Syllabus

I. Course Name: Introduction to Inverse Spectral Problems

II. Course description and objective

In science, one calls two problems inverse to each other if the formulation of each of them requires full or partial knowledge of the other. Usually, the *direct problem* is that has been studied earlier and perhaps in more detail. This kind of problems usually deals with known systems under the influence of some external excitation, and the problem consists in being able to predict the reaction of the system on a particular excitation. Respectively, *inverse problems* deals with the determination (reconstruction) of a system (characterized by unknown parameters) using the results of measurements of system's reaction on external excitations of a particular type. Mathematically, direct problems deals with seeking solutions of systems of equations (algebraic, differential, functional, etc.) while inverse problems consist in determining equations (e.g., determining unknown coefficients of the differential equation of a particular type) using certain information about solutions of the equation. In the proposed course, we are going to study inverse problems related to a particular infinite dimensional system generated by the Sturm-Liouville differential equation:

$$-u''(x)+q(x)u(x)=\lambda u(x)$$

which is considered on a finite interval, and the solutions are subject to certain conditions at the end point of the interval. First, we study the direct problem, when $q(x)$ is assumed to be known, and we are interested in the eigenvalues (the admissible values of λ) and the eigenfunctions, which are the corresponding solutions of this differential equation satisfying certain boundary conditions). A particular attention will be paid to the properties of the asymptotic distribution of the eigenvalues. Then, basing on the results obtained in the framework of the direct problem, we will study the inverse problem: what spectral information (i.e., the information about the eigenvalues and eigenfunctions) should be known in order that (apriopri unknown) $q(x)$ can be uniquely determined.

III. Elective

IV. Bachelor Program, 8th Term, 64 Hours, 4 Credits

V. Course content

Chapter 1. Direct problems for Sturm-Liouville operators.

Section 1. Differential Sturm-Liouville operator: fundamental system of solutions.

General form of differential operators of the second order. Boundary conditions for problems on a finite interval.

Section 2. The fundamental system.

Initial value problems for the Sturm-Liouville equation. The fundamental system of solutions.

Section 3. Estimates for the fundamental system.

Neumann series. Estimates for integral equations.

Section 4. Frechet derivatives of solutions of the fundamental system.

Frechet derivatives in Banach spaces. The Frechet derivatives of the eigenfunctions of the Sturm-Liouville operator with respect to the spectral parameter.

Section 5. Asymptotics of eigenvalues end eigenfunctions: primary estimates.

The characteristic function of the spectral problem. Application of Rouche's Theorem to the distribution of zeros of the characteristic functions. Rough estimates for the eigenvalue distribution.

Section 6. Asymptotics of eigenvalues end eigenfunctions: refining the estimates.

The Frechet derivative of the eigenvalues with respect to the potential. Refining the asymptotic formula for the Dirichlet problem.

Chapter 2. Inverse problems for Sturm-Liouville operators.

Section 7. Boundary value problems for hyperbolic partial differential equations.

Goursat problem. Cauchy problem. Reducing to Volterra integral equations.

Section 8. Boundary value problems of mixed type.

Reducing to Volterra integral equations.

Section 9. Transformation operators.

Gelfand-Levitan-Marchenko integral operator. Reducing to the Goursat problem.

Section 10. Completeness of the eigenfunctions of the Sturm-Liouville operators.

Reducing to equations with compact operators. Fredholm alternative.

Section 11. Inverse problems.

Inverse problem by two spectra. Inverse problem by one spectrum and norming constants

Section 12. “Half-inverse problems”.

Reconstruction of a symmetric potential by one spectrum.

VI. Pre-taken courses

Mathematical Analysis, Complex Analysis, Ordinary Differential Equations

VII. Form of the final test: examination (four-level evaluation scale)/test (two-level evaluation scale)

VIII. Teaching materials and reference books

1. Kirsch A. An Introduction to the Mathematical Theory of Inverse Problems, Springer-Verlag, New York, 1996.

2. Poschel J., Trubowitz E., Inverse Spectral Theory, Academic Press, London, 1987.

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