}}_0(z)Q_0^{\dagger}Q_1 - (z-\alpha)\dot{\mathbf{q}}_1(z) \\ \hdashline (z-\alpha)\dot{\mathbf{p}}_1(z)Q_1^{\dagger} & -(z-\alpha)\dot{\mathbf{p}}_0(z)Q_0^{\dagger}Q_1 + (z-\alpha)\dot{\mathbf{p}}_1(z) \end{array}\right] \\ &= (z-\alpha)\begin{bmatrix} -\dot{\mathbf{q}}_1(z)Q_1^{\dagger} & -\dot{\mathbf{q}}_2(z) \\ \dot{\mathbf{p}}_1(z)Q_1^{\dagger} & \dot{\mathbf{p}}_2(z) \end{bmatrix}.\end{aligned}$$

Now assume $\kappa \geq 2$ and suppose that

$$\mathfrak{V}^{[\alpha,(s_j)_{j=0}^{2\ell-1}]}(z) = (z-\alpha)^\ell \begin{bmatrix} -\dot{\mathbf{q}}_{2\ell-1}(z)Q_{2\ell-1}^\dagger & -\dot{\mathbf{q}}_{2\ell}(z) \\ \dot{\mathbf{p}}_{2\ell-1}(z)Q_{2\ell-1}^\dagger & \dot{\mathbf{p}}_{2\ell}(z) \end{bmatrix}$$

holds true for some $\ell \in \mathbb{N}$ with $2\ell \leq \kappa$. Regarding $\mathfrak{V}^{[\alpha,(s_j)_{j=0}^{2\ell}]} = \mathfrak{V}^{[\alpha,(s_j)_{j=0}^{2\ell-1}]} V_{\alpha,Q_{2\ell}}$ and Notation 14.1, we get then

$$\begin{aligned}
&\mathfrak{V}^{[\alpha,(s_j)_{j=0}^{2\ell}]}(z) \\
&= (z-\alpha)^\ell \begin{bmatrix} -\dot{\mathbf{q}}_{2\ell-1}(z)Q_{2\ell-1}^\dagger & -\dot{\mathbf{q}}_{2\ell}(z) \\ \dot{\mathbf{p}}_{2\ell-1}(z)Q_{2\ell-1}^\dagger & \dot{\mathbf{p}}_{2\ell}(z) \end{bmatrix} \begin{bmatrix} 0_{q\times q} & -Q_{2\ell} \\ (z-\alpha)Q_{2\ell}^\dagger & (z-\alpha)I_q \end{bmatrix} \\
&= (z-\alpha)^\ell \left[\begin{array}{c:c} -(z-\alpha)\dot{\mathbf{q}}_{2\ell}(z)Q_{2\ell}^\dagger & \dot{\mathbf{q}}_{2\ell-1}(z)Q_{2\ell-1}^\dagger Q_{2\ell} - (z-\alpha)\dot{\mathbf{q}}_{2\ell}(z) \\ \hdashline (z-\alpha)\dot{\mathbf{p}}_{2\ell}(z)Q_{2\ell}^\dagger & -\dot{\mathbf{p}}_{2\ell-1}(z)Q_{2\ell-1}^\dagger Q_{2\ell} + (z-\alpha)\dot{\mathbf{p}}_{2\ell}(z) \end{array}\right] \\
&= (z-\alpha)^\ell \begin{bmatrix} -(z-\alpha)\dot{\mathbf{q}}_{2\ell}(z)Q_{2\ell}^\dagger & -\dot{\mathbf{q}}_{2\ell+1}(z) \\ (z-\alpha)\dot{\mathbf{p}}_{2\ell}(z)Q_{2\ell}^\dagger & \dot{\mathbf{p}}_{2\ell+1}(z) \end{bmatrix}.
\end{aligned}$$

It remains to consider the case $2\ell + 1 \leq \kappa$. Taking into account $\mathfrak{V}^{[\alpha,(s_j)_{j=0}^{2\ell+1}]} = \mathfrak{V}^{[\alpha,(s_j)_{j=0}^{2\ell}]} V_{\alpha,Q_{2\ell+1}}$ and Notation 14.1, we obtain similarly

$$\begin{aligned}
&\mathfrak{V}^{[\alpha,(s_j)_{j=0}^{2\ell+1}]}(z) \\
&= (z-\alpha)^\ell \begin{bmatrix} -(z-\alpha)\dot{\mathbf{q}}_{2\ell}(z)Q_{2\ell}^\dagger & -\dot{\mathbf{q}}_{2\ell+1}(z) \\ (z-\alpha)\dot{\mathbf{p}}_{2\ell}(z)Q_{2\ell}^\dagger & \dot{\mathbf{p}}_{2\ell+1}(z) \end{bmatrix} \begin{bmatrix} 0_{q\times q} & -Q_{2\ell+1} \\ (z-\alpha)Q_{2\ell+1}^\dagger & (z-\alpha)I_q \end{bmatrix} \\
&= (z-\alpha)^{\ell+1} \left[\begin{array}{c:c} -\dot{\mathbf{q}}_{2\ell+1}(z)Q_{2\ell+1}^\dagger & \dot{\mathbf{q}}_{2\ell}(z)Q_{2\ell}^\dagger Q_{2\ell+1} - \dot{\mathbf{q}}_{2\ell+1}(z) \\ \hdashline \dot{\mathbf{p}}_{2\ell+1}(z)Q_{2\ell+1}^\dagger & -\dot{\mathbf{p}}_{2\ell}(z)Q_{2\ell}^\dagger Q_{2\ell+1} + \dot{\mathbf{p}}_{2\ell+1}(z) \end{array}\right] \\
&= (z-\alpha)^{\ell+1} \begin{bmatrix} -\dot{\mathbf{q}}_{2\ell+1}(z)Q_{2\ell+1}^\dagger & -\dot{\mathbf{q}}_{2\ell+2}(z) \\ \dot{\mathbf{p}}_{2\ell+1}(z)Q_{2\ell+1}^\dagger & \dot{\mathbf{p}}_{2\ell+2}(z) \end{bmatrix}.
\end{aligned}$$

Therefore, the assertion is proved by mathematical induction. □

As direct consequences of Proposition 15.1, we obtain against to the background of [21, Proposition 12.13, Theorem 13.1] now new representations for the solution sets of the Problems $\mathsf{M}[[\alpha,\infty);(s_j)_{j=0}^m,=]$ and $\mathsf{M}[[\alpha,\infty);(s_j)_{j=0}^m,\preccurlyeq]$.

Theorem 15.2 *Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n} \in \mathcal{K}_{q,2n,\alpha}^{\succcurlyeq,\mathrm{e}}$:*

(a) *Let* $G \in \mathcal{S}^{\diamond}_{q,[\alpha,\infty)}[L_n]$. *Then* $\det[(z-\alpha)\dot{\mathbf{p}}_{2n}(z)L_n^{\dagger}G(z)+\dot{\mathbf{p}}_{2n+1}(z)] \neq 0$ *for all* $z \in \mathbb{C}\setminus[\alpha,\infty)$ *and the matrix-valued function* $F\colon \mathbb{C}\setminus[\alpha,\infty) \to \mathbb{C}^{q\times q}$ *defined by*

$$F(z) = -\Big[(z-\alpha)\dot{\mathbf{q}}_{2n}(z)L_n^{\dagger}G(z)+\dot{\mathbf{q}}_{2n+1}(z)\Big] \times \Big[(z-\alpha)\dot{\mathbf{p}}_{2n}(z)L_n^{\dagger}G(z)+\dot{\mathbf{p}}_{2n+1}(z)\Big]^{-1} \tag{15.1}$$

belongs to $\mathcal{S}_{2n,q,[\alpha,\infty)}[(s_j)_{j=0}^{2n},=]$.

(b) *For each* $F \in \mathcal{S}_{2n,q,[\alpha,\infty)}[(s_j)_{j=0}^{2n},=]$, *there exists a unique* $G \in \mathcal{S}^{\diamond}_{q,[\alpha,\infty)}[L_n]$ *such that* (15.1) *is fulfilled for all* $z \in \mathbb{C}\setminus[\alpha,\infty)$.

Proof Let $\begin{bmatrix} \tilde{\mathfrak{w}}_{2n} & \tilde{\mathfrak{x}}_{2n} \\ \tilde{\mathfrak{y}}_{2n} & \tilde{\mathfrak{z}}_{2n} \end{bmatrix}$ be the $q\times q$ block representation of the restriction of $\mathfrak{V}^{[\alpha,(s_j)_{j=0}^{2n}]}$ onto $\mathbb{C}\setminus[\alpha,\infty)$. Denote by $(Q_j)_{j=0}^{2n}$ the right α-Stieltjes parametrization of $(s_j)_{j=0}^{2n}$ and by $(s_j^{[2n,\alpha]})_{j=0}^{0}$ the $2n$-th α-S-transform of $(s_j)_{j=0}^{2n}$. Because of Theorem 7.7 and Definition 5.1, we have $s_0^{[2n,\alpha]} = Q_{2n} = L_n$.

(a) Taking into account $G \in \mathcal{S}^{\diamond}_{q,[\alpha,\infty)}[s_0^{[2n,\alpha]}]$, the application of [21, Proposition 12.13(a)] yields $\det[\tilde{\mathfrak{y}}_{2n}(z)G(z)+\tilde{\mathfrak{z}}_{2n}(z)] \neq 0$ for all $z \in \mathbb{C}\setminus[\alpha,\infty)$. Regarding $G \in \mathcal{S}^{\diamond}_{q,[\alpha,\infty)}[Q_{2n}]$, from [21, Theorem 13.1(a)] we can infer that $(\tilde{\mathfrak{w}}_{2n}G+\tilde{\mathfrak{x}}_{2n})(\tilde{\mathfrak{y}}_{2n}G+\tilde{\mathfrak{z}}_{2n})^{-1}$ belongs to $\mathcal{S}_{2n,q,[\alpha,\infty)}[(s_j)_{j=0}^{2n},=]$. By virtue of Proposition 15.1, all assertions of part (a) follow.

(b) Taking into account Proposition 15.1, this is a consequence of [21, Theorem 13.1(b)].

□

Theorem 15.3 *Let* $n \in \mathbb{N}_0$ *and let* $(s_j)_{j=0}^{2n+1} \in \mathcal{K}^{\succcurlyeq,\mathrm{e}}_{q,2n+1,\alpha}$:

(a) *Let* $G \in \mathcal{S}^{\diamond}_{q,[\alpha,\infty)}[L_{\alpha\triangleright n}]$. *Then* $\det[\dot{\mathbf{p}}_{2n+1}(z)L_{\alpha\triangleright n}^{\dagger}G(z)+\dot{\mathbf{p}}_{2n+2}(z)] \neq 0$ *for all* $z \in \mathbb{C}\setminus[\alpha,\infty)$ *and the matrix-valued function* $F\colon \mathbb{C}\setminus[\alpha,\infty) \to \mathbb{C}^{q\times q}$ *defined by*

$$F(z) = -\Big[\dot{\mathbf{q}}_{2n+1}(z)L_{\alpha\triangleright n}^{\dagger}G(z)+\dot{\mathbf{q}}_{2n+2}(z)\Big]\Big[\dot{\mathbf{p}}_{2n+1}(z)L_{\alpha\triangleright n}^{\dagger}G(z)+\dot{\mathbf{p}}_{2n+2}(z)\Big]^{-1} \tag{15.2}$$

belongs to $\mathcal{S}_{2n+1,q,[\alpha,\infty)}[(s_j)_{j=0}^{2n+1},=]$.

(b) *For each* $F \in \mathcal{S}_{2n+1,q,[\alpha,\infty)}[(s_j)_{j=0}^{2n+1},=]$, *there exists a unique* $G \in \mathcal{S}^{\diamond}_{q,[\alpha,\infty)}[L_{\alpha\triangleright n}]$ *such that* (15.2) *is fulfilled for all* $z \in \mathbb{C}\setminus[\alpha,\infty)$.

Proof Let $\begin{bmatrix} \tilde{\mathfrak{w}}_{2n+1} & \tilde{\mathfrak{x}}_{2n+1} \\ \tilde{\mathfrak{y}}_{2n+1} & \tilde{\mathfrak{z}}_{2n+1} \end{bmatrix}$ be the $q \times q$ block representation of the restriction of $\mathfrak{V}^{[\alpha,(s_j)_{j=0}^{2n+1}]}$ onto $\mathbb{C} \setminus [\alpha, \infty)$. Denote by $(Q_j)_{j=0}^{2n+1}$ the right α-Stieltjes parametrization of $(s_j)_{j=0}^{2n+1}$ and by $(s_j^{[2n+1,\alpha]})_{j=0}^{0}$ the $(2n+1)$-th α-S-transform of $(s_j)_{j=0}^{2n+1}$. Because of Theorem 7.7 and Definition 5.1, we have $s_0^{[2n+1,\alpha]} = Q_{2n+1} = L_{\alpha \triangleright n}$.

(a) Taking into account $G \in \mathcal{S}^{\diamond}_{q,[\alpha,\infty)}[s_0^{[2n+1,\alpha]}]$, the application of [21, Proposition 12.13(a)] yields $\det[\tilde{\mathfrak{y}}_{2n+1}(z)G(z) + \tilde{\mathfrak{z}}_{2n+1}(z)] \neq 0$ for all $z \in \mathbb{C} \setminus [\alpha, \infty)$. Regarding $G \in \mathcal{S}^{\diamond}_{q,[\alpha,\infty)}[Q_{2n+1}]$, we can infer from [21, Theorem 13.1(a)] that $(\tilde{\mathfrak{w}}_{2n+1}G + \tilde{\mathfrak{x}}_{2n+1})(\tilde{\mathfrak{y}}_{2n+1}G + \tilde{\mathfrak{z}}_{2n+1})^{-1}$ belongs to $\mathcal{S}_{2n+1,q,[\alpha,\infty)}[(s_j)_{j=0}^{2n+1}, =]$. By virtue of Proposition 15.1, all assertions of part (a) follow.
(b) Because of Proposition 15.1, this is a consequence of [21, Theorem 13.1(b)].

□

From Theorems 15.2 and 15.3 conclusions for the localization of the zeros of the polynomials $\det \dot{\mathbf{p}}_k$ can be obtained:

Corollary 15.4 *Let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}^{\succcurlyeq,\mathrm{e}}_{q,\kappa,\alpha}$. For all $z \in \mathbb{C} \setminus [\alpha, \infty)$ and each $k \in \mathbb{Z}_{0,\kappa+1}$, then* $\det \dot{\mathbf{p}}_k(z) \neq 0$.

Proof From Notation 14.1 we see $\det \dot{\mathbf{p}}_0(z) = 1 \neq 0$ for all $z \in \mathbb{C}$. Now consider an arbitrary $k \in \mathbb{Z}_{1,\kappa+1}$. Using Remark 3.4, we can infer $(s_j)_{j=0}^{k-1} \in \mathcal{K}^{\succcurlyeq,\mathrm{e}}_{q,k-1,\alpha}$. Let $G \colon \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C}^{q \times q}$ be defined by $G(z) := 0_{q \times q}$. Obviously, $G \in \mathcal{S}^{\diamond}_{q,[\alpha,\infty)}[A]$ for all $A \in \mathbb{C}^{q \times q}$. Regarding Remark 14.3, we thus conclude from Theorems 15.2 and 15.3 that $\det \dot{\mathbf{p}}_k(z) \neq 0$ for all $z \in \mathbb{C} \setminus [\alpha, \infty)$. □

In view of Notation 14.5, we can infer from Corollary 15.4 immediately:

Remark 15.5 Let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}^{\succcurlyeq,\mathrm{e}}_{q,\kappa,\alpha}$. Then $\det \dot{\mathbf{r}}_\ell(z) \neq 0$ for all $z \in \mathbb{C} \setminus [\alpha, \infty)$ and each $\ell \in \mathbb{N}_0$ with $2\ell - 1 \leq \kappa$. Furthermore, $\det \dot{\mathbf{t}}_\ell(z) \neq 0$ for all $z \in \mathbb{C} \setminus [\alpha, \infty)$ and each $\ell \in \mathbb{N}_0$ with $2\ell \leq \kappa$.

Theorem 15.6 *Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n} \in \mathcal{K}^{\succcurlyeq,\mathrm{e}}_{q,2n,\alpha}$. Let $\tilde{\mathbf{p}}_{2n}, \tilde{\mathbf{q}}_{2n} \colon \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C}^{q \times q}$ be defined by*

$$\tilde{\mathbf{p}}_{2n}(z) := (z - \alpha)\dot{\mathbf{p}}_{2n}(z) \qquad \textit{and} \qquad \tilde{\mathbf{q}}_{2n}(z) := (z - \alpha)\dot{\mathbf{q}}_{2n}(z).$$

Denote by $\tilde{\mathbf{p}}_{2n+1}$ and $\tilde{\mathbf{q}}_{2n+1}$ the restriction of $\dot{\mathbf{p}}_{2n+1}$ and $\dot{\mathbf{q}}_{2n+1}$ onto $\mathbb{C} \setminus [\alpha, \infty)$, resp.:

(a) *Let $\Gamma \in \langle \mathcal{P}_{q,\alpha}[L_n] \rangle$ and let $(G_1, G_2) \in \Gamma$. Then* $\det(\tilde{\mathbf{p}}_{2n} L_n^\dagger G_1 + \tilde{\mathbf{p}}_{2n+1} G_2)$ *does not vanish identically in $\mathbb{C} \setminus [\alpha, \infty)$ and*

$$F = -(\tilde{\mathbf{q}}_{2n} L_n^\dagger G_1 + \tilde{\mathbf{q}}_{2n+1} G_2)(\tilde{\mathbf{p}}_{2n} L_n^\dagger G_1 + \tilde{\mathbf{p}}_{2n+1} G_2)^{-1} \qquad (15.3)$$

belongs to $\mathcal{S}_{2n,q,[\alpha,\infty)}[(s_j)_{j=0}^{2n}, \preccurlyeq]$.

(b) *For each* $F \in \mathcal{S}_{2n,q,[\alpha,\infty)}[(s_j)_{j=0}^{2n}, \preccurlyeq]$*, there exists a unique equivalence class* $\Gamma \in \langle \mathcal{P}_{q,\alpha}[L_n] \rangle$ *such that* (15.3) *is fulfilled for each* $(G_1, G_2) \in \Gamma$.

Proof Let $\begin{bmatrix} \tilde{\mathfrak{w}}_{2n} & \tilde{\mathfrak{x}}_{2n} \\ \tilde{\mathfrak{y}}_{2n} & \tilde{\mathfrak{z}}_{2n} \end{bmatrix}$ be the $q \times q$ block representation of the restriction of $\mathfrak{V}^{[\alpha,(s_j)_{j=0}^{2n}]}$ onto $\mathbb{C} \setminus [\alpha, \infty)$. Denote by $(Q_j)_{j=0}^{2n}$ the right α-Stieltjes parametrization of $(s_j)_{j=0}^{2n}$ and by $(s_j^{[2n,\alpha]})_{j=0}^{0}$ the $2n$-th α-S-transform of $(s_j)_{j=0}^{2n}$. Because of Theorem 7.7 and Definition 5.1, we have $s_0^{[2n,\alpha]} = Q_{2n} = L_n$.

(a) We have $(G_1, G_2) \in \mathcal{P}_{q,\alpha}[s_0^{[2n,\alpha]}]$. Thus, we can apply Theorem 13.1(a) to see that $\det(\tilde{\mathfrak{y}}_{2n} G_1 + \tilde{\mathfrak{z}}_{2n} G_2)$ does not vanish identically in $z \in \mathbb{C} \setminus [\alpha, \infty)$ and that $(\tilde{\mathfrak{w}}_{2n} G_1 + \tilde{\mathfrak{x}}_{2n} G_2)(\tilde{\mathfrak{y}}_{2n} G_1 + \tilde{\mathfrak{z}}_{2n} G_2)^{-1}$ belongs to $\mathcal{S}_{2n,q,[\alpha,\infty)}[(s_j)_{j=0}^{2n}, \preccurlyeq]$. By virtue of Proposition 15.1, all assertions of part (a) follow.

(b) In view of Proposition 15.1, this is a consequence of parts (b) and (c) of Theorem 13.1.

□

Theorem 15.7 *Let* $n \in \mathbb{N}_0$ *and let* $(s_j)_{j=0}^{2n+1} \in \mathcal{K}_{q,2n+1,\alpha}^{\succcurlyeq,\mathrm{e}}$. *Denote by* $\tilde{\mathbf{p}}_{2n+1}$, $\tilde{\mathbf{q}}_{2n+1}$, $\tilde{\mathbf{p}}_{2n+2}$, *and* $\tilde{\mathbf{q}}_{2n+2}$ *the restriction of* $\dot{\mathbf{p}}_{2n+1}$, $\dot{\mathbf{q}}_{2n+1}$, $\dot{\mathbf{p}}_{2n+2}$, *and* $\dot{\mathbf{q}}_{2n+2}$ *onto* $\mathbb{C} \setminus [\alpha, \infty)$, *resp.:*

(a) *Let* $\Gamma \in \langle \mathcal{P}_{q,\alpha}[L_{\alpha \triangleright n}] \rangle$ *and let* $(G_1, G_2) \in \Gamma$. *Then* $\det(\tilde{\mathbf{p}}_{2n+1} L_{\alpha \triangleright n}^{\dagger} G_1 + \dot{\mathbf{p}}_{2n+2} G_2)$ *does not vanish identically in* $\mathbb{C} \setminus [\alpha, \infty)$ *and*

$$F = -(\tilde{\mathbf{q}}_{2n+1} L_{\alpha \triangleright n}^{\dagger} G_1 + \tilde{\mathbf{q}}_{2n+2} G_2)(\tilde{\mathbf{p}}_{2n+1} L_{\alpha \triangleright n}^{\dagger} G_1 + \tilde{\mathbf{p}}_{2n+2} G_2)^{-1} \tag{15.4}$$

belongs to $\mathcal{S}_{2n+1,q,[\alpha,\infty)}[(s_j)_{j=0}^{2n+1}, \preccurlyeq]$.

(b) *For each* $F \in \mathcal{S}_{2n+1,q,[\alpha,\infty)}[(s_j)_{j=0}^{2n+1}, \preccurlyeq]$*, there exists a unique equivalence class* $\Gamma \in \langle \mathcal{P}_{q,\alpha}[L_{\alpha \triangleright n}] \rangle$ *such that* (15.4) *is fulfilled for each* $(G_1, G_2) \in \Gamma$.

Proof Let $\begin{bmatrix} \tilde{\mathfrak{w}}_{2n+1} & \tilde{\mathfrak{x}}_{2n+1} \\ \tilde{\mathfrak{y}}_{2n+1} & \tilde{\mathfrak{z}}_{2n+1} \end{bmatrix}$ be the $q \times q$ block representation of the restriction of $\mathfrak{V}^{[\alpha,(s_j)_{j=0}^{2n+1}]}$ onto $\mathbb{C} \setminus [\alpha, \infty)$. Denote by $(Q_j)_{j=0}^{2n+1}$ the right α-Stieltjes parametrization of $(s_j)_{j=0}^{2n+1}$ and by $(s_j^{[2n+1,\alpha]})_{j=0}^{0}$ the $(2n+1)$-th α-S-transform of $(s_j)_{j=0}^{2n+1}$. Because of Theorem 7.7 and Definition 5.1, we have $s_0^{[2n+1,\alpha]} = Q_{2n+1} = L_{\alpha \triangleright n}$.

(a) We have $(G_1, G_2) \in \mathcal{P}_{q,\alpha}[s_0^{[2n+1,\alpha]}]$. Thus, we can apply Theorem 13.1(a) to see that $\det(\tilde{\mathfrak{y}}_{2n+1} G_1 + \tilde{\mathfrak{z}}_{2n+1} G_2)$ does not vanish identically in $z \in \mathbb{C} \setminus [\alpha, \infty)$ and that $(\tilde{\mathfrak{w}}_{2n+1} G_1 + \tilde{\mathfrak{x}}_{2n+1} G_2)(\tilde{\mathfrak{y}}_{2n+1} G_1 + \tilde{\mathfrak{z}}_{2n+1} G_2)^{-1}$ belongs to $\mathcal{S}_{2n+1,q,[\alpha,\infty)}[(s_j)_{j=0}^{2n+1}, \preccurlyeq]$. By virtue of Proposition 15.1, all assertions of part (a) follow.

(b) In view of Proposition 15.1, this is a consequence of parts (b) and (c) of Theorem 13.1.

□

16 Further Expressions for the Upper and Lower $\mathcal{S}_{q,[\alpha,\infty)}$-Functions Associated with a Sequence $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$

Against to the background of Proposition 13.8 and formula (12.5) the application of Proposition 15.1 yields now interesting new representations for the upper and lower $\mathcal{S}_{q,[\alpha,\infty)}$-functions associated with a sequence $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\succcurlyeq,\mathrm{e}}$.

Proposition 16.1 *Let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\succcurlyeq,\mathrm{e}}$ and let $m \in \mathbb{Z}_{0,\kappa}$. Denote by $\overline{S}_m$ and $\underline{S}_m$ the upper and lower $\mathcal{S}_{q,[\alpha,\infty)}$-functions associated with $(s_j)_{j=0}^{m}$. Then:*

(a) $\det \dot{\mathbf{p}}_m(z) \neq 0$ *and* $\underline{S}_m(z) = -[\dot{\mathbf{q}}_m(z)][\dot{\mathbf{p}}_m(z)]^{-1}$ *for all* $z \in \mathbb{C} \setminus [\alpha, \infty)$.
(b) $\det \dot{\mathbf{p}}_{m+1}(z) \neq 0$ *and* $\overline{S}_m(z) = -[\dot{\mathbf{q}}_{m+1}(z)][\dot{\mathbf{p}}_{m+1}(z)]^{-1}$ *for all* $z \in \mathbb{C} \setminus [\alpha, \infty)$.

Proof Using Remark 3.4, we can infer $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$. Consider an arbitrary $z \in \mathbb{C} \setminus [\alpha, \infty)$.

(a) From Example 6.12 we can conclude $\underline{S}_0(z) = 0_{q\times q}$. Taking additionally into account Notation 14.1, we obtain then $\det \dot{\mathbf{p}}_0(z) = 1$ and $\underline{S}_0(z) = -[\dot{\mathbf{q}}_0(z)][\dot{\mathbf{p}}_0(z)]^{-1}$. If $m \geq 1$, then, regarding Remark 14.3, the all assertions of part (a) follow by combining Propositions 13.8 and 15.1.
(b) Combine Propositions 13.8 and 15.1 and take into account Remark 14.3.

□

Corollary 16.2 *Suppose $\kappa \geq 1$. Let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\succcurlyeq,\mathrm{e}}$ and let $m \in \mathbb{Z}_{1,\kappa}$. Denote by $\underline{S}_m$ and $\overline{S}_{m-1}$ the lower and the upper $\mathcal{S}_{q,[\alpha,\infty)}$-function associated with $(s_j)_{j=0}^{m}$ and with $(s_j)_{j=0}^{m-1}$, respectively. Then $\underline{S}_m = \overline{S}_{m-1}$.*

Proof This is an immediate consequence of Proposition 16.1. □

Corollary 16.3 *Let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\succcurlyeq,\mathrm{e}}$ and let $m \in \mathbb{Z}_{0,\kappa}$. Denote by $\overline{S}_m$ and $\underline{S}_m$ the upper and lower $\mathcal{S}_{q,[\alpha,\infty)}$-functions associated with $(s_j)_{j=0}^{m}$. Then:*

(a) $\det \dot{\mathbf{p}}_m(\overline{z}) \neq 0$ *and* $\underline{S}_m(z) = -[\dot{\mathbf{p}}_m(\overline{z})]^{-*}[\dot{\mathbf{q}}_m(\overline{z})]^*$ *for all* $z \in \mathbb{C} \setminus [\alpha, \infty)$.
(b) $\det \dot{\mathbf{p}}_{m+1}(\overline{z}) \neq 0$ *and* $\overline{S}_m(z) = -[\dot{\mathbf{p}}_{m+1}(\overline{z})]^{-*}[\dot{\mathbf{q}}_{m+1}(\overline{z})]^*$ *for all* $z \in \mathbb{C} \setminus [\alpha, \infty)$.

Proof In view of (9.4) and the definition of $\underline{S}_m$ and $\overline{S}_m$ as $[\alpha, \infty)$-Stieltjes transforms, we obviously have $[\underline{S}_m(w)]^* = \underline{S}_m(\overline{w})$ and $[\overline{S}_m(w)]^* = \overline{S}_m(\overline{w})$ for all $w \in \mathbb{C} \setminus [\alpha, \infty)$. The application of Proposition 16.1 completes the proof. □

17 Matricial Weyl Intervals Associated with $\mathcal{M}_q^{\succcurlyeq}[[\alpha, \infty); (s_j)_{j=0}^m, \preccurlyeq]$ and a Given Point $x \in (-\infty, \alpha)$

Our main goal in this section is the investigation of the following set of matrices from $\mathbb{C}^{q\times q}$.

Notation 17.1 Let $m \in \mathbb{N}_0$ and let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq}$. Then, for all $x \in (-\infty, \alpha)$, let $\mathcal{I}[\alpha, x, (s_j)_{j=0}^m; \preccurlyeq] := \{F(x): F \in \mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^m, \preccurlyeq]\}$.

Taking into account Definition 8.1, we infer that the set introduced in Notation 17.1 is contained in the set $\mathbb{C}_{\succcurlyeq}^{q\times q}$. Our main result in this section is Theorem 17.16, which shows that the set $\mathcal{I}[\alpha, x, (s_j)_{j=0}^m; \preccurlyeq]$ is a closed bounded matricial interval, the endpoints of which are given by the values of the particular solutions $\underline{S}_m$ and $\overline{S}_m$ at the point x. Here $\overline{S}_m$ and $\underline{S}_m$ are the upper and lower $\mathcal{S}_{q,[\alpha,\infty)}$-functions associated with $(s_j)_{j=0}^m$. For the construction of the functions $\underline{S}_m$ and $\overline{S}_m$, we refer to Sect. 13. In the so-called non-degenerate case of a sequence $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succ}$, this result was obtained in the recent thesis [27] of Benjamin Jeschke (see also [28]). B. Jeschke generalized the methods due to Yu. M. Dyukarev [7], who considered the particular case $\alpha = 0$. Our strategy of proof is completely different of that used by Yu. M. Dyukarev and B. Jeschke. Our proof is based on an inductive procedure. In the heart of our construction lies the coupled pair of Schur–Stieltjes transforms introduced in Sect. 12.

Remark 17.2 Let $\alpha, \omega \in \mathbb{R}$ with $\omega \leq \alpha$ and let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$. Then it is readily checked that $\mathcal{K}_{q,\kappa,\alpha}^{\succcurlyeq} \subseteq \mathcal{K}_{q,\kappa,\omega}^{\succcurlyeq}$ and $\mathcal{K}_{q,\kappa,\alpha}^{\succcurlyeq,\mathrm{e}} \subseteq \mathcal{K}_{q,\kappa,\omega}^{\succcurlyeq,\mathrm{e}}$.

Lemma 17.3 *Let $\omega, x \in \mathbb{R}$ with $x < \omega \leq \alpha$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq}$. Then $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\omega}^{\succcurlyeq}$ and $\mathcal{I}[\alpha, x, (s_j)_{j=0}^m; \preccurlyeq] \subseteq \mathcal{I}[\omega, x, (s_j)_{j=0}^m; \preccurlyeq]$.*

Proof From Remark 17.2 we obtain $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\omega}^{\succcurlyeq}$. In view of Theorem 10.5, let $F \in \mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^m, \preccurlyeq]$ with $[\alpha,\infty)$-Stieltjes measure σ_F. Regarding $[\alpha, \infty) \subseteq [\omega, \infty)$, then $\mu\colon \mathfrak{B}_{[\omega,\infty)} \to \mathbb{C}_{\succcurlyeq}^{q\times q}$ defined by $\mu(B) := \sigma_F(B \cap [\alpha, \infty))$ belongs to $\mathcal{M}_q^{\succcurlyeq}[[\omega, \infty); (s_j)_{j=0}^m, \preccurlyeq]$ and $\int_{[\omega,\infty)}(t-x)^{-1}\mu(\mathrm{d}t) = \int_{[\alpha,\infty)}(t-x)^{-1}\sigma_F(\mathrm{d}t)$. Hence, the $[\omega,\infty)$-Stieltjes transform S_μ of μ belongs to $\mathcal{S}_{m,q,[\omega,\infty)}[(s_j)_{j=0}^m, \preccurlyeq]$ and fulfills $S_\mu(x) = F(x)$. □

Lemma 17.4 *Let $m \in \mathbb{N}$ and let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$ with first α-S-transform $(t_j)_{j=0}^{m-1}$. For all $x \in (-\infty, \alpha)$, then $\mathcal{I}[\alpha, x, (s_j)_{j=0}^m; \preccurlyeq] = \{G^{[-,\alpha,s_0]}(x): G \in \mathcal{S}_{m-1,q,[\alpha,\infty)}[(t_j)_{j=0}^{m-1}, \preccurlyeq]\}$.*

Proof Let $x \in (-\infty, \alpha)$ and let $X \in \mathcal{I}[\alpha, x, (s_j)_{j=0}^m; \preccurlyeq]$. Then there is an $F \in \mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^m, \preccurlyeq]$ with $X = F(x)$. According to Theorem 12.5, then the (α, s_0)-Schur–Stieltjes transform G of F belongs to $\mathcal{S}_{m-1,q,[\alpha,\infty)}[(t_j)_{j=0}^{m-1}, \preccurlyeq]$. Since $(s_j)_{j=0}^{m-1} \in \mathcal{K}_{q,m-1,\alpha}^{\succcurlyeq,\mathrm{e}}$ by virtue of Remark 3.4, and $F \in \mathcal{S}_{m-1,q,[\alpha,\infty)}[(s_j)_{j=0}^{m-1}, =]$ because of Remark 10.4, the application of [21, Corollary 9.12] yields $G^{[-,\alpha,s_0]} = F$. In particular, $X = F(x) = G^{[-,\alpha,s_0]}(x)$.

Conversely, for each $G \in \mathcal{S}_{m-1,q,[\alpha,\infty)}[(t_j)_{j=0}^{m-1}, \preccurlyeq]$, we have $G^{[-,\alpha,s_0]} \in \mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^{m}, \preccurlyeq]$ by virtue of [23, Theorem 10.5], and hence $G^{[-,\alpha,s_0]}(x) \in \mathcal{I}[\alpha, x, (s_j)_{j=0}^{m}; \preccurlyeq]$. □

Remark 17.5 Let $m \in \mathbb{N}_0$, let $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq}$, and let $F \in \mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^{m}, \preccurlyeq]$. In view of Definition 8.1, Proposition 10.7, and Notation 8.4, then $F(x) \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ for all $x \in (-\infty, \alpha)$ and $\mathcal{R}(F(z)) \subseteq \mathcal{R}(s_0)$ for all $z \in \mathbb{C} \setminus [\alpha, \infty)$.

Lemma 17.6 *Let $m \in \mathbb{N}$, let $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$ with first α-S-transform $(t_j)_{j=0}^{m-1}$, and let $G \in \mathcal{S}_{m-1,q,[\alpha,\infty)}[(t_j)_{j=0}^{m-1}, \preccurlyeq]$.*

(a) *The inclusion $\mathcal{R}(G(z)) \subseteq \mathcal{R}(s_0)$, the inequality $\det((z-\alpha)[s_0^\dagger G(z) + I_q]) \neq 0$, and the equation $G^{[-,\alpha,s_0]}(z) = -s_0((z-\alpha)[s_0^\dagger G(z) + I_q])^{-1}$ hold true for all $z \in \mathbb{C} \setminus [\alpha, \infty)$.*
(b) *$G^{[-,\alpha,s_0]}(x) = (\alpha - x)^{-1} s_0 [G(x) + s_0]^\dagger s_0$ for all $x \in (-\infty, \alpha)$.*

Proof Because of $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$, Remark 3.4 implies $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq}$. Thus, Lemma 3.2 yields $s_0 \in \mathbb{C}_{\succcurlyeq}^{q\times q}$. Theorem 7.4 yields $(t_j)_{j=0}^{m-1} \in \mathcal{K}_{q,m-1,\alpha}^{\succcurlyeq,\mathrm{e}}$. According to Remark 17.5, we obtain then $G(x) \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ for all $x \in (-\infty, \alpha)$ and $\mathcal{R}(G(z)) \subseteq \mathcal{R}(t_0)$ for all $z \in \mathbb{C} \setminus [\alpha, \infty)$. Since $\mathcal{R}(t_0) \subseteq \mathcal{R}(s_0)$ holds true by virtue of Definition 7.3, then $\mathcal{R}(G(z)) \subseteq \mathcal{R}(s_0)$ follows for all $z \in \mathbb{C} \setminus [\alpha, \infty)$. In particular, $G \in \mathcal{S}_{q,[\alpha,\infty)}[s_0]$. Consequently, for all $z \in \mathbb{C} \setminus [\alpha, \infty)$, then Lemma 12.3 and (12.3) yield $G(z) \in \mathcal{Q}_{[(z-\alpha)s_0^\dagger,(z-\alpha)I_q]}$ and $G^{[-,\alpha,s_0]}(z) = \mathcal{S}_{V_{\alpha,s_0}(z)}^{(q,q)}(G(z))$, i. e., $\det[(z-\alpha)s_0^\dagger G(z) + (z-\alpha)I_q] \neq 0$ and $G^{[-,\alpha,s_0]}(z) = -s_0[(z-\alpha)s_0^\dagger G(z) + (z-\alpha)I_q]^{-1}$. Consider now an arbitrary $x \in (-\infty, \alpha)$. Then we can apply Lemma A.7 with $M = s_0$ and $X = G(x)$ to obtain $s_0[s_0^\dagger G(x) + I_q]^{-1} = s_0[G(x) + s_0]^\dagger s_0$. Hence, $G^{[-,\alpha,s_0]}(x) = (\alpha - x)^{-1} s_0 [G(x) + s_0]^\dagger s_0$. □

Lemma 17.7 *Let $m \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$ with right α-Stieltjes parametrization $(Q_j)_{j=0}^{m}$. Let $\mathfrak{V}^{[\alpha,(s_j)_{j=0}^{m}]}$ be defined via (12.4) and (12.3) and let (12.5) be the $q \times q$ block representation of $\mathfrak{V}^{[\alpha,(s_j)_{j=0}^{m}]}$. Then:*

(a) *Let $\underline{\phi}_m, \underline{\psi}_m : \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C}^{q\times q}$ be defined by $\underline{\phi}_m(z) := Q_m$ and $\underline{\psi}_m(z) := \mathbb{P}_{\mathcal{N}(Q_m)}$ where the matrix $\mathbb{P}_{\mathcal{N}(Q_m)}$ is introduced in Remark A.2. For all $z \in \mathbb{C} \setminus [\alpha, \infty)$, then $(\underline{\phi}_m(z), \underline{\psi}_m(z)) \in \tilde{\mathcal{Q}}_{[\mathfrak{v}_{21}^{[\alpha,(s_j)_{j=0}^{m}]}(z),\mathfrak{v}_{22}^{[\alpha,(s_j)_{j=0}^{m}]}(z)]}$ and $\underline{S}_m(z) = \tilde{\mathcal{S}}_{\mathfrak{V}^{[\alpha,(s_j)_{j=0}^{m}]}(z)}^{(q,q)}((\underline{\phi}_m(z), \underline{\psi}_m(z)))$.*

(b) *Let* $\overline{\phi}_m, \overline{\psi}_m : \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C}^{q\times q}$ *be defined by* $\overline{\phi}_m(z) := 0_{q\times q}$ *and* $\overline{\psi}_m(z) := I_q$. *For all* $z \in \mathbb{C} \setminus [\alpha, \infty)$, *then* $(\overline{\phi}_m(z), \overline{\psi}_m(z)) \in \tilde{\mathcal{Q}}_{[\mathfrak{v}_{21}^{[\alpha,(s_j)_{j=0}^m]}(z), \mathfrak{v}_{22}^{[\alpha,(s_j)_{j=0}^m]}(z)]}$ *and* $\overline{S}_m(z) = \tilde{\mathcal{S}}^{(q,q)}_{\mathfrak{V}^{[\alpha,(s_j)_{j=0}^m]}(z)}((\overline{\phi}_m(z), \overline{\psi}_m(z)))$. *For all* $z \in \mathbb{C} \setminus [\alpha, \infty)$, *furthermore* $\overline{\phi}_m(z) \in \mathcal{Q}_{[\mathfrak{v}_{21}^{[\alpha,(s_j)_{j=0}^m]}(z), \mathfrak{v}_{22}^{[\alpha,(s_j)_{j=0}^m]}(z)]}$ *and* $\overline{S}_m(z) = \mathcal{S}^{(q,q)}_{\mathfrak{V}^{[\alpha,(s_j)_{j=0}^m]}(z)}(\overline{\phi}_m(z))$.

Proof The assertions are an immediate consequence of Theorem 13.9. □

The following notation turns out to be very useful what concerns a unique presentation of subsequent results.

Notation 17.8 Let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}^{\succcurlyeq,\mathrm{e}}_{q,\kappa,\alpha}$. Then, for all $n \in \mathbb{N}_0$ with $2n \le \kappa$, let $S^{\langle s\rangle}_{-,2n,\alpha} := \underline{S}_{2n}$ and $S^{\langle s\rangle}_{+,2n,\alpha} := \overline{S}_{2n}$. Furthermore, for all $n \in \mathbb{N}_0$ with $2n+1 \le \kappa$, let $S^{\langle s\rangle}_{-,2n+1,\alpha} := \overline{S}_{2n+1}$ and $S^{\langle s\rangle}_{+,2n+1,\alpha} := \underline{S}_{2n+1}$.

In situations in which it is obvious which sequence $(s_j)_{j=0}^{\kappa}$ of complex matrices is meant, we will also write $S_{-,m,\alpha}$ instead of $S^{\langle s\rangle}_{-,m,\alpha}$ and $S_{+,m,\alpha}$ instead of $S^{\langle s\rangle}_{+,m,\alpha}$, $m \in \mathbb{Z}_{0,\kappa}$. Using Notation 17.8 we can rewrite Corollary 16.2 in the following way.

Lemma 17.9 *If* $(s_j)_{j=0}^{\kappa} \in \mathcal{K}^{\succcurlyeq,\mathrm{e}}_{q,\kappa,\alpha}$, *then* $S_{-,2n,\alpha} = S_{-,2n-1,\alpha}$ *for all* $n \in \mathbb{N}$ *with* $2n \le \kappa$ *and* $S_{+,2n+1,\alpha} = S_{+,2n,\alpha}$ *for all* $n \in \mathbb{N}_0$ *with* $2n+1 \le \kappa$.

Proof This is an immediate consequence of Corollary 16.2 and Notation 17.8. □

Remark 17.10 Let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}^{\succcurlyeq,\mathrm{e}}_{q,\kappa,\alpha}$. In view of Notation 17.8, Lemma 17.7, (12.4), (12.3), and (5.1), for all $z \in \mathbb{C} \setminus [\alpha, \infty)$, then $S_{-,0,\alpha}(z) = 0_{q\times q}$ and $S_{+,0,\alpha}(z) = (\alpha - z)^{-1} s_0$.

Lemma 17.11 *Let* $(s_j)_{j=0}^{\kappa} \in \mathcal{K}^{\succcurlyeq,\mathrm{e}}_{q,\kappa,\alpha}$ *and let* $m \in \mathbb{Z}_{0,\kappa}$. *Then:*

(a) $\overline{S}_m \in \mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^m, =]$ *and* $\underline{S}_m \in \mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^m, \preccurlyeq]$. *If* $m \ge 1$, *furthermore* $\underline{S}_m \in \mathcal{S}_{m-1,q,[\alpha,\infty)}[(s_j)_{j=0}^{m-1}, =]$.
(b) $\overline{S}_m(x), \underline{S}_m(x) \in \mathbb{C}^{q\times q}_{\succcurlyeq}$ *for all* $x \in (-\infty, \alpha)$.
(c) *Let* $z \in \mathbb{C} \setminus [\alpha, \infty)$. *Then* $\mathcal{R}(\overline{S}_m(z)) = \mathcal{R}(s_0)$ *and* $\mathcal{R}(\underline{S}_m(z)) \subseteq \mathcal{R}(s_0)$. *If* $m \ge 1$, *furthermore* $\mathcal{R}(\underline{S}_m(z)) = \mathcal{R}(s_0)$.

Proof In view of Remarks 3.4 and 3.3, we have $(s_j)_{j=0}^m \in \mathcal{K}^{\succcurlyeq,\mathrm{e}}_{q,m,\alpha} \subseteq \mathcal{K}^{\succcurlyeq}_{q,m,\alpha}$. Proposition 6.13 yields $\overline{\sigma}_m \in \mathcal{M}^{\succcurlyeq}_q[[\alpha, \infty); (s_j)_{j=0}^m, =]$, whereas Proposition 6.14(a) shows that $\underline{\sigma}_m \in \mathcal{M}^{\succcurlyeq}_q[[\alpha, \infty); (s_j)_{j=0}^m, \preccurlyeq]$. Hence, $\overline{S}_m \in \mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^m, =]$ and $\underline{S}_m \in \mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^m, \preccurlyeq]$. If $m \ge 1$, then Remark 10.4 yields $\underline{S}_m \in \mathcal{S}_{m-1,q,[\alpha,\infty)}[(s_j)_{j=0}^{m-1}, =]$. Thus, (a) is proved. In view of (a), parts (b) and (c) follow from Definition 8.1, [21, Proposition 5.5], and Lemma A.17. □

Lemma 17.12 *Suppose $\kappa \geq 1$. Let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\succcurlyeq,\mathrm{e}}$ with first α-S-transform $(t_j)_{j=0}^{\kappa-1}$ and let $m \in \mathbb{Z}_{1,\kappa}$. Then $\underline{S}_m^{\langle s\rangle} = (\underline{S}_{m-1}^{\langle t\rangle})^{[-,\alpha,s_0]}$ and $\overline{S}_m^{\langle s\rangle} = (\overline{S}_{m-1}^{\langle t\rangle})^{[-,\alpha,s_0]}$. In particular, $S_{-,m,\alpha}^{\langle s\rangle} = (S_{+,m-1,\alpha}^{\langle t\rangle})^{[-,\alpha,s_0]}$ and $S_{+,m,\alpha}^{\langle s\rangle} = (S_{-,m-1,\alpha}^{\langle t\rangle})^{[-,\alpha,s_0]}$.*

Proof Let $z \in \mathbb{C} \setminus [\alpha, \infty)$. From [20, Remark 7.3] we see that $(t_j)_{j=0}^{m-1}$ is the first α-S-transform of $(s_j)_{j=0}^{m}$. Because of Remark 3.4, we have $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$. Theorem 7.4 yields then

$$(t_j)_{j=0}^{m-1} \in \mathcal{K}_{q,m-1,\alpha}^{\succcurlyeq,\mathrm{e}}. \tag{17.1}$$

Denote by $(Q_j)_{j=0}^{\kappa}$ the right α-Stieltjes parametrization of $(s_j)_{j=0}^{\kappa}$ and by $(P_j)_{j=0}^{\kappa-1}$ the right α-Stieltjes parametrization of $(t_j)_{j=0}^{\kappa-1}$. According to Remark 5.5, then $(Q_j)_{j=0}^{m}$ is the right α-Stieltjes parametrization of $(s_j)_{j=0}^{m}$ and $(P_j)_{j=0}^{m-1}$ the right α-Stieltjes parametrization of $(t_j)_{j=0}^{m-1}$. From [20, Theorem 9.26] we get furthermore $P_{m-1} = Q_m$. A twofold application of Lemma 17.7 yields then

$$\underline{S}_m^{\langle s\rangle}(z) = \tilde{\mathcal{S}}^{(q,q)}_{\mathfrak{V}^{[\alpha,(s_j)_{j=0}^{m}]}(z)}\big((Q_m, \mathbb{P}_{\mathcal{N}(Q_m)})\big), \qquad \overline{S}_m^{\langle s\rangle}(z) = \mathcal{S}^{(q,q)}_{\mathfrak{V}^{[\alpha,(s_j)_{j=0}^{m}]}(z)}(0_{q\times q})$$

and

$$\underline{S}_{m-1}^{\langle t\rangle}(z) = \tilde{\mathcal{S}}^{(q,q)}_{\mathfrak{V}^{[\alpha,(t_j)_{j=0}^{m-1}]}(z)}\big((Q_m, \mathbb{P}_{\mathcal{N}(Q_m)})\big), \; \overline{S}_{m-1}^{\langle t\rangle}(z) = \mathcal{S}^{(q,q)}_{\mathfrak{V}^{[\alpha,(t_j)_{j=0}^{m-1}]}(z)}(0_{q\times q}).$$

Because of $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\succcurlyeq,\mathrm{e}}$, Remark 3.4 implies $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\succcurlyeq}$. Thus, Lemma 3.2 yields $s_0 \in \mathbb{C}_{\succcurlyeq}^{q\times q}$. In view of (17.1), from parts (a) and (c) of Lemma 17.11 we conclude that $\underline{S}_{m-1}^{\langle t\rangle}$ and $\overline{S}_{m-1}^{\langle t\rangle}$ both belong to the class $\mathcal{S}_{q,[\alpha,\infty)}$ and that $\mathcal{R}(\underline{S}_{m-1}^{\langle t\rangle}(w)) \subseteq \mathcal{R}(t_0)$ and $\mathcal{R}(\overline{S}_{m-1}^{\langle t\rangle}(w)) \subseteq \mathcal{R}(t_0)$ hold true for all $w \in \mathbb{C}\setminus[\alpha, \infty)$. Since Definition 7.3 shows that $\mathcal{R}(t_0) \subseteq \mathcal{R}(s_0)$, then $\underline{S}_{m-1}^{\langle t\rangle}, \overline{S}_{m-1}^{\langle t\rangle} \in \mathcal{S}_{q,[\alpha,\infty)}[s_0]$ follows. Consequently, Lemma 12.3 yields

$$(\underline{S}_{m-1}^{\langle t\rangle})^{[-,\alpha,s_0]}(z) = \mathcal{S}^{(q,q)}_{V_{\alpha,s_0}(z)}\Big(\underline{S}_{m-1}^{\langle t\rangle}(z)\Big)$$

and

$$(\overline{S}_{m-1}^{\langle t\rangle})^{[-,\alpha,s_0]}(z) = \mathcal{S}^{(q,q)}_{V_{\alpha,s_0}(z)}\Big(\overline{S}_{m-1}^{\langle t\rangle}(z)\Big)$$

for all $z \in \mathbb{C} \setminus [\alpha, \infty)$. In view of Remark 12.2, we have $\mathfrak{V}^{[\alpha,(s_j)_{j=0}^{m}]} = V_{\alpha,s_0}\mathfrak{V}^{[\alpha,(t_j)_{j=0}^{m-1}]}$. Thus, for each $z \in \mathbb{C} \setminus [\alpha, \infty)$, from Lemma 17.7 and Proposition C.4 we get

$$\begin{aligned}\underline{S}_m(z) &= \tilde{\mathcal{S}}^{(q,q)}_{\mathfrak{V}^{[\alpha,(s_j)_{j=0}^{m}]}(z)}\big((Q_m, \mathbb{P}_{\mathcal{N}(Q_m)})\big)\\ &= \mathcal{S}^{(q,q)}_{V_{\alpha,s_0}(z)}\left(\tilde{\mathcal{S}}^{(q,q)}_{\mathfrak{V}^{[\alpha,(t_j)_{j=0}^{m-1}]}(z)}\big((Q_m, \mathbb{P}_{\mathcal{N}(Q_m)})\big)\right)\\ &= \mathcal{S}^{(q,q)}_{V_{\alpha,s_0}(z)}\Big(\underline{S}^{\langle t\rangle}_{m-1}(z)\Big) = (\underline{S}^{\langle t\rangle}_{m-1})^{[-,\alpha,s_0]}(z)\end{aligned}$$

and

$$\begin{aligned}\overline{S}_m(z) &= \mathcal{S}^{(q,q)}_{\mathfrak{V}^{[\alpha,(s_j)_{j=0}^{m}]}(z)}(0_{q\times q}) = \mathcal{S}^{(q,q)}_{V_{\alpha,s_0}(z)}\left(\mathcal{S}^{(q,q)}_{\mathfrak{V}^{[\alpha,(t_j)_{j=0}^{m-1}]}(z)}(0_{q\times q})\right)\\ &= \mathcal{S}^{(q,q)}_{V_{\alpha,s_0}(z)}\Big(\overline{S}^{\langle t\rangle}_{m-1}(z)\Big) = (\overline{S}^{\langle t\rangle}_{m-1})^{[-,\alpha,s_0]}(z).\end{aligned}$$

□

Lemma 17.13 *Let $B \in \mathbb{C}^{q\times q}_{\mathrm{H}}$, let $x \in (-\infty, \alpha)$, and let $\sigma \in \mathcal{M}^{\succcurlyeq}_q([\alpha, \infty))$ with $\sigma([\alpha, \infty)) \preccurlyeq B$. Then $X := \int_{[\alpha,\infty)} (t - x)^{-1}\sigma(\mathrm{d}t)$ fulfills $0_{q\times q} \preccurlyeq X \preccurlyeq (\alpha - x)^{-1}B$ and $\mathcal{R}(X) = \mathcal{R}(\sigma([\alpha, \infty)))$.*

Proof For all $t \in [\alpha, \infty)$ we have $0 < \alpha - x \leq t - x$ and hence $0 < (t - x)^{-1} \leq (\alpha - x)^{-1}$. Thus, we obtain

$$\begin{aligned}0_{q\times q} &\preccurlyeq \int_{[\alpha,\infty)} \frac{1}{t-x}\sigma(\mathrm{d}t) = X\\ &\preccurlyeq \int_{[\alpha,\infty)} \frac{1}{\alpha-x}\sigma(\mathrm{d}t) = \frac{1}{\alpha-x}\sigma([\alpha, \infty)) \preccurlyeq \frac{1}{\alpha-x}B\end{aligned}$$

and, according to [17, Lemma B.2(b)], furthermore $\mathcal{R}(X) = \mathcal{R}(\sigma([\alpha, \infty)))$. □

Lemma 17.14 *Let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}^{\succcurlyeq,\mathrm{e}}_{q,\kappa,\alpha}$. Then:*

(a) *$S_{+,2n,\alpha} \in \mathcal{S}_{2n,q,[\alpha,\infty)}[(s_j)_{j=0}^{2n}, =]$ and $S_{-,2n,\alpha} \in \mathcal{S}_{2n,q,[\alpha,\infty)}[(s_j)_{j=0}^{2n}, \preccurlyeq]$ for all $n \in \mathbb{N}_0$ with $2n \leq \kappa$.*
(b) *$S_{-,2n+1,\alpha} \in \mathcal{S}_{2n+1,q,[\alpha,\infty)}[(s_j)_{j=0}^{2n+1}, =]$ and $S_{+,2n+1,\alpha} \in \mathcal{S}_{2n+1,q,[\alpha,\infty)}[(s_j)_{j=0}^{2n+1}, \preccurlyeq] \subseteq \mathcal{S}_{2n,q,[\alpha,\infty)}[(s_j)_{j=0}^{2n}, =]$ for all $n \in \mathbb{N}_0$ with $2n + 1 \leq \kappa$.*

(c) *Let* $m \in \mathbb{Z}_{0,\kappa}$. *Then* $S_{+,m,\alpha}(x), S_{-,m,\alpha}(x) \in \mathbb{C}^{q\times q}_{\succcurlyeq}$ *for all* $x \in (-\infty, \alpha)$ *and* $\mathcal{R}(S_{+,m,\alpha}(z)) = \mathcal{R}(s_0)$ *and* $\mathcal{R}(S_{-,m,\alpha}(z)) \subseteq \mathcal{R}(s_0)$ *for all* $z \in \mathbb{C} \setminus [\alpha, \infty)$. *If* $m \geq 1$, *furthermore* $\mathcal{R}(S_{-,m,\alpha}(z)) = \mathcal{R}(s_0)$ *for all* $z \in \mathbb{C} \setminus [\alpha, \infty)$.

Proof In view of Notation 17.8 and Remark 10.4, this is an immediate consequence of Lemma 17.11. □

Lemma 17.15 *Let* $(s_j)_{j=0}^{\kappa} \in \mathcal{K}^{\succcurlyeq,\mathrm{e}}_{q,\kappa,\alpha}$ *and let* $x \in (-\infty, \alpha)$. *Then* $0_{q\times q} = S_{-,0,\alpha}(x) \preccurlyeq S_{+,0,\alpha}(x)$ *and* $\mathcal{I}[\alpha, x, (s_j)_{j=0}^{0}; \preccurlyeq] = [S_{-,0,\alpha}(x), S_{+,0,\alpha}(x)]$.

Proof Let $\eta := (\alpha - x)^{-1}$ and let $M := s_0$. From Remark 17.10 we see $S_{-,0,\alpha}(x) = 0_{q\times q}$ and $S_{+,0,\alpha}(x) = \eta M$. Because of $(s_j)_{j=0}^{\kappa} \in \mathcal{K}^{\succcurlyeq,\mathrm{e}}_{q,\kappa,\alpha}$, we have $M \in \mathbb{C}^{q\times q}_{\succcurlyeq}$. Taking additionally into account $\eta \in (0, \infty)$, then Remark A.12 yields $S_{+,0,\alpha}(x) \in \mathbb{C}^{q\times q}_{\succcurlyeq}$. Consequently, $0_{q\times q} = S_{-,0,\alpha}(x) \preccurlyeq S_{+,0,\alpha}(x)$ holds true.

Consider now an arbitrary $X \in \mathcal{I}[\alpha, x, (s_j)_{j=0}^{0}; \preccurlyeq]$. Then there exists a function $F \in \mathcal{S}_{0,q,[\alpha,\infty)}[(s_j)_{j=0}^{0}, \preccurlyeq]$ with $X = F(x)$. In particular, $F \in \mathcal{S}_{0,q,[\alpha,\infty)}$ and the $[\alpha, \infty)$-Stieltjes measure σ_F of F belongs to $\mathcal{M}^{\succcurlyeq}_{q}[[\alpha, \infty); (s_j)_{j=0}^{0}, \preccurlyeq]$ and fulfills $X = \int_{[\alpha,\infty)} (t - x)^{-1} \sigma_F(\mathrm{d}t)$. Hence, $\sigma_F \in \mathcal{M}^{\succcurlyeq}_{q}([\alpha, \infty))$ and $\sigma_F([\alpha, \infty)) \preccurlyeq M$ hold true. According to Lemma 17.13, then $0_{q\times q} \preccurlyeq X \preccurlyeq \eta M$. Consequently, $X \in [0_{q\times q}, \eta M]$ by virtue of (A.3).

Conversely, consider now an arbitrary matrix $X \in [0_{q\times q}, \eta M]$. By virtue of (A.3), then X is Hermitian satisfying $0_{q\times q} \preccurlyeq X \preccurlyeq \eta M$. In view of $\eta > 0$ and Remark A.12, the matrix $N := \eta^{-1} X$ belongs to $\mathbb{C}^{q\times q}_{\succcurlyeq}$ and fulfills $N \preccurlyeq M$. Hence, $\sigma := \delta_\alpha N$ belongs to $\mathcal{M}^{\succcurlyeq}_{q}([\alpha, \infty))$ and fulfills $\sigma([\alpha, \infty)) = N \preccurlyeq M$, where δ_α denotes the Dirac measure defined on $\mathfrak{B}_{[\alpha,\infty)}$ with unit mass at α. Consequently, $\sigma \in \mathcal{M}^{\succcurlyeq}_{q}[[\alpha, \infty); (s_j)_{j=0}^{0}, \preccurlyeq]$. Thus, the $[\alpha, \infty)$-Stieltjes transform S_σ of σ belongs to $\mathcal{S}_{0,q,[\alpha,\infty)}[(s_j)_{j=0}^{0}, \preccurlyeq]$. Since furthermore $S_\sigma(x) = \int_{[\alpha,\infty)} (t - x)^{-1} \sigma(\mathrm{d}t) = \eta N = X$ holds true, then $X \in \mathcal{I}[\alpha, x, (s_j)_{j=0}^{0}; \preccurlyeq]$. □

Now we are able to formulate and prove our main result in this section. Before doing that, we explain the strategy and essential steps of our proof. The Schur analysis approach in solving Problem $\mathsf{M}[[\alpha, \infty); (s_j)_{j=0}^{m}, \preccurlyeq]$ is the main reason that our proof is based on mathematical induction. The beginning of the induction is a consequence of Lemma 17.15, whereas the step of induction is realized by a combination of Lemma 17.4 with Lemma 17.6(b). In view of Lemma 17.12, the application of Proposition B.5 implies then that the set $\mathcal{I}[\alpha, x, (s_j)_{j=0}^{m}; \preccurlyeq]$ is the closed bounded matricial interval with the asserted endpoints.

Theorem 17.16 *Let* $m \in \mathbb{N}_0$, *let* $(s_j)_{j=0}^{m} \in \mathcal{K}^{\succcurlyeq,\mathrm{e}}_{q,m,\alpha}$, *and let* $x \in (-\infty, \alpha)$. *In view Notation 17.8, then* $0_{q\times q} \preccurlyeq S_{-,m,\alpha}(x) \preccurlyeq S_{+,m,\alpha}(x)$ *and* $\mathcal{I}[\alpha, x, (s_j)_{j=0}^{m}; \preccurlyeq] = [S_{-,m,\alpha}(x), S_{+,m,\alpha}(x)]$.

Proof We use mathematical induction: In the case $m = 0$, the assertion follows from Lemma 17.15. Now assume $m \geq 1$ and suppose that, for all $\ell \in \mathbb{Z}_{0,m-1}$, the following statement holds true:

(I) If $(r_j)_{j=0}^{\ell} \in \mathcal{K}_{q,\ell,\alpha}^{\succcurlyeq,\mathrm{e}}$, then $0_{q\times q} \preccurlyeq S_{-,\ell,\alpha}^{\langle r\rangle}(x) \preccurlyeq S_{+,\ell,\alpha}^{\langle r\rangle}(x)$ and $\mathcal{I}[\alpha, x, (r_j)_{j=0}^{\ell}; \preccurlyeq] = [S_{-,\ell,\alpha}^{\langle r\rangle}(x), S_{+,\ell,\alpha}^{\langle r\rangle}(x)]$.

Denote by $(t_j)_{j=0}^{m-1}$ the first α-S-transform of $(s_j)_{j=0}^{m}$. Lemma 17.4 provides

$$\mathcal{I}[\alpha, x, (s_j)_{j=0}^{m}; \preccurlyeq] = \left\{G^{[-,\alpha,s_0]}(x) \colon G \in \mathcal{S}_{m-1,q,[\alpha,\infty)}[(t_j)_{j=0}^{m-1}, \preccurlyeq]\right\}. \tag{17.2}$$

Let $\eta := (\alpha - x)^{-1}$, let $M := s_0$, and let $\Gamma_{\eta,M} \colon \mathbb{C}^{q\times q} \to \mathbb{C}^{q\times q}$ be defined by $\Gamma_{\eta,M}(Z) := \eta M(Z + M)^{\dagger} M$. From Lemma 17.6(b) we can then conclude that

$$G^{[-,\alpha,s_0]}(x) = \Gamma_{\eta,M}(G(x)) \tag{17.3}$$

holds true for all $G \in \mathcal{S}_{m-1,q,[\alpha,\infty)}[(t_j)_{j=0}^{m-1}, \preccurlyeq]$. In view of (17.2) and Notation 17.1, consequently

$$\begin{aligned}\mathcal{I}[\alpha, x, (s_j)_{j=0}^{m}; \preccurlyeq] &= \left\{\Gamma_{\eta,M}(G(x)) \colon G \in \mathcal{S}_{m-1,q,[\alpha,\infty)}[(t_j)_{j=0}^{m-1}, \preccurlyeq]\right\}\\ &= \left\{\Gamma_{\eta,M}(X) \colon X \in \mathcal{I}[\alpha, x, (t_j)_{j=0}^{m-1}; \preccurlyeq]\right\} = \Gamma_{\eta,M}\left(\mathcal{I}[\alpha, x, (t_j)_{j=0}^{m-1}; \preccurlyeq]\right).\end{aligned} \tag{17.4}$$

Because of $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$ then Theorem 7.4 yields $(t_j)_{j=0}^{m-1} \in \mathcal{K}_{q,m-1,\alpha}^{\succcurlyeq,\mathrm{e}}$. Therefore, we can apply the induction hypothesis (I) to the sequence $(t_j)_{j=0}^{m-1}$ to obtain

$$0_{q\times q} \preccurlyeq S_{-,m-1,\alpha}^{\langle t\rangle}(x) \preccurlyeq S_{+,m-1,\alpha}^{\langle t\rangle}(x) \tag{17.5}$$

and

$$\mathcal{I}[\alpha, x, (t_j)_{j=0}^{m-1}; \preccurlyeq] = \left[S_{-,m-1,\alpha}^{\langle t\rangle}(x), S_{+,m-1,\alpha}^{\langle t\rangle}(x)\right]. \tag{17.6}$$

From parts (a) and (b) of Lemma 17.14 we can conclude that $S_{-,m-1,\alpha}^{\langle t\rangle}$ and $S_{+,m-1,\alpha}^{\langle t\rangle}$ both belong to $\mathcal{S}_{m-1,q,[\alpha,\infty)}[(t_j)_{j=0}^{m-1}, \preccurlyeq]$. Thus, Lemma 17.6(b) and (17.3) provide $(S_{-,m-1,\alpha}^{\langle t\rangle})^{[-,\alpha,s_0]}(x) = \Gamma_{\eta,M}(S_{-,m-1,\alpha}^{\langle t\rangle}(x))$ and $(S_{+,m-1,\alpha}^{\langle t\rangle})^{[-,\alpha,s_0]}(x) = \Gamma_{\eta,M}(S_{+,m-1,\alpha}^{\langle t\rangle}(x))$. Using Lemma 17.12, then

$$\Gamma_{\eta,M}\left(S_{-,m-1,\alpha}^{\langle t\rangle}(x)\right) = S_{+,m,\alpha}(x) \quad \text{and} \quad \Gamma_{\eta,M}\left(S_{+,m-1,\alpha}^{\langle t\rangle}(x)\right) = S_{-,m,\alpha}(x) \tag{17.7}$$

follow. We have $\eta \in (0, \infty)$ and, by virtue of $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$, furthermore $M \in \mathbb{C}_{\succcurlyeq}^{q\times q}$. Let $\mathcal{U} := \mathcal{R}(M)$. By virtue of Lemma 17.6(a) we can conclude then $\mathcal{R}(S_{+,m-1,\alpha}^{\langle t\rangle}(x)) \subseteq \mathcal{U}$. Taking additionally into account (17.5) and (17.7), we can infer from Proposition B.5 then $0_{q\times q} \preccurlyeq S_{-,m,\alpha}(x) \preccurlyeq S_{+,m,\alpha}(x)$ and

$$\Gamma_{\eta,M}\Big(\big[S_{-,m-1,\alpha}^{\langle t\rangle}(x), S_{+,m-1,\alpha}^{\langle t\rangle}(x)\big]\Big) = \big[S_{-,m,\alpha}(x), S_{+,m,\alpha}(x)\big]. \tag{17.8}$$

Consequently, (17.4), (17.6), and (17.8) provide us

$$\begin{aligned}\mathcal{I}[\alpha, x, (s_j)_{j=0}^m; \preccurlyeq] &= \Gamma_{\eta,M}\Big(\mathcal{I}[\alpha, x, (t_j)_{j=0}^{m-1}; \preccurlyeq]\Big)\\ &= \Gamma_{\eta,M}\Big(\big[S_{-,m-1,\alpha}^{\langle t\rangle}(x), S_{+,m-1,\alpha}^{\langle t\rangle}(x)\big]\Big) = \big[S_{-,m,\alpha}(x), S_{+,m,\alpha}(x)\big].\end{aligned}$$

□

The theme of Theorem 17.16 was opened by Yu. M. Dyukarev [7]. For the particular case of a sequence $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,0}^{\succ}$ he proved in [7, Theorem 4(c)] the inclusion

$$\mathcal{I}[0, x, (s_j)_{j=0}^m; \preccurlyeq] \subseteq \big[S_{-,m,0}(x), S_{+,m,0}(x)\big]. \tag{17.9}$$

At the end of [7, Section 3] Yu. M. Dyukarev mentioned without proof that there is even equality in (17.9). Guided by essential hints of Yu. M. Dyukarev, given during his visit at Leipzig University in January 2017, a complete proof of Theorem 17.16 for the particular case of a sequence $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succ}$ was given in the thesis [27] B. Jeschke (see also [28, Satz 4.30]). Following Yu. M. Dyukarev [7, Section 3], who considered the particular case of a sequence $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succ}$, we introduce as a consequence of Theorem 17.16 the following notation.

Definition 17.17 Let $m \in \mathbb{N}_0$, let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$, and let $x \in (-\infty, \alpha)$. Then the closed matricial interval $[S_{-,m,\alpha}(x), S_{+,m,\alpha}(x)]$, where the functions $S_{-,m,\alpha}$ and $S_{+,m,\alpha}$ are given via Notation 17.8, is called the *Weyl interval associated with* $(s_j)_{j=0}^m$ *and* x.

Proposition 17.18 *Let* $\omega, x \in \mathbb{R}$ *with* $x < \omega \leq \alpha$ *and let* $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa,\alpha}^{\succcurlyeq,\mathrm{e}}$. *Then* $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa,\omega}^{\succcurlyeq}$ *and* $S_{-,m,\omega}(x) \preccurlyeq S_{-,m,\alpha}(x) \preccurlyeq S_{+,m,\alpha}(x) \preccurlyeq S_{+,m,\omega}(x)$ *for all* $m \in \mathbb{Z}_{0,\kappa}$.

Proof Consider an arbitrary $m \in \mathbb{Z}_{0,\kappa}$. Using Remark 17.2, we infer $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa,\omega}^{\succcurlyeq,\mathrm{e}}$. From Remark 3.4 we conclude $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$ and $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\omega}^{\succcurlyeq,\mathrm{e}}$. By virtue of Theorem 17.16, then $S_{-,m,\alpha}(x) \preccurlyeq S_{+,m,\alpha}(x)$ and $\mathcal{I}[\alpha, x, (s_j)_{j=0}^m; \preccurlyeq] = [S_{-,m,\alpha}(x), S_{+,m,\alpha}(x)]$ and furthermore

$\mathcal{I}[\omega, x, (s_j)_{j=0}^m; \preccurlyeq] = [S_{-,m,\omega}(x), S_{+,m,\omega}(x)]$. In view of Lemma 17.3, we have $\mathcal{I}[\alpha, x, (s_j)_{j=0}^m; \preccurlyeq] \subseteq \mathcal{I}[\omega, x, (s_j)_{j=0}^m; \preccurlyeq]$. Consequently, $[S_{-,m,\alpha}(x), S_{+,m,\alpha}(x)] \subseteq [S_{-,m,\omega}(x), S_{+,m,\omega}(x)]$. Because of (A.3), then $S_{-,m,\omega}(x) \preccurlyeq S_{-,m,\alpha}(x)$ and $S_{+,m,\alpha}(x) \preccurlyeq S_{+,m,\omega}(x)$. □

18 On the Relation between Consecutive Matricial Weyl Intervals

In this section, we investigate relations between two consecutive closed matricial Weyl intervals, which arise as a consequence of Theorem 17.16. We start with two simple observations.

Remark 18.1 Suppose $\kappa \geq 1$. Let $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa,\alpha}^{\succcurlyeq,\mathrm{e}}$, let $x \in (-\infty, \alpha)$, and let $m \in \mathbb{Z}_{1,\kappa}$. Then Notation 17.1 shows that $\mathcal{I}[\alpha, x, (s_j)_{j=0}^m; \preccurlyeq] \subseteq \mathcal{I}[\alpha, x, (s_j)_{j=0}^{m-1}; \preccurlyeq]$.

Remark 18.2 Let $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa,\alpha}^{\succcurlyeq,\mathrm{e}}$ and let $m \in \mathbb{Z}_{0,\kappa}$. Then Theorem 17.16 shows that $S_{-,m,\alpha}(x), S_{+,m,\alpha}(x) \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ for all $x \in (-\infty, \alpha)$.

Combining Theorem 17.16, Remark 18.1, and Notation 17.8, we see that the endpoints of the closed intervals are given in alternating way by the values $\underline{S}_m(x)$ and $\overline{S}_m(x)$. The next result describes more precisely the relation between the endpoints of two consecutive intervals.

Proposition 18.3 *Suppose $\kappa \geq 1$. Let $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa,\alpha}^{\succcurlyeq,\mathrm{e}}$ and let $x \in (-\infty, \alpha)$. In view of Notation 17.8, then $S_{-,m-1,\alpha}(x) \preccurlyeq S_{-,m,\alpha}(x) \preccurlyeq S_{+,m,\alpha}(x) \preccurlyeq S_{+,m-1,\alpha}(x)$ for all $m \in \mathbb{Z}_{1,\kappa}$.*

Proof Consider an arbitrary $m \in \mathbb{Z}_{1,\kappa}$. Using Remark 3.4, we can infer $(s_j)_{j=0}^{m-1} \in \mathcal{K}_{q,m-1,\alpha}^{\succcurlyeq,\mathrm{e}}$ and $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$. By virtue of Theorem 17.16, then $\mathcal{I}[\alpha, x, (s_j)_{j=0}^{m-1}; \preccurlyeq] = [S_{-,m-1,\alpha}(x), S_{+,m-1,\alpha}(x)]$ and, furthermore, $S_{-,m,\alpha}(x) \preccurlyeq S_{+,m,\alpha}(x)$ and $\mathcal{I}[\alpha, x, (s_j)_{j=0}^m; \preccurlyeq] = [S_{-,m,\alpha}(x), S_{+,m,\alpha}(x)]$. From Remark 18.1 we infer $\mathcal{I}[\alpha, x, (s_j)_{j=0}^m; \preccurlyeq] \subseteq \mathcal{I}[\alpha, x, (s_j)_{j=0}^{m-1}; \preccurlyeq]$. Consequently, $[S_{-,m,\alpha}(x), S_{+,m,\alpha}(x)] \subseteq [S_{-,m-1,\alpha}(x), S_{+,m-1,\alpha}(x)]$. Because of (A.3), then $S_{-,m-1,\alpha}(x) \preccurlyeq S_{-,m,\alpha}(x)$ and $S_{+,m,\alpha}(x) \preccurlyeq S_{+,m-1,\alpha}(x)$ follow. □

Proposition 18.3 leads us to the consideration of the following functions.

Notation 18.4 Let $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa,\alpha}^{\succcurlyeq,\mathrm{e}}$. Then let $\mathfrak{l}_m := S_{+,m,\alpha} - S_{-,m,\alpha}$ for all $m \in \mathbb{Z}_{0,\kappa}$ and, in the case $\kappa \geq 1$, furthermore $\mathfrak{s}_{-,m} := S_{-,m,\alpha} - S_{-,m-1,\alpha}$ and $\mathfrak{s}_{+,m} := S_{+,m-1,\alpha} - S_{+,m,\alpha}$ for all $m \in \mathbb{Z}_{1,\kappa}$.

Against to the background of Theorem 17.16 and Proposition 18.3 it becomes clear that the behavior of the functions introduced in Notation 18.4 on the interval $(-\infty, \alpha)$ is of particular interest.

Remark 18.5 If $\kappa \geq 1$ and $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\succcurlyeq,\mathrm{e}}$, then from Notation 18.4 we immediately see that $\mathfrak{l}_m - \mathfrak{l}_{m+1} = \mathfrak{s}_{-,m+1} + \mathfrak{s}_{+,m+1}$ for all $m \in \mathbb{Z}_{0,\kappa-1}$.

From Notations 18.4 and 17.8, Corollary 16.2, and Lemma 17.9 we immediately see:

Remark 18.6 Let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\succcurlyeq,\mathrm{e}}$ and let $z \in \mathbb{C} \setminus [\alpha, \infty)$. Then:

(a) $\mathfrak{l}_m = (-1)^m(\overline{S}_m - \underline{S}_m)$ for all $m \in \mathbb{Z}_{0,\kappa}$.
(b) $\mathfrak{l}_m - \mathfrak{l}_{m+1} = (-1)^m(\overline{S}_{m+1} - \underline{S}_m)$ for all $m \in \mathbb{Z}_{0,\kappa-1}$.
(c) $\mathfrak{s}_{-,2n+1} = \overline{S}_{2n+1} - \underline{S}_{2n}$ and $\mathfrak{s}_{+,2n+1}(z) = 0_{q\times q}$ for all $n \in \mathbb{N}_0$ with $2n+1 \preccurlyeq \kappa$.
(d) $\mathfrak{s}_{-,2n}(z) = 0_{q\times q}$ and $\mathfrak{s}_{+,2n} = \underline{S}_{2n-1} - \overline{S}_{2n}$ for all $n \in \mathbb{N}$ with $2n \preccurlyeq \kappa$

Remark 18.7 Suppose $\kappa \geq 1$. Let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\succcurlyeq,\mathrm{e}}$. Then Remarks 18.5 and 18.6 show that $\mathfrak{l}_{2n-1} - \mathfrak{l}_{2n} = \mathfrak{s}_{+,2n}$ for all $n \in \mathbb{N}$ with $2n \preccurlyeq \kappa$ and $\mathfrak{l}_{2n} - \mathfrak{l}_{2n+1} = \mathfrak{s}_{-,2n+1}$ for all $n \in \mathbb{N}_0$ with $2n + 1 \leq \kappa$.

In our next considerations the $q \times q$ matrix polynomials introduced in Notation 14.1 will play an essential role.

Lemma 18.8 *Let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\succcurlyeq,\mathrm{e}}$ with right α-Stieltjes parametrization $(Q_j)_{j=0}^{\kappa}$. Using Notation 14.1 for all $m, \ell \in \mathbb{N}_0$ with $m + \ell \leq \kappa + 1$, let $A_{m,\ell}\colon \mathbb{C} \to \mathbb{C}^{q\times q}$ be defined by*

$$A_{m,\ell}(z) := [\dot{\mathbf{p}}_m(\overline{z})]^*[\dot{\mathbf{q}}_{m+\ell}(z)] - [\dot{\mathbf{q}}_m(\overline{z})]^*[\dot{\mathbf{p}}_{m+\ell}(z)].$$

Then:

(a) *$A_{m,0}(z) = 0_{q\times q}$ for each $m \in \mathbb{Z}_{0,\kappa+1}$ and all $z \in \mathbb{C}$.*
(b) *$A_{m,1}(z) = Q_m$ for each $m \in \mathbb{Z}_{0,\kappa}$ and all $z \in \mathbb{C}$.*
(c) *$A_{m,\ell} = \epsilon_{m+\ell-1}A_{m,\ell-1} - A_{m,\ell-2}Q_{m+\ell-2}^{\dagger}Q_{m+\ell-1}$ for all $m, \ell \in \mathbb{N}_0$ with $2 \leq m + \ell \leq \kappa + 1$, where $\epsilon_{m+\ell-1}$ is given via (14.1).*

Proof

(a) Let $m \in \mathbb{Z}_{0,\kappa+1}$. Proposition 16.1 and Corollary 16.3 yield $-[\dot{\mathbf{q}}_m(z)][\dot{\mathbf{p}}_m(z)]^{-1} = -[\dot{\mathbf{p}}_m(\overline{z})]^{-*}[\dot{\mathbf{q}}_m(\overline{z})]^*$ for all $z \in \mathbb{C}\setminus[\alpha, \infty)$. Consequently, $[\dot{\mathbf{p}}_m(\overline{z})]^*[\dot{\mathbf{q}}_m(z)] = [\dot{\mathbf{q}}_m(\overline{z})]^*[\dot{\mathbf{p}}_m(z)]$ for all $z \in \mathbb{C} \setminus [\alpha, \infty)$. Since $\dot{\mathbf{p}}_m$ and $\dot{\mathbf{q}}_m$ are complex matrix polynomials, the last equality necessarily holds true for all $z \in \mathbb{C}$. Thus, $A_{m,0}(z) = 0_{q\times q}$ for all $z \in \mathbb{C}$.

(b) For each $m \in \mathbb{Z}_{0,\kappa}$, let $B_m := A_{m,1}$. Taking into account Remark 14.2 and part (a), we obtain the recurrence relation

$$\begin{aligned} B_m(z) = A_{m,1}(z) &= [\dot{\mathbf{p}}_m(\overline{z})]^*[\dot{\mathbf{q}}_{m+1}(z)] - [\dot{\mathbf{q}}_m(\overline{z})]^*[\dot{\mathbf{p}}_{m+1}(z)] \\ &= [\dot{\mathbf{p}}_m(\overline{z})]^*\Big[\epsilon_m(z)\dot{\mathbf{q}}_m(z) - \dot{\mathbf{q}}_{m-1}(z)Q_{m-1}^\dagger Q_m\Big] \\ &\quad - [\dot{\mathbf{q}}_m(\overline{z})]^*\Big[\epsilon_m(z)\dot{\mathbf{p}}_m(z) - \dot{\mathbf{p}}_{m-1}(z)Q_{m-1}^\dagger Q_m\Big] \\ &= \epsilon_m(z)\big([\dot{\mathbf{p}}_m(\overline{z})]^*[\dot{\mathbf{q}}_m(z)] - [\dot{\mathbf{q}}_m(\overline{z})]^*[\dot{\mathbf{p}}_m(z)]\big) \\ &\quad + \big([\dot{\mathbf{q}}_m(\overline{z})]^*[\dot{\mathbf{p}}_{m-1}(z)] - [\dot{\mathbf{p}}_m(\overline{z})]^*[\dot{\mathbf{q}}_{m-1}(z)]\big)Q_{m-1}^\dagger Q_m \\ &= \epsilon_m(z)A_{m,0}(z) + \big[A_{m-1,1}(\overline{z})\big]^* Q_{m-1}^\dagger Q_m \\ &= \epsilon_m(z)\cdot 0_{q\times q} + [B_{m-1}(\overline{z})]^* Q_{m-1}^\dagger Q_m = [B_{m-1}(\overline{z})]^* Q_{m-1}^\dagger Q_m \end{aligned}$$

for each $m \in \mathbb{Z}_{1,\kappa}$ and all $z \in \mathbb{C}$. Using Remark A.4, we can infer from Theorem 5.6(b) that $Q_j^* Q_j^\dagger Q_{j+1} = Q_{j+1}$ holds true for all $j \in \mathbb{Z}_{0,\kappa-1}$. We proceed by mathematical induction. From Notation 14.1 we immediately see $B_0(z) = Q_0$ for all $z \in \mathbb{C}$. Now assume $\kappa \geq 1$ and suppose that $B_k(w) = Q_k$ holds true for some $k \in \mathbb{Z}_{0,\kappa-1}$ and all $w \in \mathbb{C}$. For all $z \in \mathbb{C}$, we then obtain

$$B_{k+1}(z) = [B_k(\overline{z})]^* Q_k^\dagger Q_{k+1} = Q_k^* Q_k^\dagger Q_{k+1} = Q_{k+1}.$$

Consequently, we have checked by mathematical induction that $B_m(z) = Q_m$ holds true for each $m \in \mathbb{Z}_{0,\kappa}$ and all $z \in \mathbb{C}$. Thus, part (b) is proved.

(c) Suppose $\kappa \geq 1$. Consider arbitrary $m, \ell \in \mathbb{N}_0$ satisfying $2 \leq m + \ell \leq \kappa + 1$. Using Remark 14.2, for all $z \in \mathbb{C}$, we infer

$$\begin{aligned} A_{m,\ell}(z) &= [\dot{\mathbf{p}}_m(\overline{z})]^*[\dot{\mathbf{q}}_{m+\ell}(z)] - [\dot{\mathbf{q}}_m(\overline{z})]^*[\dot{\mathbf{p}}_{m+\ell}(z)] \\ &= [\dot{\mathbf{p}}_m(\overline{z})]^*\Big[\epsilon_{m+\ell-1}(z)\dot{\mathbf{q}}_{m+\ell-1}(z) - \dot{\mathbf{q}}_{m+\ell-2}(z)Q_{m+\ell-2}^\dagger Q_{m+\ell-1}\Big] \\ &\quad - [\dot{\mathbf{q}}_m(\overline{z})]^*\Big[\epsilon_{m+\ell-1}(z)\dot{\mathbf{p}}_{m+\ell-1}(z) - \dot{\mathbf{p}}_{m+\ell-2}(z)Q_{m+\ell-2}^\dagger Q_{m+\ell-1}\Big] \\ &= \epsilon_{m+\ell-1}(z)\big([\dot{\mathbf{p}}_m(\overline{z})]^*[\dot{\mathbf{q}}_{m+\ell-1}(z)] - [\dot{\mathbf{q}}_m(\overline{z})]^*[\dot{\mathbf{p}}_{m+\ell-1}(z)]\big) \\ &\quad - \big([\dot{\mathbf{p}}_m(\overline{z})]^*[\dot{\mathbf{q}}_{m+\ell-2}(z)] - [\dot{\mathbf{q}}_m(\overline{z})]^*[\dot{\mathbf{p}}_{m+\ell-2}(z)]\big)Q_{m+\ell-2}^\dagger Q_{m+\ell-1} \\ &= \epsilon_{m+\ell-1}(z)A_{m,\ell-1}(z) - A_{m,\ell-2}(z)Q_{m+\ell-2}^\dagger Q_{m+\ell-1}. \end{aligned}$$

□

Lemma 18.9 *Let* $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa,\alpha}^{\succcurlyeq,\mathrm{e}}$ *with right* α*-Stieltjes parametrization* $(Q_j)_{j=0}^\kappa$ *and let* $z \in \mathbb{C}$. *Then:*

(a) $[\dot{\mathbf{p}}_m(\overline{z})]^*[\dot{\mathbf{q}}_m(z)] - [\dot{\mathbf{q}}_m(\overline{z})]^*[\dot{\mathbf{p}}_m(z)] = 0_{q\times q}$ *for each* $m \in \mathbb{Z}_{0,\kappa+1}$.

(b) $[\dot{\mathbf{p}}_m(\overline{z})]^*[\dot{\mathbf{q}}_{m+1}(z)] - [\dot{\mathbf{q}}_m(\overline{z})]^*[\dot{\mathbf{p}}_{m+1}(z)] = Q_m$ *for each* $m \in \mathbb{Z}_{0,\kappa}$.
(c) $[\dot{\mathbf{p}}_m(\overline{z})]^*[\dot{\mathbf{q}}_{m+2}(z)] - [\dot{\mathbf{q}}_m(\overline{z})]^*[\dot{\mathbf{p}}_{m+2}(z)] = \epsilon_{m+1}(z)Q_m$ *for each* $m \in \mathbb{Z}_{0,\kappa-1}$.

Proof Use Lemma 18.8. □

Lemma 18.10 *Let* $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\succcurlyeq,\mathrm{e}}$ *with right* α*-Stieltjes parametrization* $(Q_j)_{j=0}^{\kappa}$ *and let* $z \in \mathbb{C} \setminus [\alpha, \infty)$. *Then:*

(a) $\underline{S}_m(z) - \overline{S}_m(z) = [\dot{\mathbf{p}}_m(\overline{z})]^{-*}Q_m[\dot{\mathbf{p}}_{m+1}(z)]^{-1}$ *for all* $m \in \mathbb{Z}_{0,\kappa}$.
(b) $\underline{S}_m(z) - \overline{S}_{m+1}(z) = \epsilon_{m+1}(z)[\dot{\mathbf{p}}_m(\overline{z})]^{-*}Q_m[\dot{\mathbf{p}}_{m+2}(z)]^{-1}$ *for all* $m \in \mathbb{Z}_{0,\kappa-1}$.

Proof Use parts (b) and (c) of Lemma 18.9, Corollary 16.3(a), and Proposition 16.1(b). □

Lemma 18.10 leads us now to an expression for the length of the matricial Weyl interval in terms of the $q \times q$ matrix polynomials introduced in Notation 14.1.

Proposition 18.11 *Let* $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\succcurlyeq,\mathrm{e}}$ *with right* α*-Stieltjes parametrization* $(Q_j)_{j=0}^{\kappa}$ *and let* $m \in \mathbb{Z}_{0,\kappa}$. *Then:*

(a) $\mathfrak{l}_m(x) \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ *for all* $x \in (-\infty, \alpha)$.
(b) $\mathfrak{l}_m(z) = (-1)^{m+1}[\dot{\mathbf{p}}_m(\overline{z})]^{-*}Q_m[\dot{\mathbf{p}}_{m+1}(z)]^{-1}$ *for all* $z \in \mathbb{C} \setminus [\alpha, \infty)$.
(c) *The following statements are equivalent:*
  (i) $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succ}$.
  (ii) $Q_m \in \mathbb{C}_{\succ}^{q\times q}$.
  (iii) $\det Q_m \neq 0$.
  (iv) $\det \mathfrak{l}_m(z) \neq 0$ *for all* $z \in \mathbb{C} \setminus [\alpha, \infty)$.
  (v) *There exists some* $z_0 \in \mathbb{C} \setminus [\alpha, \infty)$ *with* $\det \mathfrak{l}_m(z_0) \neq 0$.
  (vi) $\mathfrak{l}_m(x) \in \mathbb{C}_{\succ}^{q\times q}$ *for all* $x \in (-\infty, \alpha)$.
  (vii) *There exists some* $x_0 \in (-\infty, \alpha)$ *with* $\mathfrak{l}_m(x_0) \in \mathbb{C}_{\succ}^{q\times q}$.

Proof

(a) In view of Notation 18.4, this is a consequence of Theorem 17.16.
(b) Combine Remark 18.6(a) and Lemma 18.10(a).
(c) The equivalence of (i)–(iii) is an immediate consequence of Corollary 5.7. In view of part (b) we see that (iii) implies (iv). The implication "(iv)⇒(v)" is trivial. From (v) and part (b) it follows (iii). Thus we have shown that (i)–(v) are equivalent. In view of part (a) we see that (iv) implies (vi). The implication "(vi)⇒(vii)" is trivial. If we assume (vii) then $\det \mathfrak{l}_m(x_0) \neq 0$, which implies (iv). Thus part (c) is proved.

□

Part (c) of Proposition 18.11 determines our next aim. Assuming $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succ}$ we are going to derive an explicit expression for the function $\mathfrak{l}_m^{-1}$.

Notation 18.12 Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices with right α-Stieltjes parametrization $(Q_j)_{j=0}^{\kappa}$ and let $m \in \mathbb{Z}_{0,\kappa}$. Then let $\mathfrak{L}_m : \mathbb{C} \to \mathbb{C}^{q \times q}$ be defined by $\mathfrak{L}_m(z) := (-1)^{m+1}[\dot{\mathbf{p}}_{m+1}(z)]Q_m^{\dagger}[\dot{\mathbf{p}}_m(\overline{z})]^*$.

From Remark 14.4 we easily see:

Remark 18.13 Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices with right α-Stieltjes parametrization $(Q_j)_{j=0}^{\kappa}$ and let $m \in \mathbb{Z}_{0,\kappa}$. Then $\mathfrak{L}_m$ is a complex $q \times q$ matrix polynomial with leading coefficient $(-1)^{m+1}Q_m^{\dagger}$ and $\deg \mathfrak{L}_m \leq m+1$ satisfying $\mathfrak{L}_m(\alpha) = 0_{q \times q}$.

Example 18.14 Let $s_0 \in \mathbb{C}^{q \times q}$. In view of Notation 14.1, (14.1), and (5.1), for all $z \in \mathbb{C}$ then

$$\mathfrak{L}_0(z) = (-1)^1 \epsilon_0(z)[\dot{\mathbf{p}}_0(z)]Q_0^{\dagger}[\dot{\mathbf{p}}_0(\overline{z})]^* = (\alpha - z)s_0^{\dagger}.$$

Using Proposition 18.11, we obtain:

Remark 18.15 Let $m \in \mathbb{N}_0$, let $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succ}$ and let $z \in \mathbb{C} \setminus [\alpha, \infty)$. Then $\det \mathfrak{L}_m(z) \neq 0$ and $[\mathfrak{L}_m(z)]^{-1} = \mathfrak{l}_m(z)$.

Lemma 18.16 *Let $m \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succ}$ with right α-Stieltjes parametrization $(Q_j)_{j=0}^{m}$. For all $z \in \mathbb{C}$, then* $\det Q_m \neq 0$ *and* $\mathfrak{L}_m(z) = (-1)^{m+1}[\dot{\mathbf{p}}_m(z)]Q_m^{-1}[\dot{\mathbf{p}}_{m+1}(\overline{z})]^*$.

Proof Theorem 5.6(c) yields $\det Q_m \neq 0$. Let $\mathfrak{R}_m : \mathbb{C} \to \mathbb{C}^{q \times q}$ be defined by $\mathfrak{R}_m(z) := (-1)^{m+1}[\dot{\mathbf{p}}_m(z)]Q_m^{-1}[\dot{\mathbf{p}}_{m+1}(\overline{z})]^*$. From Remark 14.4 we infer then that $\mathfrak{R}_m$ is a complex $q \times q$ Matrix polynomial. Using Proposition 18.11(c) we see for all $x \in (-\infty, \alpha)$ that $\mathfrak{l}_m(x)$ is invertible and Hermitian with

$$\begin{aligned}
[\mathfrak{l}_m(x)]^* &= (-1)^{m+1}[\dot{\mathbf{p}}_{m+1}(x)]^{-*}Q_m[\dot{\mathbf{p}}_m(x)]^{-1} \\
&= \left((-1)^{m+1}[\dot{\mathbf{p}}_m(x)]Q_m^{-1}[\dot{\mathbf{p}}_{m+1}(x)]^*\right)^{-1},
\end{aligned}$$

implying $[\mathfrak{l}_m(x)]^{-1} = \mathfrak{R}_m(x)$. In view of Remark 18.15, then $\mathfrak{R}_m(x) = \mathfrak{L}_m(x)$ follows for all $x \in (-\infty, \alpha)$. Taking additionally into account Remark 18.13, we can therefore infer $\mathfrak{L}_m = \mathfrak{R}_m$. □

Lemma 18.17 *Let $m \in \mathbb{N}$ and let $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succ}$ with right α-Stieltjes parametrization $(Q_j)_{j=0}^{m}$. For all $z \in \mathbb{C}$, then $\mathfrak{L}_m(z) - \mathfrak{L}_{m-1}(z) = (-1)^{m+1}\epsilon_m(z)[\dot{\mathbf{p}}_m(z)]Q_m^{-1}[\dot{\mathbf{p}}_m(\overline{z})]^*$.*

Proof According to Theorem 5.6(c) the matrices Q_m and Q_{m-1} are invertible. Consider an arbitrary $z \in \mathbb{C}$. Using Notation 18.12, Lemma 18.16, and Remark 14.2, we obtain

$$\begin{aligned}
&\mathfrak{L}_m(z) - \mathfrak{L}_{m-1}(z) \\
&= (-1)^{m+1}[\dot{\mathbf{p}}_{m+1}(z)]Q_m^{-1}[\dot{\mathbf{p}}_m(\overline{z})]^* - (-1)^m[\dot{\mathbf{p}}_{m-1}(z)]Q_{m-1}^{-1}[\dot{\mathbf{p}}_m(\overline{z})]^* \\
&= (-1)^{m+1}\Big([\dot{\mathbf{p}}_{m+1}(z)] + [\dot{\mathbf{p}}_{m-1}(z)]Q_{m-1}^{-1}Q_m\Big)Q_m^{-1}[\dot{\mathbf{p}}_m(\overline{z})]^* \\
&= (-1)^{m+1}\epsilon_m(z)[\dot{\mathbf{p}}_m(z)]Q_m^{-1}[\dot{\mathbf{p}}_m(\overline{z})]^*.
\end{aligned}$$

□

Example 18.18 Let $s_0 \in \mathbb{C}_{\succ}^{q\times q}$. In view of Lemma 18.17, (14.1), Notation 14.1, and (5.1), then $\mathfrak{L}_1(z) - \mathfrak{L}_0(z) = (z-\alpha)^2 s_{\alpha\triangleright 0}^{-1}$ for all $z \in \mathbb{C}$.

Proposition 18.19 *Let $m \in \mathbb{N}_0$ and let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succ}$ with right α-Stieltjes parametrization $(Q_j)_{j=0}^m$. For all $z \in \mathbb{C}$, then $\mathfrak{L}_m(z) = \sum_{k=0}^m (-1)^{k+1}\epsilon_k(z)[\dot{\mathbf{p}}_k(z)]$ $Q_k^{-1}[\dot{\mathbf{p}}_k(\overline{z})]^*$.*

Proof Consider an arbitrary $z \in \mathbb{C}$. Clearly

$$\mathfrak{L}_m(z) = \sum_{k=1}^m [\mathfrak{L}_k(z) - \mathfrak{L}_{k-1}(z)] + \mathfrak{L}_0(z).$$

Now the application of Lemma 18.17 to the terms in the first sum and Example 18.14 to $\mathfrak{L}_0(z)$ completes the proof. □

Corollary 18.20 *Let $m \in \mathbb{N}_0$ and let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\succ}$ with right α-Stieltjes parametrization $(Q_j)_{j=0}^m$. For all $z \in \mathbb{C} \setminus [\alpha, \infty)$, then $\mathfrak{l}_m(z) = (\sum_{k=0}^m (-1)^{k+1}\epsilon_k(z)[\dot{\mathbf{p}}_k(z)]Q_k^{-1}[\dot{\mathbf{p}}_k(\overline{z})]^*)^{-1}$.*

Proof In view of Remark 18.15, this is a consequence of Proposition 18.19. □

Corollary 18.21 *Let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\succ}$ and let $z \in \mathbb{C}$. Then:*

(a) $\mathfrak{L}_0(z) = -(z-\alpha)[\dot{\mathbf{r}}_0(z)]L_0^{-1}[\dot{\mathbf{r}}_0(\overline{z})]^*$.
(b) *Let $n \in \mathbb{N}$ with $2n \le \kappa$. Then*

$$\mathfrak{L}_{2n}(z) = (z-\alpha)^2 \sum_{k=0}^{n-1}\big[\dot{\mathbf{t}}_k(z)\big]L_{\alpha\triangleright k}^{-1}\big[\dot{\mathbf{t}}_k(\overline{z})\big]^* - (z-\alpha)\sum_{k=0}^{n}[\dot{\mathbf{r}}_k(z)]L_k^{-1}[\dot{\mathbf{r}}_k(\overline{z})]^*.$$

(c) *Let* $n \in \mathbb{N}_0$ *with* $2n+1 \leq \kappa$. *Then*

$$\mathfrak{L}_{2n+1}(z) = (z-\alpha)^2 \sum_{k=0}^{n} \big[\dot{\mathbf{t}}_k(z)\big] L_{\alpha\triangleright k}^{-1} \big[\dot{\mathbf{t}}_k(\overline{z})\big]^* - (z-\alpha) \sum_{k=0}^{n} [\dot{\mathbf{r}}_k(z)] L_k^{-1} [\dot{\mathbf{r}}_k(\overline{z})]^*.$$

Proof In view of (14.1), Notation 14.5, and Definition 5.1, this is a consequence of Proposition 18.19. □

Lemma 18.22 *Let* $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\succ}$ *and let* $z \in \mathbb{C}$. *Using Notation D.2, then:*

(a) $[\dot{\mathbf{r}}_0(z)] L_0^{-1} [\dot{\mathbf{r}}_0(\overline{z})]^* = [E_0(z)] H_0^{-1} [E_0(\overline{z})]^*$ *and*

$$[\dot{\mathbf{r}}_n(z)] L_n^{-1} [\dot{\mathbf{r}}_n(\overline{z})]^* = [E_n(z)] H_n^{-1} [E_n(\overline{z})]^* - [E_{n-1}(z)] H_{n-1}^{-1} [E_{n-1}(\overline{z})]^*$$

for all $n \in \mathbb{N}$ *with* $2n \leq \kappa$.

(b) *If* $\kappa \geq 1$, *then* $[\dot{\mathbf{t}}_0(z)] L_{\alpha\triangleright 0}^{-1} [\dot{\mathbf{t}}_0(\overline{z})]^* = [E_0(z)] H_{\alpha\triangleright 0}^{-1} [E_0(\overline{z})]^*$ *and*

$$\big[\dot{\mathbf{t}}_n(z)\big] L_{\alpha\triangleright n}^{-1} \big[\dot{\mathbf{t}}_n(\overline{z})\big]^* = [E_n(z)] H_{\alpha\triangleright n}^{-1} [E_n(\overline{z})]^* - [E_{n-1}(z)] H_{\alpha\triangleright n-1}^{-1} [E_{n-1}(\overline{z})]^*$$

for all $n \in \mathbb{N}$ *with* $2n+1 \leq \kappa$.

Proof In view of $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\succ}$ we have $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succ}$ for all $m \in \mathbb{Z}_{0,\kappa}$ and the matrices H_n and $H_{\alpha\triangleright n}$ are invertible for all $n \in \mathbb{N}_0$ with $2n \leq \kappa$ and $2n+1 \leq \kappa$, respectively.

(a) Because of Notations 14.5, 14.1 and D.2, we have $\dot{\mathbf{r}}_0(w) = \dot{\mathbf{p}}_0(w) = I_q = E_0(w)$ for all $w \in \mathbb{C}$. By virtue of (4.4) and (4.2), we see furthermore $L_0 = s_0 = H_0$. Hence, $[\dot{\mathbf{r}}_0(z)] L_0^{-1} [\dot{\mathbf{r}}_0(\overline{z})]^* = [E_0(z)] H_0^{-1} [E_0(\overline{z})]^*$ follows. Now suppose $\kappa \geq 2$ and consider an arbitrary $n \in \mathbb{N}$ with $2n \leq \kappa$. In view of Remarks 4.1 and 4.2, the invertibility of H_n and H_{n-1}, and a well-known formula for the inverse of a block matrix (see, e. g. [5, Lemma 1.1.7(a)]), the Schur complement L_n is invertible and

$$H_n^{-1} = \operatorname{diag}(H_{n-1}^{-1}, 0_{q\times q}) + \begin{bmatrix} -H_{n-1}^{-1} y_{n,2n-1} \\ I_q \end{bmatrix} L_n^{-1} [-z_{n,2n-1} H_{n-1}^{-1}, I_q]. \tag{18.1}$$

In view of Remarks 14.3 and 14.5, we can conclude from Proposition 14.8 that $(\dot{\mathbf{r}}_k)_{k=0}^{n}$ is a monic right orthogonal system with respect to $(s_j)_{j=0}^{2n}$. Taking additionally into account Proposition D.5, the invertibility of H_{n-1}, and Remark D.3, then $\dot{\mathbf{r}}_n(z) = E_n(z) \left[\begin{smallmatrix} -H_{n-1}^{-1} y_{n,2n-1} \\ I_q \end{smallmatrix}\right]$ follows. In particular, $[\dot{\mathbf{r}}_n(\overline{z})]^* = [-z_{n,2n-1} H_{n-1}^{-1}, I_q][E_n(\overline{z})]^*$, by virtue of Lemma 3.2. Multiplying (18.1) with $E_n(z)$ from the left and with $[E_n(\overline{z})]^*$ from the right and taking into account Notation D.2, we thus obtain $[E_n(z)]H_n^{-1}[E_n(\overline{z})]^* = [E_{n-1}(z)]H_{n-1}^{-1}[E_{n-1}(\overline{z})]^* + [\dot{\mathbf{r}}_n(z)]L_n^{-1}[\dot{\mathbf{r}}_n(\overline{z})]^*$.

(b) Suppose $\kappa \geq 1$. Because of Notations 14.5, 14.1 and D.2, we have $\dot{\mathbf{t}}_0(w) = I_q = E_0(w)$ for all $w \in \mathbb{C}$. By virtue of (4.7), (4.4), and (4.2), we see furthermore $L_{\alpha\triangleright 0} = s_{\alpha\triangleright 0} = H_{\alpha\triangleright 0}$. Hence, $[\dot{\mathbf{t}}_0(z)]L_{\alpha\triangleright 0}^{-1}[\dot{\mathbf{t}}_0(\overline{z})]^* = [E_0(z)]H_{\alpha\triangleright 0}^{-1}[E_0(\overline{z})]^*$ follows. Now suppose $\kappa \geq 3$ and consider an arbitrary $n \in \mathbb{N}$ with $2n+1 \leq \kappa$. In view of (4.7), Remarks 4.1 and 4.2, the invertibility of $H_{\alpha\triangleright n}$ and $H_{\alpha\triangleright n-1}$, and a well-known formula for the inverse of a block matrix (see, e. g. [5, Lemma 1.1.7(a)]), the Schur complement $L_{\alpha\triangleright n}$ is invertible and

$$H_{\alpha\triangleright n}^{-1} = \operatorname{diag}(H_{\alpha\triangleright n-1}^{-1}, 0_{q\times q}) + \begin{bmatrix} -H_{\alpha\triangleright n-1}^{-1} y_{\alpha\triangleright n,2n-1} \\ I_q \end{bmatrix} L_{\alpha\triangleright n}^{-1}[-z_{\alpha\triangleright n,2n-1}H_{\alpha\triangleright n-1}^{-1}, I_q]. \tag{18.2}$$

In view of Remarks 14.3 and 14.5, we can conclude from Proposition 14.7 that $(\dot{\mathbf{t}}_k)_{k=0}^n$ is a monic right orthogonal system with respect to $(s_{\alpha\triangleright j})_{j=0}^{2n}$. Taking additionally into account Proposition D.5, the invertibility of $H_{\alpha\triangleright n-1}$, and Remark D.3, then $\dot{\mathbf{t}}_n(z) = E_n(z)\left[\begin{smallmatrix} -H_{\alpha\triangleright n-1}^{-1} y_{\alpha\triangleright n,2n-1} \\ I_q \end{smallmatrix}\right]$ follows. In particular, $[\dot{\mathbf{t}}_n(\overline{z})]^* = [-z_{\alpha\triangleright n,2n-1}H_{\alpha\triangleright n-1}^{-1}, I_q][E_n(\overline{z})]^*$, by virtue of Lemma 3.2. Multiplying (18.2) with $E_n(z)$ from the left and with $[E_n(\overline{z})]^*$ from the right and taking into account Notation D.2, we thus obtain $[E_n(z)]H_{\alpha\triangleright n}^{-1}[E_n(\overline{z})]^* = [E_{n-1}(z)]H_{\alpha\triangleright n-1}^{-1}[E_{n-1}(\overline{z})]^* + [\dot{\mathbf{t}}_n(z)]L_{\alpha\triangleright n}^{-1}[\dot{\mathbf{t}}_n(\overline{z})]^*$.

□

Now we are able to formulate and prove a result due to Yu. M. Dyukarev [7, Theorem 4] in the case $\alpha = 0$ and B. Jeschke [28, Satz 4.27(c)] for arbitrary $\alpha \in \mathbb{R}$ who proved it in a different way.

Proposition 18.23 *Let $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa,\alpha}^{\succ}$ and let $z \in \mathbb{C}$. Using Notation D.2, then:*

(a) $\mathfrak{L}_0(z) = -(z-\alpha)[E_0(z)]H_0^{-1}[E_0(\overline{z})]^*$.

(b) *Let $n \in \mathbb{N}$ with $2n \leq \kappa$. Then*

$$\mathfrak{L}_{2n}(z) = (z-\alpha)^2[E_{n-1}(z)]H_{\alpha\triangleright n-1}^{-1}[E_{n-1}(\overline{z})]^* - (z-\alpha)[E_n(z)]H_n^{-1}[E_n(\overline{z})]^*.$$

(c) *Let $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$. Then*

$$\mathfrak{L}_{2n+1}(z) = (z-\alpha)^2[E_n(z)]H_{\alpha\triangleright n}^{-1}[E_n(\overline{z})]^* - (z-\alpha)[E_n(z)]H_n^{-1}[E_n(\overline{z})]^*.$$

Proof Combine Corollary 18.21 and Lemma 18.22. □

It should be mentioned that in the particular case $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,0}^{\succ}$ Yu. M. Dyukarev [7, Theorem 4] obtained useful algebraic expressions for the rational $q \times q$ matrix-valued function $\mathfrak{l}_m^{-1}$. This result was generalized by B. Jeschke [28, Satz 4.27(c)] (see also [27]) to sequences $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succ}$ with arbitrary real α.

Lemma 18.24 *Suppose* $\kappa \geq 1$. *Let* $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\succcurlyeq,\mathrm{e}}$ *with right* α*-Stieltjes parametrization* $(Q_j)_{j=0}^{\kappa}$. *Then:*

(a) $\mathfrak{s}_{-,m}(x) \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ *and* $\mathfrak{s}_{+,m}(x) \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ *for all* $x \in (-\infty, \alpha)$ *and all* $m \in \mathbb{Z}_{1,\kappa}$.
(b) *Let* $n \in \mathbb{N}_0$ *with* $2n+1 \preccurlyeq \kappa$ *and let* $z \in \mathbb{C} \setminus [\alpha, \infty)$. *Then* $\mathfrak{s}_{+,2n+1}(z) = 0_{q\times q}$ *and*

$$\mathfrak{s}_{-,2n+1}(z) = -[\dot{\mathbf{p}}_{2n}(\overline{z})]^{-*} Q_{2n} [\dot{\mathbf{p}}_{2n+2}(z)]^{-1} = -[\dot{\mathbf{r}}_n(\overline{z})]^{-*} L_n [\dot{\mathbf{r}}_{n+1}(z)]^{-1}.$$

In particular, $\operatorname{rank} \mathfrak{s}_{-,2n+1}(z) = \operatorname{rank} Q_{2n} = \operatorname{rank} L_n$.
(c) *Let* $n \in \mathbb{N}$ *with* $2n \preccurlyeq \kappa$ *and let* $z \in \mathbb{C} \setminus [\alpha, \infty)$. *Then* $\mathfrak{s}_{-,2n}(z) = 0_{q\times q}$ *and*

$$\mathfrak{s}_{+,2n}(z) = (z-\alpha)[\dot{\mathbf{p}}_{2n-1}(\overline{z})]^{-*} Q_{2n-1} [\dot{\mathbf{p}}_{2n+1}(z)]^{-1} = \frac{1}{z-\alpha}\left[\dot{\mathbf{t}}_{n-1}(\overline{z})\right]^{-*} L_{\alpha\triangleright n-1} \left[\dot{\mathbf{t}}_n(z)\right]^{-1}.$$

In particular, $\operatorname{rank} \mathfrak{s}_{+,2n}(z) = \operatorname{rank} Q_{2n-1} = \operatorname{rank} L_{\alpha\triangleright n-1}$.

Proof

(a) In view of Notation 18.4, this is a consequence of Proposition 18.3.
(b) Combine Remark 18.6(c) and Lemma 18.10(b) and use (14.1) and Notation 14.5.
(c) Apply Remark 18.6(d) and Lemma 18.10(b) and use (14.1) and Notation 14.5.

□

Proposition 18.25 *Suppose* $\kappa \geq 1$. *Let* $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\succcurlyeq,\mathrm{e}}$ *with right* α*-Stieltjes parametrization* $(Q_j)_{j=0}^{\kappa}$ *and let* $m \in \mathbb{Z}_{0,\kappa-1}$. *Then:*

(a) $\mathfrak{l}_m(x) \succcurlyeq \mathfrak{l}_{m+1}(x)$ *for all* $x \in (-\infty, \alpha)$.
(b) $\mathfrak{l}_m(z) - \mathfrak{l}_{m+1}(z) = (-1)^{m+1}\epsilon_{m+1}(z)[\dot{\mathbf{p}}_m(\overline{z})]^{-*} Q_m [\dot{\mathbf{p}}_{m+2}(z)]^{-1}$ *and* $\operatorname{rank}[\mathfrak{l}_m(z) - \mathfrak{l}_{m+1}(z)] = \operatorname{rank} Q_m$ *for all* $z \in \mathbb{C} \setminus [\alpha, \infty)$.

Proof

(a) Combine Remark 18.7 with Lemma 18.24(a).
(b) Apply Remark 18.6(b) with Lemma 18.10(b).

□

Corollary 18.26 *Let* $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\succcurlyeq,\mathrm{e}}$ *with right* α*-Stieltjes parametrization* $(Q_j)_{j=0}^{\kappa}$*. Then*

(a) *Let* $m \in \mathbb{Z}_{0,\kappa}$*. Then* $\operatorname{rank} \mathfrak{l}_m(z) = \operatorname{rank} \mathfrak{l}_m(w) = \operatorname{rank} Q_m$ *for all* $z, w \in \mathbb{C} \setminus [\alpha, \infty)$.
(b) *Suppose* $\kappa \geq 1$ *and let* $m \in \mathbb{Z}_{0,\kappa-1}$*. Then* $\operatorname{rank}[\mathfrak{l}_m(z) - \mathfrak{l}_{m+1}(z)] = \operatorname{rank}[\mathfrak{l}_m(w) - \mathfrak{l}_{m+1}(w)] = \operatorname{rank} Q_m$ *for all* $z, w \in \mathbb{C} \setminus [\alpha, \infty)$.
(c) *Suppose* $\kappa \geq 1$ *and let* $m \in \mathbb{Z}_{1,\kappa}$*. Then* $\operatorname{rank} \mathfrak{s}_{-,m}(z) = \operatorname{rank} \mathfrak{s}_{-,m}(w)$ *and* $\operatorname{rank} \mathfrak{s}_{+,m}(z) = \operatorname{rank} \mathfrak{s}_{+,m}(w)$ *for all* $z, w \in \mathbb{C} \setminus [\alpha, \infty)$.

Proof

(a) Use Proposition 18.11(b).
(b) Apply Proposition 18.25(b).
(c) Use parts (b) and (c) of Lemma 18.24.

□

19 Some Remarks on Limit Matricial Weyl Intervals

Considering the case of a sequence $(s_j)_{j=0}^{\infty} \in \mathcal{K}_{q,\infty,0}^{\succ}$ Yu. M. Dyukarev studied in [7] the limit behavior of the matricial Weyl intervals associated with the finite sections $(s_j)_{j=0}^{m}$. Our results obtained in Sect. 18 lead us to generalizations of some of Yu. M. Dyukarev's corresponding results to the most general case. The detailed treatment of this facts determines the context of this section. Now we are going to study the asymptotic behavior of the functions introduced in Notation 18.4.

Proposition 19.1 *Let* $(s_j)_{j=0}^{\infty} \in \mathcal{K}_{q,\infty,\alpha}^{\succcurlyeq}$*. For each* $x \in (-\infty, \alpha)$*, then the sequences* $(S_{-,j,\alpha}(x))_{j=0}^{\infty}$*,* $(S_{+,j,\alpha}(x))_{j=0}^{\infty}$*, and* $(\mathfrak{l}_j(x))_{j=0}^{\infty}$ *are convergent.*

Proof In view of Propositions 18.3, 18.11(a), and 18.25(a), all sequences in question are bounded monotone sequences of matrices belonging to $\mathbb{C}_{\succcurlyeq}^{q\times q}$. This implies all assertions. □

Notation 19.2 Let $(s_j)_{j=0}^{\infty} \in \mathcal{K}_{q,\infty,\alpha}^{\succcurlyeq}$. For each $x \in (-\infty, \alpha)$, let $S_{-,\infty,\alpha}(x) := \lim_{j\to\infty} S_{-,j,\alpha}(x)$, let $S_{+,\infty,\alpha}(x) := \lim_{j\to\infty} S_{+,j,\alpha}(x)$, and let $\mathfrak{l}_\infty(x) := \lim_{j\to\infty} \mathfrak{l}_j(x)$.

Proposition 19.3 *Let* $(s_j)_{j=0}^{\infty} \in \mathcal{K}_{q,\infty,\alpha}^{\succcurlyeq}$ *with right* α*-Stieltjes parametrization* $(Q_j)_{j=0}^{\infty}$ *and let* $x \in (-\infty, \alpha)$*. Then*

$$\lim_{m\to\infty} (-1)^{m+1} [\dot{\mathbf{p}}_m(x)]^{-*} Q_m [\dot{\mathbf{p}}_{m+1}(x)]^{-1} = \mathfrak{l}_\infty(x) = S_{+,\infty,\alpha}(x) - S_{-,\infty,\alpha}(x).$$

Furthermore, $\lim_{j\to\infty} \mathfrak{s}_{-,j}(x) = 0_{q\times q}$ *and* $\lim_{j\to\infty} \mathfrak{s}_{+,j}(x) = 0_{q\times q}$. *In particular,* $\lim_{n\to\infty}[\dot{\mathbf{r}}_n(x)]^{-*} L_n [\dot{\mathbf{r}}_{n+1}(x)]^{-1} = 0_{q\times q}$ *and* $\lim_{n\to\infty}(x-\alpha)^{-1}[\dot{\mathbf{t}}_n(x)]^{-*} L_{\alpha\rhd n}[\dot{\mathbf{t}}_{n+1}(x)]^{-1} = 0_{q\times q}$.

Proof In view of Proposition 19.1 and Notation 19.2, Proposition 18.11(b) yields $\mathfrak{l}_\infty(x) = \lim_{m\to\infty} \mathfrak{l}_m(x) = \lim_{m\to\infty}(-1)^{m+1}[\dot{\mathbf{p}}_m(x)]^{-*} Q_m [\dot{\mathbf{p}}_{m+1}(x)]^{-1}$, whereas Notation 18.4 provides $S_{+,\infty,\alpha}(x) - S_{-,\infty,\alpha}(x) = \lim_{j\to\infty} S_{+,j,\alpha}(x) - \lim_{j\to\infty} S_{-,j,\alpha}(x) = \lim_{j\to\infty}[S_{+,j,\alpha}(x) - S_{-,j,\alpha}(x)] = \lim_{j\to\infty} \mathfrak{l}_j(x) = \mathfrak{l}_\infty(x)$. Using Remark 18.7, we can conclude furthermore $0_{q\times q} = \mathfrak{l}_\infty(x) - \mathfrak{l}_\infty(x) = \lim_{n\to\infty} \mathfrak{l}_{2n}(x) - \lim_{n\to\infty} \mathfrak{l}_{2n+1}(x) = \lim_{n\to\infty}[\mathfrak{l}_{2n}(x) - \mathfrak{l}_{2n+1}(x)] = \lim_{n\to\infty} \mathfrak{s}_{-,2n+1}(x)$ and $0_{q\times q} = \mathfrak{l}_\infty(x) - \mathfrak{l}_\infty(x) = \lim_{n\to\infty} \mathfrak{l}_{2n-1}(x) - \lim_{n\to\infty} \mathfrak{l}_{2n}(x) = \lim_{n\to\infty}[\mathfrak{l}_{2n-1}(x) - \mathfrak{l}_{2n}(x)] = \lim_{n\to\infty} \mathfrak{s}_{+,2n}(x)$. Taking additionally into account that, by virtue of Remark 18.6, we have $\mathfrak{s}_{-,2n}(x) = 0_{q\times q}$ and $\mathfrak{s}_{+,2n+1}(x) = 0_{q\times q}$ for all $n \in \mathbb{N}$, then $\lim_{j\to\infty} \mathfrak{s}_{-,j}(x) = 0_{q\times q}$ and $\lim_{j\to\infty} \mathfrak{s}_{+,j}(x) = 0_{q\times q}$ follow. In view of Lemma 18.24, the proof is complete. □

Proposition 19.4 *Let* $(s_j)_{j=0}^\infty \in \mathcal{K}^{\succcurlyeq}_{q,\infty,\alpha}$ *and let* $x \in (-\infty,\alpha)$. *Then*

$$\begin{aligned}0_{q\times q} &= S_{-,0,\alpha}(x) \preccurlyeq S_{-,1,\alpha}(x) = S_{-,2,\alpha}(x) \preccurlyeq S_{-,3,\alpha}(x)\\ &= S_{-,4,\alpha}(x) \preccurlyeq \cdots \preccurlyeq S_{-,\infty,\alpha}(x) \preccurlyeq S_{+,\infty,\alpha}(x) \preccurlyeq \cdots \preccurlyeq S_{+,5,\alpha}(x)\\ &= S_{+,4,\alpha}(x) \preccurlyeq S_{+,3,\alpha}(x) = S_{+,2,\alpha}(x) \preccurlyeq S_{+,1,\alpha}(x) = S_{+,0,\alpha}(x) = \frac{1}{\alpha - x} s_0\end{aligned}$$

and

$$\frac{1}{\alpha-x} s_0 = \mathfrak{l}_0(x) \succcurlyeq \mathfrak{l}_1(x) \succcurlyeq \mathfrak{l}_2(x) \succcurlyeq \mathfrak{l}_3(x) \succcurlyeq \cdots \succcurlyeq \mathfrak{l}_\infty(x) \succcurlyeq 0_{q\times q}.$$

In particular, $[S_{-,\infty,\alpha}(x), S_{+,\infty,\alpha}(x)] = \bigcap_{m=0}^\infty [S_{-,m,\alpha}(x), S_{+,m,\alpha}(x)]$.

Proof For the first chain of inequalities, use Remark 17.10, Proposition 18.3, Lemma 17.9, and Notation 19.2. The second one follows from Notation 18.4, Remark 17.10, Propositions 18.25(a), 18.11(a), and Notation 19.2. □

Notation 19.5 Let $(s_j)_{j=0}^\infty \in \mathcal{K}^{\succcurlyeq}_{q,\infty,\alpha}$. Then, for all $x \in (-\infty,\alpha)$, let $\mathcal{I}[\alpha, x, (s_j)_{j=0}^\infty; =] := \{F(x) \colon F \in \mathcal{S}_{\infty,q,[\alpha,\infty)}[(s_j)_{j=0}^\infty, =]\}$.

Lemma 19.6 *Let* $(s_j)_{j=0}^\infty \in \mathcal{K}^{\succcurlyeq}_{q,\infty,\alpha}$ *and let* $x \in (-\infty,\alpha)$. *Then* $\mathcal{I}[\alpha, x, (s_j)_{j=0}^\infty; =] = \bigcap_{m=0}^\infty \mathcal{I}[\alpha, x, (s_j)_{j=0}^m; \preccurlyeq]$.

Proof First consider an arbitrary $X \in \mathcal{I}[\alpha, x, (s_j)_{j=0}^\infty; =]$. Then there exists a function $F \in \mathcal{S}_{\infty,q,[\alpha,\infty)}[(s_j)_{j=0}^\infty, =]$ with $X = F(x)$. Consequently, F

belongs to $\mathcal{S}_{0,q,[\alpha,\infty)}$ and the $[\alpha,\infty)$-Stieltjes measure σ_F of F belongs to $\mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty); (s_j)_{j=0}^{\infty}, =]$. In particular, we have $\sigma_F \in \mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty); (s_j)_{j=0}^{m}, =]$ implying $\sigma_F \in \mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty); (s_j)_{j=0}^{m}, \preccurlyeq]$ for all $m \in \mathbb{N}_0$. Hence, $F \in \mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^{m}, \preccurlyeq]$ for all $m \in \mathbb{N}_0$. According to Notation 17.1, then $X \in \mathcal{I}[\alpha, x, (s_j)_{j=0}^{m}; =]$ for all $m \in \mathbb{N}_0$ follows.

Conversely, consider now an arbitrary $X \in \bigcap_{m=0}^{\infty} \mathcal{I}[\alpha, x, (s_j)_{j=0}^{m}; \preccurlyeq]$. Consider an arbitrary $m \in \mathbb{Z}_{3,\infty}$. Then $X \in \mathcal{I}[\alpha, x, (s_j)_{j=0}^{m}; \preccurlyeq]$. Consequently, there exists a function $F_m \in \mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)_{j=0}^{m}, \preccurlyeq]$ with $X = F_m(x)$. Hence, $F_m \in \mathcal{S}_{0,q,[\alpha,\infty)}$ and the $[\alpha,\infty)$-Stieltjes measure σ_m of F_m belongs to $\mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty); (s_j)_{j=0}^{m}, \preccurlyeq]$ and fulfills $X = \int_{[\alpha,\infty)} f \mathrm{d}\sigma_m$, where $f\colon [\alpha,\infty) \to (0,\infty)$ is defined by $f(t) := (t-x)^{-1}$. In particular, $\sigma_m \in \mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty); (s_j)_{j=0}^{m-1}, =]$. Similar to the scalar case, which was studied in [1, Chapter 2, Paragraph 1], the application of the matricial version of the Helly–Prohorov theorem (see [13, Satz 9]) yields the existence of a subsequence $(\sigma_{m_\ell})_{\ell=1}^{\infty}$ of $(\sigma_m)_{m=3}^{\infty}$ and of a non-negative Hermitian measure $\sigma \in \mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty); (s_j)_{j=0}^{\infty}, =]$ such that $(\sigma_{m_\ell})_{\ell=1}^{\infty}$ converges weakly to σ. According to Theorem 10.3, then the $[\alpha,\infty)$-Stieltjes transform F of σ belongs to $\mathcal{S}_{\infty,q,[\alpha,\infty)}[(s_j)_{j=0}^{\infty}, =]$. Furthermore, $F(x) = \int_{[\alpha,\infty)} f \mathrm{d}\sigma$. Regarding $x \in (-\infty, \alpha)$, we have, for all $t \in [\alpha,\infty)$, obviously $0 < (t-x)^{-1} \leq (\alpha - x)^{-1}$. Thus, the function f is continuous and bounded. Therefore, $\lim_{\ell\to\infty} \int_{[\alpha,\infty)} f \mathrm{d}\sigma_{m_\ell} = \int_{[\alpha,\infty)} f \mathrm{d}\sigma$ follows by the weak convergence, implying $F(x) = X$. Consequently, X belongs to $\mathcal{I}[\alpha, x, (s_j)_{j=0}^{\infty}; =]$. □

Now we are able to prove an analogous result to Theorem 17.16 for the case that an infinite sequence $(s_j)_{j=0}^{\infty}$ of matrix moments is given.

Theorem 19.7 *Let $(s_j)_{j=0}^{\infty} \in \mathcal{K}_{q,\infty,\alpha}^{\succcurlyeq}$ and let $x \in (-\infty, \alpha)$. Then $S_{-,\infty,\alpha}(x) \preccurlyeq S_{+,\infty,\alpha}(x)$ and $\mathcal{I}[\alpha, x, (s_j)_{j=0}^{\infty}; =] = [S_{-,\infty,\alpha}(x), S_{+,\infty,\alpha}(x)]$.*

Proof From Proposition 19.4 we obtain $S_{-,\infty,\alpha}(x) \preccurlyeq S_{+,\infty,\alpha}(x)$ and $[S_{-,\infty,\alpha}(x), S_{+,\infty,\alpha}(x)] = \bigcap_{m=0}^{\infty} [S_{-,m,\alpha}(x), S_{+,m,\alpha}(x)]$. Obviously, $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\succcurlyeq,\mathrm{e}}$ for all $m \in \mathbb{N}_0$. According to Theorem 17.16, thus $\mathcal{I}[\alpha, x, (s_j)_{j=0}^{m}; \preccurlyeq] = [S_{-,m,\alpha}(x), S_{+,m,\alpha}(x)]$ for all $m \in \mathbb{N}_0$. Lemma 19.6 provides $\mathcal{I}[\alpha, x, (s_j)_{j=0}^{\infty}; =] = \bigcap_{m=0}^{\infty} \mathcal{I}[\alpha, x, (s_j)_{j=0}^{m}; \preccurlyeq]$, completing the proof. □

Corollary 19.8 *Let $(s_j)_{j=0}^{\infty} \in \mathcal{K}_{q,\infty,\alpha}^{\succcurlyeq}$ with right α-Stieltjes parametrization $(Q_j)_{j=0}^{\infty}$. Suppose that Problem $\mathsf{M}[[\alpha,\infty); (s_j)_{j=0}^{\infty}, =]$ is determinate, i. e., the set $\mathcal{M}_q^{\succcurlyeq}[[\alpha,\infty); (s_j)_{j=0}^{\infty}, =]$ consists of exactly one element. Then $\mathfrak{l}_\infty(x) = 0_{q\times q}$ for all $x \in (-\infty, \alpha)$. In particular, $\lim_{m\to\infty} (-1)^{m+1} [\dot{\mathbf{p}}_m(x)]^{-*} Q_m [\dot{\mathbf{p}}_{m+1}(x)]^{-1} = 0_{q\times q}$ for all $x \in (-\infty, \alpha)$.*

Proof By virtue of Theorem 10.3, the set $\mathcal{S}_{\infty,q,[\alpha,\infty)}[(s_j)_{j=0}^{\infty}, =]$ consists of exactly one element. Consider an arbitrary $x \in (-\infty, \alpha)$. According to Notation 19.5, then the set $\mathcal{I}[\alpha, x, (s_j)_{j=0}^{\infty}; =]$ consists of exactly one element. Using Theorem 19.7, we can thus conclude that the matricial interval $[S_{-,\infty,\alpha}(x), S_{+,\infty,\alpha}(x)]$ consists of exactly one element. Regarding (A.3), then $S_{-,\infty,\alpha}(x) = S_{+,\infty,\alpha}(x)$ easily follows. From Proposition 19.3, we can infer then $\lim_{m\to\infty}(-1)^{m+1}[\dot{\mathbf{p}}_m(x)]^{-*}Q_m[\dot{\mathbf{p}}_{m+1}(x)]^{-1} = \mathfrak{l}_{\infty}(x) = 0_{q\times q}$. □

It should be mentioned that considering the particular case $(s_j)_{j=0}^{\infty} \in \mathcal{K}_{q,\infty,0}^{\succ}$ Yu. M. Dyukarev [7, Theorem 5] obtained the inclusion

$$\mathcal{I}[0, x, (s_j)_{j=0}^{\infty}; =] \subseteq \big[S_{-,\infty,0}(x), S_{+,\infty,0}(x)\big].$$

In a short remark he added that it can be also shown that the reverse inclusion is true. Given a sequence $(s_j)_{j=0}^{\infty} \in \mathcal{K}_{q,\infty,0}^{\succ}$ he called the matricial Stieltjes moment problem $\mathsf{M}[[0,\infty); (s_j)_{j=0}^{\infty}, =]$ *indeterminate* if for all $x \in (-\infty, 0)$ the relations

$$S_{+,\infty,0}(x) - S_{-,\infty,0}(x) \in \mathbb{C}_{\succ}^{q\times q}$$

hold true. In [7, Lemma 3] he proved that for indeterminacy it is indeed sufficient that there exists some point $x_0 \in (-\infty, 0)$ with

$$S_{+,\infty,0}(x_0) - S_{-,\infty,0}(x_0) \in \mathbb{C}_{\succ}^{q\times q}.$$

Furthermore, Yu. M. Dyukarev [7, Theorem 8] found a Stieltjes-type criterion for indeterminacy of the problem $\mathsf{M}[[0,\infty); (s_j)_{j=0}^{\infty}, =]$. In this case, he studied analytic properties of the resolvent matrix [7, Section 7] and found via $[0,\infty)$-Stieltjes transform a parametrization of the set of solutions via a linear fractional transformation [7, Section 8].

Appendix A: Some Facts from Matrix Theory

In this appendix we summarize some facts from matrix theory which are used in this paper.

Remark A.1 Let $n \in \mathbb{N}$ and let $A_1, A_2, \ldots, A_n \in \mathbb{C}^{p\times q}$. For all $\eta_1, \eta_2, \ldots, \eta_n \in \mathbb{C}$, then $\mathcal{R}(\sum_{j=1}^{n} \eta_j A_j) \subseteq \sum_{j=1}^{n} \mathcal{R}(A_j)$ and $\bigcap_{j=1}^{n} \mathcal{N}(A_j) \subseteq \mathcal{N}(\sum_{j=1}^{n} \eta_j A_j)$.

Remark A.2 Let $\mathcal{U}$ be a linear subspace of the unitary space $\mathbb{C}^q$. Then $\mathbb{P}_{\mathcal{U}}$ is the unique complex $q \times q$ matrix satisfying $\mathbb{P}_{\mathcal{U}}^2 = \mathbb{P}_{\mathcal{U}}$ $\mathbb{P}_{\mathcal{U}}^* = \mathbb{P}_{\mathcal{U}}$, and $\mathcal{R}(\mathbb{P}_{\mathcal{U}}) = \mathcal{U}$. Furthermore, $\mathcal{N}(\mathbb{P}_{\mathcal{U}}) = \mathcal{U}^{\perp}$.

Remark A.3 Let $\mathcal{U}$ be a linear subspace of the unitary space $\mathbb{C}^q$ with dimension $d := \dim \mathcal{U}$ fulfilling $d \geq 1$. Let $(u_1, u_2, \ldots, u_d)$ be an orthonormal basis of $\mathcal{U}$ and let $U := [u_1, u_2, \ldots, u_d]$. Then $U^*U = I_d$ and $UU^* = \mathbb{P}_{\mathcal{U}}$.

Remark A.4 Let $A \in \mathbb{C}^{p\times q}$. Then, the following statements hold true:

(a) $(A^\dagger)^\dagger = A$, $(A^\dagger)^* = (A^*)^\dagger$, $\mathcal{N}(A^\dagger) = \mathcal{N}(A^*)$, $\mathcal{R}(A^\dagger) = \mathcal{R}(A^*)$, and $\operatorname{rank}(AA^\dagger) = \operatorname{rank} A$.
(b) Let $r \in \mathbb{N}$ and $B \in \mathbb{C}^{p\times r}$. Then $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ if and only if $AA^\dagger B = B$.
(c) Let $s \in \mathbb{N}$ and $B \in \mathbb{C}^{s\times q}$. Then $\mathcal{N}(A) \subseteq \mathcal{N}(C)$ if and only if $CA^\dagger A = C$.

Proposition A.5 *Let $A \in \mathbb{C}^{p\times q}$ and let $G \in \mathbb{C}^{q\times p}$. Then $G = A^\dagger$ if and only if $AG = \mathbb{P}_{\mathcal{R}(A)}$ and $GA = \mathbb{P}_{\mathcal{R}(G)}$ hold true. In particular, $\mathbb{P}_{\mathcal{N}(A)} = I_q - A^\dagger A$.*

A proof of Proposition A.5 is given, e. g., in [5, Theorem 1.1.1, p. 15].

Remark A.6 Let $A \in \mathbb{C}^{q\times q}$. Then $A^\dagger A = AA^\dagger$ if and only if $\mathcal{R}(A^*) = \mathcal{R}(A)$.

Lemma A.7 *Let $M, X \in \mathbb{C}^{q\times q}_{\succcurlyeq}$ with $\mathcal{R}(X) \subseteq \mathcal{R}(M)$. Then $\det(M^\dagger X + I_q) \neq 0$ and $M(M^\dagger X + I_q)^{-1} = M(X + M)^\dagger M$.*

Proof In view of $\mathcal{R}(X) \subseteq \mathcal{R}(M)$, Remark A.4(b) implies $MM^\dagger X = X$. Consequently,

$$X + M = MM^\dagger X + M = M(M^\dagger X + I_q). \tag{A.1}$$

Thus, (A.1) implies

$$\mathcal{R}(X + M) \subseteq \mathcal{R}(M) \qquad \text{and} \qquad \mathcal{N}(M^\dagger X + I_q) \subseteq \mathcal{N}(X + M). \tag{A.2}$$

Since X and M are both non-negative Hermitian, we have $\mathcal{N}(X + M) \subseteq [\mathcal{N}(X)] \cap [\mathcal{N}(M)]$. Thus, from (A.1) we get $\mathcal{N}(M^\dagger X + I_q) \subseteq \mathcal{N}(X)$. For each $v \in \mathcal{N}(X)$, we see that $v = M^\dagger Xv + v = (M^\dagger X + I_q)v = 0_{q\times 1}$. Hence, $\det(M^\dagger X + I_q) \neq 0$. Therefore, from (A.1) we infer $\operatorname{rank}(X + M) = \operatorname{rank} M$. Combining this with the first relation in (A.2), we get $\mathcal{R}(X + M) = \mathcal{R}(M)$. Consequently, Proposition A.5 yields $(X + M)(X + M)^\dagger = MM^\dagger$. From Remark A.6 we conclude then $(X + M)^\dagger(X + M) = M^\dagger M$. Combining this with (A.1) yields

$$(X + M)^\dagger M(M^\dagger X + I_q) = (X + M)^\dagger(X + M) = M^\dagger M.$$

Thus, since $M^\dagger X + I_q$ is non-singular, we get $(X + M)^\dagger M = M^\dagger M(M^\dagger X + I_q)^{-1}$ and hence

$$M(X + M)^\dagger M = MM^\dagger M(M^\dagger X + I_q)^{-1} = M(M^\dagger X + I_q)^{-1}.$$

□

Lemma A.8 (cf. [2, 12] or [5, Lemmas 1.1.9 and 1.1.10]) *Let (4.8) be the block representation of a complex $(p+q)\times(p+q)$ matrix M with $p\times p$ block A. Then M is non-negative Hermitian if and only if A and M/A are both non-negative Hermitian and furthermore $\mathcal{R}(B)\subseteq\mathcal{R}(A)$ and $C=B^*$.*

Lemma A.9 *Let (4.8) be the block representation of a complex $(p+q)\times(r+s)$ matrix M with $p\times r$ Block A. Suppose that $\mathcal{R}(B)\subseteq\mathcal{R}(A)$ and $\mathcal{N}(A)\subseteq\mathcal{N}(C)$ are fulfilled. Then $M\left[\begin{smallmatrix}-X\\ I_s\end{smallmatrix}\right]=\left[\begin{smallmatrix}0_{p\times s}\\ M/A\end{smallmatrix}\right]$ for all $X\in\mathbb{C}^{r\times s}$ with $AX=B$. Furthermore, $[-Z,I_q]M=[0_{q\times r},M/A]$ for all $Z\in\mathbb{C}^{q\times p}$ with $ZA=C$.*

Proof Consider arbitrary $X\in\mathbb{C}^{r\times s}$ and $Z\in\mathbb{C}^{q\times p}$ satisfying $AX=B$ and $ZA=C$. Observe that $AA^\dagger B=B$ and $CA^\dagger A=C$ hold true, by virtue of Remark A.4. Taking additionally into account (4.8) and (4.9), we obtain then

$$M\begin{bmatrix}-X\\ I_s\end{bmatrix}=\begin{bmatrix}A & B\\ C & D\end{bmatrix}\begin{bmatrix}-X\\ I_s\end{bmatrix}=\begin{bmatrix}B-AX\\ D-CX\end{bmatrix}$$
$$=\begin{bmatrix}B-AX\\ D-CA^\dagger AX\end{bmatrix}=\begin{bmatrix}B-B\\ D-CA^\dagger B\end{bmatrix}=\begin{bmatrix}0_{p\times s}\\ M/A\end{bmatrix}$$

and analogously $[-Z,I_q]M=[0_{q\times r},M/A]$. □

Denote by $\mathbb{C}_{\mathrm{H}}^{q\times q}:=\{A\in\mathbb{C}^{q\times q}\colon A^*=A\}$ the set of Hermitian $q\times q$ matrices.

Remark A.10 The set $\mathbb{C}_{\mathrm{H}}^{q\times q}$ is an $\mathbb{R}$-vector space.

Remark A.11 Let $A\in\mathbb{C}_{\mathrm{H}}^{q\times q}$ and let $X\in\mathbb{C}^{q\times p}$. Then $X^*AX\in\mathbb{C}_{\mathrm{H}}^{p\times p}$.

Denote by $\mathbb{C}_{\succcurlyeq}^{q\times q}:=\{A\in\mathbb{C}^{q\times q}\colon v^*Av\geq 0 \text{ for all } v\in\mathbb{C}^q\}$ the set of non-negative Hermitian $q\times q$ matrices and by $\mathbb{C}_{\succ}^{q\times q}:=\{A\in\mathbb{C}^{q\times q}\colon v^*Av>0 \text{ for all } v\in\mathbb{C}^q \text{ with } v\neq 0_{q\times 1}\}$ the set of positive Hermitian $q\times q$ matrices.

Remark A.12 The set $\mathbb{C}_{\succcurlyeq}^{q\times q}$ is a convex cone in the $\mathbb{R}$-vector space $\mathbb{C}_{\mathrm{H}}^{q\times q}$.

Remark A.13 Let $A\in\mathbb{C}_{\succcurlyeq}^{q\times q}$ and let $X\in\mathbb{C}^{q\times p}$. Then $X^*AX\in\mathbb{C}_{\succcurlyeq}^{p\times p}$.

Remark A.14 If $A\in\mathbb{C}_{\succcurlyeq}^{q\times q}$, then $\mathcal{R}(\sqrt{A})=\mathcal{R}(A)$ and $\mathcal{N}(\sqrt{A})=\mathcal{N}(A)$.

Remark A.15 If $A\in\mathbb{C}_{\succcurlyeq}^{q\times q}$, then $A^\dagger\in\mathbb{C}_{\succcurlyeq}^{q\times q}$ and $\sqrt{A^\dagger}=\sqrt{A}^\dagger$.

We use the Löwner semi-ordering in the set of Hermitian matrices, i. e., we write $A\preccurlyeq B$ or $B\succcurlyeq A$ (resp., $A\prec B$ or $B\succ A$) in order to indicate that A and B are Hermitian matrices of equal size such that the matrix $A-B$ is non-negative Hermitian (resp., positive Hermitian).

From Remarks A.11 and A.13, we obtain immediately:

Remark A.16 Let $A, B \in \mathbb{C}_{\mathrm{H}}^{q\times q}$ with $A \preccurlyeq B$ and let $X \in \mathbb{C}^{q\times p}$. Then $X^*AX \preccurlyeq X^*BX$.

Lemma A.17 (cf. [14, Lemma A.13]) *Let $A, B \in \mathbb{C}_{\mathrm{H}}^{q\times q}$ with $0_{q\times q} \preccurlyeq A \preccurlyeq B$. Then $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ and $\mathcal{N}(B) \subseteq \mathcal{N}(A)$. Furthermore, $0_{q\times q} \preccurlyeq \mathbb{P}_{\mathcal{R}(A)} B^\dagger \mathbb{P}_{\mathcal{R}(A)} \preccurlyeq A^\dagger$.*

For every choice of Hermitian complex $q \times q$ matrices A and B, let

$$[A, B] := \{X \in \mathbb{C}_{\mathrm{H}}^{q\times q} : A \preccurlyeq X \preccurlyeq B\}. \tag{A.3}$$

If $A \preccurlyeq B$, then $\{A, B\} \subseteq [A, B]$. As an immediate consequence of Remark A.12, we obtain:

Remark A.18 If $A, B \in \mathbb{C}_{\succcurlyeq}^{q\times q}$, then $[A, B] \subseteq \mathbb{C}_{\succcurlyeq}^{q\times q}$.

For each linear subspace $\mathcal{U}$ of $\mathbb{C}^q$, let $\mathbb{C}_{\succcurlyeq,\mathcal{U}}^{q\times q} := \{Z \in \mathbb{C}_{\succcurlyeq}^{q\times q} : \mathcal{R}(Z) = \mathcal{U}\}$. As an immediate consequence of Remark A.12 and Lemma A.17, we obtain:

Remark A.19 Let $\mathcal{U}$ be a linear subspace of $\mathbb{C}^q$ and let $A, B \in \mathbb{C}_{\succcurlyeq,\mathcal{U}}^{q\times q}$. Then $[A, B] \subseteq \mathbb{C}_{\succcurlyeq,\mathcal{U}}^{q\times q}$.

Lemma A.20 *Let $s_0 \in \mathbb{C}_{\mathrm{H}}^{q\times q}$ and $M \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ be such that $s_0 - M \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ and* $\operatorname{rank} s_0 = \operatorname{rank} M$ *are satisfied. Then:*

(a) $s_0 \in \mathbb{C}_{\succcurlyeq}^{q\times q}$, $\mathcal{R}(s_0) = \mathcal{R}(M)$, $s_0 s_0^\dagger = MM^\dagger$, *and* $s_0^\dagger s_0 = M^\dagger M$.
(b) $s_0 M^\dagger(s_0 - M) = (s_0 - M)^* M^\dagger (s_0 - M)^* + s_0 - M$, $s_0 M^\dagger(s_0 - M) \in \mathbb{C}_{\succcurlyeq}^{q\times q}$, *and* $s_0[s_0 + s_0 M^\dagger(s_0 - M)]^\dagger s_0 = M$.
(c) $s_0 M^\dagger(s_0 - M) = 0_{q\times q}$ *if and only if* $M = s_0$.

Proof

(a) In view of $\{s_0 - M, M\} \subseteq \mathbb{C}_{\succcurlyeq}^{q\times q}$, Lemma A.17 yields $s_0 \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ and $\mathcal{R}(M) \subseteq \mathcal{R}(s_0)$. Because of $\operatorname{rank} M = \operatorname{rank} s_0$, this implies $\mathcal{R}(M) = \mathcal{R}(s_0)$. Thus, from Proposition A.5 we get $s_0 s_0^\dagger = MM^\dagger$. By virtue of $s_0 \in \mathbb{C}_{\mathrm{H}}^{q\times q}$ and $M \in \mathbb{C}_{\succcurlyeq}^{q\times q}$, Remark A.6 yields $s_0 s_0^\dagger = s_0^\dagger s_0$ and $MM^\dagger = M^\dagger M$. Consequently, $s_0^\dagger s_0 = M^\dagger M$.
(b) Using $s_0 s_0^\dagger = MM^\dagger$, we infer

$$\begin{aligned} s_0 M^\dagger(s_0 - M) &= [(s_0 - M) + M]M^\dagger(s_0 - M) \\ &= (s_0 - M)M^\dagger(s_0 - M) + MM^\dagger(s_0 - M) \\ &= (s_0 - M)M^\dagger(s_0 - M) + MM^\dagger s_0 - M \\ &= (s_0 - M)^* M^\dagger (s_0 - M)^* + s_0 - M. \end{aligned} \tag{A.4}$$

In view of $M \in \mathbb{C}_{\succcurlyeq}^{q\times q}$, Remark A.15 provides us $M^\dagger \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ and, hence, $(s_0 - M)^* M^\dagger (s_0 - M) \in \mathbb{C}_{\succcurlyeq}^{q\times q}$. Combining this with $s_0 - M \in \mathbb{C}_{\succcurlyeq}^{q\times q}$, and (A.4), we conclude $s_0 M^\dagger (s_0 - M) \in \mathbb{C}_{\succcurlyeq}^{q\times q}$. Using $s_0^\dagger s_0 = M^\dagger M$, we obtain

$$s_0 + s_0 M^\dagger (s_0 - M) = s_0 + s_0 M^\dagger s_0 - s_0 M^\dagger M = s_0 + s_0 M^\dagger s_0 - s_0 s_0^\dagger s_0 = s_0 M^\dagger s_0.$$

Taking $s_0 s_0^\dagger = MM^\dagger$ and $s_0^\dagger s_0 = M^\dagger M$ into account, this implies

$$\begin{aligned} &s_0 \Big[s_0 + s_0 M^\dagger (s_0 - M) \Big]^\dagger s_0 = s_0 (s_0 M^\dagger s_0)^\dagger s_0 \\ &= s_0 s_0^\dagger s_0 s_0^\dagger s_0 (s_0 M^\dagger s_0)^\dagger s_0 s_0^\dagger s_0 s_0^\dagger s_0 = MM^\dagger MM^\dagger s_0 (s_0 M^\dagger s_0)^\dagger s_0 M^\dagger MM^\dagger M \\ &= M s_0^\dagger s_0 M^\dagger s_0 (s_0 M^\dagger s_0)^\dagger s_0 M^\dagger s_0 s_0^\dagger M = M s_0^\dagger s_0 M^\dagger s_0 s_0^\dagger M \\ &= MM^\dagger MM^\dagger MM^\dagger M = M. \end{aligned}$$

(c) This follows from (A.4), $s_0 - M \in \mathbb{C}_{\succcurlyeq}^{q\times q}$, and $(s_0 - M)^* M^\dagger (s_0 - M) \in \mathbb{C}_{\succcurlyeq}^{q\times q}$. □

Appendix B: Some Monotonicity Properties for Hermitian Matrices

This appendix contains an investigation of the behavior of closed matricial intervals under special transformations mapping the set $\mathbb{C}^{q\times q}$ into itself. These are just those transformations the composition of which describes the elementary step of our Schur–Stieltjes algorithm. The main result is Proposition B.5 which provides the key result for the proof of Theorem 17.16.

We consider the following four transformations: For arbitrarily given $\eta \in \mathbb{C}$ and $M \in \mathbb{C}^{q\times q}$, let $\Lambda_M, \Psi, \Pi_M, \Xi_\eta \colon \mathbb{C}^{q\times q} \to \mathbb{C}^{q\times q}$ be defined by

$$\Lambda_M(Z) := Z + M, \quad \Psi(Z) := Z^\dagger, \quad \Pi_M(Z) := MZM, \quad \text{and} \quad \Xi_\eta(Z) := \eta Z. \tag{B.1}$$

Lemma B.1 *Let $M \in \mathbb{C}^{q\times q}$, let $\mathcal{U} := \mathcal{R}(M)$, and let $\Lambda_M \colon \mathbb{C}^{q\times q} \to \mathbb{C}^{q\times q}$ be defined by $\Lambda_M(Z) := Z + M$. Then:*

(a) *$\Lambda_M \colon \mathbb{C}^{q\times q} \to \mathbb{C}^{q\times q}$ is bijective with inverse $\Lambda_{(-M)}$.*
(b) *If $M \in \mathbb{C}_{\succcurlyeq}^{q\times q}$, then $\Lambda_M(Z) \in \mathbb{C}_{\succcurlyeq,\mathcal{U}}^{q\times q}$ for all $Z \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ with $\mathcal{R}(Z) \subseteq \mathcal{U}$.*
(c) *If $M \in \mathbb{C}_{\mathrm{H}}^{q\times q}$, then $\Lambda_M(X_1) \preccurlyeq \Lambda_M(X_2)$ for all $X_1, X_2 \in \mathbb{C}_{\mathrm{H}}^{q\times q}$ with $X_1 \preccurlyeq X_2$.*

(d) *If* $M \in \mathbb{C}_{\mathrm{H}}^{q\times q}$, *then* $\Lambda_M([A, B]) = [\Lambda_M(A), \Lambda_M(B)]$ *for all* $A, B \in \mathbb{C}_{\mathrm{H}}^{q\times q}$ *with* $A \preccurlyeq B$.

Proof

(a) is obvious.

(b) Assume $M \in \mathbb{C}_{\succcurlyeq}^{q\times q}$. Consider an arbitrary $Z \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ satisfying $\mathcal{R}(Z) \subseteq \mathcal{U}$. In view of Remark A.12, then $Z + M \in \mathbb{C}_{\succcurlyeq}^{q\times q}$. By virtue of $M \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ and $(Z + M) - M = Z \in \mathbb{C}_{\succcurlyeq}^{q\times q}$, we have $0_{q\times q} \preccurlyeq M \preccurlyeq Z + M$. Hence, the application of Lemma A.17 yields $\mathcal{R}(M) \subseteq \mathcal{R}(Z + M)$. From Remark A.1, we can conclude $\mathcal{R}(Z + M) \subseteq \mathcal{R}(Z) + \mathcal{R}(M)$. Taking additionally into account $\mathcal{R}(Z) \subseteq \mathcal{U} = \mathcal{R}(M)$, we thus obtain $\mathcal{R}(M) \subseteq \mathcal{R}(Z + M) \subseteq \mathcal{R}(Z) + \mathcal{R}(M) \subseteq \mathcal{R}(M) + \mathcal{R}(M) = \mathcal{R}(M)$, implying $\mathcal{R}(Z + M) = \mathcal{R}(M)$. Consequently, $Z + M \in \mathbb{C}_{\succcurlyeq,\mathcal{U}}^{q\times q}$.

(c) Assume $M \in \mathbb{C}_{\mathrm{H}}^{q\times q}$. Consider arbitrary $X_1, X_2 \in \mathbb{C}_{\mathrm{H}}^{q\times q}$ with $X_1 \preccurlyeq X_2$. Then $X_2 - X_1 \in \mathbb{C}_{\succcurlyeq}^{q\times q}$. In view of $M \in \mathbb{C}_{\mathrm{H}}^{q\times q}$ and Remark A.10, we have $X_1 + M, X_2 + M \in \mathbb{C}_{\mathrm{H}}^{q\times q}$. Furthermore $(X_2 + M) - (X_1 + M) = X_2 - X_1$. Thus, $(X_2 + M) - (X_1 + M) \in \mathbb{C}_{\succcurlyeq}^{q\times q}$. Consequently, $X_1 + M \preccurlyeq X_2 + M$.

(d) Assume $M \in \mathbb{C}_{\mathrm{H}}^{q\times q}$. Let $A, B \in \mathbb{C}_{\mathrm{H}}^{q\times q}$ with $A \preccurlyeq B$. Consider an arbitrary $X \in [A, B]$. Then $X \in \mathbb{C}_{\mathrm{H}}^{q\times q}$ with $A \preccurlyeq X \preccurlyeq B$. By virtue of part (c), therefore $A + M \preccurlyeq X + M$ and $X + M \preccurlyeq B + M$. Consequently, $X + M \in \mathbb{C}_{\mathrm{H}}^{q\times q}$ with $A + M \preccurlyeq X + M \preccurlyeq B + M$, i.e. $X + M \in [A + M, B + M]$. Hence, we have shown $\Lambda_M([A, B]) \subseteq [\Lambda_M(A), \Lambda_M(B)]$. Since part (c) yields $\Lambda_M(A) \preccurlyeq \Lambda_M(B)$, we can apply the same reasoning to the matrices $\Lambda_M(A)$ and $\Lambda_M(B)$ and the Hermitian matrix $-M$ instead of A, B, and M to obtain $\Lambda_{(-M)}([\Lambda_M(A), \Lambda_M(B)]) \subseteq [\Lambda_{(-M)}(\Lambda_M(A)), \Lambda_{(-M)}(\Lambda_M(B))]$. According to part (a), hence $\Lambda_{(-M)}([\Lambda_M(A), \Lambda_M(B)]) \subseteq [A, B]$. Taking again into account part (a), the application of Λ_M to this inclusion yields $[\Lambda_M(A), \Lambda_M(B)] \subseteq \Lambda_M([A, B])$.

□

Lemma B.2 *Let* $\mathcal{U}$ *be a linear subspace of* $\mathbb{C}^q$ *and let* $\Psi\colon \mathbb{C}^{q\times q} \to \mathbb{C}^{q\times q}$ *be defined by* $\Psi(Z) := Z^\dagger$. *Then:*

(a) $\Psi\colon \mathbb{C}^{q\times q} \to \mathbb{C}^{q\times q}$ *is bijective with inverse* Ψ.
(b) $\Psi(Z) \in \mathbb{C}_{\succcurlyeq,\mathcal{U}}^{q\times q}$ *for all* $Z \in \mathbb{C}_{\succcurlyeq,\mathcal{U}}^{q\times q}$.
(c) $\Psi(X_2) \preccurlyeq \Psi(X_1)$ *for all* $X_1, X_2 \in \mathbb{C}_{\succcurlyeq,\mathcal{U}}^{q\times q}$ *with* $X_1 \preccurlyeq X_2$.
(d) $\Psi([A, B]) = [\Psi(B), \Psi(A)]$ *for all* $A, B \in \mathbb{C}_{\succcurlyeq,\mathcal{U}}^{q\times q}$ *with* $A \preccurlyeq B$.

Proof

(a) follows from Remark A.4(a).

(b) For each $Z \in \mathbb{C}_{\succcurlyeq,\mathcal{U}}^{q\times q}$, we have $Z^\dagger \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ according to Remark A.15 and, furthermore, $\mathcal{R}(Z^\dagger) = \mathcal{R}(Z^*) = \mathcal{R}(Z) = \mathcal{U}$ by virtue of Remark A.4(a).

(c) Consider arbitrary $X_1, X_2 \in \mathbb{C}_{\succcurlyeq,\mathcal{U}}^{q\times q}$ with $X_1 \preccurlyeq X_2$. Then $X_1, X_2 \in \mathbb{C}_{\mathrm{H}}^{q\times q}$ with $0_{q\times q} \preccurlyeq X_1 \preccurlyeq X_2$ and $\mathcal{R}(X_1) = \mathcal{U} = \mathcal{R}(X_2)$. Because of Proposition A.5 and Remark A.6, we have then $\mathbb{P}_{\mathcal{R}(X_1)} = \mathbb{P}_{\mathcal{R}(X_2)} = X_2X_2^\dagger = X_2^\dagger X_2$, whereas Lemma A.17 yields $0_{q\times q} \preccurlyeq \mathbb{P}_{\mathcal{R}(X_1)}X_2^\dagger\mathbb{P}_{\mathcal{R}(X_1)} \preccurlyeq X_1^\dagger$. Consequently, $X_2^\dagger \preccurlyeq X_1^\dagger$ follows.

(d) Let $A, B \in \mathbb{C}_{\succcurlyeq,\mathcal{U}}^{q\times q}$ with $A \preccurlyeq B$. Consider an arbitrary $X \in [A, B]$. Then $X \in \mathbb{C}_{\mathrm{H}}^{q\times q}$ and $A \preccurlyeq X \preccurlyeq B$ and furthermore $X \in \mathbb{C}_{\succcurlyeq,\mathcal{U}}^{q\times q}$, according to Remark A.19. Thus, part (c) yields $X^\dagger \preccurlyeq A^\dagger$ and $B^\dagger \preccurlyeq X^\dagger$. Consequently, $X^\dagger \in \mathbb{C}_{\mathrm{H}}^{q\times q}$ with $B^\dagger \preccurlyeq X^\dagger \preccurlyeq A^\dagger$, i.e. $X \in [A^\dagger, B^\dagger]$. Hence, we have shown $\Psi([A, B]) \subseteq [\Psi(B), \Psi(A)]$. Since parts (b) and (c) yield $\Psi(B), \Psi(A) \in \mathbb{C}_{\succcurlyeq,\mathcal{U}}^{q\times q}$ and $\Psi(B) \preccurlyeq \Psi(A)$, we can apply the same reasoning to the matrices $\Psi(B)$ and $\Psi(A)$ instead of A and B to obtain $\Psi([\Psi(B), \Psi(A)]) \subseteq [\Psi(\Psi(A)), \Psi(\Psi(B))]$. According to part (a), hence $\Psi([\Psi(B), \Psi(A)]) \subseteq [A, B]$. Taking again into account part (a), the application of Ψ to this inclusion yields $[\Psi(B), \Psi(A)] \subseteq \Psi([A, B])$.

□

Lemma B.3 *Let $M \in \mathbb{C}_{\mathrm{H}}^{q\times q}$, let $\mathcal{U} := \mathcal{R}(M)$, and let $\Pi_M\colon \mathbb{C}^{q\times q} \to \mathbb{C}^{q\times q}$ be defined by $\Pi_M(Z) := MZM$. Then:*

(a) $(\Pi_{M^\dagger} \circ \Pi_M)(Z) = Z$ *and* $(\Pi_M \circ \Pi_{M^\dagger})(Z) = Z$ *for all* $Z \in \mathbb{C}_{\succcurlyeq,\mathcal{U}}^{q\times q}$.

(b) $\Pi_M(Z) \in \mathbb{C}_{\succcurlyeq,\mathcal{U}}^{q\times q}$ *for all* $Z \in \mathbb{C}_{\succcurlyeq,\mathcal{U}}^{q\times q}$.

(c) $\Pi_M(X_1) \preccurlyeq \Pi_M(X_2)$ *for all* $X_1, X_2 \in \mathbb{C}_{\mathrm{H}}^{q\times q}$ *with* $X_1 \preccurlyeq X_2$.

(d) $\Pi_M([A, B]) = [\Pi_M(A), \Pi_M(B)]$ *for all* $A, B \in \mathbb{C}_{\succcurlyeq,\mathcal{U}}^{q\times q}$ *with* $A \preccurlyeq B$.

Proof

(a) Let $Z \in \mathbb{C}_{\succcurlyeq,\mathcal{U}}^{q\times q}$. Then $Z^* = Z$ and $\mathcal{R}(Z) = \mathcal{U} = \mathcal{R}(M)$. Consequently, using Remark A.6 and Proposition A.5, we get $Z^\dagger Z = ZZ^\dagger = \mathbb{P}_{\mathcal{U}} = MM^\dagger = M^\dagger M$. Thus, $M^\dagger MZMM^\dagger = MM^\dagger ZM^\dagger M = Z$ follows. Hence, $\Pi_{(M^\dagger)}(\Pi_M(Z)) = \Pi_M(\Pi_{(M^\dagger)}(Z)) = Z$.

(b) Consider now an arbitrary $Z \in \mathbb{C}_{\succcurlyeq,\mathcal{U}}^{q\times q}$. Then $Z \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ and $\mathcal{R}(Z) = \mathcal{R}(M)$. From Remark A.13 we can conclude $MZM \in \mathbb{C}_{\succcurlyeq}^{q\times q}$. Using Remark A.14, we infer furthermore $\mathcal{R}(MZM) = \mathcal{R}((M\sqrt{Z})(M\sqrt{Z})^*) = \mathcal{R}(M\sqrt{Z}) = M\mathcal{R}(\sqrt{Z}) = M\mathcal{R}(Z) = M\mathcal{R}(M) = M\mathcal{R}(M^*) = \mathcal{R}(MM^*) = \mathcal{R}(M) = \mathcal{U}$.

(c) Use Remark A.16.

(d) Let $A, B \in \mathbb{C}_{\succcurlyeq,\mathcal{U}}^{q\times q}$ with $A \preccurlyeq B$. Consider an arbitrary $X \in [A, B]$. Then $X \in \mathbb{C}_{\mathrm{H}}^{q\times q}$ with $A \preccurlyeq X \preccurlyeq B$. By virtue of part (c), therefore $MAM \preccurlyeq MXM$ and $MXM \preccurlyeq MBM$. Consequently, $MXM \in \mathbb{C}_{\mathrm{H}}^{q\times q}$ with $MAM \preccurlyeq MXM \preccurlyeq MBM$, i.e., $MXM \in [MAM, MBM]$. Hence, we have shown $\Pi_M([A, B]) \subseteq [\Pi_M(A), \Pi_M(B)]$. From Remark A.4(a) we can conclude $(M^\dagger)^* = (M^*)^\dagger = M^\dagger$ and $\mathcal{R}(M^\dagger) = \mathcal{R}(M^*) = \mathcal{R}(M) = \mathcal{U}$, whereas parts (b) and (c) yield $\Pi_M(A), \Pi_M(B) \in \mathbb{C}^{q\times q}_{\succcurlyeq,\mathcal{U}}$ and $\Pi_M(A) \preccurlyeq \Pi_M(B)$. Thus, we can apply the same reasoning to the matrices $\Pi_M(A)$ and $\Pi_M(B)$ and the Hermitian matrix $M^\dagger$ instead of A, B, and M to obtain $\Pi_{M^\dagger}([\Pi_M(A), \Pi_M(B)]) \subseteq [\Pi_{M^\dagger}(\Pi_M(A)), \Pi_{M^\dagger}(\Pi_M(B))]$. According to part (a), hence $\Pi_{M^\dagger}([\Pi_M(A), \Pi_M(B)]) \subseteq [A, B]$. Remark A.19 provides $[\Pi_M(A), \Pi_M(B)] \subseteq \mathbb{C}^{q\times q}_{\succcurlyeq,\mathcal{U}}$. Taking into account part (a), the application of Π_M to the inclusion $\Pi_{M^\dagger}([\Pi_M(A), \Pi_M(B)]) \subseteq [A, B]$ thus yields $[\Pi_M(A), \Pi_M(B)] \subseteq \Pi_M([A, B])$.

□

Lemma B.4 *Let $\eta \in \mathbb{C}$ and let $\Xi_\eta\colon \mathbb{C}^{q\times q} \to \mathbb{C}^{q\times q}$ be defined by $\Xi_\eta(Z) := \eta Z$. Then:*

(a) *If $\eta \neq 0$, then $\Xi_\eta\colon \mathbb{C}^{q\times q} \to \mathbb{C}^{q\times q}$ is bijective with inverse $\Xi_{1/\eta}$.*
(b) *If $\eta \in [0, \infty)$, then $\Xi_\eta(X_1) \preccurlyeq \Xi_\eta(X_2)$ for all $X_1, X_2 \in \mathbb{C}^{q\times q}_{\succcurlyeq}$ with $X_1 \preccurlyeq X_2$.*
(c) *If $\eta \in (0, \infty)$, then $\Xi_\eta([A, B]) = [\Xi_\eta(A), \Xi_\eta(B)]$ for all $A, B \in \mathbb{C}^{q\times q}_{\succcurlyeq}$ with $A \preccurlyeq B$.*

Proof

(a) is obvious.
(b) Assume $\eta \in [0, \infty)$. Consider arbitrary $X_1, X_2 \in \mathbb{C}^{q\times q}_{\succcurlyeq}$ with $X_1 \preccurlyeq X_2$. Then $X_1, X_2 \in \mathbb{C}^{q\times q}_{\mathrm{H}}$ and $X_2 - X_1 \in \mathbb{C}^{q\times q}_{\succcurlyeq}$. In view of $\eta \in \mathbb{R}$ and Remark A.10, hence $\eta X_1, \eta X_2 \in \mathbb{C}^{q\times q}_{\mathrm{H}}$. Because of $\eta \geq 0$ and Remark A.12, furthermore $\eta(X_2 - X_1) \in \mathbb{C}^{q\times q}_{\succcurlyeq}$. Thus, $\eta X_2 - \eta X_1 \in \mathbb{C}^{q\times q}_{\succcurlyeq}$. Consequently, $\eta X_1 \preccurlyeq \eta X_2$.
(c) Assume $\eta \in (0, \infty)$. Let $A, B \in \mathbb{C}^{q\times q}_{\succcurlyeq}$ with $A \preccurlyeq B$. Consider an arbitrary $X \in [A, B]$. Then $X \in \mathbb{C}^{q\times q}_{\mathrm{H}}$ with $A \preccurlyeq X \preccurlyeq B$ and furthermore $X \in \mathbb{C}^{q\times q}_{\succcurlyeq}$, according to Remark A.18. Thus, part (b) yields $\eta A \preccurlyeq \eta X$ and $\eta X \preccurlyeq \eta B$. Consequently, $\eta X \in \mathbb{C}^{q\times q}_{\mathrm{H}}$ with $\eta A \preccurlyeq \eta X \preccurlyeq \eta B$, i. e. $X \in [\eta A, \eta B]$. Hence, we have shown $\Xi_\eta([A, B]) \subseteq [\Xi_\eta(A), \Xi_\eta(B)]$. Since part (b) yields $\Xi_\eta(A) \preccurlyeq \Xi_\eta(B)$ and, in view of $\Xi_\eta(0_{q\times q}) = 0_{q\times q}$, furthermore $\Xi_\eta(A), \Xi_\eta(B) \in \mathbb{C}^{q\times q}_{\succcurlyeq}$, we can apply the same reasoning to the matrices $\Xi_\eta(A)$ and $\Xi_\eta(B)$ and the positive number $1/\eta$ instead of A, B, and η to obtain $\Xi_{1/\eta}([\Xi_\eta(A), \Xi_\eta(B)]) \subseteq [\Xi_{1/\eta}(\Xi_\eta(A)), \Xi_{1/\eta}(\Xi_\eta(B))]$. According to part (a), hence $\Xi_{1/\eta}([\Xi_\eta(A), \Xi_\eta(B)]) \subseteq [A, B]$. Taking again into account part (a), the application of Ξ_η to this inclusion yields $[\Xi_\eta(A), \Xi_\eta(B)] \subseteq \Xi_\eta([A, B])$.

□

Proposition B.5 *Let $\eta \in (0, \infty)$, let $M \in \mathbb{C}^{q\times q}_{\succcurlyeq}$, let $\mathcal{U} := \mathcal{R}(M)$, let $\Gamma_{\eta,M}\colon \mathbb{C}^{q\times q} \to \mathbb{C}^{q\times q}$ be defined by $\Gamma_{\eta,M}(Z) := \eta M(Z + M)^\dagger M$, and let $A, B \in \mathbb{C}^{q\times q}_{\mathrm{H}}$ with $\mathcal{R}(B) \subseteq \mathcal{U}$ and $0_{q\times q} \preccurlyeq A \preccurlyeq B$. Then $0_{q\times q} \preccurlyeq \Gamma_{\eta,M}(B) \preccurlyeq \Gamma_{\eta,M}(A)$ and $\Gamma_{\eta,M}([A, B]) = [\Gamma_{\eta,M}(B), \Gamma_{\eta,M}(A)]$.*

Proof Using Lemma A.17, we can conclude $\mathcal{R}(A) \subseteq \mathcal{R}(B) \subseteq \mathcal{U}$. Now observe that $\Gamma_{\eta,M} = \Xi_\eta \circ \Pi_M \circ \Psi \circ \Lambda_M$ with the notation given in (B.1) and combine Lemmas B.1, B.2, B.3, and B.4. □

Appendix C: On Linear Fractional Transformations of Matrices

In this appendix, we summarize some basic facts on linear fractional transformations of matrices. A systematic treatment of this topic was handled by V. P. Potapov in [38] (see also [5, Section 1.6]). We slightly extend the concept developed by V. P. Potapov by studying the more general version of linear fractional transformations of pairs of complex matrices. It should be mentioned that V. P. Potapov [38, pp. 80–81] observed that sometimes there are situations where linear fractional transformations of pairs of complex matrices arise, but not treated this case. We did not succeed in finding a convenient hint in the public literature. That's why we state the corresponding results.

Notation C.1 Let $E \in \mathbb{C}^{(p+q)\times(p+q)}$ and let

$$E = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{C.1}$$

be the block partition of E with $p \times p$ block a. If $\operatorname{rank}[c, d] = q$, then the linear fractional transformations $\mathcal{S}_E^{(p,q)} : \mathcal{Q}_{[c,d]} \to \mathbb{C}^{p\times q}$ and $\tilde{\mathcal{S}}_E^{(p,q)} : \tilde{\mathcal{Q}}_{[c,d]} \to \mathbb{C}^{p\times q}$ are defined by

$$\mathcal{S}_E^{(p,q)}(x) := (ax + b)(cx + d)^{-1} \text{ and } \tilde{\mathcal{S}}_E^{(p,q)}((x, y)) := (ax + by)(cx + dy)^{-1}.$$

Lemma C.2 *Let $c \in \mathbb{C}^{q\times p}$ and $d \in \mathbb{C}^{q\times q}$. Then the following statements are equivalent:*

(i) *The set* $\mathcal{Q}_{[c,d]} := \{x \in \mathbb{C}^{p\times q} : \det(cx + d) \neq 0\}$ *is non-empty.*
(ii) *The set* $\tilde{\mathcal{Q}}_{[c,d]} := \{(x, y) \in \mathbb{C}^{p\times q} \times \mathbb{C}^{q\times q} : \det(cx + dy) \neq 0\}$ *is non-empty.*
(iii) $\operatorname{rank}[c, d] = q$.

Furthermore, $\tilde{\mathcal{Q}}_{[c,d]}$ is a subset of the set $\mathcal{Q}_{p\times q}$ of all pairs $(x, y) \in \mathbb{C}^{p\times q} \times \mathbb{C}^{q\times q}$ which fulfill $\operatorname{rank}\begin{bmatrix} x \\ y \end{bmatrix} = q$.

A proof of Lemma C.2, e. g., in [5, Lemma 1.6.1, p. 52] and [23, Lemma D.2].

Proposition C.3 *Let $a_1, a_2 \in \mathbb{C}^{p\times p}$, let $b_1, b_2 \in \mathbb{C}^{p\times q}$, let $c_1, c_2 \in \mathbb{C}^{q\times p}$, and let $d_1, d_2 \in \mathbb{C}^{q\times q}$ be such that* $\operatorname{rank}[c_1, d_1] = \operatorname{rank}[c_2, d_2] = q$. *Furthermore, let*

$$E_1 := \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \qquad E_2 := \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}, \tag{C.2}$$

let $E := E_2E_1$, and let (C.1) be the block representation of E with $p \times p$ block a. Then $\mathcal{Q} := \{x \in \mathcal{Q}_{[c_1,d_1]} : \mathcal{S}_{E_1}^{(p,q)}(x) \in \mathcal{Q}_{[c_2,d_2]}\}$ is a nonempty subset of the set $\mathcal{Q}_{[c,d]}$ and $\mathcal{S}_{E_2}^{(p,q)}(\mathcal{S}_{E_1}^{(p,q)}(x)) = \mathcal{S}_{E}^{(p,q)}(x)$ holds true for all $x \in \mathcal{Q}$.

A proof of Proposition C.3 is stated, e. g., in [5, Proposition 1.6.3].

Proposition C.4 ([23, Proposition D.4]) *Let $E_1, E_2 \in \mathbb{C}^{(p+q)\times(p+q)}$ and let the block partitions of E_1 and E_2 with $p \times p$ blocks a_1 and a_2 be given by (C.2). Let $E := E_2E_1$ and let (C.1) be the block partition of E with $p \times p$ block a. Suppose that* $\operatorname{rank}[c_1, d_1] = q$ *and* $\operatorname{rank}[c_2, d_2] = q$ *hold true. Let $\tilde{\mathcal{Q}} := \{(x, y) \in \tilde{\mathcal{Q}}_{[c_1,d_1]} : \tilde{\mathcal{S}}_{E_1}^{(p,q)}((x, y)) \in \mathcal{Q}_{[c_2,d_2]}\}$. Then $\tilde{\mathcal{Q}}_{[c,d]} \cap \tilde{\mathcal{Q}}_{[c_1,d_1]} = \tilde{\mathcal{Q}}$. Furthermore, if $\tilde{\mathcal{Q}}_{[c,d]} \cap \tilde{\mathcal{Q}}_{[c_1,d_1]} \neq \emptyset$, then $\mathcal{S}_{E_2}^{(p,q)}(\tilde{\mathcal{S}}_{E_1}^{(p,q)}((x, y))) = \tilde{\mathcal{S}}_{E}^{(p,q)}((x, y))$ for all $(x, y) \in \tilde{\mathcal{Q}}_{[c,d]} \cap \tilde{\mathcal{Q}}_{[c_1,d_1]}$.*

Appendix D: Orthogonal Matrix Polynomials

We start with some notation.

Notation D.1 Let P be a complex $p \times q$ matrix polynomial. For each $n \in \mathbb{N}_0$, let $Y_n(P) := \operatorname{col}(A_j)_{j=0}^{n}$, where $(A_j)_{j=0}^{\infty}$ is the uniquely determined sequence of complex $p \times q$ matrices, such that $P(w) = \sum_{j=0}^{\infty} w^j A_j$ holds true for all $w \in \mathbb{C}$. Denote by $\deg P := \sup\{j \in \mathbb{N}_0 : A_j \neq 0_{p\times q}\}$ the *degree of* P. If $k := \deg P \geq 0$, then the matrix A_k is called the *leading coefficient of* P.

In particular, we have $\deg P = -\infty$, if $P(z) = 0_{p\times q}$ for all $z \in \mathbb{C}$.

Notation D.2 For each $n \in \mathbb{N}_0$, let $E_n : \mathbb{C} \to \mathbb{C}^{q\times(n+1)q}$ be defined by

$$E_n(z) := [z^0 I_q, z^1 I_q, z^2 I_q, \dots, z^n I_q].$$

Remark D.3 If P is a complex $q \times q$ matrix polynomial, then $P = E_n Y_n(P)$ for all $n \in \mathbb{N}_0$ with $n \geq \deg P$.

In this appendix we treat monic right orthogonal systems of $q \times q$ matrix polynomials with respect to a given sequence $(s_j)_{j=0}^{2\kappa}$ from $\mathbb{C}^{q\times q}$. For a detailed treatment of this topic under the view of the purposes of this paper we refer the reader to [15, Section 5].

Now we turn our attention to a well-known notion.

Definition D.4 Let $(s_j)_{j=0}^{2\kappa}$ be a sequence of complex $q \times q$ matrices. A sequence $(P_k)_{k=0}^{\kappa}$ of complex $q \times q$ matrix polynomials is called *monic right orthogonal system with respect to* $(s_j)_{j=0}^{2\kappa}$, if it satisfies the following conditions:

(I) For each $k \in \mathbb{Z}_{0,\kappa}$, the matrix polynomial P_k has degree k and leading coefficient I_q.

(II) $[Y_n(P_j)]^* H_n [Y_n(P_k)] = 0_{q\times q}$ for all $j, k \in \mathbb{Z}_{0,\kappa}$ with $j \neq k$, where $n := \max\{j, k\}$ and H_n is given by (4.2).

In this paper, our main interest is directed to the case $(s_j)_{j=0}^{2\kappa} \in \mathcal{H}_{q,2\kappa}^{\succcurlyeq}$.

Using Remark D.3, we can conclude from [15, Propositions 5.8(a1) and 5.9(a)] furthermore:

Proposition D.5 *Let* $(s_j)_{j=0}^{2\kappa} \in \mathcal{H}_{q,2\kappa}^{\succcurlyeq}$ *and let* $(P_k)_{k=0}^{\kappa}$ *be a sequence of complex* $q \times q$ *matrix polynomials, satisfying condition (I) of Definition D.4. Then* $(P_k)_{k=0}^{\kappa}$ *is a monic right orthogonal system with respect to* $(s_j)_{j=0}^{2\kappa}$ *if and only if* $H_{k-1}X_k = y_{k,2k-1}$ *for all* $k \in \mathbb{Z}_{1,\kappa}$*, where* X_k *is taken from the block representation* $Y_k(P_k) = \begin{bmatrix} -X_k \\ I_q \end{bmatrix}$ *and where* H_{k-1} *and* $y_{k,2k-1}$ *are given by* (4.2) *and* (4.1)*, respectively.*

Appendix E: Matrix Polynomials of the Second Kind

This appendix contains the algebraic description of the construction of the so-called matrix polynomials of second kind, which are often used in the framework of orthogonal matrix polynomials. For a detailed exposition of this topic under the view of the purposes of this paper, we refer the reader to [3, Section 4].

Remark E.1 Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. For all $m \in \mathbb{Z}_{1,\kappa}$, then the block Toeplitz matrix

$$\mathbb{S}_m := \begin{bmatrix} s_0 & s_1 & s_2 & \dots & s_m \\ 0 & s_0 & s_1 & \dots & s_{m-1} \\ 0 & 0 & s_0 & \dots & s_{m-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & s_0 \end{bmatrix}.$$

admits the block representation $\mathbb{S}_m = \begin{bmatrix} s_0 & z_{1,m} \\ 0_{mp\times q} & \mathbb{S}_{m-1} \end{bmatrix}$.

Notation E.2 Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices and let P be a complex $q \times q$ matrix polynomial with degree $k := \deg P$ satisfying $k \leq \kappa + 1$. Then let $P^{[\![s]\!]} : \mathbb{C} \to \mathbb{C}^{q\times q}$ be defined by $P^{[\![s]\!]}(z) = 0_{q\times q}$ if $k \leq 0$ and by $P^{[\![s]\!]}(z) := E_{k-1}(z)[0_{kq\times q}, \mathbb{S}_{k-1}]Y_k(P)$ if $k \geq 1$.

Remark E.3 Let P and Q be two complex $q \times q$ matrix polynomials, each having degree at most $\kappa + 1$. Then $(P + Q)^{[\![s]\!]} = P^{[\![s]\!]} + Q^{[\![s]\!]}$. Furthermore, $(PA)^{[\![s]\!]} = P^{[\![s]\!]}A$ for all $A \in \mathbb{C}^{q\times q}$ and $(\eta P)^{[\![s]\!]} = \eta P^{[\![s]\!]}$ for all $\eta \in \mathbb{C}$.

For each $m \in \mathbb{N}$, let $\Delta_{q,0,m} := I_{mq}$. Furthermore, for all $\ell, m \in \mathbb{N}$, let $\Delta_{q,\ell,m} := \begin{bmatrix} I_{mq} \\ 0_{\ell q\times mq} \end{bmatrix}$. The following result should be compared with Proposition D.5:

Lemma E.4 *Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices, let $k \in \mathbb{N}$ with $2k - 1 \leq \kappa$, and let P be a complex $q \times q$ matrix polynomial with degree k and leading coefficient I_q, satisfying $H_{k-1}X = y_{k,2k-1}$, where the matrix X is taken from the block representation $Y_k(P) = \begin{bmatrix} -X \\ I_q \end{bmatrix}$. Let the matrix polynomial Q be defined by $Q(w) := wP(w)$. For all $z \in \mathbb{C}$, then $Q^{[\![s]\!]}(z) = zP^{[\![s]\!]}(z)$.*

Proof We have $\deg Q = k + 1 \leq 2k \leq \kappa + 1$. According to Notation E.2, thus $Q^{[\![s]\!]}(z) = E_k(z)[0_{(k+1)q\times q}, \mathbb{S}_k]Y_{k+1}(Q)$ for all $z \in \mathbb{C}$. In view of Notations D.2 and D.1, hence $\deg Q^{[\![s]\!]} \leq k$ and $Y_k(Q^{[\![s]\!]}) = [0_{(k+1)q\times q}, \mathbb{S}_k]Y_{k+1}(Q)$. Observe that $Y_{k+1}(Q) = \begin{bmatrix} 0_{q\times q} \\ Y_k(P) \end{bmatrix}$, by the definition of Q. Consequently, $Y_k(Q^{[\![s]\!]}) = \mathbb{S}_k Y_k(P)$. From Remark E.1, (4.1), (4.2), and (4.3) we infer

$$\mathbb{S}_k = \begin{bmatrix} s_0 & z_{1,k} \\ 0_{kq\times q} & \mathbb{S}_{k-1} \end{bmatrix}$$

and

$$[s_0, z_{1,k}] = \Delta^*_{q,k-1,1}[y_{0,k-1}, K_{k-1}] = \Delta^*_{q,k-1,1}[H_{k-1}, y_{k,2k-1}].$$

Because of $H_{k-1}X = y_{k,2k-1}$, the latter identity implies $[s_0, z_{1,k}]\begin{bmatrix} -X \\ I_q \end{bmatrix} = 0_{q\times q}$. Thus, we obtain

$$Y_k(Q^{[\![s]\!]}) = \mathbb{S}_k Y_k(P) = \begin{bmatrix} s_0 & z_{1,k} \\ 0_{kq\times q} & \mathbb{S}_{k-1} \end{bmatrix}\begin{bmatrix} -X \\ I_q \end{bmatrix} = \begin{bmatrix} 0_{q\times q} \\ [0_{kq\times q}, \mathbb{S}_{k-1}]Y_k(P) \end{bmatrix}.$$

According to Notation E.2 and $\deg P = k \leq 2k \leq \kappa + 1$, we have $P^{[\![s]\!]}(z) = E_{k-1}(z)[0_{kq\times q}, \mathbb{S}_{k-1}]Y_k(P)$ for all $z \in \mathbb{C}$. Taking into account $\deg Q^{[\![s]\!]} \leq k$, we can conclude with Remark D.3 and Notation D.2, for all $z \in \mathbb{C}$, then

$$Q^{[\![s]\!]}(z) = E_k(z)Y_k(Q^{[\![s]\!]}) = \left[I_q, zE_{k-1}(z)\right]\begin{bmatrix} 0_{q\times q} \\ [0_{kq\times q}, \mathbb{S}_{k-1}]Y_k(P) \end{bmatrix} = zP^{[\![s]\!]}(z).$$

□

The following result can be proved completely similar to [3, Lemma 9.1]:

Lemma E.5 *Let $\alpha \in \mathbb{C}$, let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices, and let P be a complex $q \times q$ matrix polynomial with degree $k := \deg P$ satisfying*

$k \leq \kappa$. Let the matrix polynomial Q be given by $Q(z) := (z - \alpha)P(z)$. Then $Q^{[\![s]\!]} = s_0 P$ if $k \leq 0$ and $Q^{[\![s]\!]} = P^{[\![a]\!]} + s_0 P$ if $k \geq 1$, where the sequence $(a_j)_{j=0}^{\kappa-1}$ is given via (4.6).

References

1. N. I. Akhiezer. *The classical moment problem and some related questions in analysis*. Translated by N. Kemmer. Hafner Publishing Co., New York, 1965.
2. Arthur Albert. Conditions for positive and nonnegative definiteness in terms of pseudoinverses. *SIAM J. Appl. Math.*, 17:434–440, 1969.
3. Abdon E. Choque Rivero and Conrad Mädler. On Hankel positive definite perturbations of Hankel positive definite sequences and interrelations to orthogonal matrix polynomials. *Complex Anal. Oper. Theory*, 8(8):1645–1698, 2014.
4. Abdon Eddy Choque Rivero. On Dyukarev's resolvent matrix for a truncated Stieltjes matrix moment problem under the view of orthogonal matrix polynomials. *Linear Algebra Appl.*, 474:44–109, 2015.
5. Vladimir K. Dubovoj, Bernd Fritzsche, and Bernd Kirstein. *Matricial version of the classical Schur problem*, volume 129 of *Teubner-Texte zur Mathematik [Teubner Texts in Mathematics]*. B. G. Teubner Verlagsgesellschaft mbH, Stuttgart, 1992. With German, French and Russian summaries.
6. Yu. M. Dyukarev. Multiplicative and additive Stieltjes classes of analytic matrix-valued functions and interpolation problems connected with them. II (Russian). *Teor. Funktsiĭ Funktsional. Anal. i Prilozhen.*, (38):40–48, 127, 1982.
7. Yu. M. Dyukarev. Indeterminacy criteria for the Stieltjes matrix moment problem. *Mat. Zametki*, 75(1):71–88, 2004.
8. Yu. M. Dyukarev and Victor Emanuelovich Katsnelson. Multiplicative and additive Stieltjes classes of analytic matrix-valued functions and interpolation problems connected with them. I (Russian). *Teor. Funktsiĭ Funktsional. Anal. i Prilozhen.*, (36):13–27, 126, 1981.
9. Yu. M. Dyukarev and Victor Emanuelovich Katsnelson. Multiplicative and additive Stieltjes classes of analytic matrix-valued functions, and interpolation problems connected with them. III (Russian). *Teor. Funktsiĭ Funktsional. Anal. i Prilozhen.*, (41):64–70, 1984.
10. Yuriy M. Dyukarev, Bernd Fritzsche, Bernd Kirstein, and Conrad Mädler. On truncated matricial Stieltjes type moment problems. *Complex Anal. Oper. Theory*, 4(4):905–951, 2010.
11. Yuriy M. Dyukarev, Bernd Fritzsche, Bernd Kirstein, Conrad Mädler, and Helge C. Thiele. On distinguished solutions of truncated matricial Hamburger moment problems. *Complex Anal. Oper. Theory*, 3(4):759–834, 2009.
12. A. V. Efimov and V. P. Potapov. J-expanding matrix-valued functions, and their role in the analytic theory of electrical circuits. *Uspehi Mat. Nauk*, 28(1(169)):65–130, 1973.
13. Bernd Fritzsche and Bernd Kirstein. Schwache Konvergenz nichtnegativ hermitescher Borelmaße. *Wiss. Z. Karl-Marx-Univ. Leipzig Math.-Natur. Reihe*, 37(4):375–398, 1988.
14. Bernd Fritzsche, Bernd Kirstein, and Conrad Mädler. Matricial canonical moments and parametrization of matricial Hausdorff moment sequences. To appear in Complex Anal. Oper. Theory.
15. Bernd Fritzsche, Bernd Kirstein, and Conrad Mädler. On Hankel nonnegative definite sequences, the canonical Hankel parametrization, and orthogonal matrix polynomials. *Complex Anal. Oper. Theory*, 5(2):447–511, 2011.
16. Bernd Fritzsche, Bernd Kirstein, and Conrad Mädler. On a special parametrization of matricial α-Stieltjes one-sided non-negative definite sequences. In *Interpolation, Schur functions and moment problems. II*, volume 226 of *Oper. Theory Adv. Appl.*, pages 211–250. Birkhäuser/Springer Basel AG, Basel, 2012.

17. Bernd Fritzsche, Bernd Kirstein, and Conrad Mädler. On matrix-valued Herglotz-Nevanlinna functions with an emphasis on particular subclasses. *Math. Nachr.*, 285(14-15):1770–1790, 2012.
18. Bernd Fritzsche, Bernd Kirstein, and Conrad Mädler. On a simultaneous approach to the even and odd truncated matricial Hamburger moment problems. In *Recent advances in inverse scattering, Schur analysis and stochastic processes*, volume 244 of *Oper. Theory Adv. Appl.*, pages 181–285. Birkhäuser/Springer, Cham, 2015.
19. Bernd Fritzsche, Bernd Kirstein, and Conrad Mädler. An application of the Schur complement to truncated matricial power moment problems. `arXiv:1712.06864 [math.CA]`, December 2017. To appear in Oper. Theory Adv. Appl.
20. Bernd Fritzsche, Bernd Kirstein, and Conrad Mädler. On a simultaneous approach to the even and odd truncated matricial Stieltjes moment problem I: An α-Schur–Stieltjes-type algorithm for sequences of complex matrices. *Linear Algebra Appl.*, 521:142–216, 2017.
21. Bernd Fritzsche, Bernd Kirstein, and Conrad Mädler. On a simultaneous approach to the even and odd truncated matricial Stieltjes moment problem II: An α-Schur–Stieltjes-type algorithm for sequences of holomorphic matrix-valued functions. *Linear Algebra Appl.*, 520:335–398, 2017.
22. Bernd Fritzsche, Bernd Kirstein, and Conrad Mädler. On matrix-valued Stieltjes functions with an emphasis on particular subclasses. In *Large truncated Toeplitz matrices, Toeplitz operators, and related topics*, volume 259 of *Oper. Theory Adv. Appl.*, pages 301–352. Birkhäuser/Springer, Cham, 2017.
23. Bernd Fritzsche, Bernd Kirstein, Conrad Mädler, and Torsten Schröder. On the truncated matricial Stieltjes moment problem $\mathsf{M}[[\alpha, \infty); (s_j)_{j=0}^{m}, \leq]$. *Linear Algebra Appl.*, 544:30–114, 2018.
24. Bernd Fritzsche, Bernd Kirstein, Conrad Mädler, and Tilo Schwarz. On a Schur-type algorithm for sequences of complex $p \times q$-matrices and its interrelations with the canonical Hankel parametrization. In *Interpolation, Schur functions and moment problems. II*, volume 226 of *Oper. Theory Adv. Appl.*, pages 117–192. Birkhäuser/Springer Basel AG, Basel, 2012.
25. Bernd Fritzsche, Bernd Kirstein, Conrad Mädler, and Tilo Schwarz. On the concept of invertibility for sequences of complex $p \times q$-matrices and its application to holomorphic $p \times q$-matrix-valued functions. In *Interpolation, Schur functions and moment problems. II*, volume 226 of *Oper. Theory Adv. Appl.*, pages 9–56. Birkhäuser/Springer Basel AG, Basel, 2012.
26. Yong-Jian Hu and Gong-Ning Chen. A unified treatment for the matrix Stieltjes moment problem. *Linear Algebra Appl.*, 380:227–239, 2004.
27. Benjamin Jeschke. *Einige Beiträge zu vollständig nichtdegenerierten matriziellen Momentenproblemen vom α-Stieltjes-Typ*. Dissertation, Universität Leipzig, Leipzig, February 2017.
28. Benjamin Jeschke. Einige Beiträge zu vollständig nichtdegenerierten matriziellen momentenproblemen vom α-Stieltjes-Typ. `arXiv:1703.06759 [math.CA]`, March 2017.
29. I. S. Kats. On Hilbert spaces generated by monotone Hermitian matrix-functions. *Har′kov Gos. Univ. Uč. Zap. 34 = Zap. Mat. Otd. Fiz.-Mat. Fak. i Har′kov. Mat. Obšč. (4)*, 22:95–113 (1951), 1950.
30. I. S. Kats and M. G. Kreĭn. R-functions—analytic functions mapping the upper halfplane into itself (Russian). Appendix I in F. V. Atkinson. *Diskretnye i nepreryvnye granichnye zadachi*. Translated from the English by I. S. Iohvidov and G. A. Karal′nik. Edited and supplemented by I. S. Kats and M. G. Kreĭn. Izdat. "Mir", Moscow, 1968. English translation in *American Mathematical Society Translations, Series 2. Vol. 103: Nine papers in analysis*. American Mathematical Society, Providence, R.I., 1974, pages 1–18.
31. Victor Emanuelovich Katsnelson. Continual analogues of the Hamburger-Nevanlinna theorem and fundamental matrix inequalities of classical problems. I. *Teor. Funktsiĭ Funktsional. Anal. i Prilozhen.*, (36):31–48, 127, 1981.
32. Victor Emanuelovich Katsnelson. Continual analogues of the Hamburger-Nevanlinna theorem and fundamental matrix inequalities of classical problems. II. *Teor. Funktsiĭ Funktsional. Anal. i Prilozhen.*, (37):31–48, 1982.

33. Victor Emanuelovich Katsnelson. Continual analogues of the Hamburger-Nevanlinna theorem and fundamental matrix inequalities of classical problems. III. *Teor. Funktsiĭ Funktsional. Anal. i Prilozhen.*, (39):61–73, 1983.
34. Victor Emanuelovich Katsnelson. Continual analogues of the Hamburger-Nevanlinna theorem, and fundamental matrix inequalities of classical problems. IV. *Teor. Funktsiĭ Funktsional. Anal. i Prilozhen.*, (40):79–90, 1983.
35. Victor Emanuelovich Katsnelson. Methods of J-theory in continuous interpolation problems of analysis. deposited in VINITI, 1983.
36. Victor Emanuelovich Katsnelson. Integral representation of Hermitian positive kernels of mixed type and the generalized Nehari problem. I. *Teor. Funktsiĭ Funktsional. Anal. i Prilozhen.*, (43):54–70, 1985.
37. M. G. Kreĭn and A. A. Nudel′man. *The Markov moment problem and extremal problems*. American Mathematical Society, Providence, R.I., 1977. Ideas and problems of P. L. Čebyšev and A. A. Markov and their further development, Translated from the Russian by D. Louvish, Translations of Mathematical Monographs, Vol. 50.
38. V. P. Potapov. Linear-fractional transformations of matrices (Russian). In *Studies in the theory of operators and their applications (Russian)*, pages 75–97, 177. "Naukova Dumka", Kiev, 1979.
39. Milton Rosenberg. The square-integrability of matrix-valued functions with respect to a non-negative Hermitian measure. *Duke Math. J.*, 31:291–298, 1964.

A Class of Sectorial Relations and the Associated Closed Forms

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Dedicated to V.E. Katsnelson on the occasion of his 75th birthday

Abstract Let T be a closed linear relation from a Hilbert space $\mathfrak{H}$ to a Hilbert space $\mathfrak{K}$ and let $B \in \mathbf{B}(\mathfrak{K})$ be selfadjoint. It will be shown that the relation $T^*(I + iB)T$ is maximal sectorial via a matrix decomposition of B with respect to the orthogonal decomposition $\mathfrak{H} = \overline{\mathrm{dom}}\, T^* \oplus \mathrm{mul}\, T$. This leads to an explicit expression of the corresponding closed sectorial form. These results include the case where $\mathrm{mul}\, T$ is invariant under B. The more general description makes it possible to give an expression for the extremal maximal sectorial extensions of the sum of sectorial relations. In particular, one can characterize when the form sum extension is extremal.

Keywords Sectorial relation · Friedrichs extension · Kreĭn extension · Extremal extension · Form sum

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1 Introduction

A linear relation H in a Hilbert space $\mathfrak{H}$ is said to be *accretive* if $\mathrm{Re}\,(h', h) \geq 0$, $\{h, h'\} \in H$. Note that the closure of an accretive relation is also accretive. An accretive relation H in $\mathfrak{H}$ is said to be *maximal accretive* if the existence of an accretive relation H' in $\mathfrak{H}$ with $H \subset H'$ implies $H' = H$. A maximal accretive relation is automatically closed. In a similar way, a linear relation H in a Hilbert space $\mathfrak{H}$ is said to be *sectorial* with vertex at the origin and semi-angle α, $\alpha \in [0, \pi/2)$, if

$$|\mathrm{Im}\,(h', h)| \leq (\tan\alpha)\,\mathrm{Re}\,(h', h), \quad \{h, h'\} \in H. \tag{1.1}$$

The closure of a sectorial relation is also sectorial. A sectorial relation H in a Hilbert space $\mathfrak{H}$ is said to be *maximal sectorial* if the existence of a sectorial relation H' in $\mathfrak{H}$ with $H \subset H'$ implies $H' = H$. A maximal sectorial relation is automatically closed. Note that a sectorial relation is maximal sectorial if and only if it is maximal as an accretive relation; for more information, see [2, 6].

A sesquilinear form $\mathfrak{t} = \mathfrak{t}[\cdot, \cdot]$ in a Hilbert space $\mathfrak{H}$ is a mapping from $\mathrm{dom}\,\mathfrak{t} \subset \mathfrak{H}$ to $\mathbb{C}$ which is linear in its first entry and antilinear in its second entry. The adjoint $\mathfrak{t}^*$ is defined by $\mathfrak{t}^*[h, k] = \overline{\mathfrak{t}[k, h]}$, $h, k \in \mathrm{dom}\,\mathfrak{t}$; for the diagonal of $\mathfrak{t}$ the notation $\mathfrak{t}[\cdot]$ will be used. A (sesquilinear) form is said to be sectorial with vertex at the origin and semi-angle α, $\alpha \in [0, \pi/2)$, if

$$|\mathfrak{t}_{\mathrm{i}}[h]| \leq (\tan\alpha)\,\mathfrak{t}_{\mathrm{r}}[h], \quad h \in \mathrm{dom}\,\mathfrak{t}, \tag{1.2}$$

where the real part $\mathfrak{t}_{\mathrm{r}}$ and the imaginary part $\mathfrak{t}_{\mathrm{i}}$ are defined by

$$\mathfrak{t}_{\mathrm{r}} = \frac{\mathfrak{t} + \mathfrak{t}^*}{2}, \quad \mathfrak{t}_{\mathrm{i}} = \frac{\mathfrak{t} - \mathfrak{t}^*}{2i}, \quad \mathrm{dom}\,\mathfrak{t}_{\mathrm{r}} = \mathrm{dom}\,\mathfrak{t}_{\mathrm{i}} = \mathrm{dom}\,\mathfrak{t}. \tag{1.3}$$

A sesquilinear form will be called a form in the rest of this note. Observe that the form $\mathfrak{t}_{\mathrm{r}}$ is nonnegative and that the form $\mathfrak{t}_{\mathrm{i}}$ is symmetric, while $\mathfrak{t} = \mathfrak{t}_{\mathrm{r}} + i\,\mathfrak{t}_{\mathrm{i}}$. A sectorial form $\mathfrak{t}$ is said to be *closed* if

$$h_n \to h, \quad \mathfrak{t}[h_n - h_m] \to 0 \quad \Rightarrow \quad h \in \mathrm{dom}\,\mathfrak{t} \quad \text{and} \quad \mathfrak{t}[h_n - h] \to 0.$$

A sectorial form $\mathfrak{t}$ is closed if and only if its real part $\mathfrak{t}_r$ is closed; see [7].

The connection between maximal sectorial relations and closed sectorial forms is given in the so-called first representation theorem; cf. [1, 4, 7–9].

Theorem 1.1 *Let $\mathfrak{t}$ be a closed sectorial form in a Hilbert space $\mathfrak{H}$ with vertex at the origin and semi-angle α, $\alpha \in [0, \pi/2)$. Then there exists a unique maximal sectorial relation H in $\mathfrak{H}$ with vertex at the origin and semi-angle α in $\mathfrak{H}$ such that*

$$\mathrm{dom}\,H \subset \mathrm{dom}\,\mathfrak{t}, \tag{1.4}$$

and for every $\{h, h'\} \in H$ *and* $k \in \operatorname{dom} \mathfrak{t}$ *one has*

$$\mathfrak{t}[h, k] = (h', k). \tag{1.5}$$

Conversely, for every maximal sectorial relation H *with vertex at the origin and semi-angle* α, $\alpha \in [0, \pi/2)$, *there exists a unique closed sectorial form* $\mathfrak{t}$ *such that* (1.4) *and* (1.5) *are satisfied.*

This result contains as a special case the connection between nonnegative selfadjoint relations and closed nonnegative forms. The nonnegative selfadjoint relation H_r corresponding to the real part $\mathfrak{t}_{\rm r}$ of the form $\mathfrak{t}$ is called the real part of H; this notion should not to be confused with the real part introduced in [6].

In the theory of sectorial operators one encounters expressions $T^*(I + iB)T$ where T is a linear operator from a Hilbert space $\mathfrak{H}$ to a Hilbert space $\mathfrak{K}$ and $B \in \mathbf{B}(\mathfrak{K})$ is a selfadjoint operator. In the context of sectorial relations the operator T may be replaced by a linear relation T. A frequently used observation is that when T is a closed linear relation and the multivalued part $\operatorname{mul} T$ is invariant under B, then the product is a maximal sectorial relation; cf. [4]. However, in fact, the relation

$$T^*(I + iB)T \tag{1.6}$$

is maximal sectorial for any closed linear relation T. This will be shown in this note via a matrix decomposition of B with respect to the orthogonal decomposition $\mathfrak{H} = \overline{\operatorname{dom}}\, T^* \oplus \operatorname{mul} T$. In addition the closed sectorial form corresponding to $T^*(I+iB)T$ will be determined. The main argument consists of a reduction to the case where T is an operator. For the convenience of the reader the arguments in the operator case are included. Note that if T is not closed, then $T^*(I + iB)T$ is a sectorial relation which may have maximal sectorial extensions, such as $T^*(I + iB)T^{**}$ and some of these extensions have been determined in [5]; cf. [10].

It is clear that the sum of two sectorial relations is a sectorial relation and there will be maximal sectorial extensions. In [5] the Friedrichs extension has been determined in general, while the Kreĭn extension has been determined only under additional conditions. As an application of the above results for the relation in (1.6) the Kreĭn extension and, in fact, all extremal maximal sectorial extensions of the sum of two sectorial relations will be characterized in general. With this characterization one can determine when the form sum extension is extremal.

2 A Preliminary Result

The first case to be considered is the linear relation $T^*(I + iB)T$, where T a closed linear operator, which is not necessarily densely defined, and $B \in \mathbf{B}(\mathfrak{K})$ is selfadjoint. In this case one can write down a natural closed sectorial form and verify that $T^*(I + iB)T$ is the maximal sectorial relation corresponding to the form via Theorem 1.1.

Theorem 2.1 *Let T be a closed linear operator from a Hilbert space $\mathfrak{H}$ to a Hilbert space $\mathfrak{K}$ and let the operator $B \in \mathbf{B}(\mathfrak{K})$ be selfadjoint. Then the form $\mathfrak{t}$ in $\mathfrak{H}$ defined by*

$$\mathfrak{t}[h,k] = ((I+iB)Th, Tk), \quad h,k \in \operatorname{dom}\mathfrak{t} = \operatorname{dom} T, \tag{2.1}$$

is closed and sectorial with vertex at the origin and semi-angle $\alpha \le \arctan \|B\|$ and the maximal sectorial relation H corresponding to the form $\mathfrak{t}$ is given by

$$H = T^*(I+iB)T, \tag{2.2}$$

with $\operatorname{mul} H = \operatorname{mul} T^* = (\operatorname{dom} T)^{\perp}$. A subset of $\operatorname{dom}\mathfrak{t} = \operatorname{dom} T$ is a core of the form $\mathfrak{t}$ if and only if it is a core of the operator T. Moreover, the nonnegative selfadjoint relation H_r corresponding to the real part $(\mathfrak{t}_H)_{\mathrm{r}}$ of the form $\mathfrak{t}$ is given by

$$H_{\mathrm{r}} = T^*T. \tag{2.3}$$

Proof It is straightforward to check that $\mathfrak{t}$ in (2.1) is a closed sectorial form as indicated, since

$$\mathfrak{t}_r[h,k] = (Th, Tk), \quad \mathfrak{t}_{\mathrm{i}}[h,k] = (BTh, Tk).$$

Therefore, $|\mathfrak{t}_i[h]| = |(BTh, Th)| \le \|B\|\|Th\|^2 = \|B\|\mathfrak{t}_r[h]$, so that $\mathfrak{t}$ is closed and sectorial with vertex at the origin and semi-angle $\alpha \le \arctan \|B\|$. Moreover, since T is closed, it is clear that $\mathfrak{t}_r$ and hence $\mathfrak{t}$ is closed.

Now let $\{h, h'\} \in T^*(I+iB)T$, then there exists $\varphi \in \mathfrak{K}$ such that

$$\{h, \varphi\} \in T, \quad \{(I+iB)\varphi, h'\} \in T^*,$$

from which it follows that

$$(h', h) = (\varphi, \varphi) + i(B\varphi, \varphi).$$

Consequently, one sees that

$$|\mathrm{Im}\,(h'h)| = |(B\varphi, \varphi)| \le \|B\|\,\|\varphi\|^2 = \|B\|\,\mathrm{Re}\,(h', h),$$

which implies that $T^*(I+iB)T$ is a sectorial relation with vertex at the origin and semi-angle $\alpha \le \arctan \|B\|$. Furthermore, observe that the above calculation also shows that $\operatorname{mul} T^*(I+iB)T = \operatorname{mul} T^*$.

To see that $T^*(I+iB)T$ is closed, let $\{h_n, h_n'\} \in T^*(I+iB)T$ converge to $\{h, h'\}$. Then there exist $\varphi_n \in \mathfrak{K}$ such that

$$\{h_n, \varphi_n\} \in T, \quad \{(I+iB)\varphi_n, h_n'\} \in T^*,$$

and the identity Re $(h'_n, h_n) = \|\varphi_n\|^2$ shows that (φ_n) is a Cauchy sequence in $\mathfrak{K}$, so that $\varphi_n \to \varphi$ with $\varphi \in \mathfrak{K}$. Thus

$$\{h_n, \varphi_n\} \to \{h, \varphi\}, \quad \{(I + iB)\varphi_n, h'_n\} \to \{(I + iB)\varphi, h'\}.$$

Since T and T^* are closed, one concludes that $\{h, \varphi\} \in T$ and $\{(I+iB)\varphi, h'\} \in T^*$, which implies that $\{h, h'\} \in T^*(I + iB)T$. Hence $T^*(I + iB)T$ is closed.

Now let H be the maximal sectorial relation corresponding to $\mathfrak{t}$ in (2.1). Assume that $\{h, h'\} \in H$, then for all $k \in \operatorname{dom} \mathfrak{t} = \operatorname{dom} T$

$$\mathfrak{t}[h, k] = (h', k) \quad \text{or} \quad ((I + iB)Th, Tk) = (h', k),$$

which implies that

$$\{(I + iB)Th, h'\} \in T^* \quad \text{or} \quad \{h, h'\} \in T^*(I + iB)T.$$

Consequently, it follows that $H \subset T^*(I + iB)T$. Since $T^*(I + iB)T$ is sectorial and H is maximal sectorial, it follows that $H = T^*(I + iB)T$. In particular, one sees that the closed relation $T^*(I + iB)T$ is maximal sectorial. □

With the closed linear operator T from $\mathfrak{H}$ to $\mathfrak{K}$ and the selfadjoint operator $B \in \boldsymbol{B}(\mathfrak{K})$, consider the following matrix decomposition of B

$$B = \begin{pmatrix} B_{aa} & B_{ab} \\ B_{ba}^* & B_{bb} \end{pmatrix} : \begin{pmatrix} \ker T^* \\ \overline{\operatorname{ran}}\, T \end{pmatrix} \to \begin{pmatrix} \ker T^* \\ \overline{\operatorname{ran}}\, T \end{pmatrix}. \tag{2.4}$$

Then it is clear that

$$\mathfrak{t}[h, k] = ((I + iB)Th, Tk) = ((I + iB_{bb})Th, Tk), \quad h, h \in \operatorname{dom} \mathfrak{t} = \operatorname{dom} T, \tag{2.5}$$

which shows that only the compression of B to $\overline{\operatorname{ran}}\, T$ plays a role in (2.1). In applications involving Theorem 2.1, it is therefore useful to recall the following corollary.

Corollary 2.2 *Let T' be a closed linear operator from the Hilbert space $\mathfrak{H}$ to a Hilbert space $\mathfrak{K}'$ and let the operator $B' \in \mathbf{B}(\mathfrak{K}')$ be selfadjoint. Assume that the form $\mathfrak{t}$ in Theorem 2.1 is also given by*

$$\mathfrak{t}[h, k] = ((I + iB')T'h, T'k), \quad h, k \in \operatorname{dom} \mathfrak{t} = \operatorname{dom} T'.$$

Then there is a unitary mapping U from $\overline{\operatorname{ran}}\, T$ onto $\overline{\operatorname{ran}}\, T'$, such that

$$T' = UT, \quad B'_{bb} = UB_{bb}U^*,$$

where B_{bb} and B'_{bb} stand for the compressions of B and B' to $\overline{\mathrm{ran}}\, T$ and $\overline{\mathrm{ran}}\, T'$, respectively.

Proof By assumption $((I+iB')T'h, T'k) = ((I+iB)Th, Tk)$ for all $h, k \in \mathrm{dom}\, \mathfrak{t}$. This leads to

$$(T'h, T'k) = (Th, Tk) \quad \text{and} \quad (B'T'h, T'k) = (BTh, Tk)$$

for all $h, k \in \mathrm{dom}\, \mathfrak{t}$. Hence the mapping $Th \mapsto T'h$ is unitary, and denote it by U. Then $T' = UT$ and it follows that $(B'T'h, T'k) = (BU^*T'h, U^*T'k)$. □

3 A Matrix Decomposition for $T^*(I + iB)T$

To construct a useful expressions for the product relation $T^*(I + iB)T$ some basic properties of linear relations, their adjoints, and operator parts are used; see [2] for details. Let T be a linear relation from $\mathfrak{H}$ to $\mathfrak{K}$ which is closed; observe that then the subspace $\mathrm{mul}\, T$ is closed. The adjoint T^* of T is the set of all $\{h, h'\} \in \mathfrak{K} \times \mathfrak{H}$ for which

$$(h', f) = (h, f') \quad \text{for all} \quad \{f, f'\} \in T.$$

Hence, the definition of T^* depends on the Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$ in which T is assumed to act. Let $\mathfrak{K}$ have the orthogonal decomposition

$$\mathfrak{K} = \overline{\mathrm{dom}}\, T^* \oplus \mathrm{mul}\, T, \tag{3.1}$$

and let P be the orthogonal projection onto $\overline{\mathrm{dom}}\, T^*$. Observe that $PT \subset T$, since $\{0\} \times \mathrm{mul}\, T \subset T$. Therefore $T^* \subset (PT)^* = T^*P$, where the last equality holds since $P \in \mathbf{B}(\mathfrak{K})$. Then one has

$$(PT)^* = T^* \widehat{\oplus} (\mathrm{mul}\, T \times \{0\}). \tag{3.2}$$

The *orthogonal operator part* T_s of T is defined as $T_s = PT$. Hence T_s is an operator from the Hilbert space $\mathfrak{H}$ to the Hilbert space $\mathfrak{K}$ and $T_s \subset T$. Note that $\mathrm{ran}\, T_s \subset \overline{\mathrm{dom}}\, T^* = \mathfrak{K} \ominus \mathrm{mul}\, T$. Thus one may interpret T_s as an operator from the Hilbert space $\mathfrak{H}$ to the Hilbert space $\overline{\mathrm{dom}}\, T^*$ and one may also consider the adjoint $(T_s)^\times$ of T_s with respect to these spaces. It is not difficult to see the connection between these adjoints: if $\{h, h'\} \in \mathfrak{K} \times \mathfrak{H}$, then

$$\{h, h'\} \in T^* \quad \Leftrightarrow \quad \{h, h'\} \in (T_s)^\times. \tag{3.3}$$

The identity (3.2) shows the difference between $(T_s)^*$ and $(T_s)^\times$.

Let T be a closed linear relation from a Hilbert space $\mathfrak{H}$ to a Hilbert space $\mathfrak{K}$ and let $B \in \mathbf{B}(\mathfrak{K})$ be selfadjoint. In order to study the linear relation

$$T^*(I + iB)T,$$

decompose the Hilbert space $\mathfrak{K}$ as in (3.1) and decompose the selfadjoint operator $B \in \mathbf{B}(\mathfrak{K})$ accordingly:

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{pmatrix} : \begin{pmatrix} \overline{\mathrm{dom}}\, T^* \\ \mathrm{mul}\, T \end{pmatrix} \to \begin{pmatrix} \overline{\mathrm{dom}}\, T^* \\ \mathrm{mul}\, T \end{pmatrix}. \tag{3.4}$$

Here the operators $B_{11} \in \mathbf{B}(\overline{\mathrm{dom}}\, T^*)$ and $B_{22} \in \mathbf{B}(\mathrm{mul}\, T)$ are selfadjoint, while $B_{12} \in \mathbf{B}(\mathrm{mul}\, T, \overline{\mathrm{dom}}\, T^*)$ and $B_{12}^* \in \mathbf{B}(\overline{\mathrm{dom}}\, T^*, \mathrm{mul}\, T)$.

By means of the decomposition (3.4) the following auxiliary operators will be introduced. First, define the operator $C_0 \in \mathbf{B}(\overline{\mathrm{dom}}\, T^*)$ by

$$C_0 = I + B_{12}(I + B_{22}^2)^{-1} B_{12}^*. \tag{3.5}$$

Observe that $C_0 \geq I$ and that $(C_0)^{-1}$ belongs to $\mathbf{B}(\overline{\mathrm{dom}}\, T^*)$ and is a nonnegative operator. Next, define the operator $C \in \mathbf{B}(\overline{\mathrm{dom}}\, T^*)$ by

$$C = C_0^{-\frac{1}{2}} \left[B_{11} - B_{12}(I + B_{22}^2)^{-\frac{1}{2}} B_{22} (I + B_{22}^2)^{-\frac{1}{2}} B_{12}^* \right] C_0^{-\frac{1}{2}}, \tag{3.6}$$

which is clearly selfadjoint.

Lemma 3.1 *Let T be a closed linear relation from a Hilbert space $\mathfrak{H}$ to a Hilbert space $\mathfrak{K}$, let T_{s} be the orthogonal operator part of T, and let the selfadjoint operator $B \in \mathbf{B}(\mathfrak{K})$ be decomposed as in* (3.4)*. Let the operators C_0 and C be defined by* (3.5) *and* (3.6)*. Then*

$$T^*(I + iB)T = (T_{\mathrm{s}})^{\times} C_0^{1/2} (I + iC) C_0^{1/2} T_{\mathrm{s}}, \tag{3.7}$$

and, consequently, $T^*(I + iB)T$ is maximal sectorial and

$$\mathrm{mul}\, T^*(I + iB)T = \mathrm{mul}\, T^* = \mathrm{mul}\, (T_{\mathrm{s}})^{\times}. \tag{3.8}$$

Proof In order to prove the equality in (3.7), assume that $\{h, h'\} \in T^*(I + iB)T$. This means that

$$\{h, \varphi\} \in T \quad \text{and} \quad \{(I + iB)\varphi, h'\} \in T^* \tag{3.9}$$

for some $\varphi \in \mathfrak{K}$. Decompose the element φ as

$$\varphi = \varphi_1 + \varphi_2, \quad \varphi_1 \in \overline{\mathrm{dom}}\, T^*, \;\; \varphi_2 \in \mathrm{mul}\, T. \tag{3.10}$$

Since $\{0, \varphi_2\} \in T$, it is clear that

$$\{h, \varphi\} \in T \quad \Leftrightarrow \quad \{h, \varphi_1\} \in T_{\rm s}. \tag{3.11}$$

Using (3.10) and the above decomposition (3.4) of B, one observes that

$$\{(I + iB)\varphi, h'\} = \left\{ \begin{pmatrix} (I + iB_{11})\varphi_1 + iB_{12}\varphi_2 \\ iB_{12}^*\varphi_1 + (I + iB_{22})\varphi_2 \end{pmatrix}, h' \right\},$$

which implies that the condition $\{(I + iB)\varphi, h'\} \in T^*$ is equivalent to

$$\begin{cases} \{(I + iB_{11})\varphi_1 + iB_{12}\varphi_2, h'\} \in T^*, \\ iB_{12}^*\varphi_1 + (I + iB_{22})\varphi_2 = 0, \end{cases}$$

or, what is the same thing,

$$\begin{cases} \{[I + iB_{11} + B_{12}(I + iB_{22})^{-1}B_{12}^*]\varphi_1, h'\} \in T^*, \\ \varphi_2 = -i(I + iB_{22})^{-1}B_{12}^*\varphi_1. \end{cases} \tag{3.12}$$

Due to the definitions (3.5) and (3.6) and the identity

$$(I + iB_{22})^{-1} = (I + B_{22}^2)^{-\frac{1}{2}}(I - iB_{22})(I + B_{22}^2)^{-\frac{1}{2}},$$

observe that

$$\begin{aligned} I + iB_{11} &+ B_{12}(I + iB_{22})^{-1}B_{12}^* \\ &= C_0 + i[B_{11} - B_{12}(I + B_{22}^2)^{-\frac{1}{2}}B_{22}(I + B_{22}^2)^{-\frac{1}{2}}B_{12}^*] \\ &= C_0^{1/2}(I + iC)C_0^{1/2}. \end{aligned}$$

Therefore, it follows from (3.12), via the equivalence in (3.3), that

$$\{(I + iB)\varphi, h'\} \in T^* \quad \Leftrightarrow \quad \begin{cases} \{C_0^{1/2}(I + iC)C_0^{1/2}\varphi_1, h'\} \in (T_s)^\times, \\ \varphi_2 = -i(I + iB_{22})^{-1}B_{12}^*\varphi_1. \end{cases} \tag{3.13}$$

Combining (3.11) and (3.13), one sees that

$$\{h, h'\} \in (T_{\rm s})^\times C_0^{1/2}(I + iC)C_0^{1/2}T_{\rm s}.$$

Conversely, if this inclusion holds, then there exists $\varphi_1 \in \overline{\rm dom}\, T^*$, such that

$$\{h, \varphi_1\} \in T_{\rm s} \quad \text{and} \quad \{C_0^{1/2}(I + iC)C_0^{1/2}\varphi_1, h'\} \in (T_{\rm s})^\times.$$

Then define $\varphi_2 = -i(I + iB_{22})^{-1}B_{12}^*\varphi_1$, so that $\varphi_2 \in \operatorname{mul} T$. Furthermore, define $\varphi = \varphi_1 + \varphi_2$. Hence $\{h, \varphi\} \in T$, and it follows from (3.13) that

$$\{h, h'\} \in T^*(I + iB)T.$$

Therefore one can rewrite $T^*(I + iB)T$ in the form (3.7).

Observe that $C_0^{\frac{1}{2}} T_s$ is a closed linear operator from the Hilbert space $\mathfrak{H}$ to the Hilbert space $\overline{\operatorname{dom}}\, T^*$ whose adjoint is given by

$$(C_0^{1/2} T_s)^\times = (T_s)^\times C_0^{1/2}. \tag{3.14}$$

Hence, by Theorem 2.1 $(T_s)^\times C_0^{1/2}(I + iC)C_0^{1/2} T_s$ is a maximal sectorial relation in $\mathfrak{H}$ and by the identity (3.7) the same is true for $T^*(I + iB)T$.

The statement in (3.8) follows by tracing the above equivalences for an element $\{0, h'\}$. □

Remark 3.2 Let $\varphi = \varphi_1 + \varphi_2 \in \mathfrak{K}$ be decomposed as in (3.10). Then one has the following equivalence:

$$(I + iB)\varphi \in \overline{\operatorname{dom}}\, T^* \quad \Leftrightarrow \quad (I + iB)\varphi = C_0^{1/2}(I + iC)C_0^{1/2}\varphi_1.$$

To see this, let $\eta = (I + iB)\varphi$. Then $\eta \in \overline{\operatorname{dom}}\, T^*$ if and only if

$$\begin{pmatrix} I + iB_{11} & iB_{12} \\ iB_{12}^* & I + iB_{22} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} \eta \\ 0 \end{pmatrix},$$

where $\overline{\operatorname{dom}}\, T^*$ is interpreted as the subspace $\overline{\operatorname{dom}}\, T^* \times \{0\}$ of $\mathfrak{K}$. Now apply (3.13).

4 A Class of Maximal Sectorial Relations and Associated Forms

The linear relation $T^*(I + iB)T$ is maximal sectorial for any selfadjoint $B \in \mathbf{B}(\mathfrak{K})$ and any closed linear relation T from $\mathfrak{H}$ to $\mathfrak{K}$. Now the corresponding closed sectorial form will be determined. This gives the appropriate version of Theorem 2.1 in terms of relations. In fact, the general result is based on a reduction via Lemma 3.1 to Theorem 2.1.

Theorem 4.1 *Let T be a closed linear relation from a Hilbert space $\mathfrak{H}$ to a Hilbert space $\mathfrak{K}$ and let the selfadjoint operator $B \in \mathbf{B}(\mathfrak{K})$ be decomposed as in* (3.4)*. Let the operators C_0 and C be defined by* (3.5) *and* (3.6)*. Then the form $\mathfrak{t}$ defined by*

$$\mathfrak{t}[h, k] = ((I + iC)\, C_0^{\frac{1}{2}} T_s h, C_0^{\frac{1}{2}} T_s k), \quad h, k \in \operatorname{dom} \mathfrak{t} = \operatorname{dom} T, \tag{4.1}$$

is closed and sectorial with vertex at the origin and semi-angle $\gamma \le \arctan \|C\|$. *Moreover, the maximal sectorial relation* H *corresponding to the form* $\mathfrak{t}$ *is given by*

$$H = (T_{\rm s})^{\times} C_0^{1/2}(I + iC)C_0^{1/2} T_{\rm s} = T^*(I + iB)T. \tag{4.2}$$

A subset of $\operatorname{dom}\mathfrak{t} = \operatorname{dom}T$ *is a core of the form* $\mathfrak{t}$ *if and only if it is a core of the operator* $T_{\rm s}$. *Moreover, the nonnegative selfadjoint relation* H_r *corresponding to the real part* $(\mathfrak{t}_H)_r$ *of the form* $\mathfrak{t}$ *is given by*

$$H_r = (T_{\rm s})^{\times} C_0 T_{\rm s}.$$

Proof Since $C_0^{\frac{1}{2}} T_{\rm s}$ is a closed linear operator from the Hilbert space $\mathfrak{H}$ to the Hilbert space $\overline{\operatorname{dom}}\, T^*$, Theorem 2.1 (with $\mathfrak{K}$ replaced by $\overline{\operatorname{dom}}\, T^*$, B by C, and T by $C_0^{1/2}T_{\rm s}$) shows that the form $\mathfrak{t}$ in (4.1) is closed and sectorial with vertex at the origin and semi-angle $\gamma \le \arctan \|C\|$. Moreover, the maximal sectorial relation associated with the form $\mathfrak{t}$ is given by

$$(C_0^{1/2}T_{\rm s})^{\times}(I + iC)C_0^{1/2}T_{\rm s} = (T_{\rm s})^{\times}C_0^{1/2}(I + iC)C_0^{1/2}T_{\rm s},$$

cf. (2.1), (2.2), and (3.14). The identities in (4.2) are clear from Lemma 3.1. The assertion concerning the core holds, since the factor C_0 is bounded with bounded inverse. The formula (4.2) shows that

$$(\mathfrak{t}_H)_r[h, k] = (C_0^{\frac{1}{2}} T_{\rm s}h, C_0^{\frac{1}{2}} T_{\rm s}k), \quad h, k \in \operatorname{dom}\mathfrak{t} = \operatorname{dom}T,$$

and hence $H_{\rm r} = (C_0^{1/2}T_{\rm s})^{\times}C_0^{1/2}T_{\rm s} = (T_{\rm s})^{\times}C_0T_{\rm s}$ (cf. the discussion above). □

Recall that if $\{h, h'\} \in T^*(I + iB)T$, then $\{h, \varphi\} \in T$ and $\{(I + iB)\varphi, h'\} \in T^*$. The last inclusion implies the condition $(I+iB)\varphi \in \operatorname{dom}T^* \subset \overline{\operatorname{dom}}\, T^*$, giving rise to $\varphi_2 = -i(I + iB_{22})^{-1}B_{12}^*\varphi_1$. Thus, for instance, when $B = \operatorname{diag}(B_{11}, B_{22})$, it follows that $\varphi_2 = 0$, so that it is immediately clear that $\gamma \le \arctan \|B_{11}\|$, independent of B_{22}. Note that the following assertions are equivalent:

(i) $B = \operatorname{diag}(B_{11}, B_{22})$;
(ii) $B_{12} = 0$;
(iii) $C_0 = I$;
(iv) $\operatorname{mul}T$ is invariant under B,

in which case $C = B_{11}$. Hence, if $\operatorname{mul}T$ is invariant under B, i.e., if any of the assertions (i)–(iv) hold, then Theorem 4.1 gives the following corollary, which coincides with [4, Theorem 5.1]. In the case where $\operatorname{mul}T = \{0\}$ the corollary reduces to Theorem 2.1.

Corollary 4.2 *Let* T *be a closed linear relation from a Hilbert space* $\mathfrak{H}$ *to a Hilbert space* $\mathfrak{K}$, *let* $T_{\rm s}$ *be the orthogonal operator part of* T, *and let* $\operatorname{mul}T$ *be invariant* *under the selfadjoint operator* $B \in \mathbf{B}(\mathfrak{K})$*, so that* $B = \operatorname{diag}(B_{11}, B_{22})$*. Then the form* $\mathfrak{t}$ *defined by*

$$\mathfrak{t}[h,k] = ((I + iB_{11})T_{\rm s}h, T_{\rm s}k), \quad h,k \in \operatorname{dom}\mathfrak{t} = \operatorname{dom}T,$$

is closed and sectorial with vertex at the origin and semi-angle $\gamma \le \arctan\|B_{11}\|$*. Moreover, the maximal sectorial relation* H *corresponding to the form* $\mathfrak{t}$ *is given by*

$$H = (T_{\rm s})^{\times}(I + iB_{11})T_{\rm s} = T^*(I + iB)T.$$

In the case that $\operatorname{mul}T$ is not invariant under B, one has $C_0 \neq I$, and the formulas are different: for instance, the real part $(\mathfrak{t}_H)_r$ in Theorem 4.1 is of the form

$$(\mathfrak{t}_H)_r[h,k] = (C_0^{\frac{1}{2}}T_{\rm s}h, C_0^{\frac{1}{2}}T_{\rm s}k), \quad h,k \in \operatorname{dom}\mathfrak{t} = \operatorname{dom}T_{\rm s} = \operatorname{dom}T.$$

Example 4.3 Assume that $B_{11} \neq 0$ and

$$B_{11} = B_{12}(I + B_{22}^2)^{-\frac{1}{2}}B_{22}(I + B_{22}^2)^{-\frac{1}{2}}B_{12}^*,$$

so that $C = 0$. In this case the maximal sectorial relation $H = T^*(I + iB)T$ in Theorem 4.1 is selfadjoint, i.e., $H = H_r$ and the associated form $\mathfrak{t}$ is nonnegative. On the other hand, with such a choice of B the operator part of T determines the maximal sectorial relation $(T_{\rm s})^*(I + iB)T_{\rm s}$ with semi-angle $\arctan\|B_{11}\| > 0$, while $T^*(I + iB)T$ has semi-angle $\gamma = 0$.

5 Maximal Sectorial Relations and Their Representations

Let H be a maximal sectorial relation in $\mathfrak{H}$ and let the closed sectorial form $\mathfrak{t}_H$ correspond to H; cf. Theorem 1.1. Since the closed form $\mathfrak{t}_H$ is sectorial, one has the inequality

$$|(\mathfrak{t}_H)_{\rm i}[h]| \le (\tan\alpha)(\mathfrak{t}_H)_{\rm r}[h], \quad h \in \operatorname{dom}\mathfrak{t}, \tag{5.1}$$

and in this situation the real part $(\mathfrak{t}_H)_{\rm r}$ is a closed nonnegative form. Hence by the first representation theorem there exists a nonnegative selfadjoint relation $H_{\rm r}$, the so-called *real part* of H, such that $\operatorname{dom}H_{\rm r} \subset \operatorname{dom}(\mathfrak{t}_H)_{\rm r} = \operatorname{dom}\mathfrak{t}_H$ and

$$(\mathfrak{t}_H)_{\rm r}[h,k] = (h',k), \quad \{h,h'\} \in H_{\rm r}, \quad k \in \operatorname{dom}(\mathfrak{t}_H)_{\rm r} = \operatorname{dom}\mathfrak{t}_H.$$

This real part $H_{\rm r}$, not to be confused with the real part introduced in [6], will play an important role in formulating the second representation theorem below. First the case where H is a maximal sectorial operator will be considered, in which case H is automatically densely defined; see [7].

Lemma 5.1 *Let H be an maximal sectorial operator in $\mathfrak{H}$, let the closed sectorial form $\mathfrak{t}_H$ correspond to H via Theorem 1.1, and let H_r be the real part of H. Then there exists a unique selfadjoint operator $G \in \mathbf{B}(\mathfrak{H})$ with $\|G\| = \tan\alpha$, of the form*

$$G = \begin{pmatrix} 0 & 0 \\ 0 & G_{bb} \end{pmatrix} : \begin{pmatrix} \ker H_\mathrm{r} \\ \overline{\mathrm{ran}}\, H_\mathrm{r} \end{pmatrix} \to \begin{pmatrix} \ker H_\mathrm{r} \\ \overline{\mathrm{ran}}\, H_\mathrm{r} \end{pmatrix}, \tag{5.2}$$

such that

$$\mathfrak{t}_H[h,k] = ((I+iG)(H_r)^{\frac{1}{2}}h, (H_r)^{\frac{1}{2}}k), \quad h,k \in \mathrm{dom}\, \mathfrak{t}_H = \mathrm{dom}\, H_r^{\frac{1}{2}}. \tag{5.3}$$

Moreover, the corresponding maximal sectorial relation H is given by

$$H = (H_\mathrm{r})^{\frac{1}{2}}(I+iG)(H_\mathrm{r})^{\frac{1}{2}},$$

with $\mathrm{mul}\, H = \mathrm{mul}\, H_\mathrm{r}$.

Proof The inequality

$$|(\mathfrak{t}_H)_\mathrm{i}[h,k]|^2 \leq C\mathfrak{t}_\mathrm{r}[h]\mathfrak{t}_\mathrm{r}[k] = C\|H_\mathrm{r}^{\frac{1}{2}}h\|\|H_\mathrm{r}^{\frac{1}{2}}k\|, \quad h,k \in \mathrm{dom}\,,$$

shows the existence of a selfadjoint operator G in $\mathfrak{H} \ominus \ker H$ such that

$$(\mathfrak{t}_H)_\mathrm{i}[h,k] = (G(H_\mathrm{r})^{\frac{1}{2}}h, (H_\mathrm{r})^{\frac{1}{2}}k), \quad h,k \in \mathrm{dom}\,(H_\mathrm{r})^{\frac{1}{2}}. \tag{5.4}$$

Extend G to all of $\mathfrak{H}$ in a trivial way, so that the same formula remains valid; see Corollary 2.2. It follows from the decomposition $\mathfrak{t} = \mathfrak{t}_\mathrm{r} + i\mathfrak{t}_\mathrm{i}$, cf. (1.3), and the identities (5.5) and (5.4), that

$$\mathfrak{t}_H = (\mathfrak{t}_H)_\mathrm{r} + i(\mathfrak{t}_H)_\mathrm{i},$$

so that

$$\mathfrak{t}_H = [h,k] = ((H_r)_\mathrm{s}^{\frac{1}{2}}h, (H_r)_\mathrm{s}^{\frac{1}{2}}k) + i(G(H_r)_\mathrm{s}^{\frac{1}{2}}h, (H_r)_\mathrm{s}^{\frac{1}{2}}k).$$

This last identity immediately gives (5.3). The rest follows from Corollary 4.2. □

Now let H be a maximal sectorial relation, let H_r be its real part, and let $(H_r)_\mathrm{s}$ be its orthogonal operator part. Then one obtains the representation

$$(\mathfrak{t}_H)_\mathrm{r}[h,k] = (((H_r))_\mathrm{s}^{\frac{1}{2}}h, ((H_r)_\mathrm{s})^{\frac{1}{2}}k), \quad h,k \in \mathrm{dom}\,(\mathfrak{t}_H)_\mathrm{r} = \mathrm{dom}\,((H_r)_\mathrm{s})^{\frac{1}{2}}, \tag{5.5}$$

cf. Theorem 2.1. Now apply Corollary 4.2 and therefore one may formulate the second representation theorem as follows.

Theorem 5.2 *Let H be a maximal sectorial relation in $\mathfrak{H}$, let the closed sectorial form $\mathfrak{t}_H$ correspond to H via Theorem 1.1, and let $H_{\rm r}$ be the real part of H. Then there exists a selfadjoint operator $G \in \mathbf{B}(\mathfrak{H})$ with $\|G\| = \tan\alpha$, such that G is trivial on* $\ker H_{\rm r} \oplus \operatorname{mul} H_{\rm r}$, *and*

$$\mathfrak{t}_H[h,k] = ((I+iG)((H_{\rm r})_{\rm s})^{\frac{1}{2}}h, ((H_{\rm r})_{\rm s})^{\frac{1}{2}}k), \quad h,k \in \operatorname{dom}\mathfrak{t}_H = \operatorname{dom} H_{\rm r}^{\frac{1}{2}}. \tag{5.6}$$

Moreover, the maximal sectorial relation H is given by

$$H = (((H_{\rm r})_s)^{\frac{1}{2}})^{\times}(I+iG)((H_{\rm r})_s)^{\frac{1}{2}}, \tag{5.7}$$

with $\operatorname{mul} H = \operatorname{mul} H_{\rm r}$.

Next, it is assumed that H is a maximal sectorial relation of the form $H = T^*(I+iB)T$, where T is a closed linear relation from a Hilbert space $\mathfrak{H}$ to a Hilbert space $\mathfrak{K}$ and the operator $B \in \mathbf{B}(\mathfrak{K})$ is selfadjoint. Let the operators C_0 and C be defined by (3.5) and (3.6), then

$$H = (T_{\rm s})^{\times}\, C_0^{1/2}(I+iC)C_0^{1/2}\, T_{\rm s},$$

while the corresponding closed sectorial form is given

$$\mathfrak{t}[h,k] = ((I+iC)\, C_0^{\frac{1}{2}} T_{\rm s}\, h, C_0^{\frac{1}{2}} T_{\rm s}\, k), \quad h,k \in \operatorname{dom}\mathfrak{t} = \operatorname{dom} T.$$

To compare these expressions with (5.6) and (5.7), observe that

$$(C_0^{\frac{1}{2}} T_{\rm s}\, h, C_0^{\frac{1}{2}} T_{\rm s}\, k) = ((H_r)_{\rm s})^{\frac{1}{2}} h, ((H_r)_{\rm s})^{\frac{1}{2}} k)$$

and

$$(C\, C_0^{\frac{1}{2}} T_{\rm s}\, h, C_0^{\frac{1}{2}} T_{\rm s}\, k) = (G((H_r)_{\rm s})^{\frac{1}{2}} h, ((H_r)_{\rm s})^{\frac{1}{2}} k).$$

It is clear from (4.1) that only the (selfadjoint) compression of C to $\overline{\operatorname{ran}}\, C_0^{1/2} T_{\rm s}$ contributes to the form (4.2), so that it is straightforward to set up a unitary mapping; cf. Corollary 2.2.

6 Extremal Maximal Sectorial Extensions of Sums of Maximal Sectorial Relations

Let H_1 and H_2 be maximal sectorial relations in a Hilbert space $\mathfrak{H}$. Then the sum $H_1 + H_2$ is a sectorial relation in $\mathfrak{H}$ with

$$\operatorname{dom}(H_1 + H_2) = \operatorname{dom} H_1 \cap \operatorname{dom} H_2,$$

so that the sum is not necessarily densely defined. In particular, $H_1 + H_2$ and its closure need not be operators, since

$$\text{mul}\,(H_1 + H_2) = \text{mul}\,H_1 + \text{mul}\,H_2. \tag{6.1}$$

To describe the class of extremal maximal sectorial extensions of $H_1 + H_2$ some basic notations are recalled from [5], together with the description of the Friedrichs and Kreĭn extensions

$$(H_1 + H_2)_F \quad \text{and} \quad (H_1 + H_2)_K$$

of $H_1 + H_2$, respectively. In order to describe the whole class of extremal extensions of $H_1 + H_2$ and the corresponding closed forms a proper description of the closed sectorial form $\mathfrak{t}_K$ is essential. The results in Sects. 3 and 4 allow a general treatment that will relax the additional conditions in [5].

6.1 Basic Notions

Let H_1 and H_2 be maximal sectorial relations and decompose them as follows

$$H_j = A_j^{\frac{1}{2}}(I + iB_j)A_j^{\frac{1}{2}}, \quad 1 \le j \le 2, \tag{6.2}$$

where A_j (the real part of H_j), $1 \le j \le 2$, are nonnegative selfadjoint relations in $\mathfrak{H}$ and B_j, $1 \le j \le 2$, are bounded selfadjoint operators in $\mathfrak{H}$ which are trivial on $\ker A_j \oplus \text{mul}\,A_j$; cf. Theorem 5.2. Furthermore, if A_1 and A_2 are decomposed as

$$A_j = A_{js} \oplus A_{j\infty}, \quad 1 \le j \le 2,$$

where $A_{j\infty} = \{0\} \times \text{mul}\,A_j$, $1 \le j \le 2$, and A_{js}, $1 \le j \le 2$, are densely defined nonnegative selfadjoint operators (defined as orthogonal complements in the graph sense), then the uniquely determined square roots of A_j, $1 \le j \le 2$ are given by

$$A_j^{\frac{1}{2}} = A_{js}^{\frac{1}{2}} \oplus A_{j\infty}, \quad 1 \le j \le 2.$$

Associated with H_1 and H_2 is the relation Φ from $\mathfrak{H} \times \mathfrak{H}$ to $\mathfrak{H}$, defined by

$$\Phi = \left\{ \left\{\{f_1, f_2\}, f_1' + f_2'\right\} : \{f_j, f_j'\} \in A_j^{\frac{1}{2}},\ 1 \le j \le 2 \right\}. \tag{6.3}$$

Clearly, Φ is a relation whose domain and multivalued part are given by

$$\text{dom}\,\Phi = \text{dom}\,A_1^{\frac{1}{2}} \times \text{dom}\,A_2^{\frac{1}{2}}, \quad \text{mul}\,\Phi = \text{mul}\,H_1 + \text{mul}\,H_2.$$

The relation Φ is not necessarily densely defined in $\mathfrak{H} \times \mathfrak{H}$, so that in general Φ^* is a relation as $\text{mul}\,\Phi^* = (\text{dom}\,\Phi)^\perp$. Furthermore, the adjoint Φ^* of Φ is the relation from $\mathfrak{H}$ to $\mathfrak{H} \times \mathfrak{H}$, given by

$$\Phi^* = \left\{ \left\{ h, \{h'_1, h'_2\} \right\} : \{h, h'_j\} \in A_j^{\frac{1}{2}},\ 1 \le j \le 2 \right\}. \tag{6.4}$$

The identity (6.4) shows that the (orthogonal) operator part $(\Phi^*)_s$ of Φ^* is given by:

$$\begin{aligned} (\Phi^*)_s &= \left\{ \left\{ h, \{h'_1, h'_2\} \right\} : \{h, h'_j\} \in A_{js}^{\frac{1}{2}},\ 1 \le j \le 2 \right\} \\ &= \left\{ \left\{ h, \{A_{1s}^{\frac{1}{2}} h, A_{2s}^{\frac{1}{2}} h\} \right\} : h \in \text{dom}\, A_1^{\frac{1}{2}} \cap \text{dom}\, A_2^{\frac{1}{2}} \right\}. \end{aligned} \tag{6.5}$$

The identities (6.4) and (6.5) show that

$$\text{dom}\,\Phi^* = \text{dom}\, A_1^{\frac{1}{2}} \cap \text{dom}\, A_2^{\frac{1}{2}},\ \ \text{mul}\,\Phi^* = \text{mul}\, H_1 \times \text{mul}\, H_2,\ \ \text{ran}\,(\Phi^*)_s = \mathfrak{F}_0,$$

where the subspace $\mathfrak{F}_0 \subset \mathfrak{H} \times \mathfrak{H}$ is defined by

$$\mathfrak{F}_0 = \left\{ \left\{ A_{1s}^{\frac{1}{2}} h, A_{2s}^{\frac{1}{2}} h \right\} : h \in \text{dom}\, A_1^{\frac{1}{2}} \cap \text{dom}\, A_2^{\frac{1}{2}} \right\}. \tag{6.6}$$

The closure of $\mathfrak{F}_0$ in $\mathfrak{H} \times \mathfrak{H}$ will be denoted by $\mathfrak{F}$. Define the relation Ψ from $\mathfrak{H}$ to $\mathfrak{H} \times \mathfrak{H}$ by

$$\Psi = \left\{ \left\{ h, \left\{ A_{1s}^{\frac{1}{2}} h, A_{2s}^{\frac{1}{2}} h \right\} \right\} : h \in \text{dom}\, H_1 \cap \text{dom}\, H_2 \right\} \subset \mathfrak{H} \times (\mathfrak{H} \times \mathfrak{H}). \tag{6.7}$$

It follows from this definition that

$$\text{dom}\,\Psi = \text{dom}\, H_1 \cap \text{dom}\, H_2, \quad \text{mul}\,\Psi = \{0\}, \quad \text{ran}\,\Psi = \mathfrak{E}_0,$$

where the space $\mathfrak{E}_0 \subset \mathfrak{H} \times \mathfrak{H}$ is defined by

$$\mathfrak{E}_0 = \left\{ \left\{ A_{1s}^{\frac{1}{2}} f, A_{2s}^{\frac{1}{2}} f \right\} : f \in \text{dom}\, H_1 \cap \text{dom}\, H_2 \right\}. \tag{6.8}$$

Observe that $\mathfrak{E}_0 \subset \mathfrak{F}_0$. The closure of $\mathfrak{E}_0$ in $\mathfrak{H} \times \mathfrak{H}$ will be denoted by $\mathfrak{E}$. Hence,

$$\mathfrak{E} \subset \mathfrak{F}. \tag{6.9}$$

Comparison of (6.5) and (6.7) shows

$$\Psi \subset (\Phi^*)_s, \tag{6.10}$$

and thus the operator Ψ is closable and $\Psi^{**} \subset (\Phi^*)_s$. It follows from $\overline{\mathrm{dom}}\,\Psi^* = (\mathrm{mul}\,\Psi^{**})^\perp$ and $\mathrm{mul}\,\Psi^* = (\mathrm{dom}\,\Psi)^\perp$, that

$$\overline{\mathrm{dom}}\,\Psi^* = \mathfrak{H}, \quad \mathrm{mul}\,\Psi^* = (\mathrm{dom}\,H_1 \cap \mathrm{dom}\,H_2)^\perp.$$

Next, define the relation K from $\mathfrak{H} \times \mathfrak{H}$ to $\mathfrak{H}$ by

$$\begin{aligned} K = \Big\{ & \{\{(I + iB_1)A_{1s}^{\frac{1}{2}} f, (I + iB_2)A_{2s}^{\frac{1}{2}} f\}, f_1' + f_2'\} : \\ & \{(I + iB_1)A_{1s}^{\frac{1}{2}} f, f_1'\} \in A_1^{\frac{1}{2}}, \{(I + iB_2)A_{2s}^{\frac{1}{2}} f, f_2'\} \in A_2^{\frac{1}{2}} \Big\} \\ \subset \ & (\mathfrak{H} \times \mathfrak{H}) \times \mathfrak{H}. \end{aligned} \tag{6.11}$$

Clearly, the domain and multivalued part of K are given by

$$\mathrm{dom}\,K = \mathfrak{D}_0, \quad \mathrm{mul}\,K = \mathrm{mul}\,(H_1 + H_2),$$

where

$$\mathfrak{D}_0 = \left\{ \{(I + iB_1)A_{1s}^{1/2} f, (I + iB_2)A_{2s}^{1/2} f\} : f \in \mathrm{dom}\,H_1 \cap \mathrm{dom}\,H_2 \right\}. \tag{6.12}$$

The closure of $\mathfrak{D}_0$ in $\mathfrak{H} \times \mathfrak{H}$ will be denoted by $\mathfrak{D}$.

Lemma 6.1 *The relations K, Φ, and Ψ satisfy the following inclusions:*

$$K \subset \Phi \subset \Psi^*, \quad \Psi \subset \Phi^* \subset K^*. \tag{6.13}$$

Proof To see this note that $K \subset \Phi$ follows from (6.3) and (6.11), and that $\Psi \subset \Phi^*$ follows from (6.4) and (6.7). Therefore, also $\Phi^* \subset K^*$ and $\Phi \subset \Phi^{**} \subset \Psi^*$. □

6.2 The Friedrichs and the Kreĭn Extensions of $H_1 + H_2$

The descriptions of the Friedrichs extension and the Kreĭn extension $(H_1+H_2)_F$ and $(H_1 + H_2)_K$ of $H_1 + H_2$ are now recalled from [5]. For this, define the orthogonal sum of the operators B_1 and B_2 in $\mathfrak{H} \times \mathfrak{H}$ by

$$B_\oplus := B_1 \oplus B_2 = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}.$$

The descriptions of $(H_1 + H_2)_F$ and $(H_1 + H_2)_K$ incorporate the initial data on the factorizations (6.2) of H_1 and H_2 via the mappings Φ, Ψ, and K in Sect. 6.1. The construction of the Friedrichs extension was given in [5, Theorem 3.2], where some further details and a proof of the following result can be found. The new additions in the next theorem are the second representations for $(H_1 + H_2)_F$ and $\mathfrak{t}_F$ that will be needed in the rest of this paper.

Theorem 6.2 *Let H_1 and H_2 be maximal sectorial and let Ψ be defined by* (6.7). *Then the Friedrichs extension of $H_1 + H_2$ has the expression*

$$(H_1 + H_2)_F = \Psi^*(I + iB_\oplus)\Psi^{**} = \Psi^* C_0^{1/2}(I + iC)C_0^{1/2} P_{\mathfrak{D}}(\Psi^{**})_s. \tag{6.14}$$

The closed sectorial form $\mathfrak{t}_F$ associated with $(H_1 + H_2)_F$ is given by

$$\mathfrak{t}_F[f, g] = ((I + iB_\oplus)\Psi^{**} f, \Psi^{**} g) = (C_0^{1/2}(I + iC)C_0^{1/2} P_{\mathfrak{D}}(\Psi^{**})_s f, P_{\mathfrak{D}}(\Psi^{**})_s g), \tag{6.15}$$

for all $f, g \in \operatorname{dom} \mathfrak{t}_F = \operatorname{dom} \Psi^{**}$.

Proof As indicated the first expressions for $(H_1 + H_2)_F$ in (6.14) and $\mathfrak{t}_F$ in (6.15) have been proved in [5, Theorem 3.2] and, hence, it suffices to derive the second expressions in (6.14) and (6.15).

By definition, one has $\operatorname{ran} \Psi = \mathfrak{E}_0$ (see (6.7), (6.8)), and by Lemma 6.1 one has $\Psi \subset \Psi^{**} \subset K^*$, which after projection onto $\mathfrak{D} = \overline{\operatorname{dom}} K$ yields

$$P_{\mathfrak{D}} \Psi^{**} \subset P_{\mathfrak{D}} K^* = (K^*)_s.$$

Notice that $\mathfrak{D}_0 = \operatorname{dom} K = (I + iB_\oplus)\mathfrak{E}_0$ (see (6.8), (6.12)). Since the operator $I + iB_\oplus$ is bounded with bounded inverse, one has the equality

$$\mathfrak{D} = (I + iB_\oplus)\mathfrak{E}. \tag{6.16}$$

It follows that the range of $(I + iB_\oplus)\Psi^{**}$ belongs to $\mathfrak{D} = \overline{\operatorname{dom}} K$. Now by Remark 3.2 this implies that for all $f \in \operatorname{dom} \Psi^{**}$ one has the equality

$$(I + B_\oplus)(\Psi^{**})_s f = C_0^{1/2}(I + iC)C_0^{1/2} P_{\mathfrak{D}}(\Psi^{**})_s f. \tag{6.17}$$

This leads to

$$\Psi^*(I + iB_\oplus)\Psi^{**} = \Psi^* C_0^{1/2}(I + iC)C_0^{1/2} P_{\mathfrak{D}}(\Psi^{**})_s,$$

which proves (6.14). Similarly by substituting (6.17) into the first formula for $\mathfrak{t}_F$ and noting that $P_{\mathfrak{D}} C_0^{1/2} = P_{\mathfrak{D}}$, one obtains the second formula in (6.15). □

Also the construction of the Kreĭn extension for the sum $H_1 + H_2$ can be found in [5, Theorem 3.2]. However, the corresponding form $\mathfrak{t}_K$ was described only under additional conditions to prevent the difficulty that appears by the fact that the multivalued part of $(H_1 + H_2)_K$ is in general not invariant under the mapping $B_\oplus$. Theorem 4.1 allows a removal of these additional conditions and leads to a description of the form $\mathfrak{t}_K$ in the general situation.

For this purpose, decompose the Hilbert space $\mathfrak{H} \times \mathfrak{H}$ as follows

$$\mathfrak{H} \times \mathfrak{H} = \overline{\mathrm{dom}}\, K \oplus \mathrm{mul}\, K^*, \tag{6.18}$$

and let P be the orthogonal projection onto $\overline{\mathrm{dom}}\, K$. Moreover, decompose the selfadjoint operator $B_\oplus \in \mathbf{B}(\mathfrak{H} \times \mathfrak{H})$ accordingly:

$$B_\oplus = \begin{pmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{pmatrix} : \begin{pmatrix} \overline{\mathrm{dom}}\, K \\ \mathrm{mul}\, K^* \end{pmatrix} \to \begin{pmatrix} \overline{\mathrm{dom}}\, K \\ \mathrm{mul}\, K^* \end{pmatrix}. \tag{6.19}$$

Next define the operator $C_0 \in \mathbf{B}(\overline{\mathrm{dom}}\, K^*)$ by

$$C_0 = I + B_{12}(I + B_{22}^2)^{-1} B_{12}^*, \tag{6.20}$$

and the operator $C \in \mathbf{B}(\overline{\mathrm{dom}}\, K^*)$ by

$$C = C_0^{-\frac{1}{2}} \left[B_{11} - B_{12}(I + B_{22}^2)^{-\frac{1}{2}} B_{22} (I + B_{22}^2)^{-\frac{1}{2}} B_{12}^* \right] C_0^{-\frac{1}{2}}, \tag{6.21}$$

which is clearly selfadjoint.

Theorem 6.3 *Let H_1 and H_2 be maximal sectorial relations in a Hilbert space $\mathfrak{H}$, let K be defined by (6.11), and let C_0 and C be given by (6.20) and (6.21), respectively. Then the Kreĭn extension of $H_1 + H_2$ has the expression*

$$(H_1 + H_2)_K = K^{**}(I + iB_\oplus)K^* = ((K^*)_{\mathrm{s}})^\times\, C_0^{1/2}(I + iC)C_0^{1/2}\, (K^*)_{\mathrm{s}}.$$

The closed sectorial form $\mathfrak{t}_K$ associated with $(H_1 + H_2)_K$ is given by

$$\mathfrak{t}_K[f, g] = ((I + iC)C_0^{1/2}(K^*)_{\mathrm{s}} f, C_0^{1/2}(K^*)_{\mathrm{s}} g), \quad f, g \in \mathrm{dom}\, \mathfrak{t}_K = \mathrm{dom}\, K^*.$$

Proof The first equality in the first statement is proved in [5, Theorem 3.2]. The second equality is obtained by applying Theorem 4.1 to the sectorial relation $K^{**}(I + iB_\oplus)K^*$.

The statement concerning the form $\mathfrak{t}_K$ is a consequence of this second representation of $(H_1 + H_2)_K$, since $C_0^{1/2}(K^*)_s$ is a closed operator and hence one can apply Theorem 2.1 to get the desired expression for the corresponding form $\mathfrak{t}_K$. □

The form $\mathfrak{t}_K$ described in Theorem 6.3 can be used to give a complete description of all *extremal maximal sectorial extensions* of the sum $H_1 + H_2$. Namely, a maximal sectorial extension $\widetilde{H}$ of a sectorial relation S is extremal precisely when the corresponding closed sectorial form $\mathfrak{t}_{\widetilde{H}}$ is a restriction of the closed sectorial form $\mathfrak{t}_K$ generated by the Kreĭn extension S_K of S; see e.g. [4, Definition 7.7, Theorems 8.2, 8.4, 8.5]. Therefore, Theorem 6.3 implies the following description of all extremal maximal sectorial extensions of $H_1 + H_2$.

Theorem 6.4 *Let H_1 and H_2 be maximal sectorial relations in $\mathfrak{H}$, let Ψ and K be defined by (6.7) and (6.11), respectively, and let $P_{\mathfrak{D}}$ be the orthogonal projection from $\mathfrak{H} \times \mathfrak{H}$ onto $\mathfrak{D} = \overline{\mathrm{dom}}\, K$. Then the following statements are equivalent:*

(i) *$\widetilde{H}$ is an extremal maximal sectorial extension of $H_1 + H_2$;*
(ii) *$\widetilde{H} = R^*(I + iC)R$, where R is a closed linear operator satisfying*

$$C_0^{1/2} P_{\mathfrak{D}} \Psi^{**} \subset R \subset C_0^{1/2} (K^*)_{\mathrm{s}}.$$

Proof For comparison with the abstract results this statement will be proved by means of the constructions used in [4]. Let $S = H_1 + H_2$ then the sectorial relation S gives rise to a Hilbert space $\mathfrak{H}_S$ and a selfadjoint operator $B_S \in \mathbf{B}(\mathfrak{H}_S)$ such that the Friedrichs extension S_F and the Kreĭn extension S_K of S are given by

$$S_F = Q^*(I + iB_S)Q^{**}, \quad t_F = J^{**}(I + iB_S)J^*,$$

with corresponding forms

$$\mathfrak{t}_F[f, g] = ((I + iB_S)Q^{**} f, Q^{**} g), \quad f, g \in \mathrm{dom}\, Q^{**},$$

and

$$\mathfrak{t}_K[f, g] = ((I + iB_S)J^* f, J^* g), \quad f, g \in \mathrm{dom}\, J^*;$$

see [4, Theorem 8.3]. Here $Q : \mathfrak{H} \to \mathfrak{H}_S$ is an operator and $J : \mathfrak{H}_S \to \mathfrak{H}$ is a densely defined linear relation such that

$$J \subset Q^*, \quad Q \subset J^*;$$

in particular, the adjoint J^* is an operator.

Recall from Theorem 6.3 that

$$\mathfrak{t}_K[f, g] = ((I + iC)C_0^{1/2}(K^*)_{\mathrm{s}} f, C_0^{1/2}(K^*)_{\mathrm{s}} g),$$

while Theorem 6.2 gives

$$\mathfrak{t}_F[f, g] = ((I + iB_{\oplus})\Psi^{**} f, \Psi^{**} g) = (C_0^{1/2}(I + iC)C_0^{1/2} P_{\mathfrak{D}}(\Psi^{**})_s f, P_{\mathfrak{D}}(\Psi^{**})_s g).$$

Now apply [4, Theorem 8.4] and Corollary 2.2. □

6.3 The Form Sum Construction

The maximal sectorial relations H_1 and H_2 generate the following closed sectorial form

$$((I+iB_1)A_{1s}^{\frac{1}{2}}h, A_{1s}^{\frac{1}{2}}k) + ((I+iB_2)A_{2s}^{\frac{1}{2}}h, A_{2s}^{\frac{1}{2}}k), \quad h,k \in \operatorname{dom} A_1^{\frac{1}{2}} \cap \operatorname{dom} A_2^{\frac{1}{2}}. \tag{6.22}$$

Observe that the restriction of this form to $\operatorname{dom} \Psi^{**}$ is equal to

$$(\Psi^{**}h, \Psi^{**}k) = ((I+iB_1)A_{1s}^{\frac{1}{2}}h, A_{1s}^{\frac{1}{2}}k) + ((I+iB_2)A_{2s}^{\frac{1}{2}}h, A_{2s}^{\frac{1}{2}}k), \quad h,k \in \operatorname{dom} \Psi^{**}, \tag{6.23}$$

since $\Psi^{**} \subset (\Phi^*)_s$, cf. (6.5). Thus, the form in (6.22) has a natural domain which is in general larger than $\operatorname{dom} \Psi^{**}$.

Theorem 6.5 *Let H_1 and H_2 be maximal sectorial relations in $\mathfrak{H}$, let Φ be given by (6.3), and let $\mathfrak{E} = \operatorname{clos} \mathfrak{E}_0$ and $\mathfrak{F} = \operatorname{clos} \mathfrak{F}_0$ be defined by (6.8) amd (6.6). Then the maximal sectorial relation*

$$\Phi^{**}(I+iB_\oplus)\Phi^*$$

is an extension of the relation $H_1 + H_2$, which corresponds to the closed sectorial form in (6.22).

Moreover, the following statements are equivalent:

(i) $\Phi^{**}(I+iB_\oplus)\Phi^*$ *is extremal;*
(ii) $\mathfrak{E} = \mathfrak{F}$.

Proof The first statement is proved in [5, Theorem 3.5]. For the proof of the equivalence of (i) and (ii) appropriate modifications are needed in the arguments used in the proof of [5, Theorem 3.5]. The special case treated there was based on the additional assumption that $\mathfrak{D} = \mathfrak{E}$, where $\mathfrak{D} = \overline{\operatorname{dom}}\, K$; a condition which implies the invariance of $\operatorname{mul} K^*$ under the operator $B_\oplus$. In the present general case such an invariance property cannot be assumed. Now for simplicity denote the form sum extension of $H_1 + H_2$ briefly by $\widehat{H} = \Phi^{**}(I+iB_\oplus)\Phi^*$.

(i) $\Rightarrow$ (ii) Assume that $\widehat{H}$ is extremal. Since $\mathfrak{E} \subset \mathfrak{F}$ by (6.9), it is enough to prove the inclusion $\mathfrak{F} \subset \mathfrak{E}$. By Theorem 6.4 and $\operatorname{mul} \Phi^* = \operatorname{mul} H_1 \times \operatorname{mul} H_2$ one sees

$$\widehat{H} = ((\Phi^*)_s)^*(I+iB_\oplus)(\Phi^*)_s = R^*(I+iC)R, \tag{6.24}$$

for some closed operator R satisfying

$$C_0^{1/2} P_{\mathfrak{D}} \Psi^{**} \subset R \subset C_0^{1/2}(K^*)_s, \tag{6.25}$$

where $P_{\mathfrak{D}}$ is the orthogonal projection of $\mathfrak{H} \times \mathfrak{H}$ onto $\mathfrak{D} = \overline{\mathrm{dom}}\, K$. Recall that $(\Phi^*)_s \subset \Phi^* \subset K^*$ and hence $P_{\mathfrak{D}}(\Phi^*)_s \subset P_{\mathfrak{D}} K^* = (K^*)_s$. Moreover, one has $\mathrm{dom}\, P_{\mathfrak{D}}(\Phi^*)_s = \mathrm{dom}\,(\Phi^*)_s = \mathrm{dom}\, R$, since by assumption these two domains coincide with the corresponding joint form domain. Denoting $\widehat{R} = C_0^{-1/2} R$, one has $\mathrm{dom}\, \widehat{R} = \mathrm{dom}\, P_{\mathfrak{D}}(\Phi^*)_s$ and (6.24) can be rewritten as

$$\widehat{H} = ((\Phi^*)_s)^*(I + iB_{\oplus})(\Phi^*)_s = \widehat{R}^* C_0^{1/2}(I + iC)C_0^{1/2}\widehat{R}, \tag{6.26}$$

where $\widehat{R}$ satisfies $P_{\mathfrak{D}}\Psi^{**} \subset \widehat{R} \subset (K^*)_s$. One concludes that $P_{\mathfrak{D}}(\Phi^*)_s = \widehat{R}$, since both operators are restrictions of $(K^*)_s$, and thus

$$((\Phi^*)_s)^* P_{\mathfrak{D}} = \widehat{R}^*. \tag{6.27}$$

Now one obtains from (6.26) the equalities

$$\begin{aligned} ((\Phi^*)_s)^*(I + iB_{\oplus})(\Phi^*)_s &= \widehat{R}^* C_0^{1/2}(I + iC)C_0^{1/2}\widehat{R} \\ &= ((\Phi^*)_s)^* P_{\mathfrak{D}} C_0^{1/2}(I + iC)C_0^{1/2}\widehat{R} \\ &= ((\Phi^*)_s)^* C_0^{1/2}(I + iC)C_0^{1/2}\widehat{R}. \end{aligned}$$

Hence, for every $f \in \mathrm{dom}\, \widehat{H}$ one has

$$(I + iB_{\oplus})(\Phi^*)_s f - C_0^{1/2}(I + iC)C_0^{1/2}\widehat{R} f \in \ker((\Phi^*)_s)^*.$$

Here $C_0^{1/2}(I + iC)C_0^{1/2}\widehat{R} f \in \mathfrak{D} = \overline{\mathrm{dom}}\, K$ and $\mathfrak{D} = \overline{\mathrm{dom}}\, K = (I + iB_{\oplus})\mathfrak{E}$; see (6.16). Therefore, there exists $\varphi \in \mathfrak{E}$ such that

$$C_0^{1/2}(I + iC)C_0^{1/2}\widehat{R} f = (I + iB_{\oplus})\varphi.$$

On the other hand, $(\Phi^*)_s f \in \mathfrak{F} = \overline{\mathrm{ran}}\,(\Phi^*)_s = (\ker((\Phi^*)_s)^*)^{\perp}$, see (6.5), (6.6). Since $\varphi \in \mathfrak{E} \subset \mathfrak{F}$, this yields

$$\big((I + iB_{\oplus})((\Phi^*)_s f - \varphi), (\Phi^*)_s f - \varphi\big) = 0,$$

and thus $(\Phi^*)_s f - \varphi = 0$. Consequently, for all $f \in \mathrm{dom}\, \widehat{H}$ one has

$$(\Phi^*)_s f \in \mathfrak{E}.$$

Since $\mathrm{dom}\, \widehat{H}$ is a core for the corresponding closed form, or equivalently, the closure of $(\Phi^*)_s {\upharpoonright} \mathrm{dom}\, \widehat{H}$ is equal to $(\Phi^*)_s$, the claim follows: $\mathfrak{F} = \overline{\mathrm{ran}}\,(\Phi^*)_s \subset \mathfrak{E}$.

(ii) ⇒ (i) Assume that $\mathfrak{E} = \mathfrak{F}$. Then $\mathfrak{F}_0 = \mathrm{ran}\,(\Phi^*)_s \subset \mathfrak{E}$ and hence for all $f \in \mathrm{dom}\,(\Phi^*)_s$ one has $(I + B_{\oplus})(\Phi^*)_s f \in \overline{\mathrm{dom}}\, K$. By Remark 3.2 this implies that

$$(I + B_{\oplus})(\Phi^*)_s f = C_0^{1/2}(I + iC)C_0^{1/2} P_{\mathfrak{D}}(\Phi^*)_s f. \tag{6.28}$$

On the other hand, as shown above $P_{\mathfrak{D}}(\Phi^*)_s \subset P_{\mathfrak{D}} K^* = (K^*)_s$. Let $\widehat{R}$ be the closure of $(K^*)_s \upharpoonright \operatorname{dom}(\Phi^*)_s$. Then $\widehat{R}^*$ satisfies the identity (6.27). Since $\Psi^{**} \subset (\Phi^*)_s$ (see (6.10)) one obtains $P_{\mathfrak{D}} \Psi^{**} \subset \widehat{R}$. The identities (6.27) and (6.28) imply that for all $f \in \operatorname{dom} \widehat{H}$ the equalities

$$\begin{aligned}((\Phi^*)_s)^*(I + B_\oplus)(\Phi^*)_s f &= ((\Phi^*)_s)^* P_{\mathfrak{D}} C_0^{1/2}(I + iC)C_0^{1/2} P_{\mathfrak{D}}(\Phi^*)_s f \\ &= \widehat{R}^* C_0^{1/2}(I + iC)C_0^{1/2}\widehat{R} f\end{aligned}$$

hold. Then the closed operator $R = C_0^{1/2}\widehat{R}$ satisfies the inclusions (6.25) as well as the desired identity $((\Phi^*)_s)^*(I + B_\oplus)(\Phi^*)_s = R^*(I + iC)R$, and thus $\widehat{H}$ is extremal, cf. Theorem 6.4. □

Theorem 6.5 is a generalization of [5, Theorem 3.5], where an additional invariance of $\operatorname{mul} K^*$ under the operator $B_\oplus$ was used. Moreover, Theorem 6.5 generalizes a corresponding result for the form sum of two closed nonnegative forms established earlier in [3, Theorem 4.1].

The present result relies on Theorem 4.1, where the description of the closed sectorial form generated by a general maximal sectorial relation of the form $H = T^*(I + iB)T$ where T is a closed relation. This generality implies that with special choices of B the relation H can be taken to be nonnegative and selfadjoint, i.e., the corresponding closed form $\mathfrak{t}$ becomes nonnegative; see Example 4.3.

References

1. Yu.M. Arlinskiĭ, “Maximal sectorial extensions and closed forms associated with them”, Ukrainian Math. J., 48 (1996), 723–739.
2. J. Behrndt, S. Hassi, and H.S.V. de Snoo, *Boundary value problems, Weyl functions, and differential operators*, Monographs in Mathematics 108, Birkhäuser, 2020.
3. S. Hassi, A. Sandovici, H.S.V. de Snoo, and H. Winkler, “Extremal extensions for the sum of nonnegative selfadjoint relations”, Proc. Amer. Math. Soc., 135 (2007), 3193–3204.
4. S. Hassi, A. Sandovici, H.S.V. de Snoo, and H. Winkler, “Extremal maximal sectorial extensions of sectorial relations”, Indag. Math., 28 (2017), 1019–1055.
5. S. Hassi, A. Sandovici, and H.S.V. de Snoo, “Factorized sectorial relations, their maximal sectorial extensions, and form sums”, Banach J. Math. Anal., 13 (2019), 538–564.
6. S. Hassi, H.S.V. de Snoo, and F.H. Szafraniec, “Componentwise and Cartesian decompositions of linear relations”, Dissertationes Mathematicae, 465 (2009), 59 pages.
7. T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, Berlin, 1980.
8. F.S. Rofe-Beketov, “Numerical range of a linear relation and maximal relations”, Teor. Funktsii Funktsional. Anal. i Prilozhen, 44 (1985), 103–112 (Russian) [English translation: J. Math. Sciences, 48 (1990), 329–336].
9. A. Sandovici, *Contributions to the extension theory in Hilbert spaces*, Doctoral dissertation, University of Groningen, 2006.
10. Z. Sebestyén and Zs. Tarcsay, “T^*T always has a positive selfadjoint extension”, Acta Math. Hungar., 135 (2012), 116–129.

Spectral Decompositions of Selfadjoint Relations in Pontryagin Spaces and Factorizations of Generalized Nevanlinna Functions

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Dedicated to V.E. Katsnelson on the occasion of his 75th birthday

Abstract Selfadjoint relations in Pontryagin spaces do not possess a spectral family completely characterizing them in the way that selfadjoint relations in Hilbert spaces do. Here it is shown that a combination of a factorization of generalized Nevanlinna functions with the standard spectral family of selfadjoint relations in Hilbert spaces can function as a spectral family for selfadjoint relations in Pontryagin spaces. By this technique additive decompositions are established for generalized Nevanlinna functions and selfadjoint relations in Pontryagin spaces.

Keywords Generalized Nevanlinna functions · Selfadjoint (multi-valued) operators · (Minimal) Realizations

Mathematics Subject Classification (2000) Primary: 47B50; Secondary: 46C20, 47A10, 47A15

1 Introduction

It is well known that the class of generalized Nevanlinna functions can be realized by means of selfadjoint relations in Pontryagin spaces (cf. Sect. 2.2 below). In [16] it has been shown that there is a strong connection between the factorization result for scalar generalized Nevanlinna functions and the invariant subspace properties of selfadjoint relations in Pontryagin spaces. Here that approach is extended to the case of operator-valued generalized Nevanlinna functions whose values are bounded

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operators on a Hilbert space $\mathcal{H}$; in what follows this class is denoted by $\mathfrak{N}_\kappa(\mathcal{H})$, where $\kappa \in \mathbb{N}$ refers to the number of negative squares of the associated Nevanlinna kernel; see [11, 12]. More precisely, by combining the multiplicative factorization for operator-valued generalized Nevanlinna functions established in [14] with the well-known spectral family results for selfadjoint operators in Hilbert spaces the following additive decomposition is obtained.

Theorem 1.1 *Let $F \in \mathfrak{N}_\kappa(\mathcal{H})$ and let Δ be a measurable subset of $\mathbb{R} \cup \{\infty\}$ or a closed symmetric subset of $\mathbb{C} \setminus \mathbb{R}$. Then F can be written as $F_\Delta + F_R$, where*

(i) $\sigma(F_\Delta) \subseteq \operatorname{clos} \Delta$ *and* $\operatorname{int} \Delta \subseteq \rho(F_R)$;
(ii) $F_\Delta \in \mathfrak{N}_{\kappa_\Delta}(\mathcal{H})$, $F_R \in \mathfrak{N}_{\kappa_R}(\mathcal{H})$ *and* $\kappa_\Delta + \kappa_R \geq \kappa$.

If $\partial\Delta \cap \mathrm{GPNT}\,(F) = (\operatorname{clos}(\Delta) \setminus \operatorname{int}(\Delta)) \cap \mathrm{GPNT}\,(F) = \emptyset$, then the decomposition may be chosen such that F_Δ and F_R do not have a generalized pole in common. In this case, $\kappa_\Delta + \kappa_R = \kappa$.

In Theorem 1.1 $\rho(F)$ denotes the set of holomorphy of $F \in \mathfrak{N}_\kappa(\mathcal{H})$ in $\mathbb{C} \cup \{\infty\}$ and $\sigma(F)$ stands for its complement in $\mathbb{C} \cup \{\infty\}$. For the definition of generalized poles and generalized poles not of positive type (GPNTs), see Sect. 2.2 below. It should be mentioned that Theorem 1.1 generalizes a result obtained for matrix-valued generalized Nevanlinna functions by K. Daho and H. Langer in [2, Prop. 3.3].

For the proof of Theorem 1.1 spectral families for Pontryagin space selfadjoint relations are replaced by factorizations of generalized Nevanlinna functions in combination with the standard spectral decompositions of selfadjoint Hilbert space operators (or relations); this is the main contribution of this paper. Such an approach is needed because spectral families for Pontryagin space selfadjoint relations do not exist in an appropriate form to establish Theorem 1.1; cf. [13]. This approach can be extended to decompose for instance definitizable functions (and operators) in a Kreĭn space setting. Starting from the essentially multiplicative representation of an definitizable function F in [10, Thm. 3.6] one can for example show that F can be written as the sum of two definitizable functions F_+ and F_-, where F_+ has no points of negative type and F_- has no points of positive type.

The intimate connection between generalized Nevanlinna functions and selfadjoint relations in Pontryagin spaces, see e.g. Sect. 2.2 below, means that the following analogue of Theorem 1.1 holds for selfadjoint relations in Pontryagin spaces. For the notation $\mathrm{ENT}\,(A)$ in the following theorem, see Sect. 2.1 below.

Theorem 1.2 *Let A be a selfadjoint relation in a Pontryagin space $\{\Pi, [\cdot,\cdot]\}$ with $\rho(A) \neq \emptyset$ and let Δ be either a measurable subset of $\mathbb{R} \cup \{\infty\}$ or a closed symmetric subset of $\mathbb{C} \setminus \mathbb{R}$. Then there exists a selfadjoint relation A_e in a Pontryagin space $\{\Pi_e, [\cdot,\cdot]_e\}$ with $\mathrm{gr}(A) \subseteq \mathrm{gr}(A_e)$ and a decomposition $\Pi_\Delta[+]\Pi_R$ of Π_e such that*

(i) $\{\Pi_\Delta, [\cdot,\cdot]\}$ *and* $\{\Pi_R, [\cdot,\cdot]\}$ *are Pontryagin spaces;*
(ii) Π_Δ *and* Π_R *are* A_e*-invariant;*
(iii) $\sigma(A_e \restriction_{\Pi_\Delta}) \subseteq \operatorname{clos} \Delta$ *and* $\operatorname{int} \Delta \subseteq \rho(A_e \restriction_{\Pi_R})$.

If $\partial\Delta \cap \text{ENT}(A) = \emptyset$, *then* A_e *and* Π_e *can be taken to be* A *and* Π, *respectively, and the decomposition can be taken such that*

$$\sigma_p(A\restriction_{\Pi_\Delta}) \cap \sigma_p(A\restriction_{\Pi_R}) = \emptyset.$$

In the particular case that Δ is a closed symmetric subset of $\mathbb{C}\setminus\mathbb{R}$ the decomposition in Theorem 1.2 is readily obtained by means of Riesz projection operators; see e.g. [1, Ch. 2: Thm 2.20 & Cor. 3.12]. However, Theorem 1.2 cannot always be established by means of spectral families of selfadjoint relations in Pontryagin spaces if $\partial\Delta\cap\text{ENT}(A) \neq \emptyset$. Indeed the eigenspaces of ENTs can be neutral or even degenerate; in such cases the corresponding eigenvalues are critical points and the spectral family might not be extendable to sets having these points as their endpoints; cf. [13, Comments following Thm. 5.7].

To mention another example of decompositions included in Theorem 1.2 consider $\Delta = (-\infty, a) \cup (b, \infty) \cup \{\infty\}$, where $a, b \in \mathbb{R}\setminus\text{ENT}(A)$ and $a < b$. Then Theorem 1.2 says that a selfadjoint relation in a Pontryagin spaces can be decomposed into the sum of an unbounded selfadjoint relation in a Pontryagin space and a bounded selfadjoint operator in a Pontryagin space; for selfadjoint operators this last result can be found in [11]; see also the references therein. Note that intervals Δ of the given type naturally arise in connection with rational functions; for instance when considering definitizable operators or the products of (generalized) Nevanlinna functions with rational functions, see e.g. [8].

Finally the contents of the paper are shortly outlined. The first half of Sect. 2 consists of an introduction to selfadjoint relations (multi-valued operators) in Pontryagin spaces together with a short overview of minimal operator realizations of (operator-valued) generalized Nevanlinna functions. In the latter half of this section we recall some results about how non-minimal realizations can be reduced to minimal ones and also consider the (minimality of the) realization for the sum of generalized Nevanlinna functions. In Sect. 3 we first establish the connection between a factorization of a generalized Nevanlinna function and the spectral properties of its operator realization. This result is a key tool for using the factorization of generalized Nevanlinna functions as a replacement for a spectral decomposition of selfadjoint relations in Pontryagin spaces. Finally, in the second and third subsections of Sect. 3 Theorems 1.1 and 1.2 are proven, respectively.

2 Preliminaries

The first two subsections contain introductions to (unbounded) operators or, more generally, linear relations in Pontryagin spaces and (minimal) operator realizations for generalized Nevanlinna functions, respectively. In the third subsection it is shown how non-minimal realizations may be reduced to minimal ones. Finally, in the fourth subsection the sum of generalized Nevanlinna functions is considered.

2.1 Linear Relations in Pontryagin Spaces

A linear space Π together with a sesqui-linear form $[\cdot,\cdot]$ defined on it is a *Pontryagin space* if there exists an orthogonal decomposition $\Pi^+ + \Pi^-$ of Π such that $\{\Pi^+, [\cdot,\cdot\}$ and $\{\Pi^-, -[\cdot,\cdot]\}$ are Hilbert spaces either of which is finite-dimensional; here orthogonal means that $[f^+, f^-] = 0$ for all $f^+ \in \Pi^+$ and $f^- \in \Pi^-$. For our purposes it suffices to consider only Pontryagin spaces for which Π^- is finite-dimensional; its dimension (which is independent of the orthogonal decomposition $\Pi^+ + \Pi^-$) is *the negative index* of Π.

A *(linear) relation* H in $\{\Pi, [\cdot,\cdot]\}$ is a *multi-valued (linear) operator* whose domain is a linear subspace of Π, denoted by $\operatorname{dom} H$, and which linearly maps each element $x \in \operatorname{dom} H$ to a subset $Hx := H(x)$ of Π. (Graphs of) linear relations on Π can be identified with subspaces of $\Pi \times \Pi$; in what follows this identification will tacitly be used. The linear subspace $H(0)$ is called the *multi-valued part* of H and is denoted by $\operatorname{mul} H$.

A relation H is closed if (the graph of) H is a closed subspace of $\Pi \times \Pi$. For any relation H in $\{\Pi, [\cdot,\cdot]\}$, its adjoint, denoted as $H^{[*]}$, is defined via its graph:

$$\operatorname{gr}(H^{[*]}) = \{\{f, f'\} \in \Pi \times \Pi : [f, g'] = [f', g], \quad \forall \{g, g'\} \in \operatorname{gr}(H)\}.$$

A relation A in $\{\Pi, [\cdot,\cdot]\}$ is *symmetric* if $A \subseteq A^{[*]}$ and *selfadjoint* if $A = A^{[*]}$. An operator V from (a Pontryagin space) $\{\Pi_1, [\cdot,\cdot]_1\}$ to (a Pontryagin space) $\{\Pi_2, [\cdot,\cdot]_2\}$ is *isometric* if $[f, g]_1 = [Vf, Vg]_2$ for all $f, g \in \operatorname{dom} V$. An isometric operator U from $\{\Pi_1, [\cdot,\cdot]_1\}$ to $\{\Pi_2, [\cdot,\cdot]_2\}$ is a *standard unitary operator* if $\operatorname{dom} U = \Pi_1$ and $\operatorname{ran} U = \Pi_2$.

For a closed relation H in $\{\Pi, [\cdot,\cdot]\}$, the resolvent set, $\rho(H)$, and the spectrum, $\sigma(H)$, are defined as usual:

$$\rho(H) = \{z \in \mathbb{C} : \ker(H-z) = \{0\},\ \operatorname{ran}(H-z) = \Pi\} \quad \text{and} \quad \sigma(H) = \mathbb{C}\setminus\rho(H).$$

Moreover, the point spectrum $\sigma_p(H)$ is defined as the set

$$\sigma_p(H) = \{z \in \mathbb{C} \cup \{\infty\} : \exists\, x\ (\neq 0) \in \Pi \text{ s.t. } \{x, zx\} \in \operatorname{gr}(H)\}.$$

These sets have the normal properties, see e.g. [4]. Below we also use the convention that $\infty \in \sigma_p(H)$ if and only if $\operatorname{mul} H \neq \{0\}$ or, equivalently, $0 \in \sigma_p(H^{-1})$, where H^{-1} stands for the inverse (linear relation) of H. Similarly, $\infty \in \rho(H)$ means that $0 \in \rho(H^{-1})$ or, equivalently, that H is a bounded everywhere defined operator, i.e., $H \in \mathbf{B}(\Pi)$.

A subspace $\mathfrak{L}$ of Π is said to be *invariant* under a relation H with $\rho(H) \neq \emptyset$, or H-invariant for short, if

$$(H-z)^{-1}\mathfrak{L} \subseteq \mathfrak{L}, \qquad \forall z \in \rho(H).$$

Here $(H-z)^{-1} \in \mathbf{B}(\Pi)$ is defined via its graph as

$$\operatorname{gr}((H-z)^{-1}) = \{\{f' - zf, f\} \in \Pi \times \Pi : \{f, f'\} \in \operatorname{gr}(H)\}.$$

Recall that the spectrum and resolvent set $\sigma(A)$ and $\rho(A)$ of a selfadjoint relation A in a Pontryagin space are symmetric with respect to the real line:

$$\rho(A) = \overline{\rho(A)}, \quad \sigma(A) = \overline{\sigma(A)} \quad \text{and} \quad \sigma_p(A) = \overline{\sigma_p(A)}. \tag{2.1}$$

Moreover, if $\rho(A) \neq \emptyset$, then $\rho(A)$ contains $\mathbb{C} \setminus \mathbb{R}$ except finitely many points; see [4].

Finally, $\alpha \in \mathbb{C} \cup \{\infty\}$ is an *eigenvalue not of positive type*, or ENT for short, of a selfadjoint relation A in a Pontryagin space, if there exists a non-trivial non-positive A-invariant subspace $\mathfrak{L}$ such that $\sigma(A \upharpoonright_{\mathfrak{L}}) = \alpha$. Recall that selfadjoint relations in Pontryagin spaces possess at most finitely many ENTs, see e.g. [9, Thm. 12.1']. The set of all ENTs of a selfadjoint relation A in $\mathbb{C} \cup \{\infty\}$ is denoted by ENT (A).

2.2 Minimal Realizations of Generalized Nevanlinna Functions

The concept of an operator-valued generalized Nevanlinna function has been introduced and studied by M. G. Kreĭn and H. Langer; see [11, 12]. In particular, with some additional analytic assumptions, operator-valued generalized Nevanlinna functions were described as so-called *Q-functions* of symmetric operators in a Pontryagin space. Those additional conditions were removed by allowing selfadjoint relations in model spaces; cf. [3] for the case of matrix functions and [7] for operator-valued functions.

If A is a selfadjoint relation in (a Pontryagin space) $\{\Pi, [\cdot,\cdot]\}$ with a nonempty resolvent set $\rho(A)$, C is a bounded selfadjoint operator in a Hilbert space $\{\mathcal{H}, (\cdot,\cdot)\}$ and Γ is an everywhere defined operator from $\mathcal{H}$ to Π, then F defined by

$$F(z) = C + \overline{z_0}\Gamma^{[*]}\Gamma + (z - \overline{z_0})\Gamma^{[*]}\left(I + (z - z_0)(A - z)^{-1}\right)\Gamma, \qquad z, z_0 \in \rho(A), \tag{2.2}$$

is a generalized Nevanlinna function. Conversely, if F is a generalized Nevanlinna function, then there exist $A = A^{[*]}$ with $\rho(A) \neq \emptyset$, Γ and C as above such that (2.2) holds; in this case $C + \overline{z_0}\Gamma^{[*]}\Gamma = F(z_0)^* = F(\overline{z_0})$.

If (2.2) holds for some generalized Nevanlinna function F, then the pair $\{A, \Gamma\}$ *realizes* F (at z_0). In particular, in the term realization the *realizing space* $\{\Pi, [\cdot,\cdot]\}$ is suppressed; also the selection of the arbitrarily fixed point z_0 is suppressed when it doesn't play a role. With a realizing pair $\{A, \Gamma\}$ (at z_0) we associate a bounded operator-valued function Γ_z, called the *γ-field* associated with $\{A, \Gamma\}$, via

$$\Gamma_z := \left(I + (z - z_0)(A - z)^{-1}\right)\Gamma, \qquad z \in \rho(A). \tag{2.3}$$

Using the γ-field and the resolvent identity, (2.2) can be rewritten into a symmetric form:

$$\frac{F(z) - F(w)^*}{z - \overline{w}} = \Gamma_w^{[*]}\Gamma_z, \qquad z, w \in \rho(A). \tag{2.4}$$

The pair $\{A, \Gamma\}$ is said to realize F (as in (2.2)) *minimally* if

$$\Pi = \text{c.l.s.}\,\{\Gamma_z h : z \in \rho(A), \quad h \in \mathcal{H}\}.$$

For the existence of a minimal realization for any generalized Nevanlinna function see e.g. [7, Thm. 4.2].

By means of a minimal realization the index of a generalized Nevanlinna function can be characterized: F is a generalized Nevanlinna function with index κ, $F \in \mathfrak{N}_\kappa(\mathcal{H})$, if the negative index of the realizing (Pontryagin) space for any minimal realization is κ. In fact, all minimal realizations are connected by means of (standard) unitary operators.

Proposition 2.1 ([7, Thm. 3.2]) *Let $\{A_i, \Gamma_i\}$ realize $F \in \mathfrak{N}_\kappa(\mathcal{H})$ minimally for $i = 1, 2$. Then there exists a standard unitary operator from $\{\Pi_1, [\cdot, \cdot]_1\}$ to $\{\Pi_2, [\cdot, \cdot]_2\}$ such that $A_2 = UA_1U^{-1}$ and $\Gamma_2 = U\Gamma_1$.*

For a generalized Nevanlinna function F the notation $\rho(F)$ and $\sigma(F)$ is used to denote the *domain of holomorphy* of F in $\mathbb{C}\cup\{\infty\}$ and its complement (in $\mathbb{C}\cup\{\infty\}$), respectively. In particular, (2.2) implies that

$$\rho(A) \subseteq \rho(F) \quad \text{and} \quad \sigma(F) \subseteq \sigma(A). \tag{2.5}$$

For minimal realizations the reverse inclusions also hold.

Theorem 2.2 ([11, Satz 4.4]) *Let $F \in \mathfrak{N}_\kappa(\mathcal{H})$ be minimally realized by $\{A, \Gamma\}$. Then $\rho(A) = \rho(F)$.*

Finally, $\alpha \in \mathbb{C} \cup \{\infty\}$ is a *generalized pole* of a generalized Nevanlinna function F if $\alpha \in \sigma_p(A)$ for any minimal realization $\{A, \Gamma\}$ of F. Furthermore, the set of *generalized poles of not of positive type of* F, GPNT (F), is defined to be ENT (A) for any minimal realization $\{A, \Gamma\}$ of F (see Sect. 2.1). Note that Proposition 2.1 guarantees that these concepts are well-defined.

2.3 Reduction of Non-minimal Realizations

Realizations for a generalized Nevanlinna function need not be minimal. For instance, if the negative index of the realizing Pontryagin space is greater than the negative index of a generalized Nevanlinna function, then the realization is not minimal. Even if the negative index of the realizing space is equal to the negative index of a generalized Nevanlinna function, then the realization might still be non-minimal; cf. Sect. 2.4 below. The following operator-valued analog of [16, Prop. 2.2] shows how non-minimal realizations can be reduced to minimal ones; see also [11] and [7, Section 2].

Proposition 2.3 *Let* $\{A, \Gamma\}$ *realize* $F \in \mathfrak{N}_\kappa(\mathcal{H})$ *and let* κ_m *denote the negative index of the realizing Pontryagin space* $\{\Pi, [\cdot,\cdot]\}$. *Moreover, with*

$$\mathfrak{M} := \operatorname{span}\{\left(I + (z - z_0)(A - z)^{-1}\right)\Gamma h : z \in \rho(A), \quad h \in \mathcal{H}\},$$

define $\mathfrak{L}$, Π_s *and* Π_r *as*

$$\mathfrak{L} = (\operatorname{clos}\mathfrak{M}) \cap \mathfrak{M}^{[\perp]}, \quad \Pi_s = (\operatorname{clos}\mathfrak{M})/\mathfrak{L} \quad \text{and} \quad \Pi_r = \mathfrak{M}^{[\perp]}/\mathfrak{L}.$$

Then the following statements hold:

(i) $\mathfrak{L}$ *is an* A*-invariant neutral subspace of* $\{\Pi, [\cdot,\cdot]\}$ *with* $\kappa_{\mathfrak{L}} := \dim\mathfrak{L} \le \kappa_m$*;*

(ii) A_s *and* A_r*, defined via*

$$\begin{aligned} \operatorname{gr} A_s &= \{\{f + [\mathfrak{L}], f' + [\mathfrak{L}]\} : \{f, f'\} \in \operatorname{gr} A \cap (\Pi_s \times \Pi_s)\}; \\ \operatorname{gr} A_r &= \{\{f + [\mathfrak{L}], f' + [\mathfrak{L}]\} : \{f, f'\} \in \operatorname{gr} A \cap (\Pi_r \times \Pi_r)\}, \end{aligned}$$

are selfadjoint relations in the Pontryagin spaces $\{\Pi_s, [\cdot,\cdot]\}$ *and* $\{\Pi_r, [\cdot,\cdot]\}$ *with negative index* κ *and* $\kappa_m - \kappa - \kappa_{\mathfrak{L}}$*, respectively;*

(iii) $\{A_s, \Gamma + [\mathfrak{L}]\}$ *realizes* f *minimally;*

(iv) $\mathfrak{M}^{[\perp]}$ *is the largest* A*-invariant subspace contained in* $\ker \Gamma^{[*]}$.

Proof

(i) Let $\mathfrak{M}$ be as in the statement, then $(A - \xi)^{-1}\mathfrak{M} \subseteq \mathfrak{M}$ for every $\xi \in \rho(A)$ by the resolvent identity. From the preceding inclusion it follows by elementary arguments that $\left((A - \xi)^{-1}\right)^{[*]}\mathfrak{M}^{[\perp]} \subseteq \mathfrak{M}^{[\perp]}$ or, equivalently, using the selfadjointness of A that $(A - \overline{\xi})^{-1}\mathfrak{M}^{[\perp]} \subseteq \mathfrak{M}^{[\perp]}$. Another application of the same argument yields that $(A - \xi)^{-1}\operatorname{clos}\mathfrak{M} \subseteq \operatorname{clos}\mathfrak{M}$. Since $\rho(A)$ is symmetric with respect to the real line for selfadjoint relations, see (2.1), $\mathfrak{M}$, $\operatorname{clos}\mathfrak{M}$ and $\mathfrak{M}^{[\perp]}$ are A-invariant and, hence, $\mathfrak{L}$ is A-invariant too.

(ii) Since $\mathfrak{L}$ is neutral in a Pontryagin space, it is a finite-dimensional (closed) subspace. Therefore $\{\mathfrak{L}^{[\perp]}/\mathfrak{L}, [\cdot,\cdot]\}$ is a Pontryagin space with negative index $\kappa_m - \kappa_{\mathfrak{L}}$, see [1, Ch. 1: Cor. 9.14]. A calculation, using the A-invariance and neutrality of $\mathfrak{L}$, shows that $A_{\mathfrak{L}}$, defined via

$$\mathrm{gr}(A_{\mathfrak{L}}) = \left\{ \{f + [\mathfrak{L}], f' + [\mathfrak{L}]\} \in \mathfrak{L}^{[\perp]}/\mathfrak{L} \times \mathfrak{L}^{[\perp]}/\mathfrak{L} : \{f, f'\} \in \mathrm{gr}(A) \cap (\mathfrak{L}^{[\perp]} \times \mathfrak{L}^{[\perp]}) \right\}$$

is a symmetric linear relation in the introduced quotient space. To establish that A is selfadjoint, it suffices by Dijksma and de Snoo [4, Thm. 4.6] to show that

$$\rho(A) \subseteq \rho(A_{\mathfrak{L}}). \tag{2.6}$$

Let $z \in \rho(A)$ be arbitrary. Since $\mathfrak{L}$ is A-invariant (see (i)), $\mathfrak{L}^{[\perp]}$ is also A-invariant and $\mathfrak{L}^{[\perp]} \subseteq \mathrm{ran}\,(A - z)$, because $z \in \rho(A)$ by assumption. Thus for every $g \in \mathfrak{L}^{[\perp]}$ there exists $\{f, f'\} \in A$, such that $g = f' - zf$. Now the A-invariance of $\mathfrak{L}^{[\perp]}$ implies that $f = (A - z)^{-1} g \in \mathfrak{L}^{[\perp]}$ and thus also $f' \in \mathfrak{L}^{[\perp]}$. Therefore,

$$\mathfrak{L}^{[\perp]} \subseteq \{ f' - zf : \{f, f'\} \in \mathrm{gr}(A) \cap (\mathfrak{L}^{[\perp]} \times \mathfrak{L}^{[\perp]})\}.$$

Consequently, $\mathrm{ran}\,(A_{\mathfrak{L}} - z) = \mathfrak{L}^{[\perp]}/\mathfrak{L}$ and this implies that $z \in \rho(A_{\mathfrak{L}})$. Since $z \in \rho(A)$ was arbitrary, the above argument shows that (2.6) holds.
Now $\Gamma_{\mathfrak{L}}$, defined via $\Gamma_{\mathfrak{L}} h := \Gamma h + [\mathfrak{L}]$ for $h \in \mathcal{H}$, is an everywhere defined mapping from $\mathcal{H}$ to $\mathfrak{L}^{[\perp]}/\mathfrak{L}$. Using $\Gamma_{\mathfrak{L}}$ and $A_{\mathfrak{L}}$ define the subspace $\mathfrak{M}_{\mathfrak{L}}$ of $\mathfrak{L}^{[\perp]}/\mathfrak{L}$ as

$$\mathfrak{M}_{\mathfrak{L}} := \mathrm{span}\,\{\left(I + (z - z_0)(A_{\mathfrak{L}} - z)^{-1}\right) \Gamma_{\mathfrak{L}} h : z \in \rho(A_{\mathfrak{L}}), \quad h \in \mathcal{H}\}. \tag{2.7}$$

By means of $\mathfrak{M}_{\mathfrak{L}}$ introduce in $\{\mathfrak{L}^{[\perp]}/\mathfrak{L}, [\cdot,\cdot]\}$ the subspaces $\Pi_s := \mathrm{clos}\,\mathfrak{M}_{\mathfrak{L}}$ and $\Pi_r := \Pi_s^{[\perp]_{\mathfrak{L}}}$; here $[\perp]_{\mathfrak{L}}$ denotes the orthogonal complement in $\{\mathfrak{L}^{[\perp]}/\mathfrak{L}, [\cdot,\cdot]\}$. Then clearly $\Pi_s = \mathrm{clos}\,(\mathfrak{M})/\mathfrak{L}$ and $\Pi_r = \mathfrak{M}^{[\perp]}/\mathfrak{L}$. Since $\mathfrak{L} = \mathrm{clos}\,(\mathfrak{M}) \cap \mathfrak{M}^{[\perp]}$, Π_s and Π_r are non-degenerate. Therefore $\{\Pi_s, [\cdot,\cdot]\}$ and $\{\Pi_r, [\cdot,\cdot]\}$ are Pontryagin spaces, see [1, Ch. 1: Thm. 7.16 & Thm. 9.9]. The same arguments used in (i) yield

$$(A_{\mathfrak{L}} - \xi)^{-1}\Pi_s \subseteq \Pi_s \quad \text{and} \quad (A_{\mathfrak{L}} - \xi)^{-1}\Pi_r \subseteq \Pi_r, \quad \xi \in \rho(A_{\mathfrak{L}}) \supseteq \rho(A). \tag{2.8}$$

Let A_s and A_r be as in (ii) with Π_s and Π_r as defined following (2.7), then A_s and A_r, being restrictions of the selfadjoint relation $A_{\mathfrak{L}}$, are symmetric. Moreover, (2.8) together with the decomposition $\mathfrak{L}^{[\perp]}/\mathfrak{L} = \Pi_s[\dot{+}]\Pi_r$ implies that $\rho(A_s) \cap \mathbb{C}_+$, $\rho(A_s) \cap \mathbb{C}_-$, $\rho(A_r) \cap \mathbb{C}_+$ and $\rho(A_r) \cap \mathbb{C}_-$ are all non-empty. Therefore A_s and A_r are selfadjoint relations; again cf. [4]. The last assertion on the negative indices of the Pontryagin spaces is a consequence of the result in (iii) combined with the fact that the negative index of the Pontryagin space $\{\mathfrak{L}^{[\perp]}/\mathfrak{L}, [\cdot,\cdot]\}$ is $\kappa_m - \dim \mathfrak{L} = \kappa_m - \kappa_{\mathfrak{L}}$.

(iii) Let Γ_z be the γ-field associated with the realization $\{A, \Gamma\}$ as in (2.3). Then for every $\omega_g, \omega_h \in \mathfrak{L}$ we have by definition of $\mathfrak{L}$ that

$$[\Gamma_z h + \omega_h, \Gamma_w g + \omega_g] = [\Gamma_z h, \Gamma_w g] = g^* \frac{F(z) - F(w)^*}{z - \overline{w}} h, \quad g, h \in \mathcal{H}.$$

Hence, $\{A_s, \Gamma_{\mathfrak{L}}\}$ realizes F, see (2.4). Moreover, this realization is minimal by construction, see the proof of (ii). Therefore the negative index of $\{\Pi_s, [\cdot,\cdot]\}$ is κ by Proposition 2.1 and the discussion preceding it.

(iv) In (i) it has been established that $\mathfrak{M}^{[\perp]}$ is A-invariant. The inclusion $\mathfrak{M}^{[\perp]} \subseteq \ker \Gamma^{[*]}$ follows directly from the fact that $\operatorname{ran} \Gamma \subseteq \mathfrak{M}$. To prove the assertion it therefore suffices to show that all A-invariant subspaces $\mathfrak{N}$ contained in $\ker \Gamma^{[*]}$ are orthogonal to $\mathfrak{M}$. Let $\mathfrak{N}$ be any such subspace. Then for all $h \in \mathcal{H}$ and $z \in \rho(A)$

$$[(I + (z - z_0)(A - z)^{-1})\Gamma h, \mathfrak{N}] = (h, \Gamma^{[*]}(I + (\overline{z} - \overline{z_0})(A - \overline{z})^{-1})\mathfrak{N}) = 0.$$

This shows that $\mathfrak{N} \subseteq \mathfrak{M}^{[\perp]}$. □

Corollary 2.4 *Let $F \in \mathfrak{N}_\kappa(\mathcal{H})$ be realized by $\{A, \Gamma\}$ and let κ_m denote the negative index of the realizing Pontryagin space $\{\Pi, [\cdot,\cdot]\}$. Then*

$$\kappa_m - \kappa = \max_{\mathfrak{N}} \{\dim \mathfrak{N} : \mathfrak{N}\ is\ A\text{-}invariant, \quad \mathfrak{N} \subseteq \ker \Gamma^{[*]}\};$$

here the maximum is over all nonpositive subspaces $\mathfrak{N}$ of $\{\Pi, [\cdot,\cdot]\}$.

Proof Using the notation as in Proposition 2.3, Proposition 2.3 (ii) shows that $\kappa_m - \kappa = \dim \mathfrak{L} + \kappa_r$; here κ_r is defined to be the negative index of the Pontryagin space $\{\Pi_r, [\cdot,\cdot]\}$. Since the negative index of the subspace $\mathfrak{M}^{[\perp]}$ of $\{\Pi, [\cdot,\cdot]\}$ is equal to $\dim \mathfrak{L} + \kappa_r$ and $\{\mathfrak{N} : \mathfrak{N} \text{ is } A\text{−invariant}, \quad \mathfrak{N} \subseteq \ker \Gamma^{[*]}\} \subseteq \mathfrak{M}^{[\perp]}$ by Proposition 2.3 (iv), the statement is proven if the existence of a nonpositive A-invariant subspace of dimension $\dim \mathfrak{L} + \kappa_r$ contained in $\ker \Gamma^{[*]}$ is established.

Since A_r, the restriction of A to $\{\Pi_r, [\cdot,\cdot]\}$ (see Proposition 2.3 (ii)), is a selfadjoint relation in the Pontryagin space $\{\Pi_r, [\cdot,\cdot]\}$, the invariant subspace theorem states that there exists a κ_r-dimensional A_r-invariant nonpositive subspace $\mathfrak{L}_r$ of $\{\Pi_r, [\cdot,\cdot]\}$, see e.g. [9, Thm. 12.1']. Therefore $\mathfrak{N} := \mathfrak{L}_r + \mathfrak{L}$ is a $(\dim \mathfrak{L} + \kappa_r)$-dimensional nonpositive A-invariant subspace contained in $\mathfrak{M}^{[\perp]}$. □

2.4 The Sum of Generalized Nevanlinna Functions

A particular situation where non-minimal realizations may be encountered is when the sum of generalized Nevanlinna functions is considered; cf. [6]. Let $F_i \in \mathfrak{N}_{\kappa_i}(\mathcal{H})$

be (minimally) realized by $\{A_i, \Gamma_i\}$, for $i = 1, 2$. Then the sum $F_1 + F_2$ is realized by $\{A_1 \widehat{\oplus} A_2, \mathrm{col}\,(\Gamma_1, \Gamma_2)\}$, where

$$\begin{aligned} \mathrm{gr}(A_1 \widehat{\oplus} A_2) &= \{\{\{f_1, f_2\}, \{f_1', f_2'\}\} : \{f_i, f_i'\} \in \mathrm{gr}(A_i)\}; \\ \mathrm{col}\,(\Gamma_1, \Gamma_2)h &= \begin{pmatrix} \Gamma_1 h \\ \Gamma_2 h \end{pmatrix}. \end{aligned} \tag{2.9}$$

Here the realizing space is $\{\Pi_{\mathrm{sum}}, [\cdot, \cdot]_{\mathrm{sum}}\}$ where $\Pi_{\mathrm{sum}} = \Pi_1 \times \Pi_2$ and

$$[\{f_1, f_2\}, \{g_1, g_2\}]_{\mathrm{sum}} = [f_1, g_1]_1 + [f_2, g_2]_2, \quad \{f_1, f_2\}, \{g_1, g_2\} \in \Pi_1 \times \Pi_2. \tag{2.10}$$

To see this note that

$$\begin{aligned} &(\mathrm{col}\,(\Gamma_1, \Gamma_2))^{[*]}(I + (z - z_0)(A_1 \widehat{\oplus} A_2 - z)^{-1})\mathrm{col}\,(\Gamma_1, \Gamma_2) \\ =&\begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix}^{[*]} \begin{pmatrix} I + (z - z_0)(A_1 - z)^{-1} & 0 \\ 0 & I + (z - z_0)(A_2 - z)^{-1} \end{pmatrix} \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix} \\ =&\Gamma_1^{[*]}(I + (z - z_0)(A_1 - z)^{-1})\Gamma_1 + \Gamma_2^{[*]}(I + (z - z_0)(A_2 - z)^{-1})\Gamma_2 \\ =&\frac{F_1(z) - F_1(\overline{z_0})}{z - \overline{z_0}} + \frac{F_2(z) - F_2(\overline{z_0})}{z - \overline{z_0}} = \frac{F_1(z) + F_2(z) - (F_1(\overline{z_0}) + F_2(\overline{z_0}))}{z - \overline{z_0}}, \end{aligned}$$

where in the third step (2.2) was used. In view of (2.2) this calculation shows that $\{A_1 \widehat{\oplus} A_2, \mathrm{col}\,(\Gamma_1, \Gamma_2)\}$ realizes $F_1 + F_2$; cf. [6, Prop. 4.1]. In particular, $F_1 + F_2 \in \mathfrak{N}_{\kappa_{\mathrm{sum}}}(\mathcal{H})$ where $\kappa_{\mathrm{sum}} \le \kappa_1 + \kappa_2$; cf. Proposition 2.3.

Notice, conversely, that if $\{A, \Gamma\}$ realizes the function $F \in \mathfrak{N}_\kappa(\mathcal{H})$ and there exists a decomposing (regular) subspace Π_1 of Π, i.e. $\Pi = \Pi_1 [\dot{+}] \Pi_2$ with $\Pi_2 = \Pi_1^{[\perp]}$, which also reduces A, $A = A_1 \widehat{\oplus} A_2$, then $\{A_1, P_1\Gamma\}$ and $\{A_2, P_2\Gamma\}$, where P_j with $j = 1, 2$ is the Π-orthogonal projection onto Π_j, produce realizations for generalized Nevanlinna functions F_1 and F_2 such that $F = F_1 + F_2$.

Proposition 2.5 below contains sufficient conditions for the index of $F_1 + F_2$ to be the sum of the indices of F_1 and F_2; see [2, Prop. 3.2] for a similar statement for matrix-valued generalized Nevanlinna functions.

Proposition 2.5 *Let $F_1 \in \mathfrak{N}_{\kappa_1}(\mathcal{H})$, $F_2 \in \mathfrak{N}_{\kappa_2}(\mathcal{H})$ and assume that F_1 and F_2 do not have a generalized pole in common. Then $F_1 + F_2 \in \mathfrak{N}_{\kappa_1 + \kappa_2}(\mathcal{H})$.*

Proof Let $\{A_i, \Gamma_i\}$ be a minimal realization for F_i where the realizing space is $\{\Pi_i, [\cdot, \cdot]_i\}$, for $i = 1, 2$. Then, as the discussion preceding this statement demonstrated, $F_1 + F_2$ is realized by $\{A, \Gamma\} := \{A_1 \widehat{\oplus} A_2, \mathrm{col}\,(\Gamma_1, \Gamma_2)\}$ where the realizing space is the Pontryagin space $\{\Pi, [\cdot, \cdot]\} := \{\Pi_{\mathrm{sum}}, [\cdot, \cdot]_{\mathrm{sum}}\}$ whose negative index is $\kappa_1 + \kappa_2$, see (2.10). Hence $F_1 + F_2$ is a generalized Nevanlinna function. In order to establish that its index is $\kappa_1 + \kappa_2$, the non-minimal part of its realization $\{A, \Gamma\}$ should be investigated; cf. Proposition 3.1. But first note that if

P_1 and P_2 are the orthogonal projections onto Π_1 and Π_2 in Π_{sum}, then

$$(A_1 \widehat{\oplus} A_2 - z)^{-1} P_i = (A_i - z)^{-1} P_i = P_i (A_1 \widehat{\oplus} A_2 - z)^{-1}, \qquad i = 1, 2, \tag{2.11}$$

see (2.9). Denote by $\mathfrak{M}^{[\perp]}$ the non-minimal part of the realization $\{A, \Gamma\}$ as in Proposition 2.3. If $\mathfrak{L} := \operatorname{clos}(\mathfrak{M}) \cap \mathfrak{M}^{[\perp]} \neq \{0\}$, then $\mathfrak{L}$, being finite-dimensional and A-invariant (see Proposition 2.3 (i)), contains an eigenvector x for $A = A_1 \widehat{\oplus} A_2$ such that $x \in \ker \Gamma^{[*]}$. But, then (2.11) implies that $P_1 x$ and $P_2 x$ are eigenvectors for A_1 and A_2, respectively. Since $\sigma_p(A_1) \cap \sigma_p(A_2) = \emptyset$ by assumption, this implies that either of the two vectors is zero; say $P_2 x = 0$. Thus $P_1 x$ is an eigenvector for A_1, $P_2 x = 0$ and $x \in \ker \Gamma^{[*]}$. The last two conditions together yield that $P_1 x \in \ker \Gamma_1^{[*]}$; cf. (2.9). But then the realization $\{A_1, \Gamma_1\}$ for F_1 is not minimal by Proposition 2.3; in contradiction to the assumption. I.e., $\mathfrak{L} = \{0\}$.

Therefore $\mathfrak{M}^{[\perp]}$ is A-invariant and $\{\mathfrak{M}^{[\perp]}, [\cdot,\cdot]\}$ is a Pontryagin space, see Proposition 2.3 (ii). The exact same argument as used in the preceding paragraph shows that $\sigma_p(A \upharpoonright_{\mathfrak{M}^{[\perp]}}) = \emptyset$. Hence Pontryagin's invariant subspace theorem (applied to the selfadjoint relation $A \upharpoonright_{\mathfrak{M}^{[\perp]}}$ in $\{\mathfrak{M}^{[\perp]}, [\cdot,\cdot]\}$) implies that $\{\mathfrak{M}^{[\perp]}, [\cdot,\cdot]\}$ is a Hilbert space, see e.g. [9, Thm. 12.1']. Consequently, the statement holds by Proposition 2.3 (ii); cf. Corollary 2.4. □

The following extension of Proposition 2.5 shows when a minimal realization for $F_1 + F_2$ is obtained when starting from minimal realizations for F_1 and F_2.

Proposition 2.6 *Let $\{A_i, \Gamma_i\}$ minimally realize the generalized Nevanlinna function $F_i \in \mathfrak{N}_{\kappa_i}(\mathcal{H})$, for $i = 1, 2$, and assume that*

$$\sigma_p(A_1) \cap \sigma_p(A_2) = \emptyset \quad \text{and} \quad \sigma(A_1) \cap \sigma(A_2) = \{\gamma_1, \ldots, \gamma_n\} \subseteq \mathbb{R} \cup \{\infty\}.$$

Then $F_1 + F_2 \in \mathfrak{N}_{\kappa_1+\kappa_2}(\mathcal{H})$ is minimally realized by $\{A_1 \widehat{\oplus} A_2, \operatorname{col}(\Gamma_1, \Gamma_2)\}$.

Proof As the above discussion demonstrated, $F_1 + F_2$ is realized by $\{A, \Gamma\} := \{A_1 \widehat{\oplus} A_2, \operatorname{col}(\Gamma_1, \Gamma_2)\}$ where the realizing space is the Pontryagin space $\{\Pi, [\cdot,\cdot]\} := \{\Pi_{\text{sum}}, [\cdot,\cdot]_{\text{sum}}\}$ whose negative index is $\kappa_1 + \kappa_2$, see (2.10). To prove the minimality of the realization for $F_1 + F_2$ let $\mathfrak{M}$ be as in Proposition 2.3.

Since the index of $F_1 + F_2$ is equal to the negative index of $\{\Pi, [\cdot,\cdot]\}$ by Propositions 2.5, 2.3 yields that $\{\mathfrak{M}^{[\perp]}, [\cdot,\cdot]\}$ is a Hilbert space and that A_r, defined via $\operatorname{gr}(A_r) = \operatorname{gr}(A) \cap (\mathfrak{M}^{[\perp]} \times \mathfrak{M}^{[\perp]})$, is a selfadjoint relation in $\{\mathfrak{M}^{[\perp]}, [\cdot,\cdot]\}$. In particular, $\sigma(A_r) \subseteq \mathbb{R} \cup \{\infty\}$. We claim that

$$\sigma(A_r) \subseteq \sigma(A_1) \quad \text{and} \quad \sigma(A_r) \subseteq \sigma(A_2). \tag{2.12}$$

If the first inclusion does not hold, then, since $\sigma(A_1) \cap (\mathbb{R} \cup \{\infty\})$ and $\sigma(A_r)$ are closed subsets of $\mathbb{R} \cup \{\infty\}$, there exists a closed interval $\Delta = [a, b]$ of $\mathbb{R}$ such that

$$\Delta \cap \sigma(A_r) \neq \emptyset \quad \text{and} \quad \Delta \subseteq \rho(A_1). \tag{2.13}$$

Let E_t be the spectral family of A_r and let P_i be the orthogonal projections onto Π_i in Π, for $i = 1, 2$. Then the assumption $\Delta \cap \sigma(A_r) \neq \emptyset$ implies that

$$\mathfrak{L} := (E_b - E_a)\mathfrak{M}^{[\perp]} \neq \{0\}.$$

Consider $\mathfrak{L}_1 := P_1\mathfrak{L} \subseteq \Pi_1$. Then, on the one hand,

$$\sigma(A \restriction_{\mathfrak{L}_1}) \subseteq \sigma(A \restriction_{\mathfrak{L}}) \subseteq \Delta \subseteq \rho(A_1).$$

On the other hand, the A_1-invariance of $\mathfrak{L}_1$ implies that $\sigma(A \restriction_{\mathfrak{L}_1}) \subseteq \sigma(A_1)$. The preceding two results together imply that $\mathfrak{L}_1 = \{0\}$; cf. (2.13). In other words, $\mathfrak{L} \subseteq \{0\} \times \Pi_2$. But then $\mathfrak{L} \subseteq \ker \Gamma_2^{[*]}$, because $\mathfrak{L} \subseteq \mathfrak{M}^{[\perp]} \subseteq \ker \Gamma^{[*]}$. Consequently, the realization $\{A_2, \Gamma_2\}$ is not minimal. This contradiction shows that the first inclusion in (2.12) holds. By symmetry the second inclusion also holds.

Combining the inclusions from (2.12) together with the assumption on $\sigma(A_1) \cap \sigma(A_2)$ yields that $\sigma(A_r)$ consists at most of isolated points. I.e., all the spectrum of A_r is point spectrum. Let x be an eigenvector for A_r. Then P_1x and P_2x are eigenvectors for A_1 and A_2, respectively. Since $\sigma_p(A_1) \cap \sigma_p(A_2) = \emptyset$, either P_1x or P_2x should be equal to zero. Assume the latter. Since $x \in \mathfrak{M}^{[\perp]} \subseteq \ker \Gamma^{[*]}$ (see Proposition 2.3), it follows that $x = P_1x \subseteq \ker \Gamma_1^{[*]}$; but this is in contradiction to the assumed minimality of the realization $\{A_1, \Gamma_1\}$ for F_1. □

3 Decompositions of Generalized Nevanlinna Functions

For $\alpha, \beta \in \mathbb{C} \cup \{\infty\}$, with $\alpha \neq \beta$, and for non-orthogonal vectors η and ξ in a Hilbert space $\mathcal{H}$ define the operator-valued rational function R as:

$$R(z; \alpha, \beta, \eta, \xi) = I - P + \frac{z-\alpha}{z-\beta}P, \qquad P = \frac{\xi\eta^*}{\eta^*\xi}, \quad \eta^*\xi \neq 0; \tag{3.1}$$

here $R(z; \infty, \beta, \eta, \xi)$ and $R(z; \alpha, \infty, \eta, \xi)$ should be interpreted to be $I - P + (z - \beta)^{-1}P$ and $I - P + (z - \alpha)P$, respectively. Note that

$$(R(z; \alpha, \beta, \eta, \xi))^\# = R(z; \overline{\alpha}, \overline{\beta}, \xi, \eta) \quad \text{and} \quad (R(z; \alpha, \beta, \eta, \xi))^{-1} = (R(z; \beta, \alpha, \eta, \xi));$$

here for any operator-valued function $Q(z)$, $Q^\#(z)$ is defined to be $Q(\overline{z})^*$.

With this notation, (realizations for) products of the form $R^{\#}FR$, where $R(z) = R(z;\alpha,\beta,\eta,\xi)$, are investigated in the first subsection. In the second subsection these considerations are combined with a factorization from [14] to decompose generalized Nevanlinna functions with respect to their analytic behavior as stated in Theorem 1.1. These results are in turn used to prove Theorem 1.2 in the third and final subsection.

3.1 Multiplication with an Order One Term

Here an explicit realization for $R^{\#}FR$, where R is as in (3.1), is generated from any given realization for $F \in \mathfrak{N}_\kappa(\mathcal{H})$. This realization can be seen as a modification and extension of [16, Thm. 1.3] from scalar-valued to operator-valued functions. Note that the explicit resolvent formula in Proposition 3.1 reflects how the invariant subspaces of the realizing relation for $R^{\#}FR$ are connected to the invariant subspaces of the realizing relation for the original function F.

Proposition 3.1 *Let $F \in \mathfrak{N}_\kappa(\mathcal{H})$ be realized by $\{A,\Gamma\}$ at $z_0 \in \rho(A)\setminus\{\beta,\overline{\beta}\}$, where $\alpha,\beta \in \mathbb{C}\cup\{\infty\}$ satisfy $\alpha \neq \overline{\beta}$, and let $\xi,\eta \in \mathcal{H}$ satisfy $\eta^*\xi \neq 0$. Then $F_R := R^{\#}FR$, where $R(z) = R(z;\overline{\alpha},\beta,\eta,\xi)$ as in (3.1), is realized by $\{A_R,\Gamma_R\}$ which are defined for $z \in \rho(A)\setminus\{\beta,\overline{\beta}\}$ via*

$$(A_R - z)^{-1} = \begin{pmatrix} \frac{1}{\overline{\beta}-z} & \frac{\xi^*\Gamma_{\overline{z}}^{[*]}}{\overline{\beta}-z} & \frac{\xi^* F(z)\xi}{(\overline{\beta}-z)(\beta-z)} \\ 0 & (A-z)^{-1} & \frac{\Gamma_z\xi}{\beta-z} \\ 0 & 0 & \frac{1}{\beta-z} \end{pmatrix}, \quad \Gamma_R = \begin{pmatrix} \frac{\xi^* F(z_0)}{\overline{\beta}-z_0} R(z_0) \\ \Gamma R(z_0) \\ \frac{\overline{\alpha}-\beta}{\beta-z_0}\frac{\eta^*}{\eta^*\xi} \end{pmatrix}. \tag{3.2}$$

Here the realizing space $\{\Pi_2,[\cdot,\cdot]_2\}$ of $\{A_R,\Gamma_R\}$ is defined as

$$[g,h]_2 := [g_c,h_c] + g_r\overline{h_l} + g_l\overline{h_r}, \quad g = \{g_l,g_c,g_r\}, h = \{h_l,h_c,h_r\} \in \Pi_2 := \mathbb{C}\times\Pi\times\mathbb{C},$$

where $\{\Pi,[\cdot,\cdot]\}$ is the realizing space of $\{A,\Gamma\}$.

Recall that Γ_z in Proposition 3.1 is the γ-field associated with the realization $\{A,\Gamma\}$ for F, see (2.3). Furthermore, if $\alpha = \infty$, then Γ_R should be interpreted to be

$$\Gamma_R = \begin{pmatrix} \frac{\xi^* F(z_0)}{\overline{\beta}-z_0} R(z_0) & \Gamma R(z_0) & -\frac{1}{\beta-z_0}\frac{\eta^*}{\eta^*\xi} \end{pmatrix}^T,$$

and if $\beta = \infty$, then $\{A_R,\Gamma_R\}$ should be interpreted to be

$$(A_R - z)^{-1} = \begin{pmatrix} 0 & \xi^*\Gamma_{\overline{z}}^{[*]} & \xi^* F(z)\xi \\ 0 & (A-z)^{-1} & \Gamma_z\xi \\ 0 & 0 & 0 \end{pmatrix}, \quad \Gamma_R = \begin{pmatrix} \xi^* F(z_0) R(z_0) \\ \Gamma R(z_0) \\ \frac{\eta^*}{\eta^*\xi} \end{pmatrix}.$$

Proof Here only the case $\alpha, \beta \in \mathbb{C}$ is treated; the cases $\alpha = \infty$ or $\beta = \infty$ follow by analogous arguments.

First the selfadjointness of A_R is established. Therefore let $H(z) := (A_R - z)^{-1}$. Then the formula in (3.2) shows that $H(z)$ is an everywhere defined operator for $z \in \rho(A) \setminus \{\beta, \overline{\beta}\}$. In particular, since $\rho(A) \neq \emptyset$, $\rho(A)$ contains all of $\mathbb{C} \setminus \mathbb{R}$ except finitely many points; for all those points $H(z)$ is an everywhere defined bounded operator. Moreover, a direct calculation shows that $H(z)^{[*]} = H(\overline{z})$. Next we establish that H satisfies the resolvent identity. Therefore note that a calculation shows that $H(z)H(w)$ is equal to

$$\begin{pmatrix} \frac{1}{(\overline{\beta}-z)(\overline{\beta}-w)} & \frac{\xi^*}{\overline{\beta}-z}\left(\frac{\Gamma_{\overline{w}}^{[*]}}{\overline{\beta}-w} + \Gamma_{\overline{z}}^{[*]}(A-w)^{-1}\right) & \xi^* \frac{\frac{F(w)}{\overline{\beta}-w}+\Gamma_{\overline{z}}^{[*]}\Gamma_w+\frac{F(z)}{\beta-z}}{(\overline{\beta}-z)(\beta-w)}\xi \\ 0 & (A-z)^{-1}(A-w)^{-1} & \left((A-z)^{-1}\Gamma_w + \frac{\Gamma_z}{\beta-z}\right)\frac{\xi}{\beta-w} \\ 0 & 0 & \frac{1}{(\beta-z)(\beta-w)} \end{pmatrix}.$$

Using (2.3) and the resolvent identity for A we have that

$$\begin{aligned} \Gamma_{\overline{z}}^{[*]}(A-w)^{-1} &= \Gamma^{[*]}\left(I + (z-\overline{z_0})(A-z)^{-1}\right)(A-w)^{-1} \\ &= \Gamma^{[*]}\left((A-w)^{-1} + \frac{z-\overline{z_0}}{z-w}\left((A-z)^{-1} - (A-w)^{-1}\right)\right) \\ &= \frac{\Gamma^{[*]}}{z-w}\left((z-\overline{z_0})(A-z)^{-1} - (w-\overline{z_0})(A-w)^{-1}\right) = \frac{\Gamma_{\overline{z}}^{[*]} - \Gamma_{\overline{w}}^{[*]}}{z-w}. \end{aligned}$$

Moreover, using (2.4) we have that

$$\begin{aligned} \frac{F(w)}{\overline{\beta}-w} + \Gamma_{\overline{z}}^{[*]}\Gamma_w + \frac{F(z)}{\beta-z} &= F(z)\left(\frac{1}{\beta-z} + \frac{1}{z-w}\right) + F(w)\left(\frac{1}{\overline{\beta}-w} - \frac{1}{z-w}\right) \\ &= \frac{1}{z-w}\left(\frac{\beta-w}{\beta-z}F(z) - \frac{\overline{\beta}-z}{\overline{\beta}-w}F(w)\right). \end{aligned}$$

Combining the three preceding expressions and using the resolvent identity for A yields that $H(z)H(w) = \frac{H(z)-H(w)}{z-w}$. Consequently, A_R is a selfadjoint relation in $\{\Pi_2, [\cdot,\cdot]_2\}$, see [4, Prop. 3.4 and Cor. on p. 162].

As the second step towards proving that $\{A_R, \Gamma_R\}$ realizes F_R, the γ-field associated with $\{A_R, \Gamma_R\}$ is determined. Using

$$\frac{z_0-\overline{\alpha}}{z_0-\beta} + \frac{z-z_0}{\beta-z}\frac{\overline{\alpha}-\beta}{\beta-z_0} = \frac{\overline{\alpha}-\beta}{\beta-z} + 1 = \frac{z-\overline{\alpha}}{z-\beta} \tag{3.3}$$

and the identity $(z - z_0)\Gamma_{\overline{z}}^{[*]}\Gamma = F(z) - F(z_0)$, see (2.4), a straight-forward calculation shows that

$$(\Gamma_R)_z := (I + (z - z_0)(A_R - z)^{-1})\Gamma_R = \begin{pmatrix} \frac{\xi^* F(z)}{\overline{\beta}-z} R(z) & \Gamma_z R(z) & \frac{\overline{\alpha}-\beta}{\beta-z}\frac{\eta^*}{\eta^*\xi} \end{pmatrix}^{\top}.$$

Combining this last result with (3.3) and the identity $(z - \overline{z_0})\Gamma^{[*]}\Gamma_z = F(z) - F(\overline{z_0})$ from (2.4) leads to

$$\begin{aligned}(z - \overline{z_0})\Gamma_R^{[*]}(\Gamma_R)_z &= \frac{z - \overline{z_0}}{\overline{\beta} - z}\frac{\alpha - \overline{\beta}}{\overline{\beta} - \overline{z_0}}\frac{\eta\xi^*}{\xi^*\eta}F(z)R(z) + R(z_0)^* F(\overline{z_0})\frac{z - \overline{z_0}}{\beta - \overline{z_0}}\frac{\overline{\alpha} - \beta}{\beta - z}\frac{\xi\eta^*}{\eta^*\xi} \\ &\quad + \left[(I - P^*) + \frac{\overline{z_0} - \alpha}{\overline{z_0} - \overline{\beta}}P^*\right](F(z) - F(\overline{z_0}))[(I - P) + \frac{z - \overline{\alpha}}{z - \beta}P] \\ &= R^{\#}(z)F(z)R(z) - R^{\#}(\overline{z_0})F(\overline{z_0})R(\overline{z_0}) = F_R(z) - F_R(\overline{z_0}).\end{aligned}$$

This shows that $\{A_R, \Gamma_R\}$ realizes F_R, see (2.4). □

3.2 *Decomposing Generalized Nevanlinna Functions*

Recall that $\rho(F)$ and $\sigma(F)$ denote the set of holomorphy of a generalized Nevanlinna function F in $\mathbb{C} \cup \{\infty\}$ and its complement, respectively. When F is minimally realized by $\{A, \Gamma\}$, then $\rho(F)$ and $\sigma(F)$ coincide with $\rho(A)$ and $\sigma(A)$, respectively, see Theorem 2.2.

Proof Let $z_0 \in \rho(F) \cap (\mathbb{C} \setminus \mathbb{R})(\neq \emptyset)$, then there exists an everywhere defined selfadjoint operator C in $\mathcal{H}$ such that $\operatorname{ran}(F(z_0) + C) = \mathcal{H}$, i.e., that $F + C$ is boundedly invertible at z_0. Since the statement clearly holds for F if it holds for $F + C$, we may w.l.o.g. assume that F is boundedly invertible at a point $z_0 \in \rho(F) \cap (\mathbb{C} \setminus \mathbb{R})$, cf. [14, Prop. 2.1]; such operator-valued generalized Nevanlinna functions are called *regular*.

Let $\{\alpha_1, \ldots, \alpha_\kappa\}$ and $\{\beta_1, \ldots, \beta_\kappa\}$ be the sets of all GPNTs of F and $-F^{-1}$ in $\mathbb{C}_+ \cup \mathbb{R} \cup \{\infty\}$, respectively; here each GPNT occurs in accordance with its multiplicity. Since F is assumed to be regular, [14, Thm. 5.2 and Cor. 5.3] yield the existence of $\eta_1, \xi_1, \tilde{\eta}_1, \tilde{\xi}_1 \in \mathcal{H}$ satisfying $\eta_1^*\xi_1 \neq 0$ and $\tilde{\eta}_1^*\tilde{\xi}_1 \neq 0$ such that $F_1 := R_1^{\#} F R \in \mathfrak{N}_{\kappa-1}(\mathcal{H})$, where

$$R_1(z) = R(z; \beta_1, \overline{\gamma}, \tilde{\eta}_1, \tilde{\xi}_1)R(z; \gamma, \overline{\alpha_1}, \eta_1, \xi_1);$$

here γ is an arbitrary element of $\mathbb{C}\setminus(\mathbb{R}\cup\mathrm{GPNT}\,(F)\cup\mathrm{GPNT}\,(-F^{-1}))$. Moreover, the cited statements yield that $\{\alpha_2, \ldots, \alpha_\kappa\}$ and $\{\beta_2, \ldots, \beta_\kappa\}$ are the sets of all GPNTs of F_1 and $-F_1^{-1}$ in $\mathbb{C}_+ \cup \mathbb{R} \cup \{\infty\}$, respectively. Since F_1 is evidently regular, inductively applying this argument yields that F can be factorized as $R^{\#} F_0 R$, where

$F_0 \in \mathfrak{N}_0(\mathcal{H})$ and

$$R(z) = \prod_{j=1}^{\kappa} R(z; \overline{\alpha}_i, \gamma, \eta_i, \xi_i) R(z; \overline{\gamma}, \beta_i, \tilde{\eta}_i, \tilde{\xi}_i); \tag{3.4}$$

here γ is any element of $\mathbb{C} \setminus (\mathbb{R} \cup \mathrm{GPNT}\,(F) \cup \mathrm{GPNT}\,(-F^{-1}))$ and $\eta_i, \xi_i, \tilde{\eta}_i, \tilde{\xi}_i \in \mathcal{H}$ satisfy $\eta_i^* \xi_i \neq 0 \neq \tilde{\eta}_i^* \tilde{\xi}_i$ for $i = 1, \ldots, \kappa$. For later usage introduce the set $\mathcal{P}_0$ as

$$\mathcal{P}_0 := \{\gamma, \overline{\gamma}\} \cup \mathrm{GPNT}\,(F) = \{\gamma, \overline{\gamma}\} \cup \{\alpha_1, \ldots, \alpha_\kappa, \overline{\alpha_1}, \ldots, \overline{\alpha_\kappa}\}. \tag{3.5}$$

Let $\{A_0, \Gamma_0\}$ realize F_0 minimally, then the corresponding realizing space is a Hilbert space $\{\mathfrak{H}, (\cdot, \cdot)\}$, see e.g. [15]. Using the spectral family of A_0, $\mathfrak{H}$ can be decomposed as $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ such that, with A_i defined via $\mathrm{gr}(A_i) = \mathrm{gr}(A) \cap (\mathfrak{H}_i \times \mathfrak{H}_i)$,

(a) $\{\mathfrak{H}_i, (\cdot, \cdot)\}$ is a Hilbert space and $\mathfrak{H}_i$ is A-invariant for $i = 1, 2$;
(b) $\sigma(A_1) \subseteq \mathrm{clos}\,\Delta$ and $\mathrm{int}\,\Delta \subseteq \rho(A_2)$;
(c) $\sigma_p(A_1) \cap \sigma_p(A_2) = \emptyset$.

For instance, if $\Delta = (a, b) \subseteq \mathbb{R}$, then the desired decomposition with the properties (a)–(c) can be obtained by taking $\mathfrak{H}_1$ to be $E_b - E_a$, where $\{E_x\}_{x \in \mathbb{R}}$ is the spectral family associated with the Hilbert space selfadjoint relation A_0.

Since $\{A_0, \Gamma_0\}$ is a minimal realization for F_0, the decomposition with the properties (a)–(c) induces an additive representation $F_0 = F_1 + F_2$, where F_j is an ordinary Nevanlinna function realized by $\{A_j, P_j \Gamma_0\}$; here P_j is the Π-orthogonal projection onto $\mathfrak{H}_j$ for $j = 1, 2$, see the discussion following (2.10). Notice that the realizations $\{A_1, P_1 \Gamma_0\}$ and $\{A_2, P_2 \Gamma_0\}$ are automatically minimal, because the realization $\{A_0, \Gamma_0\}$ is assumed to be minimal. Inserting this additive representation $F_0 = F_1 + F_2$ into the factorization $F = R^\# F_0 R$ produces the following decomposition for F:

$$F(z) = R^\#(z) F_0(z) R(z) = R^\#(z) F_1 R(z) + R^\#(z) F_2(z) R(z). \tag{3.6}$$

Next the terms $R^\# F_1 R$ and $R^\# F_2 R$ are considered separately. In order to treat them, divide $\mathcal{P}_0$, see (3.5), into the following three sets:

$$\mathcal{P}_\Delta = \mathcal{P}_0 \cap \mathrm{int}\,\Delta, \quad \mathcal{P}_c = \mathcal{P}_0 \cap \partial\Delta \quad \text{and} \quad \mathcal{P}_r = \mathcal{P}_0 \setminus (\mathcal{P}_\Delta \cup \mathcal{P}_c). \tag{3.7}$$

$R^\# F_1 R$**:** By Proposition 3.1 there exist an extension $A_{1,R}$ of A_1 in a Pontryagin space $\{\Pi_{1,R}, [\cdot, \cdot]_{1,R}\}$ (with at most 2κ negative squares since, in addition to the poles $\overline{\alpha_i}$, R in (3.4) can have at most κ additional poles located at $\overline{\gamma}$) and a mapping $\Gamma_{1,R}$ such that $\{A_{1,R}, \Gamma_{1,R}\}$ realizes $R^\# F_1 R$. Furthermore, Proposition 3.1 shows that

$$\sigma(A_{1,R}) \subseteq \sigma(A_1) \cup \mathcal{P}_0 = \sigma(A_1) \cup \mathcal{P}_\Delta \cup \mathcal{P}_c \cup \mathcal{P}_r.$$

By definition, see (b) and (3.7), $\mathcal{P}_r$ consists of (finitely many) isolated points of the spectrum $\sigma(A_{1,R})$. Therefore $\{\Pi_{1,R}, [\cdot,\cdot]\}$ can by means of Riesz projections (contour integrals of the resolvent, see e.g. [1, Ch. 2: Thm. 2.20]) be decomposed as $\Pi^1_{1,R}[+]\Pi^2_{1,R}$, such that, with $A_{1,R,i}$ defined by $\mathrm{gr}(A_{1,R,i}) = \mathrm{gr}(A_{1,R}) \cap (\Pi^i_{1,R} \times \Pi^i_{1,R})$, the following statements hold:

(a_1) $\{\Pi^i_{1,R}, [\cdot,\cdot]_{1,R}\}$ is a Pontryagin space and $\Pi^i_{1,R}$ is $A_{1,R}$-invariant for $i = 1, 2$;
(b_1) $\sigma(A_{1,R,1}) \subseteq \sigma(A_1) \cup \mathcal{P}_\Delta \cup \mathcal{P}_c \subseteq \mathrm{clos}\,\Delta$;
(c_1) $\sigma(A_{1,R,2}) \subseteq \mathcal{P}_r$ and, hence, $\mathrm{int}\,\Delta \subseteq \rho(A_{1,R,2})$.

$R^\# F_2 R$: By Proposition 3.1 there exist an extension $A_{2,R}$ of A_2 in a Pontryagin space $\{\Pi_{2,R}, [\cdot,\cdot]_{2,R}\}$ (again with at most 2κ negative squares) and a mapping $\Gamma_{2,R}$ such that $\{A_{2,R}, \Gamma_{2,R}\}$ realizes $R^\# F_2 R$. Again Proposition 3.1 shows that

$$\sigma(A_{2,R}) \subseteq \sigma(A_2) \cup \mathcal{P}_0 = \sigma(A_2) \cup \mathcal{P}_\Delta \cup \mathcal{P}_c \cup \mathcal{P}_r.$$

Hence, by construction (see (b)) there exist an open neighborhood $\mathcal{O}$ (in $\mathbb{C}$) containing $\mathcal{P}_\Delta$ such that $\mathcal{O} \setminus \mathcal{P}_\Delta \subseteq \rho(A_{2,R})$. Thus $\{\Pi_{2,R}, [\cdot,\cdot]\}$ can by means of Riesz projections (see [1, Ch. 2: Thm. 2.20]) be decomposed as $\Pi^1_{2,R}[+]\Pi^2_{2,R}$, where, with $A_{2,R,i}$ defined via $\mathrm{gr}(A_{2,R,i}) = \mathrm{gr}(A_{2,R}) \cap (\Pi^i_{2,R} \times \Pi^i_{2,R})$, the following statements hold:

(a_2) $\{\Pi^i_{2,R}, [\cdot,\cdot]_{2,R}\}$ is a Pontryagin space and $\Pi^i_{2,R}$ is $A_{2,R}$-invariant for $i = 1, 2$;
(b_2) $\sigma(A_{2,R,1}) \subseteq \mathcal{P}_\Delta \subseteq \Delta$;
(c_2) $\sigma(A_{2,R,2}) \subseteq \sigma(A_2) \cup \mathcal{P}_c \cup \mathcal{P}_r$ and, hence, $\mathrm{int}\,\Delta \subseteq \rho(A_{2,R,2})$.

Now we reconsider F and decompose it as claimed in Theorem 1.1. Therefore let $F_{i,j}$ be the function realized by $\{A_{i,R,j}, \Gamma_{i,R,j}\}$ for $i, j = 1, 2$. By means of these functions define F_Δ and F_R as

$$F_\Delta := F_{1,1} + F_{2,1} \quad \text{and} \quad F_R := F_{1,2} + F_{2,2}.$$

We claim that these functions satisfy all the criteria in Theorem 1.1. Indeed, by construction the functions F_Δ and F_R are (possibly non-minimally) realized by $\{A_\Delta, \Gamma_\Delta\} := \{A_{1,R,1} \oplus A_{2,R,1}, \mathrm{col}\,(\Gamma_{1,R,1}, \Gamma_{2,R,1})\}$ and $\{A_R, \Gamma_R\} := \{A_{1,R,2} \oplus A_{2,R,2}, \mathrm{col}\,(\Gamma_{1,R,2}, \Gamma_{2,R,2})\}$, see Sect. 2.4. Therefore (b)–(c), (b_1)–(c_1) and (b_2)–(c_2) show that

$$\sigma(A_\Delta) \subseteq \mathrm{clos}\,\Delta, \quad \mathrm{int}\,(\Delta) \subseteq \rho(A_R) \quad \text{and} \quad \sigma_p(A_\Delta) \cap \sigma_p(A_r) \subseteq \mathcal{P}_c.$$

Since $\mathcal{P}_c$ is by definition equal to $(\{\gamma, \overline{\gamma}\} \cup \mathrm{GPNT}\,(F)) \cap \partial\Delta$, cf. (3.5) and (3.7), (2.5) and Proposition 2.3 show that all the assertions in Theorem 1.1 hold except the assertions about the sum of the indices κ_Δ and κ_R of F_Δ and F_R. The fact that $\kappa_\Delta + \kappa_R \geq \kappa$ is indicated in the discussion preceding (2.9). The final assertion in Theorem 1.1 that $\kappa_\Delta + \kappa_R = \kappa$ if $\mathrm{GPNT}\,(F) \cap (\mathrm{clos}\,(\Delta) \setminus \Delta) = \emptyset$ is now a consequence of Proposition 2.5. □

Inductively applying the preceding statement to the case when Δ is an interval of $\mathbb{R} \cup \{\infty\}$ containing precisely one GPNT in its interior yields Corollary 3.2 below. Note in connection with Corollary 3.2 that since non-real poles of a generalized Nevanlinna function are isolated, we can always write a generalized Nevanlinna function as the sum of a generalized Nevanlinna function holomorphic in $\mathbb{C} \setminus \mathbb{R}$ with rational functions each having a pole only at a non-real point and its conjugate.

Corollary 3.2 *Let* $F \in \mathfrak{N}_\kappa(\mathcal{H})$ *and let* $\mathrm{GPNT}\,(F) = \{\alpha_1, \ldots, \alpha_n, \overline{\alpha_1}, \ldots, \overline{\alpha_n}\}$ *where* $\alpha_1, \ldots, \alpha_n$ *are distinct elements of* $\mathbb{C} \cup \{\infty\}$. *Then* $F = \sum_{i=1}^n F_i$, *where*

(i) $F_i \in \mathfrak{N}_{\kappa_i}(\mathcal{H})$, *for* $i = 1, \ldots, n$, *and* $\sum_{i=1}^n \kappa_i = \kappa$;
(ii) $\mathrm{GPNT}\,(F_i) = \{\alpha_i, \overline{\alpha_i}\}$, *for* $i = 1, \ldots, n$;
(iii) $\sigma(F_i) \cap \sigma(F_j)$ *contains at most two points and any point contained in the intersection is not both a generalized pole for* F_i *and* F_j, *for* $1 \le i \ne j \le n$.

3.3 Decomposing Selfadjoint Relations in Pontryagin Spaces

In order to prove Theorem 1.2, the result from the preceding section is lifted to the setting of selfadjoint relations by associating to (the resolvent of) selfadjoint relations an (operator-valued) generalized Nevanlinna function.

Proof Let J be any canonical symmetry for the Pontryagin space $\{\Pi, [\cdot,\cdot]\}$ appearing in Theorem 1.2. Then $\{\mathcal{H}, (\cdot,\cdot)\} := \{\Pi, [J\cdot,\cdot]\}$ defines a Hilbert space, see e.g. [1, Ch. 1, § 3]. In addition to the given selfadjoint relation A in the Pontryagin space Π introduce the operator $\Gamma : \mathcal{H}(=\Pi) \to \Pi$ as the identity mapping. Then the pair $\{A, \Gamma\}$ provides a minimal realization for the following generalized Nevanlinna function:

$$F(z) = \overline{z_0}J + (z - \overline{z_0})J\left(I + (z - z_0)(A - z)^{-1}\right), \quad z_0, z \in \rho(A); \tag{3.8}$$

cf. (2.2). Let $F_\Delta + F_R$ be the additive decomposition of F provided by Theorem 1.1 with respect to Δ as in Theorem 1.2. In particular,

$$\sigma(F_\Delta) \subseteq \mathrm{clos}\,\Delta \quad \text{and} \quad \mathrm{int}\,\Delta \subseteq \rho(F_R). \tag{3.9}$$

If $\{A_\Delta, \Gamma_\Delta\}$ and $\{A_R, \Gamma_R\}$ are arbitrary minimal realizations for F_Δ and F_R, respectively, then $\{A_\Delta \widehat{\oplus} A_R, \mathrm{col}\,(\Gamma_\Delta, \Gamma_R)\}$ is a realization for F. Moreover, by Theorem 2.2 $\sigma(A_\Delta) = \sigma(F_\Delta)$ and $\rho(A_R) = \rho(F_R)$. In view of (3.9), the first statement in Theorem 1.2 now holds by Propositions 2.3 and 2.1.

Finally, if $\partial\Delta \cap \mathrm{ENT}\,(A) = \emptyset$, then by definition $\partial\Delta \cap \mathrm{GPNT}\,(F) = \emptyset$. Thus the additive decomposition $F_\Delta + F_R$ of F with respect to Δ provided by Theorem 1.1 has the following properties:

(a) $\sigma(F_\Delta) \subseteq \mathrm{clos}\,\Delta$ and $\mathrm{int}\,\Delta \subseteq \rho(F_R)$;
(b) no point of $\mathrm{clos}\,(\Delta) \setminus \Delta$ is both a generalized pole of F_Δ and F_R.

Let $\{A_\Delta, \Gamma_\Delta\}$ and $\{A_R, \Gamma_R\}$ be arbitrary minimal realizations for the function F_Δ and F_R, respectively. By Theorem 2.2 and the definition of generalized poles (see Sect. 2.2) the preceding two properties imply that

(a') $\sigma(A_\Delta) \subseteq \text{clos}\,\Delta$ and $\text{int}\,\Delta \subseteq \rho(A_R)$;
(b') $\sigma_p(A_\Delta) \cap \sigma_p(A_R) = \emptyset$.

Thus Proposition 2.6 implies that $\{A_\Delta \widehat{\oplus} A_R, \text{col}\,(\Gamma_\Delta, \Gamma_R)\}$, see (2.9), is a minimal realization for F in (3.8). Therefore the statement has been proven, because all minimal realizations for the same generalized Nevanlinna function are unitarily equivalent by Proposition 2.1. □

The assumption $\rho(A) \neq \emptyset$ in Theorem 1.2 is needed, because there exist selfadjoint relations A (even in finite-dimensional) Pontryagin spaces for which $\sigma_p(A) = \mathbb{C} \cup \{\infty\}$; see [4, p. 155–156].

Applying Theorem 1.2 inductively leads to the following decomposition results for selfadjoint relations. Note that from Corollary 3.3 the so-called canonical form of selfadjoint operators in finite-dimensional Pontryagin spaces, see [5, Thm. 5.1.1.], can be derived.

Corollary 3.3 *Let A be a selfadjoint relation in a Pontryagin space $\{\Pi, [\cdot,\cdot]\}$ with $\sigma(A) \cap (\mathbb{C}_+ \cup \mathbb{R} \cup \{\infty\}) = \{\alpha_1, \dots, \alpha_n\}$. Then there exists a decomposition $\Pi_1[+]\dots[+]\Pi_n$ of Π such that*

(i) *$\{\Pi_i, [\cdot,\cdot]\}$ is a Pontryagin space for $i = 1, \dots, n$;*
(ii) *Π_i is A-invariant for $i = 1, \dots, n$;*
(iii) *$\sigma(A\restriction_{\Pi_i}) = \{\alpha_i, \overline{\alpha_i}\}$ for $i = 1, \dots, n$.*

Corollary 3.4 *Let A be a selfadjoint relation in a Pontryagin space $\{\Pi, [\cdot,\cdot]\}$ with $\rho(A) \neq \emptyset$ and let $\text{ENT}\,(A) = \{\alpha_1, \dots, \alpha_n, \overline{\alpha_1}, \dots, \overline{\alpha_n}\}$. Then there exists a decomposition $\Pi_1[+]\dots[+]\Pi_n$ of Π such that*

(i) *$\{\Pi_i, [\cdot,\cdot]\}$ is a Pontryagin space for $i = 1, \dots, n$;*
(ii) *Π_i is A-invariant for $i = 1, \dots, n$;*
(iii) *$\{\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots \alpha_n\} \in \rho(A\restriction_{\Pi_i})$ for $i = 1, \dots, n$;*
(iv) *$\sigma_p(A\restriction_{\Pi_i}) \cap \sigma_p(A\restriction_{\Pi_j}) = \emptyset$ and $\sigma(A\restriction_{\Pi_i}) \cap \sigma(A\restriction_{\Pi_j})$ contains at most finitely many points, for $1 \leq i \neq j \leq n$.*

Observe that condition (iii) in Corollary 3.4 implies that α_i and $\overline{\alpha_i}$ are the only ENTs of A restricted to Π_i for $i = 1, \dots, n$.

References

1. T. Ya. Azizov and I. S. Iokhvidov, *Linear operators in spaces with an indefinite metric*, John Wiley and Sons, New York, 1989.
2. K. Daho, and H. Langer, "Matrix functions of the class $\mathfrak{N}_\kappa$", *Math. Nachr.*, **120** (1985), 275–294.

3. A. Dijksma, H. Langer, and H.S.V. de Snoo, "Eigenvalues and pole functions of Hamiltonian systems with eigenvalue depending boundary conditions", *Math. Nachr.*, 161 (1993), 107–154.
4. A. Dijksma and H. S. V. de Snoo, "Symmetric and selfadjoint relations in Kreĭn spaces I", *Oper. Theory Adv. Appl.*, **24** (1987), 145–166.
5. I. Gohberg, P. Lancaster, and L. Rodman, *Indefinite linear algebra and applications*, Birkhäuser Verlag, Basel, 2005.
6. S. Hassi, M. Kaltenbäck, and H.S.V. de Snoo, "The sum of matrix Nevanlinna functions and selfadjoint extensions in exit spaces", *Oper. Theory Adv. Appl.*, 103 (1998), 137–154.
7. S. Hassi, H.S.V. de Snoo, and H. Woracek, "Some interpolation problems of Nevanlinna-Pick type", *Oper. Theory Adv. Appl.*, 106 (1998), 201–216.
8. S. Hassi and H.L. Wietsma, "Products of generalized Nevanlinna functions with symmetric rational functions", *J. Funct. Anal.* **266** (2014), 3321–3376.
9. I.S. Iohvidov, M.G. Kreĭn and H. Langer, *Introduction to the spectral theory of operators in spaces with an indefinite metric*, Akademie-Verlag, Berlin, 1982.
10. P. Jonas, "Operator representations of definitizable functions", *Ann. Acad. Sci. Fenn. Math.* **25** (2000), 41–72.
11. M.G. Kreĭn and H. Langer, "Über die Q-Funktion eines π-hermiteschen Operators im Raume Π_κ", *Acta Sci. Math. (Szeged)* **34** (1973), 191–230.
12. M.G. Kreĭn and H. Langer, "Über einige Fortsetzungsprobleme, die eng mit der Theorie hermitescher Operatoren im Raum Π_κ zusammenhängen, I. Einige Funktionenklassen und ihre Darstellungen", *Math. Nachr.* **77** (1977), 187–236.
13. H. Langer, *Spectral functions of definitizable operators in Kreĭn spaces*, Functional analysis, Proceedings, Dubrovnik 1981, Lecture Notes in Mathematics 948, Berlin 1982.
14. A. Luger, "A factorization of regular generalized Nevanlinna functions", *Integral Equations Operator Theory* **43** (2002), 326–345.
15. H. Langer and B. Textorius, "On generalized resolvents and Q-functions of symmetric linear relations (subspaces) in Hilbert space", *Pacific J. of Math.*, **72** (1977), 135–165.
16. H.L. Wietsma, "Factorization of generalized Nevanlinna functions and the invariant subspace property", *Indagationes Mathematicae* **30** (2019), 26–38.

Martin Functions of Fuchsian Groups and Character Automorphic Subspaces of the Hardy Space in the Upper Half Plane

A. Kheifets and P. Yuditskii

Dedicated to our teacher Prof. V. E. Katsnelson[1] *on the occasion of his 75-th birthday*

Abstract We establish exact conditions for non triviality of all subspaces of the standard Hardy space in the upper half plane, that consist of the character automorphic functions with respect to the action of a discrete subgroup of $SL_2(\mathbb{R})$. Such spaces are the natural objects in the context of the spectral theory of almost periodic differential operators and in the asymptotics of the approximations by entire functions. A naive idea: it should be completely parallel to the celebrated Widom characterization for Hardy spaces on Riemann surfaces with a minor modification, namely, one has to substitute the Green function of the domain with the Martin function. Basically, this is correct, but…

[1]Following the Russian tradition, we use the initials V.E. for Victor Emmanuilovich. V.E. was the most unusual and therefore the most attractive person among professors in the math department for students of our generation. The method he used to bring us in mathematics was also very much unusual for our time and our country: V.E. took us (four first year Master students) to a REAL mathematical conference. Actually, it was a school, but definitely of the highest conference level. At this school, we were learning the J-theory for 2 weeks at least 6 h a day. V.E. was one of the lecturers, highly enthusiastic. We studied the theory together with the most prominent professors of our department. At the lunchtime, during a ski trip, or at the night lectures we were able to meet the authors of practically all popular textbooks of that time. V.E. was our guide to this new world. If indeed a personality is completely determined by the first 3 years of our life, our mathematical personalities definitely were determined by these first 2 weeks of our mathematical childhood.

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1 Introduction

Harold Widom discovered that the asymptotic behaviour of orthogonal polynomials associated with a system of curves in the complex plane can be expressed in terms of the *reproducing kernels* of the Hardy spaces of *character automorphic* functions on the complementary domain (containing infinity component) [20]. Later on in [21] he found a condition that guaranties non triviality of all these spaces on infinitely connected domains. Essentially this created a foundation for the most comprehensive currently available function theory on multiply connected domains (and Riemann surfaces) [8].

In its turn the theory of character automorphic Hardy spaces appeared to be the most efficient tool in solving inverse spectral and scattering problems for ergodic, almost periodic difference/differential operators and for their perturbations, see e.g. [16, 17], see also [22]. We mention here that a certain reverse influence also took place, see [18].

Another broad field of research to be naturally mentioned here is the spectral theory of commuting non-self adjoint operators and the interpolation theory on the Riemann surfaces, see e.g. [1, 2, 12].

Viewing the Hardy spaces on the Riemann surfaces in terms of the *universal covering* is extremely convenient for analysts, see e.g. [14]. Under this approach we realize the corresponding Hilbert space on the Riemann surface as a subspace of the standard H^2 in the disc consisting of the functions automorphic (character automorphic with a prescribed character) with respect to the action of a certain Fuchsian group Γ. Effectively, an essential part of the book [8] can be substituted with a single paper [15] by Christian Pommerenke if one uses this approach.

In this paper we give precise conditions for non triviality of all subspaces $\mathcal{H}^2(\alpha)$ (see Definition 1.6). Such spaces are natural in the context of the spectral theory of differential operators and in the asymptotics of approximations by entire functions.

A naive idea: it should be completely parallel to the corresponding Widom characterization with a minor modification, namely, one has to substitute the Green function of the domain with the Martin function. Basically, this is correct ..., but, as we will see, one more condition (see condition (B) in our main Theorem 1.8) should be surprisingly added to Widom type condition (A) in this case. Moreover, in Remark 1.11 we will show why the approach of [15] cannot work here in principle without essential modification.

Going to the precise statement we will introduce some notations, recall definitions and some facts. We restrict ourself to *Denjoy domains* (complements to real closed sets), which are *regular* in the sense of the potential theory.

Let $E = \mathbb{R} \setminus \cup_{j\in\mathbb{Z}}(a_j, b_j)$ be a closed subset of $\mathbb{R}$, $E \neq \mathbb{R}$, unbounded in the following sense

$$\forall N > 0\ \exists \lambda_\pm \in E : \lambda_+ > N \text{ and } \lambda_- < -N. \tag{1.1}$$

Regularity of E means that there exists a positive harmonic function in $\Omega = \mathbb{C} \setminus E$ with the only singularity at a point $\lambda_0 \in \Omega$ that is continuous up to the boundary of Ω and vanishes there. This function is called the Green function and is denoted by $G(\lambda, \lambda_0)$.

One can give a parametric description of regular Denjoy domains in terms of the special conformal mappings that were introduced by Akhiezer and Levin, see [11], and that are extensively used in the spectral theory, see [13]; for a modern point of view see [6], in particular for the proofs of Theorem 1.1, Propositions 1.2 and 1.7. Let

$$\Pi = \{\xi + i\eta, \eta > 0, \xi \in (0, \pi)\} \setminus \cup_{j\neq 0}\{\omega_j + i\eta : \eta \in (0, h_j)\}, \tag{1.2}$$

where $\{(\omega_k, h_k)\}_{k\neq 0}$ is any collection of numbers such that

$$\omega_k \in (0, \pi), \quad \omega_k \neq \omega_j \quad \text{for} \quad k \neq j, \tag{1.3}$$

and

$$h_k > 0, \quad \lim_{k\to\infty} h_k = 0. \tag{1.4}$$

Domains Π of this type are called the *regular combs*.

Theorem 1.1 *Let* $(a_0, b_0) \subset \mathbb{R}$ *and* $\lambda_* \in (a_0, b_0)$. *Let* Π *be an arbitrary comb of the form* (1.2)–(1.4) *with parameters*

$$(\omega_k, h_k), \quad k \neq 0,$$

and let

$$\theta_{\lambda_*} : \mathbb{C}_+ \to \Pi$$

be the conformal mapping[2] *of* $\mathbb{C}_+$ *onto* Π, *normalized as follows*

$$\theta_{\lambda_*}(\lambda_*) = \infty,\ \theta_{\lambda_*}(b_0) = 0,\ \theta_{\lambda_*}(a_0) = \pi. \tag{1.5}$$

[2] Here $\mathbb{C}_+$ is considered as a subset of $\mathbb{C} \setminus E$.

Then θ_{λ_*} can be extended by continuity to the real axis and the set $E := \theta_{\lambda_*}^{-1}([0,\pi])$ is regular. Moreover, $\operatorname{Im}\theta_{\lambda_*}(\lambda)$ *can be extended to the domain $\Omega := \mathbb{C} \setminus E$ as a single-valued function, and for this extension we have*

$$\operatorname{Im}\theta_{\lambda_*}(\lambda) = G(\lambda, \lambda_*), \tag{1.6}$$

where $G(\lambda, \lambda_*)$ is the Green function of Ω. Due to normalization (1.5), $\theta_{\lambda_*}(\infty) \in (0,\pi)$. If it does not coincide with the base point of a slit, i.e., $\theta_{\lambda_*}(\infty) \neq \omega_j$, $j \in \mathbb{Z}$, then the set E has property (1.1).

Conversely, let E be a regular set, let (a_0, b_0) be a component of $\mathbb{R} \setminus E$ and let $\lambda_* \in (a_0, b_0)$. Then there exists a comb Π_{λ_*} of the form (1.2)–(1.4) with parameters

$$(\omega_{\lambda_*,k}, h_{\lambda_*,k}), \quad k \neq 0,$$

such that E corresponds to the base $[0,\pi]$ for the conformal mapping $\theta_{\lambda_*} : \mathbb{C}_+ \to \Pi_{\lambda_*}$, normalized as in (1.5). Moreover, (1.6) holds. If E has property (1.1), then $\theta_{\lambda_*}(\infty)$ does not coincide with the base point of a slit, i.e., $\theta_{\lambda_*}(\infty) \neq \omega_j$, $j \in \mathbb{Z}$.

The function $\theta_{\lambda_*}(\lambda)$ admits a Schwarz-Christoffel type representation (an infinite analogue of the conformal mapping onto a polygon).

Proposition 1.2 *Assume that E is regular and that $\theta_{\lambda_*}(\lambda)$ is the conformal mapping on the corresponding comb domain. Let*

$$\mu_{\lambda_*,k} := \theta_{\lambda_*}^{-1}(\omega_{\lambda_*,k} + i h_{\lambda_*,k}) \in (a_k, b_k).$$

Then for $\lambda \in \Omega$

$$\theta'_{\lambda_*}(\lambda) = \frac{i}{\lambda_* - \lambda} \frac{\sqrt{(\lambda_* - a_0)(\lambda_* - b_0)}}{\sqrt{(\lambda - a_0)(\lambda - b_0)}} \prod_{k \neq 0} \frac{\lambda - \mu_{\lambda_*,k}}{\lambda_* - \mu_{\lambda_*,k}} \frac{\sqrt{(\lambda_* - a_k)(\lambda_* - b_k)}}{\sqrt{(\lambda - a_k)(\lambda - b_k)}}. \tag{1.7}$$

In particular, $\{\mu_{\lambda_*,k}\}_{k \neq 0}$ is the complete list of the critical points of the function $G(\lambda, \lambda_*)$, that is, the points where $\nabla G(\lambda, \lambda_*) = 0$.

Definition 1.3 A regular domain is said to be of the Widom type if

$$\sum_{\mu : \nabla G(\mu, \lambda_*) = 0} G(\mu, \lambda_*) = \sum_{k \neq 0} G(\mu_{\lambda_*,k}, \lambda_*) < \infty. \tag{1.8}$$

Note that, by (1.6), $G(\mu_{\lambda_*,k}, \lambda_*) = h_{\lambda_*,k}$ and (1.8) is the same as

$$\sum_{k \neq 0} h_{\lambda_*,k} < \infty. \tag{1.9}$$

Thus, all Denjoy domains of the Widom type are represented by the conformal mappings on the comb domains, where (1.4) should be substituted with a stronger condition (1.9).

According to the *uniformization theorem* there exists an analytic function $\Lambda(z)$ on the upper half plane $\mathbb{C}_+$ that sets a one to one correspondence between the domain Ω and the factor of $\mathbb{C}_+$ under the action of a discrete group $\Gamma \subset SL_2(\mathbb{R})$, that is, $\Lambda(z) \in \Omega$, and for every $\lambda \in \Omega$ there exists $z \in \mathbb{C}_+$ such that $\Lambda(z) = \lambda$. Moreover,

$$\Lambda(\gamma(z)) = \Lambda(z), \text{ where } \gamma(z) = \frac{\gamma^{11}z + \gamma^{12}}{\gamma^{21}z + \gamma^{22}} \text{ for all } \gamma = \begin{bmatrix} \gamma^{11} & \gamma^{12} \\ \gamma^{21} & \gamma^{22} \end{bmatrix} \in \Gamma,$$

and $\Lambda(z_1) = \Lambda(z_2)$ implies that there exists $\gamma \in \Gamma$ such that $z_1 = \gamma(z_2)$.

In terms of the universal covering the Green function $G(\lambda, \lambda_*)$ admits the following representation. Let us fix z_* such that $\Lambda(z_*) = \lambda_*$. Consider[3]

$$g(z, z_*) = \prod_{\gamma \in \Gamma} \frac{z - \gamma(z_*)}{z - \overline{\gamma(z_*)}} C_\gamma, \quad z \in \mathbb{C}_+, \tag{1.10}$$

where $C_{1_\Gamma} = 1$ and for all $\gamma \neq 1_\Gamma$

$$C_\gamma = \left| \frac{z_* - \gamma(z_*)}{z_* - \overline{\gamma(z_*)}} \right| \frac{z_* - \overline{\gamma(z_*)}}{z_* - \gamma(z_*)} .$$

Then

$$G(\Lambda(z), \Lambda(z_*)) = -\ln |g(z, z_*)|. \tag{1.11}$$

For this reason $g(z, z_*)$ is called the (complex) Green function of the group Γ, see [15]. Combining (1.6) and (1.11), we get

$$\theta_{\Lambda(z_*)}(\Lambda(z)) = -i \ln g(z, z_*). \tag{1.12}$$

Therefore,

$$\theta'_{\Lambda(z_*)}(\Lambda(z)) \cdot \Lambda'(z) = -i \frac{g'(z, z_*)}{g(z, z_*)}.$$

[3] If E is regular, then Γ is of the convergent type. That is, the orbit of every point in $\mathbb{C}_+$ satisfies the Blaschke condition.

From here we see that the critical points $\{\mu_{\lambda_*,k}\}_{k\neq 0}$ of the Green function $G(\lambda,\lambda_*)$ are images of the zeros of $g'(z,z_*)$ under Λ. Then Widom condition (1.8) can be written in terms of $g(z,z_*)$ as follows

$$\prod_{k\neq 0}|g(c_{z_*,k},z_*)|>0, \tag{1.13}$$

where $c_{z_*,k}$ are zeros of $g'(z,z_*)$ in the fundamental domain of Γ, which is the Blaschke condition on *all* the zeros of $g'(z,z_*)$ in the upper half plane.

By Γ^* we denote the group of the unimodular characters of Γ, that is, the functions

$$\alpha:\Gamma\to\mathbb{T}\quad\text{such that}\quad\alpha(\gamma_1\gamma_2)=\alpha(\gamma_1)\alpha(\gamma_2),\,\gamma_j\in\Gamma.$$

Note $g(z,z_*)$ is an example of the character automorphic function, that is, there exists $\beta_*=\beta_{g(\cdot,z_*)}\in\Gamma^*$ such that

$$g(\gamma(z),z_*)=\beta_*(\gamma)g(z,z_*),$$

respectively,

$$|g(\gamma(z),z_*)|=|g(z,z_*)|.$$

Passing by a linear fractional transformation from the unit disk $\mathbb{D}$ to the upper half plane $\mathbb{C}_+$, we introduce the classical Hardy space H^2 of holomorphic functions on $\mathbb{C}_+$, with the norm

$$\|f\|^2=\|f\|^2_{H^2}=\int_{\mathbb{R}}|f(x)|^2dm(x),\quad dm(x)=\frac{dx}{1+x^2}. \tag{1.14}$$

Definition 1.4 For a fixed character $\alpha\in\Gamma^*$ we define

$$H^2(\alpha)=\{f\in H^2:\ f(\gamma(z))=\alpha(\gamma)f(z),\forall\gamma\in\Gamma\}.$$

The following statement is the Pommerenke version [15] of the Widom theorem (recall, in this paper we discuss only Denjoy domains).

Theorem 1.5 *Let $\Omega=\mathbb{C}\setminus E$ be a regular Denjoy domain and $\Lambda:\mathbb{C}_+/\Gamma\simeq\Omega$ be its uniformization. The following conditions are equivalent*

(i) *For every $\alpha\in\Gamma^*$ the space $H^2(\alpha)$ contains a non constant function.*
(ii) *The derivative $g'(z,z_*)$ of the Green function is a function of bounded characteristic in $\mathbb{C}_+$ (a ratio of two bounded holomorphic functions)*
(iii) *Widom condition* (1.8) (*equivalently* (1.13)) *holds.*

Let now $\mathcal{H}^2$ be the standard Hardy space in the upper half plane, that is, Smirnov class functions f with finite norm

$$\|f\|^2 = \|f\|^2_{\mathcal{H}^2} = \int_{\mathbb{R}} |f(x)|^2 dx.$$

Definition 1.6 For a fixed character $\alpha \in \Gamma^*$ we introduce

$$\mathcal{H}^2(\alpha) = \{f \in \mathcal{H}^2 : \ f(\gamma(z)) = \alpha(\gamma) f(z), \forall \gamma \in \Gamma\}.$$

We express a similar property of non triviality of all $\mathcal{H}^2(\alpha)$ spaces in terms of the Martin function $M(\lambda)$ in Ω (associated to the infinity).

To be more precise, by $M(\lambda)$, $\lambda \in \Omega$, we denote the symmetric Martin function with respect to the infinity, see e.g. [4, 6, 11], and the references therein. That is, $M(\lambda)$ is a positive harmonic in Ω function, continuous up to the boundary with the only exception at the infinity and vanishing at every finite point of the boundary. Symmetry means that

$$M(\overline{\lambda}) = M(\lambda).$$

Such a function is unique up to a positive constant factor. It also admits a Schwarz-Christoffel type representation.

Proposition 1.7 *Assume that Ω is a regular Danjoy domain. All critical points μ_k of the symmetric Martin function are real, moreover it has exactly one critical point in each gap*

$$\mu_k \in (a_k, b_k), \quad k \in \mathbb{Z}.$$

Then for $\lambda \in \mathbb{C}_+$ and a fixed normalization point $\lambda_* \in (a_0, \mu_0)$

$$\theta(\lambda) := i(\partial_x M)(\lambda_*) \int_{a_0}^{\lambda} \prod_{k \in \mathbb{Z}} \frac{\xi - \mu_k}{\lambda_* - \mu_k} \frac{\sqrt{(\lambda_* - a_k)(\lambda_* - b_k)}}{\sqrt{(\xi - a_k)(\xi - b_k)}} d\xi \tag{1.15}$$

and $M(\lambda) = \operatorname{Im} \theta(\lambda)$.

Note that $\theta(\lambda)$ also generates a conformal mapping of the upper half plane on a special comb domain [6]. It can be extended to Ω as an (additive) character automorphic function. This property is convenient to state in terms of the uniformization: let $m(z) = \theta(\Lambda(z))$, then

$$M(\Lambda(z)) = \operatorname{Im} m(z), \quad m(\gamma(z)) = m(z) + \eta(\gamma), \tag{1.16}$$

where $\eta(\gamma) \in \mathbb{R}$, $\eta(\gamma_1\gamma_2) = \eta(\gamma_1) + \eta(\gamma_2)$. Similar to $g(z, z_*)$, the function $m(z)$ can be called the *(symmetric) complex Martin function of the group* Γ.

Now we can state our main result.

Theorem 1.8 *Let $\Omega = \mathbb{C} \setminus E$ be a regular Denjoy domain and $\Lambda : \mathbb{C}_+/\Gamma \simeq \Omega$ be its uniformization. The following conditions are equivalent*

(i) *For every $\alpha \in \Gamma^*$ the space $\mathcal{H}^2(\alpha) \neq \{0\}$.*
(ii) *(a) The derivative $m'(z)$ of the Martin function of the group Γ is a function of bounded characteristic;*
  (b) The Riesz-Herglotz measure correspondent to $m(z)$ is a pure point one.
(iii) *The symmetric Martin function $M(\lambda)$ of the domain Ω possesses the following two properties*
  (A) $\sum_{j\in\mathbb{Z}} G(\mu_j, \lambda_*) < \infty$, *where μ_j are the critical points of the* Martin function*;*
  (B) $\lim_{\eta\to+\infty} M(i\eta)/\eta > 0$.

Remark 1.9 First of all we note that in our theorem condition (a), as an expected counterpart of (ii) in the Pommerenke theorem, should be accompanied by the second condition (b), respectively, the Widom type condition (A) in our case is accompanied by condition (B) that characterizes a special behaviour of the Martin function at infinity.

Remark 1.10 (B) is the well known Akhiezer-Levin condition, see e.g. [4]. As soon as (B) holds $M(\lambda)$ is also called the *Phragmén-Lindelöf function*, see [9] and especially Theorem on p. 407 in this book. Condition (b) was discussed in [19], see especially Theorem 5 and Lemma 1 there. It can be equivalently stated in a form similar to condition (B)

$$(b_1) \qquad \lim_{y\to+\infty} M(\Lambda(iy))/y > 0. \tag{1.17}$$

We point out that in condition (B) $i\eta$ belongs to the upper half plane of the domain $\mathbb{C} \setminus E$, whereas in condition (b_1) iy is in the universal cover. It appears that the equivalence of the Akhiezer-Levin condition (B) and property (b), proved in Sect. 4.2 below, is a new result.

Remark 1.11 Pommerenke's proof of implication (iii) to (ii) in Theorem 1.5 is based on an exhaustion of the given domain Ω by subdomains Ω_ϵ: connected components of the set

$$\{\lambda : G(\lambda, \lambda_*) > \epsilon\},$$

containing λ_*. It is highly important in the proof that such domains are finitely connected. To follow this line in our proof and to keep under control the critical points of the Martin function one has to make a similar exhaustion generated by the sets

$$\{\lambda : M(\lambda) > \epsilon\}.$$

But the simplest example

$$\Omega := \left\{\lambda : |\cos\lambda| > \frac{1}{2}\right\}$$

shows that the corresponding domains Ω_ϵ remain possibly infinitely connected for *all* sufficiently small ϵ. Thus, another kind of approximation of the given domain is needed, respectively the proof should be essentially reorganised.

In this paper we choose the approximation of the group Γ by its finitely generated subgroups. The corresponding construction is discussed in Sect. 2. In Sect. 3 we partially reprove Pommerenke Theorem 1.5 (equivalence of (ii) and (iii)) using this approach. In this part, it is an essential simplification of his original construction. Note, though, that we are restricted in our setting to Denjoy domains only, while Pommerenke's proof is valid for arbitrary Riemann surfaces. Section 4.1 describes the Martin functions $m(z)$ that possess property (b) (equivalently (b_1) of (1.17), by Proposition 4.1). In Sect. 4.2 we prove that condition (b_1) and Akhiezer-Levin condition (B) are equivalent. Finally, in Sect. 5 we prove our main Theorem 1.8. The proof is broken into several steps, each one corresponds to a certain implication between assertions (i)–(iii). For the reader's convenience in the Appendix we give proofs of the Carathéodory and Frostman theorems, that were essential components of the original Pommernke's proof [15] (given there as references).

2 Preliminaries

The Blaschke condition on a set $\{z_k\}$ for the upper half plane can be written as

$$\sum_k \frac{\operatorname{Im} z_k}{|z-\overline{z_k}|^2} < \infty, \quad \operatorname{Im} z_k > 0, \tag{2.1}$$

where z is an arbitrary fixed point in the upper half plane. The convergence in (2.1) is uniform in z on compact subsets of the open upper half plane, since

$$\frac{|z-w|}{|\tilde{z}-w|}$$

is continuous and, therefore, is bounded when z and $\tilde{z}$ are in a compact subset of the open upper half plane and w is in the closed lower half plane (including infinity). Hence, the corresponding Blaschke product

$$\prod_k \frac{z-z_k}{z-\overline{z_k}} C_k,$$

converges uniformly on the compact subsets of $\mathbb{C}_+$, where constants C_k are chosen to make the factors positive at one point of the upper half plane.

Since Γ is of convergent type, the Blaschke condition holds for the orbit of an arbitrary point z_* in the upper half plane

$$\sum_{\gamma\in\Gamma} \frac{\operatorname{Im}\gamma(z_*)}{|z-\overline{\gamma(z_*)}|^2} < \infty, \quad \operatorname{Im} z > 0. \tag{2.2}$$

Hence, $g(z, z_*)$ is well defined by this formula

$$g(z, z_*) = \prod_{\gamma\in\Gamma} \frac{z-\gamma(z_*)}{z-\overline{\gamma(z_*)}}\, C_\gamma, \quad z \in \mathbb{C}_+, \tag{2.3}$$

and the convergence is uniform on the compact subsets of $\mathbb{C}_+$. Equivalently, $g(z, z_*)$ can be defined as

$$g(z, z_*) = \prod_{\gamma\in\Gamma} \frac{\gamma(z)-z_*}{\gamma(z)-\overline{z_*}}\, \tilde{C}_\gamma, \quad z \in \mathbb{C}_+. \tag{2.4}$$

For the logarithmic derivative of $g(z, z_*)$ we get

$$\frac{g'(z, z_*)}{g(z, z_*)} = (z_* - \overline{z_*}) \sum_{\gamma\in\Gamma} \frac{\gamma'(z)}{(\gamma(z)-z_*)(\gamma(z)-\overline{z_*})}, \quad z \in \mathbb{C}_+. \tag{2.5}$$

From here we see that

$$g'(z, z_*) = (z_* - \overline{z_*}) \sum_{\gamma\in\Gamma} \frac{g(z, z_*)\gamma'(z)}{(\gamma(z)-z_*)(\gamma(z)-\overline{z_*})}, \quad z \in \mathbb{C}_+. \tag{2.6}$$

The convergence in (2.6) is absolute and uniform on compact subsets of $\mathbb{C}_+$ due to the uniform convergence in (2.2), see also (2.3), (2.4).

We consider domain $\mathcal{F}$ that is obtained from the universal covering space $\mathbb{C}_+$ by removing countably (or finitely) many semi-disks with real centers. We choose one of them to be of radius 1 with center at 0 and we label it with index 0. The universal covering map carries $\mathcal{F}$ conformally onto the upper half plane in $\mathbb{C} \setminus E$. The semi-circles are mapped onto the gaps, the real part of the boundary of $\mathcal{F}$ is mapped onto E. The fundamental domain of the group Γ can be obtained by taking the union of $\mathcal{F}$ with its reflection about the 0-th semi-circle. We also mention here that generators of the group Γ are the compositions of this reflection with the reflections about the other boundary semi-circles of $\mathcal{F}$.

We consider domain $\mathcal{F}_n$ that is obtained from $\mathcal{F}$ by keeping a finite number of the semi-circles and replacing the others with their diameters on the real line. We have that

$$\mathcal{F} = \bigcap_n \mathcal{F}_n.$$

Group Γ_n is generated by the compositions of pairs of the reflections about the boundary semi-circles of $\mathcal{F}_n$. Γ_n is a subgroup of Γ and

$$\Gamma = \bigcup_n \Gamma_n.$$

We consider the complex Green function for Γ_n similar to the one for Γ with the same z_*

$$g_n(z, z_*) = \prod_{\gamma \in \Gamma_n} \frac{\gamma(z) - z_*}{\gamma(z) - \overline{z_*}} \widetilde{C}_\gamma, \quad z \in \mathbb{C}_+.$$

g_n is a divisor of g. Therefore,

$$|g_n(z)| \geq |g(z)|, \quad z \in \mathbb{C}_+. \tag{2.7}$$

We also mention here that

$$\frac{g_n'(z, z_*)}{g_n(z, z_*)} = (z_* - \overline{z_*}) \sum_{\gamma \in \Gamma_n} \frac{\gamma'(z)}{(\gamma(z) - z_*)(\gamma(z) - \overline{z_*})}, \quad z \in \mathbb{C}_+,$$

and

$$g_n'(z, z_*) = (z_* - \overline{z_*}) \sum_{\gamma \in \Gamma_n} \frac{g_n(z, z_*)\gamma'(z)}{(\gamma(z) - z_*)(\gamma(z) - \overline{z_*})}. \tag{2.8}$$

Again, the convergence is absolute and uniform on the compact subsets of $\mathbb{C}_+$, since this is true even for the whole group Γ (see (2.6)).

Lemma 2.1 *As n goes to ∞, $g_n(z, z_*)$ converges to $g(z, z_*)$ uniformly on the compact subsets in $\mathbb{C}_+$ and $g_n'(z, z_*)$ converges to $g'(z, z_*)$ uniformly on the compact subsets in $\mathbb{C}_+$. Let $c_{z_*,k}^{(n)}$ be the zero of $g_n'(z, z_*)$ on the k-th semicircle and $c_{z_*,k}$ be the zero of $g'(z, z_*)$ on the k-th semicircle. Then*

$$c_{z_*,k}^{(n)} \to c_{z_*,k}$$

for every $k \neq 0$. Moreover, $g_n(c_{z_*,k}^{(n)}, z_*)$ converges to $g(c_{z_*,k}, z_*)$ for every $k \neq 0$.

Proof The uniform convergence of $g_n(z, z_*)$ follows from the convergent type of Γ (see (2.2), (2.3)). The uniform convergence of $g_n'(z, z_*)$ follows from the uniform convergence of $g_n(z, z_*)$ (by local Cauchy integral formula). The convergence of $c^{(n)}_{z_*,k}$ follows from the *regularity of E* (g and g_n vanish at the endpoints of the gaps) and from the uniform convergence of $g_n'(z, z_*)$, by the Rouche's Theorem. The last assertion is obtained by combining the uniform convergence of $g_n(z, z_*)$ with the convergence of $c^{(n)}_{z_*,k}$. □

3 Pommerenke Theorem

Theorem 3.1 *Let $c_{z_*,k}$ be the zeros of $g'(z, z_*)$, one on each semicircle on the boundary of $\mathcal{F}$, except for the 0-th one. Assume that they satisfy the Widom condition* (1.13)

$$\prod_{k\neq 0} |g(c_{z_*,k}, z_*)| > 0. \tag{3.1}$$

Then $g'(z, z_*)$ is of bounded characteristic, that is, it is a ratio of two bounded analytic functions.

Proof Let B_k be the Blaschke product over the orbit of $c_{z_*,k}$, $k \neq 0$

$$B_k(z) = \prod_{\gamma\in\Gamma} \frac{\gamma(z) - c_{z_*,k}}{\gamma(z) - \overline{c_{z_*,k}}}\, d_\gamma, \quad z \in \mathbb{C}_+,$$

where $|d_\gamma| = 1$ are chosen so that the factors in B_k are positive at z_*. It converges since Γ is of the convergent type. We now consider

$$B(z) = \prod_{k\neq 0} B_k(z).$$

This product converges due to assumption (3.1). Moreover, it converges uniformly on the compact subsets of $\mathbb{C}_+$.

The goal here is to prove that

$$\frac{1}{(z - \overline{z_*})^2} \frac{B(z)}{g'(z, z_*)}$$

is a bounded analytic function on $\mathbb{C}_+$. Then $g'(z, z_*)$ will be the ratio of the following two bounded analytic functions

$$\frac{1}{(z - \overline{z_*})^2} B(z) \quad \text{and} \quad \frac{1}{(z - \overline{z_*})^2} \frac{B(z)}{g'(z, z_*)}.$$

More precisely, we will prove that

$$\left|\frac{1}{(z-\overline{z_*})^2}\frac{B(z)}{g'(z,z_*)}\right| \le 1, \quad z \in \mathbb{C}_+. \tag{3.2}$$

It turns out that it is easier to prove even a stronger inequality

$$\mathfrak{f}(z) \le 1, \quad z \in \mathbb{C}_+, \quad \text{where} \quad \mathfrak{f}(z) = \left|\frac{B(z)}{g'(z,z_*)}\right| \sum_{\gamma\in\Gamma} \frac{|\gamma'(z)|}{|\gamma(z)-\overline{z_*}|^2}. \tag{3.3}$$

It is easier because of the automorphic property of the latter function. Note that the series in (3.3) converges to a function continuous on $\mathbb{C}_+$ for any group Γ of convergent type. So, we are going to prove that

$$\mathfrak{f}(z) = \sum_{\gamma\in\Gamma}\left|\frac{B(z)\gamma'(z)}{g'(z,z_*)(\gamma(z)-\overline{z_*})^2}\right| \le 1, \quad z \in \mathbb{C}_+. \tag{3.4}$$

Observe that

$$\frac{B(z)\gamma'(z)}{g'(z,z_*)(\gamma(z)-\overline{z_*})^2}$$

is holomorphic on $\mathbb{C}_+$. Therefore, its absolute value is a subharmonic function on $\mathbb{C}_+$. Hence $\mathfrak{f}(z)$ is a subharmonic function, which is automorphic with respect to Γ.

We consider first the finitely generated approximation described in Sect. 2. Let Ω_n be the Denjoy domain corresponding to the subgroup Γ_n, $\Lambda_n : \mathbb{C}_+/\Gamma_n \simeq \Omega_n$. Let $c^{(n)}_{z_*,k}$ be the zero of $g_n'(z,z_*)$ on the k-th semicircle. Let $B^{(n)}_k$ be the Blaschke product over the orbit of $c^{(n)}_{z_*,k}$ under Γ_n

$$B^{(n)}_k(z) = \prod_{\gamma\in\Gamma_n} \frac{\gamma(z)-c^{(n)}_{z_*,k}}{\gamma(z)-\overline{c^{(n)}_{z_*,k}}}\, d_\gamma, \quad z \in \mathbb{C}_+,$$

if k-th semicircle is a part of the boundary of $\mathcal{F}_n$, and let $B^{(n)}_k(z) = 1$ otherwise. We now consider

$$B^{(n)}(z) = \prod_{k\neq 0} B^{(n)}_k(z).$$

We are going to prove this approximative version of (3.4)

$$\mathfrak{f}_n(z) \le 1, \quad z \in \mathbb{C}_+, \quad \text{where} \quad \mathfrak{f}_n(z) = \sum_{\gamma\in\Gamma_n}\left|\frac{B^{(n)}(z)\gamma'(z)}{g_n'(z,z_*)(\gamma(z)-\overline{z_*})^2}\right|. \tag{3.5}$$

The advantage of the series in (3.5) over the series in (3.4) is that it converges also on the boundary of the domain $\mathcal{F}_n$ and that the sum in (3.5) is continuous on $\mathcal{F}_n$ and up to the boundary, since Γ_n is finitely generated. The same is true for the fundamental domain of Γ_n, which is the union of $\mathcal{F}_n$ and the reflection of $\mathcal{F}_n$ about the 0-th semicircle.

Due to the automorphic property of $\mathfrak{f}_n(z)$, it possesses the representation

$$\mathfrak{f}_n(z) = \mathfrak{F}_n(\Lambda_n(z)),$$

where $\mathfrak{F}_n(\lambda)$ is still subharmonic in Ω_n and continuous up to the boundary of the domain. Therefore, its maximum is on the boundary of Ω_n. Thus, going back to the function $\mathfrak{f}_n(z)$, we get that its maximum is on the part of the boundary of the fundamental domain that lies on the real axis. Recall that on the boundary of the fundamental domain of Γ_n all series below converge to continuous functions. Therefore, for real z on the boundary of the fundamental domain of Γ_n we have, by (2.8),

$$|g_n'(z, z_*)| = \left| g_n(z, z_*) \sum_{\gamma \in \Gamma_n} \frac{\gamma'(z)}{(\gamma(z) - z_*)(\gamma(z) - \overline{z_*})} \right| = \sum_{\gamma \in \Gamma_n} \frac{\gamma'(z)}{|\gamma(z) - \overline{z_*}|^2}. \tag{3.6}$$

Here we used the fact that $\gamma(z)$ is real for every real z and that $\gamma'(z)$ is positive for every real z. Therefore, for every real z on the boundary of the fundamental domain

$$\mathfrak{f}_n(z) = \sum_{\gamma \in \Gamma_n} \left| \frac{B^{(n)}(z)\gamma'(z)}{g_n'(z, z_*)(\gamma(z) - \overline{z_*})^2} \right| = \frac{1}{|g_n'(z, z_*)|} \sum_{\gamma \in \Gamma_n} \frac{\gamma'(z)}{|\gamma(z) - \overline{z_*}|^2} = 1. \tag{3.7}$$

Hence, (3.5) follows. Thus we have this approximative version of (3.3)

$$\left| \frac{B^{(n)}(z)}{g_n'(z, z_*)} \right| \sum_{\gamma \in \Gamma_n} \frac{|\gamma'(z)|}{|\gamma(z) - \overline{z_*}|^2} \le 1, \quad z \in \mathbb{C}_+. \tag{3.8}$$

Now we want to pass to the limit in (3.8) for arbitrary fixed $z \in \mathbb{C}_+$ as n goes to infinity. By Lemma 2.1, $g_n'(z, z_*)$ converges to $g'(z, z_*)$. The sum over Γ_n converges to the sum over Γ. It remains to show that $|B^{(n)}(z)|$ converges to $|B(z)|$. Note that $|B_k^{(n)}(z)| = |g_n(c_{z_*,k}^{(n)}, z)|$ converges to $|g(c_{z_*,k}, z)| = |B_k(z)|$, by Lemma 2.1. Further,

$$|B_k^{(n)}(z_*)| = |g_n(c_{z_*,k}^{(n)}, z_*)| \ge |g(c_{z_*,k}^{(n)}, z_*)| \ge |g(c_{z_*,k}, z_*)|.$$

The first inequality holds since g_n is a divisor of g, the second does since $c_{z_*,k}$ is the point of minimum of $|g|$ on the k-th semi-circle. By assumption (3.1) the product

$$\prod_{k\neq 0} |g(c_{z_*,k}, z_*)|$$

converges (that is greater than 0). Then, by the Dominated Convergence theorem,[4]

$$\lim_{n\to\infty} |B^{(n)}(z_*)| = \lim_{n\to\infty} \prod_{k\neq 0} |B_k^{(n)}(z_*)| = \prod_{k\neq 0} \lim_{n\to\infty} |B_k^{(n)}(z_*)|$$
$$= \prod_{k\neq 0} |B_k(z_*)| = |B(z_*)|.$$

There exists a subsequence n_j such that $B^{(n_j)}(z)$ converges for all $z \in \mathbb{C}_+$. Let

$$\widetilde{B}(z) = \lim_{j\to\infty} B^{(n_j)}(z).$$

Pick and hold any $z \in \mathbb{C}_+$. Then, by the Fatou's lemma,[5]

$$|\widetilde{B}(z)| = \lim_{j\to\infty} |B^{(n_j)}(z)| = \lim_{j\to\infty} \prod_{k\neq 0} |B_k^{(n_j)}(z)|$$
$$\leq \prod_{k\neq 0} \lim_{j\to\infty} |B_k^{(n_j)}(z)| = \prod_{k\neq 0} |B_k(z)| = |B(z)|.$$

Thus

$$|\widetilde{B}(z)| \leq |B(z)|, \quad z \in \mathbb{C}_+.$$

Since

$$|\widetilde{B}(z_*)| = |B(z_*)|,$$

the equality must hold

$$|\widetilde{B}(z)| = |B(z)|, \quad z \in \mathbb{C}_+. \tag{3.9}$$

[4] This case reduces to the standard Dominated Convergence by applying $(-\log)$ to the products.
[5] Same explanation as in the previous footnote.

Since (3.9) holds for every subsequential limit $\widetilde{B}(z)$ of $B^{(n)}(z)$, we get

$$B(z) = \lim_{n\to\infty} B^{(n)}(z).$$

Thus, we get (3.3) and, therefore, (3.2). □

Remark 3.2 Since the function

$$\frac{1}{(z-\overline{z_*})^2}\frac{B(z)}{g'(z,z_*)}$$

is bounded, it can be written as

$$\frac{1}{(z-\overline{z_*})^2}\frac{B(z)}{g'(z,z_*)} = I(z)\cdot O_2(z),$$

where I is an inner function and O_2 is a bounded outer function. Moreover, I is a singular inner function, since the left hand side does not have zeros in $\mathbb{C}_+$. Therefore,

$$g'(z,z_*) = \frac{O_1(z)}{O_2(z)}\frac{B(z)}{I(z)},$$

where $O_1(z) = \frac{1}{(z-\overline{z_*})^2}$ is also a bounded outer function. Thus,

$$g'(z,z_*) = O(z)\frac{B(z)}{I(z)}, \tag{3.10}$$

where $O(z) = \frac{O_1(z)}{O_2(z)}$ is a ratio of two bounded outer functions.

Theorem 3.3 (Pommerenke) *The function*

$$\frac{1}{(z-\overline{z_*})^2}\frac{B(z)}{g'(z,z_*)}$$

is outer. That is, $I(z) = 1$.

Lemma 3.4 *Let* $x \in \mathbb{R}$. *The nontangential limits* $g(x,z_*)$ *and* $g'(x,z_*)$ *exist with* $|g(x,z_*)| = 1$, $g'(x,z_*)$ *finite if and only if*

$$\sum_{\gamma\in\Gamma}\frac{|\gamma'(x)|}{|\gamma(x)-\overline{z_*}|^2} < \infty. \tag{3.11}$$

In this case

$$\frac{1}{i}\frac{g'(x,z_*)}{g(x,z_*)} = |g'(x,z_*)| = 2\text{Im}\, z_* \sum_{\gamma\in\Gamma} \frac{|\gamma'(x)|}{|\gamma(x)-\overline{z_*}|^2}. \tag{3.12}$$

Hence, in our case (g is a Blaschke product, g′ is of bounded characteristic) (3.12) *holds almost everywhere on* $\mathbb{R}$.

Proof This lemma is Corollary 6.6 of the Appendix with $w = g(z, z_*)$, which is a product of these Blaschke factors

$$B_\gamma(z) = \frac{\gamma(z)-z_*}{\gamma(z)-\overline{z_*}}.$$

(3.12) follows since in this case

$$B'_\gamma(z) = 2i\,\text{Im}\, z_* \frac{\gamma'(z)}{(\gamma(z)-\overline{z_*})^2}.$$

□

Lemma 3.5 *For every $z \in \mathbb{C}_+$ the following inequality holds*

$$\frac{1}{\pi}\int_{\mathbb{R}} \log \sum_{\gamma\in\Gamma} \frac{|\gamma'(x)|}{|\gamma(x)-\overline{z_*}|^2} \frac{\text{Im}\, z\, dx}{|x-z|^2} \geq \log \sum_{\gamma\in\Gamma} \frac{|\gamma'(z)|}{|\gamma(z)-\overline{z_*}|^2}. \tag{3.13}$$

Proof Since

$$\gamma'(z) = \frac{1}{(\gamma^{21}z+\gamma^{22})^2},$$

one can write

$$\sum_{\gamma\in\Gamma} \frac{|\gamma'(z)|}{|\gamma(z)-\overline{z_*}|^2} = \sum_{\gamma\in\Gamma} \overline{\phi_\gamma(z)}\phi_\gamma(z),$$

where

$$\phi_\gamma(z) = \frac{1}{\gamma^{21}z+\gamma^{22}} \frac{1}{\gamma(z)-\overline{z_*}}.$$

We enumerate the elements of the group Γ, $\Gamma = \{\gamma_k\}$, and consider functions $u_n(z)$ defined by the finite sums

$$u_n(z) = \sum_{k=1}^{n} \frac{|\gamma_k'(z)|}{|\gamma_k(z)-\overline{z_*}|^2} = \sum_{k=1}^{n} \overline{\phi_{\gamma_k}(z)}\phi_{\gamma_k}(z), \quad \text{Im}\, z > 0.$$

From here we see that u_n is a subharmonic function since

$$\frac{\partial^2}{\partial z \partial \overline{z}} u_n(z) = \sum_{k=1}^{n} \overline{\phi'_{\gamma_k}(z)} \phi'_{\gamma_k}(z) \geq 0.$$

Also $\log u_n(z)$ is subharmonic, since

$$\frac{\partial^2}{\partial z \partial \overline{z}} \log u_n(z) = -\frac{1}{u_n^2(z)} \frac{\partial u_n}{\partial z} \frac{\partial u_n}{\partial \overline{z}} + \frac{1}{u_n} \frac{\partial^2 u_n}{\partial z \partial \overline{z}} =$$

$$\frac{1}{u_n^2(z)} \left\{ \sum_{k=1}^{n} \overline{\phi_{\gamma_k}(z)} \phi_{\gamma_k}(z) \sum_{k=1}^{n} \overline{\phi'_{\gamma_k}(z)} \phi'_{\gamma_k}(z) - \sum_{k=1}^{n} \overline{\phi_{\gamma_k}(z)} \phi'_{\gamma_k}(z) \sum_{k=1}^{n} \overline{\phi'_{\gamma_k}(z)} \phi_{\gamma_k}(z) \right\},$$

which is nonnegative by the Cauchy-Schwarz inequality. Therefore,

$$\frac{1}{\pi} \int_{\mathbb{R}} \log \sum_{k=1}^{n} \frac{|\gamma'_k(x)|}{|\gamma_k(x) - \overline{z_*}|^2} \frac{\operatorname{Im} z \, dx}{|x - z|^2} \geq \log \sum_{k=1}^{n} \frac{|\gamma'_k(z)|}{|\gamma_k(z) - \overline{z_*}|^2}.$$

We now pass to the limit in this inequality. Since all integrands here have lower summable bound $\log \frac{1}{|x - \overline{z_*}|^2}$, the Monotone Convergence Theorem applies and we get (3.13). □

Proof **(Theorem 3.3)** It follows from

$$g'(z, z_*) = (z_* - \overline{z_*}) \sum_{\gamma \in \Gamma} \frac{g(z, z_*) \gamma'(z)}{(\gamma(z) - z_*)(\gamma(z) - \overline{z_*})}$$

that for $z \in \mathbb{C}_+$

$$\begin{aligned} |g'(z, z_*)| =& 2 \operatorname{Im} z_* \left| \sum_{\gamma \in \Gamma} \frac{g(z, z_*) \gamma'(z)}{(\gamma(z) - z_*)(\gamma(z) - \overline{z_*})} \right| \\ \leq& 2 \operatorname{Im} z_* \sum_{\gamma \in \Gamma} \left| \frac{g(z, z_*) \gamma'(z)}{(\gamma(z) - z_*)(\gamma(z) - \overline{z_*})} \right| \leq 2 \operatorname{Im} z_* \sum_{\gamma \in \Gamma} \frac{|\gamma'(z)|}{|\gamma(z) - \overline{z_*}|^2}. \end{aligned} \tag{3.14}$$

We used here only one of the factors of $g(z, z_*)$ in every term. Now, by Lemmas 3.4 and 3.5,

$$\frac{1}{\pi}\int_{\mathbb{R}} \log \frac{|g'(x, z_*)|}{2\operatorname{Im} z_*} \frac{\operatorname{Im} z}{|x-z|^2} dx = \frac{1}{\pi}\int_{\mathbb{R}} \log \sum_{\gamma\in\Gamma} \frac{|\gamma'(x)|}{|\gamma(x)-\overline{z_*}|^2} \frac{\operatorname{Im} z}{|x-z|^2} dx$$
$$\geq \log \sum_{\gamma\in\Gamma} \frac{|\gamma'(z)|}{|\gamma(z)-\overline{z_*}|^2}.$$

That is,

$$\frac{1}{\pi}\int_{\mathbb{R}} \log |g'(x, z_*)| \frac{\operatorname{Im} z}{|x-z|^2} dx \geq \log \left(2\operatorname{Im} z_* \sum_{\gamma\in\Gamma} \frac{|\gamma'(z)|}{|\gamma(z)-\overline{z_*}|^2} \right).$$

On the other hand

$$\frac{1}{\pi}\int_{\mathbb{R}} \log |g'(x, z_*)| \frac{\operatorname{Im} z}{|x-z|^2} dx = \frac{1}{\pi}\int_{\mathbb{R}} \log |O(x)| \frac{\operatorname{Im} z}{|x-z|^2} dx = \log |O(z)|,$$

since O is a ratio of two bounded outer functions. Thus,

$$2\operatorname{Im} z_* \sum_{\gamma\in\Gamma} \frac{|\gamma'(z)|}{|\gamma(z)-\overline{z_*}|^2} \leq |O(z)|, \quad z \in \mathbb{C}_+. \tag{3.15}$$

Combining (3.15) and (3.14), we get

$$|g'(z, z_*)| \leq |O(z)|, \quad z \in \mathbb{C}_+.$$

That is, in view of (3.10),

$$\left|\frac{B(z)}{I(z)}\right| = \left|\frac{g'(z, z_*)}{O(z)}\right| \leq 1.$$

The latter implies that $I(z) = 1$. □

4 Conditions (*b*) and (B) in Theorem 1.8

4.1 Martin Function with a Pure Point Measure

Recall that (see (1.15), (1.16)) $M(\lambda) = \operatorname{Im} \theta(\lambda)$, $\lambda \in \Omega$ and that $m(z) = \theta(\Lambda(z))$. Thus,

$$M(\Lambda(z)) = \operatorname{Im} m(z).$$

$m(z)$ is a single-valued holomorphic function defined in $\mathbb{C}_+$, additively character automorphic with respect to Γ. $\operatorname{Im} m(z) \geq 0$ for all $z \in \mathbb{C}_+$. Therefore, $m(z)$ admits a Riesz-Herglotz representation

$$m(z) = az + b + \int_{\mathbb{R}} \left(\frac{1}{x-z} - \frac{x}{1+x^2} \right) \sigma(dx),$$

where $a \geq 0$, b is real and σ is a singular measure on $\mathbb{R}$ with

$$\int_{\mathbb{R}} \frac{\sigma(dx)}{1+x^2} < \infty.$$

Let us mention that $e^{i\ell m(z)}$ is a singular inner character automorphic function, for all $\ell > 0$.

We observe (see, e.g. [19]) that for Martin functions in Denjoy domains there are two options: either $a > 0$ (that is, (b_1) of (1.17) holds) and σ is a pure point measure (that is, (b) of Theorem 1.8 holds), supported by orbits of ∞ and 0; or $a = 0$ (that is, (b_1) of (1.17) fails) and σ is a continuous singular measure (that is, (b) of Theorem 1.8 fails). For further references we state it as

Proposition 4.1 *Properties (b) of Theorem 1.8 and (b_1) of (1.17) are equivalent.*

We point out that the orbits $\{\gamma(0)\}_{\gamma\in\Gamma}$ and $\{\gamma(\infty)\}_{\gamma\in\Gamma}$ cannot intersect due to the structure of the generators of the group Γ.

We start with a singular function supported by the orbit of ∞,

$$m_+(z) = z + \sum_{\gamma\in\Gamma\ \gamma\neq 1} \left(\frac{1}{\gamma(\infty) - z} - \frac{\gamma(\infty)}{1+\gamma(\infty)^2} \right) \sigma_\gamma \tag{4.1}$$

where

$$\sum_{\gamma\in\Gamma\ \gamma\neq 1} \frac{\sigma_\gamma}{1+\gamma(\infty)^2} < \infty. \tag{4.2}$$

Lemma 4.2 *The function $m_+(z)$ defined in (4.1) is additive character automorphic with respect to the group Γ if and only if*

$$\sigma_\gamma = \frac{1}{(\gamma^{21})^2}. \tag{4.3}$$

Respectively,

$$\sum_{\gamma\in\Gamma} \frac{\sigma_\gamma}{1+\gamma(\infty)^2} = \sum_{\gamma\in\Gamma} \frac{1}{(\gamma^{11})^2 + (\gamma^{21})^2} < \infty, \tag{4.4}$$

and

$$m_+(z) = \sum_{\gamma\in\Gamma} (\gamma(z) - \operatorname{Re}\gamma(i))\,. \tag{4.5}$$

Proof Since

$$\gamma(z) = \frac{\gamma^{11}z + \gamma^{12}}{\gamma^{21}z + \gamma^{22}}, \quad \gamma \in SL_2(\mathbb{R}),$$

we have $\gamma(\infty) = \gamma^{11}/\gamma^{21}$. Note that for every $\gamma \in \Gamma$, $\gamma^{11} \neq 0$ (since ∞ is not carried to 0) and $\gamma^{22} \neq 0$ (since 0 is not carried to ∞); also for $\gamma \neq 1$, $\gamma^{12} \neq 0$ (since 0 is not a fixed point) and $\gamma^{21} \neq 0$ (since ∞ is not a fixed point). So, let $m_+(z)$ be defined by (4.1)

$$m_+(z) = z + \sum_{\widetilde{\gamma}\in\Gamma\ \widetilde{\gamma}\neq 1} \left(\frac{1}{\widetilde{\gamma}(\infty) - z} - \frac{\widetilde{\gamma}(\infty)}{1 + \widetilde{\gamma}(\infty)^2} \right) \sigma_{\widetilde{\gamma}}. \tag{4.6}$$

Let $\gamma \in \Gamma$, then $m_+(\gamma(z))$ is the same as $m_+(z)$ up to a real additive constant. Let $\gamma \neq 1_\Gamma$. We substitute $\gamma(z)$ instead of z in (4.6) and we consider the term with $\widetilde{\gamma} = \gamma$. We have

$$\frac{1}{\gamma(\infty) - \gamma(z)} = \left(\frac{\gamma^{11}}{\gamma^{21}} - \frac{\gamma^{11}z + \gamma^{12}}{\gamma^{21}z + \gamma^{22}} \right)^{-1} = (\gamma^{21})^2 z + \gamma^{21}\gamma^{22}.$$

Since the coefficient of z in $m(\gamma(z))$ must be equal to 1, we get (4.3); then (4.4) follows from (4.2). Thus, we can write

$$\begin{aligned} m_+(z) &= z + \sum_{\gamma\in\Gamma\ \gamma\neq 1} \left(\frac{1}{\gamma(\infty) - z} - \frac{\gamma(\infty)}{1 + \gamma(\infty)^2} \right) \frac{1}{(\gamma^{21})^2} \qquad (4.7) \\ &= z + \sum_{\gamma\in\Gamma\ \gamma\neq 1} \left(\frac{1}{\gamma^{11} - \gamma^{21}z} \frac{1}{\gamma^{21}} - \frac{\gamma(\infty)}{1 + \gamma(\infty)^2} \frac{1}{(\gamma^{21})^2} \right). \end{aligned}$$

Since $\gamma \in SL_2(\mathbb{R})$, we have

$$\gamma^{-1} = \begin{bmatrix} \gamma^{22} & -\gamma^{12} \\ -\gamma^{21} & \gamma^{11} \end{bmatrix} \quad \text{for} \quad \gamma = \begin{bmatrix} \gamma^{11} & \gamma^{12} \\ \gamma^{21} & \gamma^{22} \end{bmatrix}.$$

Then we can further rewrite

$$m_+(z) = z + \sum_{\gamma\in\Gamma\ \gamma\neq 1} \left(\gamma^{-1}(z) - \gamma^{-1}(\infty) - \frac{\gamma(\infty)}{1+\gamma(\infty)^2}\frac{1}{(\gamma^{21})^2}\right)$$

$$= \sum_{\gamma\in\Gamma}\left(\gamma^{-1}(z) - c(\gamma^{-1})\right) = \sum_{\gamma\in\Gamma}(\gamma(z) - c(\gamma)), \tag{4.8}$$

where $c(1_\Gamma) = 0$ and for $\gamma \neq 1_\Gamma$

$$c(\gamma) = \gamma(\infty) + \frac{\gamma^{-1}(\infty)}{1+\gamma^{-1}(\infty)^2}\frac{1}{(-\gamma^{21})^2} = \frac{\gamma^{11}}{\gamma^{21}} - \frac{\gamma^{22}}{\gamma^{21}}\cdot\frac{1}{(\gamma^{22})^2 + (\gamma^{21})^2}.$$

Actually, since

$$\operatorname{Re}\left(\frac{1}{\gamma(\infty) - i} - \frac{\gamma(\infty)}{1+\gamma(\infty)^2}\right) = \operatorname{Re}\left(\frac{\gamma(\infty)+i}{1+\gamma(\infty)^2} - \frac{\gamma(\infty)}{1+\gamma(\infty)^2}\right) = 0,$$

we get from (4.7) and (4.8) that

$$c(\gamma) = \operatorname{Re}\gamma(i) = \frac{\gamma^{12}\gamma^{22} + \gamma^{11}\gamma^{21}}{(\gamma^{22})^2 + (\gamma^{21})^2}.$$

That is, we get (4.5). The fact that the function $m_+(z)$ is additively character automorphic follows directly from representation (4.5). □

The convergence in (4.5) is absolute and uniform as long as z is bounded away from the orbit of ∞. We also see that

$$m_+'(z) = \sum_{\gamma\in\Gamma}\gamma'(z).$$

the convergence here is also absolute and uniform as long as z is bounded away from the orbit of ∞. By (4.5), we get

$$\operatorname{Im} m_+(z) = \sum_{\gamma\in\Gamma}\operatorname{Im}\gamma(z) = \sum_{\gamma\in\Gamma}\frac{\operatorname{Im} z}{|\gamma^{21}z + \gamma^{22}|^2} = \operatorname{Im} z\sum_{\gamma\in\Gamma}|\gamma'(z)|.$$

Thus

$$\frac{\operatorname{Im} m_+(z)}{\operatorname{Im} z} = \sum_{\gamma\in\Gamma}\frac{1}{|\gamma^{21}z + \gamma^{22}|^2} = \sum_{\gamma\in\Gamma}|\gamma'(z)|. \tag{4.9}$$

The antiholomorphic automorphism $\lambda \mapsto \overline{\lambda}$ on Ω acts as $z \mapsto 1/\overline{z}$ on the universal covering $\mathbb{C}_+$. Thus, the symmetric Martin function $m(z)$ of the group Γ possesses the following property

$$\operatorname{Im} m(1/\overline{z}) = \operatorname{Im} m(z).$$

Let us define $m_-(z)$ so that

$$\operatorname{Im} m_-(z) = \operatorname{Im} m_+(1/\overline{z})$$

In this case,

$$\operatorname{Im} m_-(z) = \operatorname{Im} m_+(1/\overline{z}) = \sum_{\gamma \in \Gamma} \frac{\operatorname{Im} 1/\overline{z}}{\left|\frac{\gamma^{21}}{\overline{z}} + \gamma^{22}\right|^2} = \sum_{\gamma \in \Gamma} \frac{\operatorname{Im} z}{|\gamma^{21} + \gamma^{22} z|^2}. \tag{4.10}$$

Lemma 4.3 *Group* Γ *has the following symmetry. Let*

$$\tau(z) = \frac{1}{\overline{z}}. \tag{4.11}$$

This is a reflection about the unit circle. Also $\tau : \mathbb{C}_+ \to \mathbb{C}_+$. *Then*

$$\gamma \mapsto \widetilde{\gamma} = \tau \gamma \tau \tag{4.12}$$

is an automorphism (one-to-one and onto) of Γ. *In the matrix form this automorphism reads as follows*

$$\gamma = \begin{bmatrix} \gamma^{11} & \gamma^{12} \\ \gamma^{21} & \gamma^{22} \end{bmatrix} \mapsto \widetilde{\gamma} = \begin{bmatrix} \gamma^{22} & \gamma^{21} \\ \gamma^{12} & \gamma^{11} \end{bmatrix}. \tag{4.13}$$

Proof Lemma follows from the observation that every generator of the group Γ is a composition of two reflections about the boundary semicircles of $\mathcal{F}$. □

In view of this lemma, we may continue (4.10) (compare with (4.9)) as

$$\operatorname{Im} m_-(z) = \sum_{\gamma \in \Gamma} \frac{\operatorname{Im} z}{|\gamma^{21} + \gamma^{22} z|^2} = -\sum_{\gamma \in \Gamma} \operatorname{Im} \frac{1}{\widetilde{\gamma}(z)} = -\sum_{\gamma \in \Gamma} \operatorname{Im} \frac{1}{\gamma(z)}.$$

Therefore,

$$m_-(z) = -\sum_{\gamma \in \Gamma} \left(\frac{1}{\gamma(z)} - \operatorname{Re} \frac{1}{\gamma(i)} \right).$$

This function also admits representation of type (4.1)

$$m_-(z) = \sum_{\gamma\in\Gamma} \left(\frac{1}{\gamma(0) - z} - \frac{\gamma(0)}{1+\gamma(0)^2} \right) \beta_\gamma, \quad \text{where } \beta_\gamma = \frac{1}{(\gamma^{22})^2}. \tag{4.14}$$

The corresponding convergence condition reads as follows

$$\sum_{\gamma\in\Gamma} \frac{\beta_\gamma}{1+\gamma(0)^2} = \sum_{\gamma\in\Gamma} \frac{1}{(\gamma^{12})^2 + (\gamma^{22})^2} < \infty, \tag{4.15}$$

which is equivalent to (4.4) in view of Lemma 4.3. Finally, we arrive at the following proposition.

Proposition 4.4 *(b) in Theorem 1.8 holds if and only if*

$$\sum_{\gamma\in\Gamma} \operatorname{Im} \gamma(i) = \sum_{\gamma\in\Gamma} |\gamma'(i)| < \infty. \tag{4.16}$$

In this case the symmetric Martin function of the group Γ *is given by*

$$m(z) = m_+(z) + m_-(z) = \sum_{\gamma\in\Gamma} \left(\gamma(z) - \frac{1}{\gamma(z)} \right) - \operatorname{Re} \left(\gamma(i) - \frac{1}{\gamma(i)} \right). \tag{4.17}$$

Moreover,

$$m'(z) = \sum_{\gamma\in\Gamma} \gamma'(z) + \sum_{\gamma\in\Gamma} \frac{\gamma'(z)}{\gamma^2(z)} = \sum_{\gamma\in\Gamma} \frac{1}{(\gamma^{21}z + \gamma^{22})^2} + \sum_{\gamma\in\Gamma} \frac{1}{(\gamma^{11}z + \gamma^{12})^2} \tag{4.18}$$

and

$$\frac{\operatorname{Im} m(z)}{\operatorname{Im} z} = \sum_{\gamma\in\Gamma} |\gamma'(z)| + \sum_{\gamma\in\Gamma} \left| \frac{\gamma'(z)}{\gamma^2(z)} \right| = \sum_{\gamma\in\Gamma} \frac{1}{|\gamma^{21}z + \gamma^{22}|^2} + \sum_{\gamma\in\Gamma} \frac{1}{|\gamma^{11}z + \gamma^{12}|^2}. \tag{4.19}$$

Proof We consider

$$\gamma = \begin{bmatrix} \gamma^{11} & \gamma^{12} \\ \gamma^{21} & \gamma^{22} \end{bmatrix}, \quad \widetilde{\gamma} = \begin{bmatrix} \gamma^{22} & \gamma^{21} \\ \gamma^{12} & \gamma^{11} \end{bmatrix},$$

$$\gamma^{-1} = \begin{bmatrix} \gamma^{22} & -\gamma^{12} \\ -\gamma^{21} & \gamma^{11} \end{bmatrix}, \quad \widetilde{\gamma}^{-1} = \begin{bmatrix} \gamma^{11} & -\gamma^{21} \\ -\gamma^{12} & \gamma^{22} \end{bmatrix}.$$

The meaning of $\widetilde{\gamma}$ is explained in (4.11)–(4.13). By looking at the first columns of those matrices, we can conclude that one of the four convergence conditions below implies the others

$$\sum_{\gamma\in\Gamma}\frac{1}{(\gamma^{11})^2+(\gamma^{21})^2}<\infty, \quad \sum_{\gamma\in\Gamma}\frac{1}{(\gamma^{12})^2+(\gamma^{22})^2}<\infty, \tag{4.20}$$

$$\sum_{\gamma\in\Gamma}\frac{1}{(\gamma^{22})^2+(\gamma^{21})^2}<\infty, \quad \sum_{\gamma\in\Gamma}\frac{1}{(\gamma^{11})^2+(\gamma^{12})^2}<\infty. \tag{4.21}$$

In particular, (4.15) and (4.4) are equivalent. Equation (4.16) corresponds to the first condition in (4.21). Due to Lemma 4.2 we get representation (4.5) for $m_+(z)$. By (4.15) we have (4.14) for $m_-(z)$, and therefore (4.17)–(4.19). Also in this formulas γ can be replaced with γ^{-1}, $\widetilde{\gamma}$ or $\widetilde{\gamma}^{-1}$ if needed. □

Lemma 4.5 *Assume that convergence condition (4.15) (or any equivalent) holds. Let m be the symmetric Martin function of the group Γ defined as in (4.17). Let $m_n(z)$ be the symmetric Martin function of the group Γ_n. All critical points of $m(z)$ and $m_n(z)$ are located on the boundary semi-circles of $\mathcal{F}$ and $\mathcal{F}_n$, respectively. Let $c_k^{(n)}$ be the zero of $m_n'(z)$ on the k-th semicircle and let c_k be the zero of $m'(z)$ on the k-th semicircle. Then, as n goes to ∞, $m_n(z)$ converges to $m(z)$ uniformly on the compact subsets in $\mathbb{C}_+$ and $m_n'(z)$ converges to $m'(z)$ uniformly on the compact subsets in $\mathbb{C}_+$. Also*

$$c_k^{(n)} \to c_k$$

for every k. Moreover, $m_n(c_k^{(n)})$ converges to $m(c_k)$ and $g_n(c_k^{(n)}, z_*)$ converges to $g(c_k, z_*)$ for every k.

Proof Completely parallel to the proof of Lemma 2.1. □

4.2 Akhiezer–Levin Condition

In this subsection we prove

Theorem 4.6 *Properties (B) of Theorem 1.8 and (b_1) of (1.17) are equivalent.*

We start with the following lemma.

Lemma 4.7 *Let $u(z)$ be a function with positive imaginary part on the upper half plane. We assume that u is not a real constant. Let*

$$f(z)=\frac{u(z)-i}{u(z)+i}, \quad \textit{respectively}, \quad u(z)=i\frac{1+f(z)}{1-f(z)}.$$

Let

$$\zeta = \frac{z-i}{z+i}, \quad \textit{respectively}, \quad z = i\,\frac{1+\zeta}{1-\zeta}.$$

Let also

$$w(\zeta) = f\left(i\frac{1+\zeta}{1-\zeta}\right). \tag{4.22}$$

Then

$$a := \lim_{z=iy, y\to\infty} \frac{\operatorname{Im} u(z)}{\operatorname{Im} z} > 0$$

if and only if

$$\lim_{\zeta>0, \zeta\to 1} w(\zeta) = 1 \quad \textit{and} \quad d := \lim_{\zeta>0, \zeta\to 1} \frac{1-|w(\zeta)|^2}{1-|\zeta|^2} < \infty.$$

In this case

$$a = \frac{1}{d}.$$

Proof Note that $\zeta \to 1$ as $z \to \infty$. Compute

$$\frac{\operatorname{Im} u(z)}{\operatorname{Im} z} = \frac{1-|f(z)|^2}{|1-f(z)|^2}\frac{|1-\zeta|^2}{1-|\zeta|^2} = \frac{1-|w(\zeta)|^2}{1-|\zeta|^2}\left|\frac{1-\zeta}{1-w(\zeta)}\right|^2.$$

The backwards computation gives

$$\frac{1-|w(\zeta)|^2}{1-|\zeta|^2} = \frac{\operatorname{Im} u(z)}{\operatorname{Im} z}\frac{|z+i|^2}{|u(z)+i|^2}. \tag{4.23}$$

The limit

$$a = \lim_{z=iy, y\to\infty} \frac{\operatorname{Im} u(z)}{\operatorname{Im} z} \geq 0$$

always exists and is finite (a is the coefficient of z in the Riesz-Herglotz representation of u). Assume that $a > 0$. Then

$$\lim_{z=iy, y\to\infty} u(z) = \infty \text{ and, therefore, } \lim_{\zeta>0, \zeta\to 1} w(\zeta) = 1.$$

Further,

$$d = \lim_{\zeta>0,\zeta\to 1} \frac{1-|w(\zeta)|^2}{1-|\zeta|^2} = \lim_{z=iy,y\to\infty} \frac{\operatorname{Im} u(z)}{\operatorname{Im} z} \frac{|z+i|^2}{|u(z)+i|^2}$$
$$= \lim_{z=iy,y\to\infty} \frac{\operatorname{Im} u(z)}{\operatorname{Im} z} \frac{(\operatorname{Im} z+1)^2}{(\operatorname{Im} u(z)+1)^2 + (\operatorname{Re} u(z))^2}$$
$$\leq \lim_{z=iy,y\to\infty} \frac{\operatorname{Im} u(z)}{\operatorname{Im} z} \frac{(\operatorname{Im} z+1)^2}{(\operatorname{Im} u(z)+1)^2} = \frac{1}{a} < \infty.$$

Conversely, let

$$\lim_{\zeta>0,\zeta\to 1} w(\zeta) = 1 \quad \text{and} \quad \lim_{\zeta>0,\zeta\to 1} \frac{1-|w(\zeta)|^2}{1-|\zeta|^2} = d < \infty.$$

$d > 0$, since w is not a unimodular constant (since u is not a real constant). Then, by the Julia Theorem (see Theorem 6.1 in Appendix),

$$\frac{1-|w(\zeta)|^2}{1-|\zeta|^2} \left| \frac{1-\zeta}{1-w(\zeta)} \right|^2 \geq \frac{1}{d} > 0.$$

Therefore,

$$\frac{\operatorname{Im} u(z)}{\operatorname{Im} z} = \frac{1-|w(\zeta)|^2}{1-|\zeta|^2} \left| \frac{1-\zeta}{1-w(\zeta)} \right|^2 \geq \frac{1}{d} > 0$$

and

$$a = \lim_{z=iy,y\to\infty} \frac{\operatorname{Im} u(z)}{\operatorname{Im} z} \geq \frac{1}{d} > 0.$$

□

Combining Lemma 4.7 and the Carathéodory–Julia theorem (Theorem 6.1 in Appendix) we get

Corollary 4.8 *Assume that $u(z)$ is not a real constant, then the following are equivalent*

(1) *There exists a sequence z_k, $\operatorname{Im} z_k > 0$, $\lim z_k = \infty$ such that*

$$\lim u(z_k) = \infty \quad \textit{and} \quad d_1 := \lim \frac{\operatorname{Im} u(z_k)}{\operatorname{Im} z_k} \frac{|z_k+i|^2}{|u(z_k)+i|^2} < \infty;$$

(2) $\displaystyle \lim_{z=iy,y\to\infty} u(z) = \infty \quad \textit{and} \quad d_2 := \lim_{z=iy,y\to\infty} \frac{\operatorname{Im} u(z)}{\operatorname{Im} z} \frac{|z+i|^2}{|u(z)+i|^2} < \infty;$

(3) $\displaystyle a = \lim_{z=iy,y\to\infty} \frac{\operatorname{Im} u(z)}{\operatorname{Im} z} > 0;$

When these conditions hold, we have $d_1 = d_2 = \frac{1}{a}$. *Function u is a real constant if and only if* $d_1 = d_2 = 0$.

***Proof* (Theorem 4.6)** Note first that, in view of formulas (4.22) and (4.23), (1) and (2) are equivalent, respectively, to

(1′) There exists a sequence ζ_k, $|\zeta_k| < 1$, $\lim \zeta_k = 1$ such that

$$\lim w(\zeta_k) = 1 \quad \text{and} \quad \lim \frac{1 - |w(\zeta_k)|^2}{1 - |\zeta_k|^2} < \infty;$$

(2′) $$\lim_{\zeta > 0, \zeta \to 1} w(\zeta) = 1 \quad \text{and} \quad \lim_{\zeta > 0, \zeta \to 1} \frac{1 - |w(\zeta)|^2}{1 - |\zeta|^2} < \infty.$$

By Lemma 4.7, (3) is equivalent to (2') and, therefore (3) and (2) are equivalent. (2) obviously implies (1). It remains to show that (1) implies (2), equivalently, that (1') implies (2'). By Theorem 6.1, the second part of (1') implies the second part of (2'). Also (1') implies (6.1) of Theorem 6.1 with $w_0 = 1$. Hence, $w_0 = 1$ is the limit of $w(\zeta)$ as ζ, $|\zeta| < 1$, approaches $t_0 = 1$ nontangentially. □

Proof First we show that (B) of Theorem 1.8 implies (b_1) of (1.17). Property (B) of Theorem 1.8 says that

$$\lim_{\lambda = i\eta, \eta \to \infty} \frac{\operatorname{Im} \theta(\lambda)}{\operatorname{Im} \lambda} > 0. \tag{4.24}$$

By Corollary 4.8, we have that

$$\lim_{\lambda = i\eta, \eta \to \infty} \theta(\lambda) = \infty. \tag{4.25}$$

and

$$\lim_{\lambda = i\eta, \eta \to \infty} \frac{\operatorname{Im} \theta(\lambda)}{\operatorname{Im} \lambda} \frac{|\lambda + i|^2}{|\theta(\lambda) + i|^2} < \infty. \tag{4.26}$$

Consider $z = \Lambda^{-1}(\lambda)$ as a mapping from $\mathbb{C}_+ \subset \mathbb{C} \setminus E$ to the fundamental domain (in $\mathbb{C}_+$). Since $\Lambda^{-1}(\lambda)$ is a nonconstant function with positive imaginary part, we get, by Corollary 4.8, that

$$\lim_{\lambda = i\eta, \eta \to \infty} \frac{\operatorname{Im} \Lambda^{-1}(\lambda)}{\operatorname{Im} \lambda} \frac{|\lambda + i|^2}{|\Lambda^{-1}(\lambda) + i|^2} > 0. \tag{4.27}$$

Combining (4.26) and (4.27), we get

$$\lim_{\lambda = i\eta, \eta \to \infty} \frac{\operatorname{Im} \theta(\lambda)}{\operatorname{Im} \Lambda^{-1}(\lambda)} \frac{|\Lambda^{-1}(\lambda) + i|^2}{|\theta(\lambda) + i|^2} < \infty. \tag{4.28}$$

Substituting $\lambda = \Lambda(z)$, we get that

$$\lim \frac{\operatorname{Im} m(z)}{\operatorname{Im} z} \frac{|z+i|^2}{|m(z)+i|^2} < \infty \tag{4.29}$$

as z goes to ∞ along $\Lambda^{-1}(\{i\eta, \eta > 0\})$. We also have from (4.25) that

$$\lim m(z) = \infty$$

as z goes to ∞ along $\Lambda^{-1}(\{i\eta, \eta > 0\})$. By Corollary 4.8,

$$\lim_{z=iy, y\to\infty} \frac{\operatorname{Im} m(z)}{\operatorname{Im} z} > 0,$$

which is (b_1) of (1.17).

Now we will show that condition (b_1) of (1.17) implies (B) of Theorem 1.8. In $\Omega = \mathbb{C} \setminus E$ we have two different Martin functions $M_\pm$ such that $M_+(\Lambda(z)) = \operatorname{Im} m_+(z)$, where the measure of m_+ is supported by the orbit $\{\gamma(\infty)\}$, and $M_-(\lambda) = M_+(\overline{\lambda})$. It is known that if the cone of Martin functions in the domain is two dimensional, then (B) of Theorem 1.8 holds. Indeed, consider functions $M_\pm(\lambda)$ on the upper half plane in $\mathbb{C} \setminus E$. As positive harmonic functions continuous up to the real line, they admit Poisson representations in terms of their values on the real line. On E they both vanish, on $\mathbb{R} \setminus E$ they coincide. Therefore, coefficients $a_\pm \geq 0$ of $\operatorname{Im} \lambda$ in their Poisson representations are not equal. Hence at least one of them is positive. Finally we conclude that the coefficient $a_+ + a_-$ of $\operatorname{Im} \lambda$ in the Poisson representation of the symmetric Martin function $M(\lambda) = M_+(\lambda) + M_-(\lambda)$ is positive, that is, (B) holds. □

Remark 4.9 Observe that actually $a_- = 0$, for otherwise, by Corollary 4.8,

$$\lim_{z=iy, y\to\infty} \frac{\operatorname{Im} m_-(z)}{\operatorname{Im} z} > 0,$$

which is not the case. Therefore, $a_+ > 0$. Now we can write

$$M_+(\lambda) - M_-(\lambda) = a_+ \operatorname{Im} \lambda$$

or, after substituting $\lambda = \Lambda(z)$,

$$\operatorname{Im} m_+(z) - \operatorname{Im} m_-(z) = a_+ \operatorname{Im} \Lambda(z).$$

Hence,

$$\frac{1}{\lim_{z=iy, y\to\infty} \frac{\operatorname{Im} \Lambda(z)}{\operatorname{Im} z}} = a_+ > 0.$$

5 Proof of the Main Theorem (Theorem 1.8)

In this section we will use restatement of condition (A) in terms of the universal cover

$$(A) \qquad \prod_{k\neq 0} |g(c_k, z_*)| > 0, \tag{5.1}$$

where c_k are the zeros of $m'(z)$ (one on each boundary semicircle of $\mathcal{F}$). This is the Blaschke condition on *all* the zeros of $m'(z)$ in the upper half plane (orbits of c_k under the action of the group Γ).

5.1 *Proof of the Implication $(ii) \Rightarrow (iii)$*

By Proposition 4.1, (b) is equivalent to (b_1) of (1.17) and, by Theorem 4.6, (b_1) is equivalent to (B) of Theorem 1.8. Property (a) implies (A), since zeros of a function of bounded characteristic satisfy the Blaschke condition.

5.2 *Proof of the Implication $(iii) \Rightarrow (ii)$*

Theorem 5.1 *Assume that condition (iii) holds. That is, we assume that* (5.1) *holds and that (B) of Theorem* 1.8 (*equivalently* (b_1) *of*(1.17)) *holds. Then $m'(z)$ is of bounded characteristic, that is, it is a ratio of two bounded analytic functions.*

Proof Let B_k be the Blaschke product over the orbit of c_k

$$B_k(z) = \prod_{\gamma\in\Gamma} \frac{\gamma(z) - c_k}{\gamma(z) - \overline{c_k}} d_\gamma, \quad z \in \mathbb{C}_+,$$

where $|d_\gamma| = 1$ are chosen so that the factors in B_k are positive at z_*. It converges since Γ is of convergent type. We now consider

$$B(z) = \prod_k B_k(z).$$

This product converges due to the assumption (5.1). Moreover, it converges uniformly on compact subsets of $\mathbb{C}_+$.

Our goal is to prove that $\frac{B(z)}{m'(z)}$ is a bounded analytic function on $\mathbb{C}_+$. More precisely, that

$$\left|\frac{B(z)}{m'(z)}\right| \le 1, \quad z \in \mathbb{C}_+. \tag{5.2}$$

It turns out that it is easier to prove even stronger inequality

$$\left|\frac{B(z)}{m'(z)}\right| \sum_{\gamma\in\Gamma} |\gamma'(z)| \left(1 + \frac{1}{|\gamma(z)|^2}\right) \le 1, \quad z \in \mathbb{C}_+. \tag{5.3}$$

Easier because of the automorphic property of the latter function. Recall here that the series in (5.3) converges to a function continuous on $\mathbb{C}_+$, due to the assumption (B) of Theorem 1.8 (equivalently (b_1) of (1.17)). In other words we will prove that

$$\sum_{\gamma\in\Gamma} \left|\frac{B(z)\gamma'(z)}{m'(z)}\right| + \left|\frac{B(z)\gamma'(z)}{m'(z)\gamma^2(z)}\right| \le 1, \quad z \in \mathbb{C}_+. \tag{5.4}$$

Observe that

$$\frac{B(z)\gamma'(z)}{m'(z)} \quad \text{and} \quad \frac{B(z)\gamma'(z)}{m'(z)\gamma^2(z)}$$

are holomorphic on $\mathbb{C}_+$. Therefore, their absolute values are subharmonic functions on $\mathbb{C}_+$. Hence, the sum in (5.4) is a subharmonic function. Also the sum is automorphic with respect to Γ.

We consider first the finitely generated approximation described in Sect. 2. Recall that

$$\Lambda_n : \mathbb{C}_+/\Gamma_n \simeq \Omega_n.$$

Let $c_k^{(n)}$ be the zero of $m_n'(z)$ on the k-th semicircle. Let $B_k^{(n)}$ be the Blaschke product over the orbit of $c_k^{(n)}$ under Γ_n

$$B_k^{(n)}(z) = \prod_{\gamma\in\Gamma_n} \frac{\gamma(z) - c_k^{(n)}}{\gamma(z) - \overline{c_k^{(n)}}} d_\gamma, \quad z \in \mathbb{C}_+,$$

if k-th semicircle is a part of the boundary of $\mathcal{F}_n$, and $B_k^{(n)}(z) = 1$ otherwise. We now consider

$$B^{(n)}(z) = \prod_k B_k^{(n)}(z).$$

We are going to prove this approximative version of (5.4)

$$\mathfrak{f}_n(z) := \sum_{\gamma\in\Gamma_n} \left|\frac{B^{(n)}(z)\gamma'(z)}{m_n'(z)}\right| + \left|\frac{B^{(n)}(z)\gamma'(z)}{m_n'(z)\gamma^2(z)}\right| \le 1, \quad z \in \mathbb{C}_+. \tag{5.5}$$

Advantage of the function in (5.5) over the function in (5.4) is that the series in (5.5) converges in $\mathcal{F}_n$ and also on the boundary of $\mathcal{F}_n$ to a function continuous on $\mathcal{F}_n$ and up to the boundary of $\mathcal{F}_n$ (including infinity), since Γ_n is finitely generated. The same is true for the fundamental domain of Γ_n, which is the union of $\mathcal{F}_n$ and the reflection of $\mathcal{F}_n$ about the 0-th semicircle.

Similar to what we did in Sect. 3, we define a subharmonic function $\mathfrak{F}_n(\lambda)$, $\lambda \in \Omega_n$, by

$$\mathfrak{f}_n(z) = \mathfrak{F}_n(\Lambda_n(z)).$$

The function $\mathfrak{F}_n(\lambda)$ is continuous in $\Omega_n = \mathbb{C} \setminus E_n$ and also up to E_n. By subharmonicity, it attains its maximum on the boundary of Ω_n. Thus, the maximum of $\mathfrak{f}_n(z)$ is attained on the part of the boundary of the fundamental domain that lies on the real axis. Recall that on the boundary of the fundamental domain all the series below converge to continuous functions. Therefore, for real z on the boundary of the fundamental domain of Γ_n we have, by (4.18), (4.17), (4.1) and (4.14), that

$$\frac{1}{|m_n'(z)|}\sum_{\gamma\in\Gamma_n} |B^{(n)}(z)\gamma'(z)| + \left|\frac{B^{(n)}(z)\gamma'(z)}{\gamma^2(z)}\right| = 1.$$

Here we used the fact that $\gamma(z)$ is real for real z and that $\gamma'(z)$ is positive for real z. Hence, (5.5) follows, which is the approximative version of (5.4).

Now we want to pass to the limit in (5.5) for arbitrary fixed $z \in \mathbb{C}_+$ as n goes to infinity. By Lemma 4.5, $m_n'(z)$ converges to $m'(z)$. The sum over Γ_n converges to the sum over Γ. It remains to show that $|B^{(n)}(z)|$ converges to $|B(z)|$. Note that $|B_k^{(n)}(z)| = |g_n(c_k^{(n)}, z)|$ converges to $|g(c_k, z)| = |B_k(z)|$, by Lemmas 2.1 and 4.5. Further, $k \ne 0$,

$$|B_k^{(n)}(z_*)| = |g_n(c_k^{(n)}, z_*)| \ge |g(c_k^{(n)}, z_*)| \ge |g(c_k, z_*)|.$$

By assumption (5.1) the product

$$\prod_{k\ne 0} |g(c_k, z_*)|$$

converges (that is, it is greater than 0). Then, by the Dominated Convergence theorem,[6]

$$\lim_{n\to\infty} |B^{(n)}(z_*)| = \lim_{n\to\infty} \prod_{k\neq 0} |B_k^{(n)}(z_*)| = \prod_{k\neq 0} \lim_{n\to\infty} |B_k^{(n)}(z_*)|$$

$$= \prod_{k\neq 0} |B_k(z_*)| = |B(z_*)|.$$

There exists a subsequence n_j such that $B^{(n_j)}(z)$ converges for all $z \in \mathbb{C}_+$. Let

$$\widetilde{B}(z) = \lim_{j\to\infty} B^{(n_j)}(z).$$

Fix any $z \in \mathbb{C}_+$. Then by Fatou's lemma,[7]

$$|\widetilde{B}(z)| = \lim_{j\to\infty} |B^{(n_j)}(z)| = \lim_{j\to\infty} \prod_k |B_k^{(n_j)}(z)|$$

$$\leq \prod_k \lim_{j\to\infty} |B_k^{(n_j)}(z)| = \prod_k |B_k(z)| = |B(z)|.$$

Thus

$$|\widetilde{B}(z)| \leq |B(z)|, \quad z \in \mathbb{C}_+.$$

Since

$$|\widetilde{B}(z_*)| = |B(z_*)|,$$

the equality must hold

$$|\widetilde{B}(z)| = |B(z)|, \quad z \in \mathbb{C}_+. \tag{5.6}$$

Thus we get (5.3) and, therefore, (5.2). Since (5.6) holds for every subsequential limit $\widetilde{B}(z)$ of $B^{(n)}(z)$, we get

$$B(z) = \lim_{n\to\infty} B^{(n)}(z).$$

□

[6]This case reduces to the standard Dominated Convergence by applying (− log) to the products.
[7]Same explanation as in the previous footnote.

Corollary 5.2 *$m'(z)$ is of bounded characteristic as the ratio of the following two bounded analytic functions*

$$B(z) \quad \textit{and} \quad \frac{B(z)}{m'(z)}.$$

Remark 5.3 Since function $B(z)/m'(z)$ is bounded, it can be written as

$$\frac{B(z)}{m'(z)} = I(z) \cdot O(z),$$

where $I(z)$ is an inner function and O is a bounded outer function. Moreover, $I(z)$ is a singular inner function, since the left hand side does not have zeros in $\mathbb{C}_+$. Therefore,

$$m'(z) = \frac{B(z)}{O(z)I(z)}. \tag{5.7}$$

Theorem 5.4 *Function $B(z)/m'(z)$ is outer. That is, $I(z) = 1$.*

The following facts are used to prove Theorem 5.4.

Lemma 5.5 (Corollary 6.7 of Appendix) *Let $x \in \mathbb{R}$. Then a finite nontangential limits $m(x)$ and $m'(x)$ exist, $m(x)$ is real, if and only if*

$$\sum_{\gamma\in\Gamma} \gamma'(x) + \frac{\gamma'(x)}{\gamma^2(x)} < \infty.$$

In this case

$$m'(x) = \sum_{\gamma\in\Gamma} \gamma'(x)\left(1 + \frac{1}{\gamma^2(x)}\right). \tag{5.8}$$

Hence, in our case (m is a pure point and m' is of bounded characteristic) (5.8) holds almost everywhere on $\mathbb{R}$.

Lemma 5.6 *For every $z \in \mathbb{C}_+$ the following inequality holds*

$$\frac{1}{\pi}\int_{\mathbb{R}} \log \sum_{\gamma\in\Gamma}\left(\gamma'(x) + \frac{\gamma'(x)}{\gamma(x)^2}\right)\frac{\operatorname{Im} z}{|x-z|^2}dx \geq \log \sum_{\gamma\in\Gamma}\left(|\gamma'(z)| + \frac{|\gamma'(z)|}{|\gamma(z)|^2}\right). \tag{5.9}$$

Proof Since

$$\gamma'(z) = \frac{1}{(\gamma^{21}z + \gamma^{22})^2},$$

one can write

$$\sum_{\gamma\in\Gamma} |\gamma'(z)| \left(1 + \frac{1}{|\gamma(z)|^2}\right) = \sum_{\gamma\in\Gamma} \phi_\gamma(z)^* \phi_\gamma(z),$$

where

$$\phi_\gamma(z) = \begin{bmatrix} \dfrac{1}{\gamma^{21}z + \gamma^{22}} \\ \dfrac{1}{\gamma^{21}z + \gamma^{22}} \cdot \dfrac{1}{\gamma(z)} \end{bmatrix}.$$

We consider functions

$$u_n(z) = \sum_{k=1}^{n} |\gamma_k'(z)| \left(1 + \frac{1}{|\gamma_k(z)|^2}\right) = \sum_{k=1}^{n} \phi_{\gamma_k}(z)^* \phi_{\gamma_k}(z), \quad \operatorname{Im} z > 0.$$

From here we see that u_n is a subharmonic function since

$$\frac{\partial^2}{\partial z \partial \overline{z}} u_n(z) = \sum_{k=1}^{n} \phi'_{\gamma_k}(z)^* \phi'_{\gamma_k}(z) \geq 0.$$

Also $\log u_n(z)$ is subharmonic, since

$$\frac{\partial^2}{\partial z \partial \overline{z}} \log u_n(z) = -\frac{1}{u_n^2(z)} \frac{\partial u_n}{\partial z} \frac{\partial u_n}{\partial \overline{z}} + \frac{1}{u_n} \frac{\partial^2 u_n}{\partial z \partial \overline{z}} =$$

$$\frac{1}{u_n^2(z)} \left\{ \sum_{k=1}^{n} \phi_{\gamma_k}(z)^* \phi_{\gamma_k}(z) \sum_{k=1}^{n} \phi'_{\gamma_k}(z)^* \phi'_{\gamma_k}(z) - \sum_{k=1}^{n} \phi_{\gamma_k}(z)^* \phi'_{\gamma_k}(z) \sum_{k=1}^{n} \phi'_{\gamma_k}(z)^* \phi_{\gamma_k}(z) \right\},$$

which is nonnegative by Cauchy-Schwarz inequality. Therefore,

$$\frac{1}{\pi} \int_{\mathbb{R}} \log \sum_{k=1}^{n} \left({\gamma_k}'(x) + \frac{{\gamma_k}'(x)}{\gamma_k(x)^2} \right) \frac{\operatorname{Im} z}{|x - z|^2} dx \geq \log \sum_{k=1}^{n} \left(|{\gamma_k}'(z)| + \frac{|{\gamma_k}'(z)|}{|\gamma_k(z)|^2} \right).$$

We now pass to the limit in this inequality. Since all integrands are nonnegative, the Monotone Convergence Theorem applies and we get (5.9). □

Proof **(Theorem 5.4)** By (4.18), for $z \in \mathbb{C}_+$ we have

$$|m'(z)| \leq \sum_{\gamma\in\Gamma} |\gamma'(z)| + \frac{|\gamma'(z)|}{|\gamma(z)|^2}. \tag{5.10}$$

Now, by Lemmas 5.5 and 5.6,

$$\frac{1}{\pi}\int_{\mathbb{R}} \log m'(x)\frac{\operatorname{Im} z}{|x-z|^2}dx = \frac{1}{\pi}\int_{\mathbb{R}} \log \sum_{\gamma\in\Gamma}\left(\gamma'(x)+\frac{\gamma'(x)}{\gamma(x)^2}\right)\frac{\operatorname{Im} z}{|x-z|^2}dx$$

$$\geq \log \sum_{\gamma\in\Gamma}\left(|\gamma'(z)|+\frac{|\gamma'(z)|}{|\gamma(z)|^2}\right).$$

On the other hand (see (5.7))

$$\frac{1}{\pi}\int_{\mathbb{R}} \log m'(x)\frac{\operatorname{Im} z}{|x-z|^2}dx = -\frac{1}{\pi}\int_{\mathbb{R}} \log|O(x)|\frac{\operatorname{Im} z}{|x-z|^2}dx = -\log|O(z)|,$$

since O is a bounded outer function. Thus,

$$\sum_{\gamma\in\Gamma}|\gamma'(z)|+\frac{|\gamma'(z)|}{|\gamma(z)|^2}\leq\frac{1}{|O(z)|},\quad z\in\mathbb{C}_+. \tag{5.11}$$

Combining (5.11) with (5.10) and (5.7), we get

$$\left|\frac{B(z)}{O(z)I(z)}\right| = |m'(z)|\leq\frac{1}{|O(z)|},\quad z\in\mathbb{C}_+.$$

That is,

$$\left|\frac{B(z)}{I(z)}\right|\leq 1.$$

The latter implies that $I(z)=1$. □

5.3 *Proof of the Implication* (*i*) ⇒ (*iii*)

Let $\mathcal{H}^2(\alpha)$ be non trivial for all $\alpha\in\Gamma^*$. Then $H^2(\alpha)$ is non trivial for all α, that is, the Widom condition holds

$$\sum_{\mu:\nabla G(\mu,\lambda_*)=0} G(\mu,\lambda_*)<\infty.$$

This in turn implies (A) of condition (iii) in Theorem 1.8, since if μ_{λ_*} is the critical point of $G(\lambda,\lambda_*)$ in the k-th gap and μ is any point in this gap, then

$$G(\mu,\lambda_*)\leq G(\mu_{\lambda_*},\lambda_*).$$

Now, since the Widom condition holds, Γ acts on $\mathbb{R}$ dissipatively, that is, there exists a measurable fundamental set $\mathbb{E} \subset \mathbb{R}$. Let f be a non trivial function from $\mathcal{H}^2(\alpha)$. Then

$$\int_{\mathbb{R}} |f(x)|^2 dx = \sum_{\gamma \in \Gamma} \int_{\mathbb{E}} |f(x)|^2 \gamma'(x) dx = \int_{\mathbb{E}} \sum_{\gamma \in \Gamma} |f(x)|^2 \gamma'(x) dx,$$

the latter equality is due to Fubini's theorem. Therefore,

$$|f(x)|^2 \sum_{\gamma \in \Gamma} \gamma'(x) < \infty$$

almost everywhere on $\mathbb{E}$. Since $f \neq 0$ (almost everywhere) we have

$$\sum_{\gamma \in \Gamma} \gamma'(x) < \infty \quad \text{for a.e. } x \in \mathbb{E}.$$

We fix one such x, then for $z_0 = x + i$ we obtain

$$\sum_{\gamma \in \Gamma} |\gamma'(z_0)| < \infty.$$

By the Harnack inequality we have (4.16). By Proposition 4.4, (b) holds and it is equivalent to condition (B) of Theorem 1.8.

5.4 Proof of the Implication $(ii) \Rightarrow (i)$

Lemma 5.7 *If (a) and (b) hold, then $H^2(\alpha)$ are non trivial for all $\alpha \in \Gamma^*$.*

Proof By Lemma 5.5, we have that under assumptions (a) and (b)

$$m'(x) = \sum_{\gamma \in \Gamma} \gamma'(x) \left(1 + \frac{1}{\gamma^2(x)}\right)$$

almost everywhere on $\mathbb{R}$ and that

$$0 < \int_{\mathbb{R}} \frac{\log m'(x)}{1 + x^2} dx < \infty.$$

Then

$$\rho(x) = \sum_{\gamma \in \Gamma} \gamma'(x).$$

also converges almost everywhere on $\mathbb{R}$ and

$$0 < \int\limits_{\mathbb{R}} \frac{\log \rho(x)}{1+x^2} dx < \infty. \tag{5.12}$$

Consider the following function on $\mathbb{R}$ (compare to (2.3), (2.5), also to (3.7))

$$\rho_i(x) = \sum_{\gamma \in \Gamma} \frac{\gamma'(x)}{1+\gamma(x)^2}.$$

Since

$$\frac{1}{1+x^2} \le \rho_i(x) \le \rho(x),$$

we have

$$-\infty < \int \frac{\log \rho_i(x) dx}{1+x^2} < \infty. \tag{5.13}$$

Combining inequality (3.13) of Lemma 3.5 with inequality (3.14), we conclude that $\log^+ |g'(z,i)|$ has a harmonic majorant in the upper half plane. This means that $g'(z,i)$ is of bounded type on the upper half plane. Therefore, by Theorem 1.5, all $H^2(\alpha)$ are non-trivial. □

Inequalities (5.12) and (5.13) allow to define an outer function $\phi(z)$ by

$$|\phi(x)|^2 = \frac{\rho_i(x)}{\rho(x)} \le 1. \tag{5.14}$$

We denote by α_ϕ the character associated to this function.

Proposition 5.8 *If (a) and (b) hold, then* $\mathcal{H}^2(\alpha) = \phi H^2(\alpha_\phi^{-1}\alpha)$.

Proof We first show that $\mathcal{H}^2(\alpha) \subseteq \phi H^2(\alpha_\phi^{-1}\alpha)$. If $f \in \mathcal{H}^2(\alpha)$, then $h = f/\phi$ is of Smirnov class. Recall that $\mathbb{E}$ is the fundamental measurable set for the action of Γ on $\mathbb{R}$. In view of (5.14), we have

$$\int_{\mathbb{R}} |h|^2 \frac{dx}{1+x^2} = \int_{\mathbb{E}} |h(x)|^2 \rho_i(x) dx = \int_{\mathbb{E}} |f(x)|^2 \rho(x) dx = \int_{\mathbb{R}} |f|^2 dx.$$

Then, by the Smirnov maximum principle, $h \in H^2$, and, therefore, $h \in H^2(\alpha_\phi^{-1}\alpha)$. The converse inclusion is proved the same way. □

Corollary 5.9 *If (a) and (b) hold, then* $\mathcal{H}^2(\alpha)$ *are non trivial for all* $\alpha \in \Gamma^*$.

Proof This is a straightforward combination of Lemma 5.7 and Proposition 5.8. □

Remark 5.10 We point out that an analogue of the space $\mathcal{H}^2(\alpha)$ still can be defined for Widom domains (see the definition below) even if condition (b) is violated. Indeed, the density outer function (compare to (5.14))

$$|\Phi(\lambda)|^2 = \frac{\theta'_{\lambda_*}(\lambda)}{\theta'(\lambda)}, \quad \lambda \in E,$$

where θ_{λ_*} and θ are defined in (1.7) and (1.15), is always well defined in Widom domains due to Theorem D [16]. This suggests the following

Definition 5.11 Let Ω be of Widom type. Let $\pi_1(\Omega) \simeq \Gamma$ be the fundamental group of this domain. For a character $\alpha \in \pi_1(\Omega)^*$ we say that a function F belongs to $\mathcal{H}^2_\Omega(\alpha)$ if it is a character-automorphic multivalued function in the domain, i.e.,

$$F(\tilde{\gamma}(\lambda)) = \alpha(\tilde{\gamma})F(\lambda), \quad \tilde{\gamma} \in \pi_1(\Omega),$$

and $|F(\lambda)/\Phi(\lambda)|^2$ possesses a harmonic majorant in Ω.

6 Appendix: Carathéodory and Frostman Theorems

Theorems of Carathéodory and Frostmant that are used in the proofs of Pommerenke theorem (Theorem 3.1) and in the most important part (Theorem 5.1) of our main theorem depend on the following theorem due to Carathéodory and Julia [5], for a modern exposition see, e.g., [3] and further references there.

Theorem 6.1 (Carathéodory–Julia [5]) *Let function w be analytic in the unit disk and bounded in modulus by* 1. *Let t_0 be a point on the unit circle. The following are equivalent:*

(1) $d_1 := \liminf_{z\to t_0} \dfrac{1-|w(z)|^2}{1-|z|^2} < \infty \quad (|z|<1, z \text{ approaches } t_0 \text{ in an arbitrary way});$

(2) $d_2 := \lim_{z\to t_0} \dfrac{1-|w(z)|^2}{1-|z|^2} < \infty \quad (z \text{ approaches } t_0 \text{ nontangentially});$

(3) *Finite nontangential limits*

$$w(t_0) := \lim_{z\to t_0} w(z) \ \textit{and} \ d_3 := \lim_{z\to t_0} \frac{1-w(z)\overline{w(t_0)}}{1-z\bar{t}_0}$$

exist, $|w(t_0)| = 1$.

(4) *Finite nontangential limits*

$$w(t_0) := \lim_{z\to t_0} w(z) \ \ \textit{and} \ \ w'(t_0) = \lim_{z\to t_0} \frac{w(z)-w(t_0)}{z-t_0}$$

exist, $|w(t_0)| = 1$. $w'(t_0)$ *is called the angular derivative at* t_0.

(5) *Finite nontangential limits*

$$w(t_0) := \lim_{z\to t_0} w(z) \ \ \textit{and} \ \ w_0' := \lim_{z\to t_0} w'(z)$$

exist, $|w(t_0)| = 1$.

(6) *There exist a constant* w_0, $|w_0| = 1$ *and a constant* $d \geq 0$ *such that the boundary Schwarz-Pick inequality holds*

$$\left|\frac{w(z)-w_0}{z-t_0}\right|^2 \leq d \cdot \frac{1-|w(z)|^2}{1-|z|^2}, \quad |z| < 1; \tag{6.1}$$

inequality (6.1) *implies that the following nontangential limit*

$$w(t_0) := \lim_{z\to t_0} w(z) \ \ \textit{exists and} \ \ w(t_0) = w_0;$$

we denote the smallest constant d *that works for* (6.1) *by* d_4.

When these conditions hold, we have $w_0' = w'(t_0)$ *and*

$$d_1 = d_2 = d_3 = d_4 = t_0 \frac{w'(t_0)}{w(t_0)} = |w'(t_0)|.$$

This number is equal to 0 *if and only if* w *is a unimodular constant.*

Theorem 6.2 (Carathéodory [5]) *Let* w, w_n *be analytic functions bounded in modulus by* 1 *on the unit disk. Assume that* $w_n(z)$ *converges to* $w(z)$ *for every* $|z| < 1$. *Let* $|t_0| = 1$. *Assume that nontangential boundary values* $w_n(t_0)$, $w_n'(t_0)$ *exist and that* $|w_n(t_0)| = 1$, $w_n'(t_0)$ *are finite. We assume that*

$$\underline{\lim}\, |w_n'(t_0)| < \infty.$$

Then the nontangential boundary values $w(t_0)$, $w'(t_0)$ *exist*, $|w(t_0)| = 1$ *and*

$$|w'(t_0)| \leq \underline{\lim}\, |w_n'(t_0)|. \tag{6.2}$$

Proof Let $|w_{n_k}'(t_0)|$ converge to $\underline{\lim}\, |w_n'(t_0)|$. By (6.1), we have

$$\left|\frac{w_{n_k}(z)-w_{n_k}(t_0)}{z-t_0}\right|^2 \leq |w_{n_k}'(t_0)| \cdot \frac{1-|w_{n_k}(z)|^2}{1-|z|^2}, \quad |z| < 1.$$

$w_{n_k}(t_0)$ is a sequence of complex numbers of modulus one. Therefore, there exists a convergent subsequence $w_{n_{k_j}}(t_0)$. We denote the limit by w_0, $|w_0| = 1$. Since $w_{n_{k_j}}(z)$ converge to $w(z)$ for every $|z| < 1$, we get (by passing to the limit as $j \to \infty$)

$$\left|\frac{w(z) - w_0}{z - t_0}\right|^2 \le \underline{\lim}\, |w_n'(t_0)| \cdot \frac{1 - |w(z)|^2}{1 - |z|^2}, \quad |z| < 1.$$

From here we see that $w_0 = w(t_0)$ and we get

$$\left|\frac{w(z) - w(t_0)}{z - t_0}\right|^2 \le \underline{\lim}\, |w_n'(t_0)| \cdot \frac{1 - |w(z)|^2}{1 - |z|^2}, \quad |z| < 1.$$

By Theorem 6.1, the latter inequality implies that $w'(t_0)$ exists and that it is finite. Since the smallest constant that works for this inequality is $|w'(t_0)|$, (6.2) follows. □

Theorem 6.3 (Frostman [7]) *In addition to assumptions of Theorem 6.2, assume that* $|w_n(z)| \ge |w(z)|$ *for every* z, $|z| < 1$. *Then*

$$w(t_0) = \lim w_n(t_0) \quad \textit{and} \quad w'(t_0) = \lim w_n'(t_0).$$

Proof By assumption,

$$\frac{1 - |w_n(z)|^2}{1 - |z|^2} \le \frac{1 - |w(z)|^2}{1 - |z|^2}.$$

Therefore,

$$\lim_{z \to t_0} \frac{1 - |w_n(z)|^2}{1 - |z|^2} \le \lim_{z \to t_0} \frac{1 - |w(z)|^2}{1 - |z|^2}.$$

That is, in view of Theorem 6.1,

$$|w_n'(t_0)| \le |w'(t_0)|.$$

Hence, we get

$$\overline{\lim} |w_n'(t_0)| \le |w'(t_0)|.$$

Combining this with Theorem 6.2, we get

$$|w'(t_0)| = \lim |w_n'(t_0)|.$$

The first assertion of the theorem follows from the observation that now one does not need to choose a subsequence at the beginning of the proof of Theorem 6.2. This implies that every subsequential limit of $w_n(t_0)$ is $w(t_0)$. After that, the second assertion is a consequence of the relation

$$|w'(t_0)| = t_0 \frac{w'(t_0)}{w(t_0)}.$$

□

By a simple substitution

$$z := \frac{z-i}{z+i},$$

that maps upper half plane onto the unit disk, Theorems 6.1 and 6.3 can be restated for functions on the upper half plane.

Theorem 6.4 (Carathéodory–Julia) *Let w be analytic on the upper half plane and bounded in modulus by* 1. *Let $x \in \mathbb{R}$ be a point on the real axis. The following are equivalent:*

(1) $d_1 := \liminf_{z \to x} \dfrac{1-|w(z)|^2}{2\operatorname{Im} z} < \infty \quad (\operatorname{Im} z > 0,\ z \textit{ approaches } x \textit{ in an arbitrary way});$

(2) $d_2 := \lim_{z \to x} \dfrac{1-|w(z)|^2}{2\operatorname{Im} z} < \infty \quad (\operatorname{Im} z > 0,\ z \textit{ approaches } x \textit{ nontangentially});$

(3) *Finite nontangential limits*

$$w(x) := \lim_{z\to x} w(z) \ \textit{ and } \ d_3 := \lim_{z\to x} \frac{1 - w(z)\overline{w(x)}}{i(x-z)}$$

exist, $|w(x)| = 1$.

(4) *Finite nontangential limits*

$$w(x) := \lim_{z\to x} w(z) \ \textit{ and } \ w'(x) = \lim_{z\to x} \frac{w(z)-w(x)}{z-x}$$

exist, $|w(x)| = 1$. $w'(x)$ *is called the angular derivative at* x.

(5) *Finite nontangential limits*

$$w(x) := \lim_{z\to x} w(z) \ \textit{ and } \ w'_0 := \lim_{z\to x} w'(z)$$

exist, $|w(x)| = 1$.

(6) *There exist a constant* w_0, $|w_0| = 1$ *and a constant* $d \geq 0$ *such that the boundary Schwarz-Pick inequality holds*

$$\left|\frac{w(z) - w_0}{z - x}\right|^2 \leq d \cdot \frac{1 - |w(z)|^2}{2\operatorname{Im} z}, \quad \operatorname{Im} z > 0; \tag{6.3}$$

inequality (6.3) *implies that the following nontangential limit*

$$w(x) := \lim_{z \to x} w(z) \ \text{ exists and } \ w(x) = w_0;$$

we denote the smallest constant that works for (6.3) *by* d_4.

When these conditions hold, we have $w_0' = w'(x)$ *and*

$$|w'(x)| = \frac{1}{i}\frac{w'(x)}{w(x)} = d_1 = d_2 = d_3 = d_4.$$

The next theorem is a version of Theorem 6.3 for the upper half plane.

Theorem 6.5 (Frostman [7]) *Let* w, w_n *be analytic functions on the upper half plane bounded in modulus by* 1. *Assume that* w_n *converge to* w *for every* $z \in \mathbb{C}_+$ *and that* $|w_n(z)| \geq |w(z)|$ *for every* $z \in \mathbb{C}_+$. *Let* $x \in \mathbb{R}$. *Let* $w_n(x)$, *and* $w_n'(x)$ *be the nontangential boundary values,* $|w_n(x)| = 1$, $w_n'(x)$ *is finite. Assume that*

$$\underline{\lim}\, |w_n'(x)| < \infty.$$

Then nontangential boundary values $w(x)$ *and* $w'(x)$ *exist,* $|w(x)| = 1$, $w'(x)$ *is finite and*

$$w(x) = \lim w_n(x), \qquad w'(x) = \lim w_n'(x).$$

Corollary 6.6 (Frostman [7]) *Let* w *be a Blaschke product on the upper half plane*

$$w(z) = \prod_k B_k(z).$$

Let $x \in \mathbb{R}$. *Then* $w(x)$ *and* $w'(x)$ *exist with* $|w(x)| = 1$, $w'(x)$ *finite if and only if*

$$\sum_k |B_k'(x)| < \infty.$$

In this case

$$|w'(x)| = \sum_k |B_k'(x)|.$$

Proof Let w_n be a finite Blaschke product

$$w_n(z) = \prod_{k=1}^{n} B_k(z).$$

Then

$$\frac{w_n'(z)}{w_n(z)} = \sum_{k=1}^{n} \frac{B_k'(z)}{B_k(z)}.$$

Observe that (compare to Theorem 6.4)

$$|B_k(x)| = 1, \text{ and } \frac{1}{i} \cdot \frac{B_k'(x)}{B_k(x)} = |B_k'(x)|.$$

Same is true for w_n at point x. Therefore, we get

$$|w_n'(x)| = \sum_{k=1}^{n} |B_k'(x)|.$$

Since w_n is a divisor of w the following inequality holds

$$|w_n(z)| \geq |w(z)|$$

for every $z \in \mathbb{C}_+$. Also $w_n(z)$ converge to $w(z)$ for every $z \in \mathbb{C}_+$. If $|w(x)| = 1$, $w'(x)$ exists and it is finite, then (like in Theorem 6.3)

$$|w_n'(x)| \leq |w'(x)|.$$

Therefore, for every n

$$\infty > |w'(x)| \geq |w_n'(x)| = \sum_{k=1}^{n} |B_k'(x)|$$

and

$$\sum_{k=1}^{\infty} |B_k'(x)| < \infty.$$

Conversely, if the latter sum converges, then $|w_n'(x)|$ are bounded and we are in the situation of Theorem 6.5. □

Corollary 6.7 *Let $m(z)$ be the symmetric Martin function with a pure point measure, defined as in* (4.17)

$$m(z) = \sum_{\gamma\in\Gamma}\left(\gamma(z) - \frac{1}{\gamma(z)}\right) - \mathrm{Re}\left(\gamma(i) - \frac{1}{\gamma(i)}\right).$$

Recall that by (4.18)

$$m'(z) = \sum_{\gamma\in\Gamma}\gamma'(z) + \frac{\gamma'(z)}{\gamma^2(z)}$$

Let $x \in \mathbb{R}$. Then a finite nontangential limits $m(x)$ and $m'(x)$ exist, $m(x)$ is real, if and only if

$$\sum_{\gamma\in\Gamma}\gamma'(x) + \frac{\gamma'(x)}{\gamma^2(x)} < \infty.$$

In this case

$$m'(x) = \sum_{\gamma\in\Gamma}\gamma'(x) + \frac{\gamma'(x)}{\gamma^2(x)}.$$

Proof Consider the following inner function

$$w(z) = e^{im(z)}.$$

Observe that

$$\frac{w'(z)}{w(z)} = im'(z).$$

Consider

$$m_n(z) = \sum_{k=1}^{n}\left(\gamma_k(z) - \frac{1}{\gamma_k(z)}\right) - \mathrm{Re}\left(\gamma_k(i) - \frac{1}{\gamma_k(i)}\right)$$

and the corresponding inner function

$$w_n(z) = e^{im_n(z)}.$$

Then

$$\frac{w_n'(z)}{w_n(z)} = im_n'(z).$$

In view of formula (4.19), $\operatorname{Im} m_n(z)$ increases in n for every $\operatorname{Im} z > 0$. Therefore, $|w_n(z)|$ decreases in n. If finite nontangential limits $m(x)$ and $m'(x)$ exist, $m(x)$ is real, then finite nontangential limits $w(x)$ and $w'(x)$ exist, $|w(x)| = 1$. Therefore, we are in the situation of Theorem 6.5. Hence,

$$m'(x) = \frac{1}{i}\frac{w'(x)}{w(x)} = |w'(x)| = \lim |w_n'(x)| = \lim \frac{1}{i}\frac{w_n'(x)}{w_n(x)}$$

$$= \lim m_n'(x) = \lim \sum_{k=1}^{n} \gamma_k'(x) + \frac{\gamma_k'(x)}{\gamma_k^2(x)} = \sum_{\gamma\in\Gamma} \gamma'(x) + \frac{\gamma'(x)}{\gamma^2(x)}.$$

Conversely, if

$$\sum_{\gamma\in\Gamma} \gamma'(x) + \frac{\gamma'(x)}{\gamma^2(x)} < \infty,$$

then $|w_n'(x)| = m_n'(x)$ are bounded and we are again in the situation of Theorem 6.5.
□

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References

1. D. Alpay, V. Vinnikov, *Indefinite Hardy spaces on finite bordered Riemann surfaces*, J. Funct. Anal. 172 (2000), no. 1, 221–248.
2. J. Ball, V. Vinnikov, *Zero-pole interpolation for matrix meromorphic functions on a compact Riemann surface and a matrix Fay trisecant identity*, Amer. J. Math. 121 (1999), no. 4, 841–888.
3. V. Bolotnikov, A. Kheifets, *A higher order analogue of the Caratheodory-Julia theorem*, Journal of Functional Analysis, 237, no. 1 (2006), 350–371
4. A. Borichev, M. Sodin, *Krein's entire functions and Bernstein approximation problem*, Illinois Journal of Mathematics, 45, no. 1 (2001), 167–185.
5. C. Carathéodory, *Theory of Functions of a Complex Variable*, Engl. Translation, Chelsea Publishing Company, NY, 1960.
6. A. Eremenko, P. Yuditskii, *Comb functions.* Recent advances in orthogonal polynomials, special functions, and their applications, Contemp. Math., **578** (2012), 99–118.
7. O. Frostman, *Sur les produits de Blaschke*, Kungl. Fysiografiska Sälskapets I Lund Förhandlingar, 12, no.15 (1943), 169–182.
8. M. Hasumi, *Hardy Classes on Infinitely Connected Riemann Surfaces*, LNM **1027**, Springer, New York, Berlin, 1983.
9. P. Koosis, *The logarithmic integral. I*, Cambridge Studies in Advanced Mathematics, vol. 12, Cambridge University Press, Cambridge, 1998, Corrected reprint of the 1988 original.
10. N. K. Nikolskii, *Treatise on the shift operator*, A Series of Comprehensive Studies in Mathematics, Springer-Verlag, Berlin Heidelberg New York Tokyo, 1986.

11. B. Ya. Levin, *Majorants in classes of subharmonic functions, II, The relation between majorants and conformal mapping, III, The classification of the closed sets on* $\mathbb{R}$ *and the representation of the majorants*, Teor. Funktsii Funktsional. Anal. i Prilozhen. 52 (1989), 3–33; English translation: J. Soviet Math. 52 (1990), 3351–3372.
12. M. S. Livšic, N. Kravitsky, A. S. Markus, V. Vinnikov, *Theory of Commuting Nonselfadjoint Operators*, Mathematics and Its Applications, 332, Kluwer Academic, Dordrecht (1995)
13. V. Marchenko, *Sturm-Liouville operators and applications*, Birkhäuser Verlag, Basel, 1986.
14. B. S. Pavlov, S. I. Fedorov, *Shift group and harmonic analysis on a Riemann surface of genus one*, Algebra i Analiz, 1:2 (1989), 132–168; Leningrad Math. J., 1:2 (1990), 447–490
15. Ch. Pommerenke, *On the Green's function of Fuchsian groups,* Ann. Acad. Sci. Fenn., 2 (1976), 409–427.
16. M. Sodin and P. Yuditskii, *Almost periodic Jacobi matrices with homogeneous spectrum, infinite-dimensional Jacobi inversion, and Hardy spaces of character-automorphic functions*, J. Geom. Anal. **7** (1997), 387–435.
17. A. Volberg and P. Yuditskii, *On the inverse scattering problem for Jacobi matrices with the spectrum on an interval, a finite system of Intervals or a Cantor set of positive length*, Communications in Mathematical Physics, **226** (2002), 567–605.
18. A. Volberg and P. Yuditskii, *Kotani-Last problem and Hardy spaces on surfaces of Widom type*. Invent. Math., 197 (2014), No. 3, 683–740.
19. A. Volberg and P. Yuditskii, *Mean type of functions of bounded characteristic and Martin functions in Denjoy domains*, Adv. in Math., 290 (2016), 860–887.
20. H. Widom, *Extremal polynomials associated with a system of curves in the complex plane*, Adv. in Math., **3** (1969), 127–232.
21. H. Widom, H^p *sections of vector bundles over Riemann surfaces*, Ann. of Math., **94** (1971), 304–324.
22. J. You, *Quantitative almost reducibility and its applications*, Proc. Int. Cong. of Math. - 2018, Rio de Janeiro, Vol. 2, 2107–2128.