Syllabus

I. Course Name: Linear Algebra

II. Course description and objective

This 2-term course covers the following topics: determinants, vector spaces, matrix operations, systems of linear equations, linear operators in vector spaces, Euclidean spaces, linear operators in Euclidean spaces, polynomials in several variables over a field, quadratic forms, and Jordan normal form of linear operators.

III. Compulsory

IV. Bachelor Program, 2nd and 3rd Terms, 360 Hours, 12 Credits

V. Course content

Chapter 1. Determinants

1. Determinants: definition and examples.
2. Properties of determinants.
3. The Laplace expansion. Expansion along a row or column.
4. Determinants of block diagonal and other special matrices.
5. The Vandermonde determinant.

Chapter 2. Vector spaces

1. Definition and properties of vector spaces.
2. Linear independence (linear dependence) of vectors.
3. Maximal linearly independent systems and Gauss theorem.
4. Basis and dimension of a vector space.
5. Subspaces of a vector space. Linear span of a set of vectors, its dimension.
6. Completion of a linear independent set of vectors to a basis of the space.
7. Sum and intersection of linear subspaces. The Grassmann formula.
8. Direct sum of subspaces. A complement to a subspace, direct complement.

Chapter 3. Matrix operations.

1. Multiplication of matrices, its properties.
2. Determinant of a product of square matrices.
3. Inverse of a matrix.

4. Column, row and minor ranks of a matrix. Matrix rank theorem.
5. Coordinates of a vector in a basis. Change of basis and transformation matrix.

Chapter 4. Systems of linear equations.

1. Systems of linear equations. Rouché-Capelli theorem.
2. Cramer's rule.
3. General theory of systems of linear equations.
4. Homogeneous systems of linear equations. Connection between solutions of general system and corresponding homogeneous system of linear equations.

Chapter 5. Linear operators in vector spaces.

1. Linear operators: definition and examples.
2. Image and kernel of a linear operator.
3. Matrix of a linear operator in a basis.
4. Matrix of a linear operator transformation under change of basis.
5. Eigenvalues and eigenvectors of a linear operator.
6. Invariance of the characteristic polynomial of a linear operator under change of basis. Trace and determinant of a linear operator.
7. Linear independence of eigenvectors corresponding to different eigenvalues.
8. Diagonalizable linear operators. Algebraic and geometric multiplicities of an eigenvalue.

Chapter 6. Euclidean spaces.

1. Scalar product: definition and properties.
2. Cauchy–Schwarz inequality.
3. Vector norm: definition and properties. Vector norm induced by a scalar product.
4. Gram–Schmidt process.
5. Orthonormal bases of Euclidean spaces.
6. Orthogonal complement of a subspace. Euclidean space as the direct sum of any subspace and its orthogonal complement

Chapter 7. Linear operators in Euclidean spaces.

1. Adjoint of a linear operator: definition and properties.
2. Connection between matrices of a linear operator and its adjoint operator in an orthonormal basis.
3. Self-adjoint linear operators in Euclidean spaces.
4. Spectral theorem for self-adjoint linear operators in complex Euclidean spaces.
5. Spectral theorem for self-adjoint linear operators in real Euclidean spaces.
6. Unitary linear operators in Euclidean spaces: equivalent definitions.
7. Existence of a common eigenvector for commute linear operators in a complex linear space.
8. Spectral theorem for unitary linear operators in complex Euclidean spaces.
9. Unitary linear operators in a 2-dimensional real Euclidean space.
10. Spectral theorem for unitary linear operators in real Euclidean spaces.
11. Normal linear operators in complex Euclidean spaces.

Chapter 8. Polynomials in several variables over a field.

1. Multivariate polynomial ring over a field: definition and properties.
2. Lexicographic order. The leading term of a polynomial in the lexicographic order.
3. The leading term of the product of polynomials in the lexicographic order.
4. Symmetric polynomials. Elementary symmetric polynomials
5. Fundamental theorem on symmetric polynomials.
6. Vieta's formulas.
7. Resultant and discriminant.

Chapter 9. Quadratic forms.

1. Quadratic forms definition and properties.
2. Diagonalization of a quadratic form by the Lagrange method.
3. Decomposable quadratic forms.
4. Sylvester's law of inertia for quadratic forms.
5. Unitary Diagonalization of Quadratic Form.
6. Positive-definite quadratic forms. Sylvester's criterion.
7. Simultaneous diagonalization of arbitrary and positive-definite quadratic forms.

Chapter 10. Jordan normal form of linear operators.

1. Polynomials in a linear operator. Annulling polynomials.
2. The Cayley–Hamilton theorem.
3. The minimal polynomial of a linear operator. Its connection to the characteristic polynomial.
4. Root subspaces of a linear operator.
5. Jordan normal form of a linear operator.
6. Connection between a minimal polynomial of a linear operator and its Jordan normal form.
7. The spectral mapping theorem.
8. Functions of a linear operator argument.

VI. Pre-taken courses

Elements of Algebra and Number Theory

VII. Form of the final test: examination (four-level evaluation scale)

VIII. Teaching materials and reference books

1. I. M. Gelfand, Lectures on Linear Algebra, Dover Publications Ink. New York, 1989.
2. G.E. Shilov, Linear algebra, Dover Publications Ink. New York, 1977.
3. Jim Hefferon, Linear Algebra, the open-source textbook http://joshua.smcvt.edu/linearalgebra/#current_version.
4. Peter J. Cameron, Notes on Linear Algebra, the open-source textbook http://www.maths.qmul.ac.uk/~pjc/notes/linalg.pdf

Written by: Anna Vishnyakova